Independence Distribution Preserving Covariance Structures for the Multivariate Linear Model

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Consider the multivariate linear model for the random matrix $Y_{n \times p} \sim MN(XB, V \otimes \Sigma)$, where B is the parameter matrix, X is a model matrix, not necessarily of full rank, and $V \otimes \Sigma$ is an $np \times np$ positive-definite dispersion matrix. This paper presents sufficient conditions on the positive-definite matrix V such that the statistics for testing $H_0 : CB = 0$ vs $H_a : CB \neq 0$ have the same distribution as under the i.i.d. covariance structure $I \otimes \Sigma$. © 1999 Academic Press

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1. INTRODUCTION

This paper is concerned with testing linear hypotheses about regression coefficients in the multivariate linear model. For tests of such hypotheses, one typically assumes multivariate normality of the error terms with an i.i.d. covariance structure. We are interested in the extent to which the i.i.d. assumption can be violated without changing the standard i.i.d.-based sampling distribution properties of test statistics for linear hypotheses. We shall call covariance structures that achieve this property *independence distribution-preserving* (IDP).

The IDP dependency structures established in this paper yield insight into the robustness of commonly used statistics for testing multivariate linear hypotheses. The existence of such covariance structures implies that error terms in the multivariate linear model need not be independent nor must the marginal covariance structures be equal in order for the usual

i.i.d.-induced properties of statistics for linear hypotheses to hold. Hence, under the i.i.d.-normality assumption, some degree of robustness against dependent error terms exists. Furthermore, under the general IDP error structure derived in this paper, the marginal covariance matrices of the error terms need not be identical.

In this section we formulate the problem to be addressed and introduce notation. We also briefly review the literature and preview our main results.

Let $\mathbf{Y}_{n \times p}$ represent the random matrix $[\mathbf{Y}_1, \mathbf{Y}_2, ..., \mathbf{Y}_n]'$, where each $\mathbf{Y}_i$ is a $p \times 1$ random vector. Denote by $\text{vec}(\mathbf{Y}')$ the $np \times 1$ vector formed by the vertical concatenation of the $\mathbf{Y}_i$s. The random matrix $\mathbf{Y}$ is said to have a *matrix normal distribution* with mean $\mu_{n \times p} = [\mu_1, \mu_2, ..., \mu_n]'$ and covariance matrix $\varXi_{np \times np}$ if $\text{vec}(\mathbf{Y}')$ has a multivariate normal distribution with mean $\text{vec}(\mu')$ and covariance matrix $\varXi$. We write $\mathbf{Y} \sim \text{MN}(\mu, \varXi)$. Note that μ_i is a $p \times 1$ vector for $i = 1, ..., n$. Let $W_p(k, \Sigma, \varDelta)$ denote a noncentral Wishart distribution with k degrees of freedom, parameter matrix Σ, and noncentrality parameter $\varDelta$. A central Wishart distribution ($\varDelta = 0$) with k degrees of freedom is denoted by $W_p(k, \Sigma)$.

Let $\mathbf{Y}$ be an $n \times p$ matrix of n observations on p characteristics and let $\mathbf{X}$ be an $n \times q$ model (design) matrix of fixed independent variables of rank r. Let $\mathbf{B}$ be the $q \times p$ matrix of coefficient parameters, and let $\mathbf{E}$ be an $n \times p$ matrix of random errors. Then, the multivariate linear model is

$$\mathbf{Y} = \mathbf{XB} + \mathbf{E}, \tag{1.1}$$

where $E(\mathbf{Y}) = \mathbf{XB}$ and the first column of the known design matrix $\mathbf{X}$ is a vector of ones. We assume that $\text{Var}(\mathbf{Y}) = \text{Var}(\mathbf{E}) = \mathbf{W}$, where the positive-definite (p.d.) covariance matrix, $\mathbf{W}$, is unknown. Note that in the model (1.1) we assume that the parameter matrix $\mathbf{B}$ is unknown, $\mathbf{E} \sim \text{MN}(\mathbf{0}, \mathbf{W})$, and $\mathbf{Y} \sim \text{MN}(\mathbf{XB}, \mathbf{W})$. Let $\mathbf{Y}_i$ be the ith column of $\mathbf{Y}'$ and let $\text{Var}(\mathbf{Y}_i) = \Sigma$, where Σ is a p.d. matrix. If we assume the usual i.i.d. covariance structure, then

$$\mathbf{W} = \mathbf{I}_n \otimes \Sigma, \tag{1.2}$$

where the notation $\mathbf{A} \otimes \mathbf{B}$ represents the Kronecker matrix product defined by $(a_{ij}\mathbf{B})$, as given in Anderson [1, p. 599], and $\mathbf{I}_n$ is the $n \times n$ identity matrix.

Consider the test of the linear hypothesis

$$H_0: \mathbf{CB} = \mathbf{0} \qquad \text{vs} \qquad H_a: \mathbf{CB} \neq \mathbf{0}, \tag{1.3}$$

where $\mathbf{C}$ is an $s \times q$ constraint matrix of rank s such that $\mathbf{CB}$ is estimable. Let

$$\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^- \mathbf{X}' \tag{1.4}$$

and

$$\mathbf{P}_0 = (\mathbf{X}(\mathbf{X}'\mathbf{X})^- \mathbf{C}')(\mathbf{C}(\mathbf{X}'\mathbf{X})^- \mathbf{C}')^{-1} (\mathbf{C}(\mathbf{X}'\mathbf{X})^- \mathbf{X}'). \qquad (1.5)$$

Assuming the i.i.d. covariance structure (1.2), we have that

$$\mathbf{Q} = \mathbf{Y}'(\mathbf{I}_n - \mathbf{P}) \qquad \mathbf{Y} \sim \mathrm{W}_p(n - r, \Sigma) \qquad (1.6)$$

and

$$\mathbf{R} = \mathbf{Y}'\mathbf{P}_0\mathbf{Y} \sim \mathrm{W}_p(s, \Sigma, \Delta), \qquad (1.7)$$

where $r = \mathrm{rank}(\mathbf{X})$ and $\Delta = (\mathbf{CB})'(\mathbf{C}(\mathbf{X}'\mathbf{X})^- \mathbf{C}')^{-1}(\mathbf{CB})$ is the noncentrality matrix. Under the matrix-normal i.i.d. model defined in (1.1) and (1.2), common statistics for testing the hypothesis (1.3) are functions of $\mathbf{Q}$ and $\mathbf{R}$. These include, for example, the Lawley–Hotelling trace statistic, $\mathrm{LH} = (n - r)\,\mathrm{tr}[(\mathbf{QR}^{-1})]$; Wilks' lambda statistic, $\Lambda = |\mathbf{Q}|/|\mathbf{Q} + \mathbf{R}|$; and Pillai's trace statistic, $\mathrm{PT} = \mathrm{tr}[\mathbf{R}(\mathbf{Q} + \mathbf{R})^{-1}]$, among others. We shall denote an arbitrary member of this group of test statistics by $f(\mathbf{Q}, \mathbf{R})$.

Consider the covariance structure

$$\mathbf{W} = \mathbf{V} \otimes \Sigma, \qquad (1.8)$$

where $\mathbf{V}$ is an $n \times n$ p.d. matrix. In this paper we provide necessary and sufficient conditions on the matrix $\mathbf{V}$ such that, given the linear constraint matrix $\mathbf{C}$ in (1.3), a test statistic $f(\mathbf{Q}, \mathbf{R})$ is distributed exactly as under the i.i.d. dispersion structure (1.2). In doing so, we also provide conditions on $\mathbf{V}$ which insure that $\mathbf{W}$ is p.d.

Our result extends the IDP-related results for univariate linear models in Mathew and Bhimasankaram [8], Tranquilli and Baldesarri [11], Jeyaratnam [4], Khatri [5], and Ghosh and Sinha [3]. Our work also extends the IDP results for the multivariate general linear model by Pavur [10] and Meaux, Young, and Seaman [9]. Pavur [10] has obtained a sufficient general form of the IDP covariance matrix for some multivariate analysis of variance test statistics, assuming a particular design matrix $\mathbf{X}$ and a particular form for the constraint matrix $\mathbf{C}$. Meaux *et al.* [9] have formulated a general IDP covariance structure for testing hypothesis (1.3), where $\mathbf{C} = [\mathbf{0}_{(q-1) \times 1} : \mathbf{I}_{(q-1) \times (q-1)}]$ under the multivariate general linear model.

In Section 2 we provide some notation and preliminary results which we utilize in our characterization of the general form of the IDP dependency structure for testing the coefficients of the multivariate linear model. We present our main result in Section 3 and conclude with brief comments in Section 4.

2. MATHEMATICAL PRELIMINARIES

In this section we establish notation, present two lemmas, and derive a theorem that is utilized in the proof of the main results. The linear space of all $m \times n$ matrices over the complex field is denoted by $\mathbb{C}_{m \times n}$ and the set of real-valued matrices in $\mathbb{C}_{m \times n}$ is represented by $\mathbb{R}_{m \times n}$. Denote the cone consisting of all nonnegative-definite (n.d.) matrices in $\mathbb{C}_{p \times p}$ ($\mathbb{R}_{p \times p}$) by $\mathbb{C}_p^{\geqslant}$ ($\mathbb{R}_p^{\geqslant}$). Let $\mathbb{C}_p^{>}$ ($\mathbb{R}_p^{>}$) be the interior of $\mathbb{C}_p^{\geqslant}$ ($\mathbb{R}_p^{\geqslant}$), which consists of all p.d. matrices in $\mathbb{C}_{p \times p}$ ($\mathbb{R}_{p \times p}$). Also, let $\mathbb{R}_p^{S}$ represent the set of all symmetric matrices in $\mathbb{R}_{p \times p}$. The symbol $\mathbf{A}^*$ denotes the conjugate transpose of a matrix $\mathbf{A} \in \mathbb{C}_{m \times n}$. Let $\mathbf{U}^+$ denote the Moore-Penrose pseudo-inverse of the matrix $\mathbf{U} \in \mathbb{C}_{m \times n}(\mathbb{R}_{m \times n})$, and use $\mathcal{N}(\mathbf{A})$ and $\mathcal{R}(\mathbf{A})$ to denote the null space and range space, respectively, of $\mathbf{A} \in \mathbb{R}_{m \times n}$.

The following two lemmas and theorem are utilized in the proof of our main theorems. The first lemma is a combination of Corollaries 2.3.2 and 2.4.2 in Wong, Masaro, and Wang [12].

LEMMA 1. *Let* $\mathbf{Y} \sim \mathrm{MN}(\mu, \mathbf{V} \otimes \Sigma)$, *where* $\mu \in \mathbb{R}_{n \times p}$, $\mathbf{V} \in \mathbb{R}_n^{\geqslant}$, *and* $\Sigma \in \mathbb{R}_p^{\geqslant}$. *Consider the multivariate quadratic forms* $\mathbf{Y}' \mathbf{A}_i \mathbf{Y}$, $i = 1, 2$, *where* $\mathbf{A}_i \in \mathbb{R}_n^{S}$. *Then, for* $i = 1, 2$, $\mathbf{Y}' \mathbf{A}_i \mathbf{Y} \sim \mathrm{W}_p(k_i, \Sigma, \Delta_i)$ *with* $k_i = \mathrm{rank}(\mathbf{V} \mathbf{A}_i)$ *degrees of freedom and noncentrality parameter* $\Delta_i = \mu' \mathbf{A}_i \mu$ *if and only if the following conditions hold*:

   (i) $\mathbf{V} \mathbf{A}_i \mathbf{V} \mathbf{A}_i \mathbf{V} = \mathbf{V} \mathbf{A}_i \mathbf{V}$;

   (ii) $\mu' \mathbf{A}_i \mathbf{V} \mathbf{A}_i \mathbf{V} \mathbf{A}_i \mu = \mu' \mathbf{A}_i \mathbf{V} \mathbf{A}_i \mu = \mu' \mathbf{A}_i \mu$;

   (iii) $\mathbf{V} \mathbf{A}_1 \mathbf{V} \mathbf{A}_2 \mathbf{V} = \mathbf{0}$;

   (iv) $\mathbf{V} \mathbf{A}_1 \mathbf{V} \mathbf{A}_2 \mu = \mathbf{0}$;

and

   (v) $\mu' \mathbf{A}_1 \mathbf{V} \mathbf{A}_2 \mu = \mathbf{0}$.

Moreover, given $\mathbf{Y}' \mathbf{A}_i \mathbf{Y} \sim \mathrm{W}_p(k_i, \Sigma, \Delta_i)$, *then* $\mathbf{Y}' \mathbf{A}_i \mathbf{Y} \sim \mathrm{W}_p(k_i, \Sigma)$ *if and only if* $\mathbf{A}_i \mu = \mathbf{0}$, $i = 1, 2$.

The following lemma is well known and, therefore, stated without proof.

LEMMA 2. *For* $\mathbf{F}, \mathbf{G} \in \mathbb{C}_{m \times n}$, *the matrix equation* $\mathbf{F} \mathbf{F}^* = \mathbf{G} \mathbf{G}^*$ *holds if and only if* $\mathbf{F} = \mathbf{G} \mathbf{Q}$ *for some unitary matrix* $\mathbf{Q} \in \mathbb{C}_{n \times n}$.

The following theorem is an extension of a result proved in Baksalary [2] which gives a representation of the general n.d. solution to the single matrix equation $\mathbf{A} \mathbf{X} \mathbf{A}^* = \mathbf{K}$ where $\mathbf{A} \in \mathbb{C}_{m \times n}$ and $\mathbf{K} \in \mathbb{C}_m^{\geqslant}$, provided such a solution exists. The theorem gives a representation of the general n.d.

common solution to the pair of matrix equations $\mathbf{A}_i \mathbf{X} \mathbf{A}_i^* = \mathbf{K}_i$, $i = 1, 2$, where $\mathbf{K}_1$ and $\mathbf{K}_2$ are n.d. matrices.

THEOREM 1. *Let* $\mathbf{A}_i \in \mathbb{C}_{m \times n}$, *let* $\mathbf{K}_i \in \mathbb{C}_m^{\geq}$, *and let* $\mathbf{K}_i = \mathbf{B}_i \mathbf{B}_i^*$, *where* $\mathbf{B}_i$ *is an arbitrary but fixed* $m \times n$ *matrix*, $i = 1, 2$. *The matrix equations*

$$\mathbf{A}_i \mathbf{X} \mathbf{A}_i^* = \mathbf{K}_i; \qquad i = 1, 2, \tag{2.1}$$

have a common n.d. solution if and only if

$$\mathbf{A}_1 \mathbf{A}_1^+ \mathbf{B}_1 = \mathbf{B}_1 \tag{2.2}$$

and

$$[\mathbf{A}_2(\mathbf{I}_n - \mathbf{A}_1^+ \mathbf{A}_1)][\mathbf{A}_2(\mathbf{I}_n - \mathbf{A}_1^+ \mathbf{A}_1)]^+ (\mathbf{B}_2 - \mathbf{A}_2 \mathbf{A}_1^+ \mathbf{B}_1) = (\mathbf{B}_2 - \mathbf{A}_2 \mathbf{A}_1^+ \mathbf{B}_1). \tag{2.3}$$

If a common n.d. solution exists, a representation of the general common n.d. solution is $\mathbf{X} = \mathbf{U} \mathbf{U}^*$, *where*

$$\mathbf{U} = \mathbf{A}_1^+ \mathbf{B}_1 + (\mathbf{I}_n - \mathbf{A}_1^+ \mathbf{A}_1)[\mathbf{A}_2(\mathbf{I}_n - \mathbf{A}_1^+ \mathbf{A}_1)]^+ (\mathbf{B}_2 - \mathbf{A}_2 \mathbf{A}_1^+ \mathbf{B}_1)$$
$$+ (\mathbf{I}_n - \mathbf{A}_1^+ \mathbf{A}_1)(\mathbf{I}_n - [\mathbf{A}_2(\mathbf{I}_n - \mathbf{A}_1^+ \mathbf{A}_1)]^+ [\mathbf{A}_2(\mathbf{I}_n - \mathbf{A}_1^+ \mathbf{A}_1)]) \mathbf{Z} \tag{2.4}$$

and $\mathbf{Z}$ *is free to vary over* $\mathbb{C}_{n \times n}$.

Proof. Consider the two matrix equations

$$\mathbf{A}_1 \mathbf{U} = \mathbf{B}_1 \tag{2.5}$$

and

$$\mathbf{A}_2 \mathbf{U} = \mathbf{B}_2. \tag{2.6}$$

Equation (2.5) has a solution if and only if condition (2.2) holds and a representation of the general solution is

$$\mathbf{U} = \mathbf{A}_1^+ \mathbf{B}_1 + (\mathbf{I}_n - \mathbf{A}_1^+ \mathbf{A}_1) \mathbf{Z}, \tag{2.7}$$

where $\mathbf{Z} \in \mathbb{C}_{n \times n}$ is arbitrary. Substituting (2.7) into (2.6), we have that (2.5) and (2.6) have a common solution if and only if (2.2) and (2.3) hold. We note that a common solution to (2.5) and (2.6) exists if and only if a common n.d. solution to the equations (2.1) exists.

Now, consider the pair of matrix equations (2.1). Provided a common n.d. solution exists, one can readily show that the matrix $\mathbf{X} = \mathbf{U} \mathbf{U}^*$, where $\mathbf{U}$, defined in (2.4), is a common n.d. solution to the two matrix equations (2.1), regardless of the choice of $\mathbf{Z}$. To prove that $\mathbf{X} = \mathbf{U} \mathbf{U}^*$ yields a

representation of the general common n.d. solution to the pair of matrix equations (2.1), assume that $\mathbf{X}_0$ is an arbitrary common n.d. solution to the matrix equations (2.1). Let $\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$. If the matrix $\mathbf{X}_0$ is a common n.d. solution to the pair of matrix equations (2.1), then there exists a matrix $\mathbf{U}_0$ such that $\mathbf{X}_0 = \mathbf{U}_0 \mathbf{U}_0^*$ and $\mathbf{A} \mathbf{U}_0 \mathbf{U}_0^* \mathbf{A}^* = \mathbf{B} \mathbf{B}^*$. From Lemma 2 it follows that $\mathbf{A} \mathbf{U}_0 \mathbf{U}_0^* \mathbf{A}^* = \mathbf{B} \mathbf{B}^*$ if and only if $\mathbf{A} \mathbf{U}_0 = \mathbf{B} \mathbf{Q}$ for some unitary matrix $\mathbf{Q}$. Now, let $\mathbf{Z} = \mathbf{U}_0 \mathbf{Q}^*$ in (2.4). Then,

$$
\begin{aligned}
\mathbf{U} &= \mathbf{A}_1^+ \mathbf{B}_1 + (\mathbf{I}_n - \mathbf{A}_1^+ \mathbf{A}_1)[\mathbf{A}_2(\mathbf{I}_n - \mathbf{A}_1^+ \mathbf{A}_1)]^+ [\mathbf{B}_2 - \mathbf{A}_2 \mathbf{A}_1^+ \mathbf{B}_1] \\
&\quad + (\mathbf{I}_n - \mathbf{A}_1^+ \mathbf{A}_1)(\mathbf{I}_n - [\mathbf{A}_2(\mathbf{I}_n - \mathbf{A}_1^+ \mathbf{A}_1)]^+ [\mathbf{A}_2(\mathbf{I}_n - \mathbf{A}_1^+ \mathbf{A}_1)]) \, \mathbf{U}_0 \mathbf{Q}^* \\
&= \mathbf{A}^+ \mathbf{B} + (\mathbf{I}_n - \mathbf{A}^+ \mathbf{A}) \, \mathbf{U}_0 \mathbf{Q}^* \\
&= \mathbf{A}^+ \mathbf{B} + \mathbf{U}_0 \mathbf{Q}^* - \mathbf{A}^+ \mathbf{A} \mathbf{U}_0 \mathbf{Q}^* \\
&= \mathbf{A}^+ \mathbf{B} + \mathbf{U}_0 \mathbf{Q}^* - \mathbf{A}^+ \mathbf{B} \\
&= \mathbf{U}_0 \mathbf{Q}^*.
\end{aligned}
$$

Hence, we have $\mathbf{X}_0 = \mathbf{U}_0 \mathbf{U}_0^* = \mathbf{U}_0 \mathbf{Q}^* \mathbf{Q} \mathbf{U}_0^* = \mathbf{U} \mathbf{U}^* = \mathbf{X}$. Thus, $\mathbf{X}$ is a representation of the general n.d. solution to the pair of matrix equations (2.1). ∎

3. THE MAIN RESULTS

The main results of this paper are contained in the following two theorems. In the first theorem, assuming the error covariance structure $\mathbf{W} = \mathbf{V} \otimes \Sigma$, we provide an explicit characterization of the general p.d. matrix $\mathbf{V}$ such that a pair of multivariate quadratic forms are distributed as independent noncentral Wishart random matrices.

THEOREM 2. *Let* $\mathbf{Y} \sim \mathrm{MN}(\mu, \mathbf{V} \otimes \Sigma)$ *where* $\mathbf{V} \in \mathbb{R}_n^>$ *and* $\Sigma \in \mathbb{R}_p^>$, *and let* $\mathbf{G}_i \in \mathbb{R}_n^\geqslant$, $i = 1, 2$, *be idempotent matrices such that* $\mathbf{G}_1 \mathbf{G}_2 = \mathbf{0}$. *Then,* $\mathbf{Y}' \mathbf{G}_i \mathbf{Y} \sim \mathrm{W}_p(k_i, \Sigma, \Delta_i)$ *and are independent with* $k_i = \mathrm{rank}(\mathbf{G}_i)$ *and* $\Delta_i = \mu' \mathbf{G}_i \mu$, $i = 1, 2$, *if and only if*

$$
\mathbf{V} = \sum_{i=1}^2 \mathbf{G}_i + \left(\mathbf{I}_n - \sum_{i=1}^2 \mathbf{G}_i \right) \mathbf{H} + \mathbf{H}' \left(\mathbf{I}_n - \sum_{i=1}^2 \mathbf{G}_i \right), \tag{3.1}
$$

where $\mathbf{H} = \mathbf{Z}[\sum_{i=1}^2 \mathbf{G}_i + \frac{1}{2} \mathbf{Z}'(\mathbf{I}_n - \sum_{i=1}^2 \mathbf{G}_i)]$ *and* $\mathbf{Z}$ *is free to vary over the set* $\{\mathbf{Z} \in \mathbb{R}_{n \times n} : \mathcal{N}(\mathbf{Z}') \cap \mathcal{N}(\sum_{i=1}^2 \mathbf{G}_i) = \{\mathbf{0}\}$ *and* $\mathcal{R}(\sum_{i=1}^2 \mathbf{G}_i) \cap \mathcal{R}[\mathbf{Z}'(\mathbf{I}_n - \sum_{i=1}^2 \mathbf{G}_i)] = \{\mathbf{0}\}\}$.

Proof. We first prove necessity. Assume that $\mathbf{Y}'\mathbf{G}_i\mathbf{Y} \sim \mathrm{W}_p(k_i, \Sigma, \Delta_i)$, $i = 1, 2$, that $\mathbf{Y}'\mathbf{G}_1\mathbf{Y}$ is independent of $\mathbf{Y}'\mathbf{G}_2\mathbf{Y}$, and that $\mathbf{V} \in \mathbb{R}_n^>$. Then, from Lemma 1 we have that $\mathbf{V}$ is a common p.d. solution of the matrix equations $\mathbf{V}\mathbf{G}_i\mathbf{V}\mathbf{G}_i\mathbf{V} = \mathbf{V}\mathbf{G}_i\mathbf{V}$, $i = 1, 2$. Because $\mathbf{V} \in \mathbb{R}_n^>$ and $\mathbf{G}_i \in \mathbb{R}_n^{\geq}$, $i = 1, 2$, it follows that $\mathbf{V}$ is a common p.d. solution to the pair of matrix equations

$$\mathbf{G}_i\mathbf{V}\mathbf{G}_i = \mathbf{G}_i, \tag{3.2}$$

$i = 1, 2$. By Theorem 1 a representation of the general n.d. matrix $\mathbf{V}$ which simultaneously satisfies the matrix equations (3.2) is of the form

$$
\begin{aligned}
\mathbf{V} &= \left(\sum_{i=1}^2 \mathbf{G}_i + \left[\prod_{i=1}^2 (\mathbf{I}_n - \mathbf{G}_i) \right] \mathbf{Z} \right) \left(\sum_{i=1}^2 \mathbf{G}_i + \left[\prod_{i=1}^2 (\mathbf{I}_n - \mathbf{G}_i) \right] \mathbf{Z} \right)' \\
&= \left[\sum_{i=1}^2 \mathbf{G}_i + \left(\mathbf{I}_n - \sum_{i=1}^2 \mathbf{G}_i \right) \mathbf{Z} \right] \left[\sum_{i=1}^2 \mathbf{G}_i + \left(\mathbf{I}_n - \sum_{i=1}^2 \mathbf{G}_i \right) \mathbf{Z} \right]' \\
&= \sum_{i=1}^2 \mathbf{G}_i + \left(\sum_{i=1}^2 \mathbf{G}_i \right) \mathbf{Z}' \left(\mathbf{I}_n - \sum_{i=1}^2 \mathbf{G}_i \right) + \left(\mathbf{I}_n - \sum_{i=1}^2 \mathbf{G}_i \right) \mathbf{Z} \left(\sum_{i=2}^2 \mathbf{G}_i \right) \\
&\quad + \left(\mathbf{I}_n - \sum_{i=1}^2 \mathbf{G}_i \right) \mathbf{Z}\mathbf{Z}' \left(\mathbf{I}_n - \sum_{i=1}^2 \mathbf{G}_i \right) \\
&= \sum_{i=1}^2 \mathbf{G}_i + \left(\mathbf{I}_n - \sum_{i=1}^2 \mathbf{G}_i \right) \mathbf{Z} \left[\sum_{i=1}^2 \mathbf{G}_i + \tfrac{1}{2}\mathbf{Z}' \left(\mathbf{I}_n - \sum_{i=1}^2 \mathbf{G}_i \right) \right] \\
&\quad + \left[\sum_{i=1}^2 \mathbf{G}_i + \tfrac{1}{2} \left(\mathbf{I}_n - \sum_{i=1}^2 \mathbf{G}_i \right) \mathbf{Z} \right] \mathbf{Z}' \left(\mathbf{I}_n - \sum_{i=1}^2 \mathbf{G}_i \right) \\
&= \sum_{i=1}^2 \mathbf{G}_i + \left(\mathbf{I}_n - \sum_{i=1}^2 \mathbf{G}_i \right) \mathbf{H} + \mathbf{H}' \left(\mathbf{I}_n - \sum_{i=1}^2 \mathbf{G}_i \right),
\end{aligned}
$$

where

$$\mathbf{H} = \mathbf{Z} \left[\sum_{i=1}^2 \mathbf{G}_i + \tfrac{1}{2}\mathbf{Z}' \left(\mathbf{I}_n - \sum_{i=1}^2 \mathbf{G}_i \right) \right].$$

We now give conditions on $\mathbf{Z}$ which insure that $\mathbf{V} \in \mathbb{R}_n^>$. Following Baksalary's [2] method for restricting the arbitrary matrix $\mathbf{Z}$ to insure that $\mathbf{V} \in \mathbb{R}_n^>$, we have that $\mathrm{rank}(\mathbf{V}) = n$ if and only if $\mathrm{rank}[\sum_{i=1}^2 \mathbf{G}_i + (\mathbf{I}_n - \sum_{i=1}^2 \mathbf{G}_i) \mathbf{Z}] = n$ if and only if

$$\mathrm{rank}\left[\left(\mathbf{I}_n - \sum_{i=1}^2 \mathbf{G}_i \right) \mathbf{Z} \right] = \mathrm{rank}\left[\left(\mathbf{I}_n - \sum_{i=1}^2 \mathbf{G}_i \right) \right] \tag{3.3}$$

and

$$\text{rank} \left[\sum_{i=1}^{2} \mathbf{G}_i + \left(\mathbf{I}_n - \sum_{i=1}^{2} \mathbf{G}_i \right) \mathbf{Z} \right]$$

$$= \text{rank} \left(\sum_{i=1}^{2} \mathbf{G}_i \right) + \text{rank} \left[\left(\mathbf{I}_n - \sum_{i=1}^{2} \mathbf{G}_i \right) \mathbf{Z} \right]. \qquad (3.4)$$

Corollary 6.2 in Marsaglia and Styan [7] implies that (3.3) holds if and only if $\mathcal{N}(\mathbf{Z}') \cap \mathcal{N}(\sum_{i=1}^{2} \mathbf{G}_i) = \{\mathbf{0}\}$. Also, from a result in Marsaglia and Styan [6] restriction (3.4) holds if and only if

$$\mathcal{R} \left(\sum_{i=1}^{2} \mathbf{G}_i \right) \cap \mathcal{R} \left[\left(\mathbf{I}_n - \sum_{i=1}^{2} \mathbf{G}_i \right) \mathbf{Z} \right] = \{\mathbf{0}\} \qquad (3.5)$$

and

$$\mathcal{R} \left(\sum_{i=1}^{2} \mathbf{G}_i \right) \cap \mathcal{R} \left[\mathbf{Z}' \left(\mathbf{I}_n - \sum_{i=1}^{2} \mathbf{G}_i \right) \right] = \{\mathbf{0}\}.$$

Restriction (3.5) is trivially true.

To prove sufficiency, note that $\mathbf{V}$, defined in (3.1), is a common p.d. solution to the matrix equations $\mathbf{G}_i \mathbf{V} \mathbf{G}_i = \mathbf{G}_i$, $i = 1, 2$, and $\mathbf{G}_1 \mathbf{V} \mathbf{G}_2 = \mathbf{0}$. Thus, from Lemma 1, $\mathbf{Y}' \mathbf{G}_i \mathbf{Y} \sim W_p(k_i, \Sigma, \Delta_i)$, with $k_i = \text{rank}(\mathbf{G}_i)$ and $\Delta_i = \mu' \mathbf{G}_i \mu$, $i = 1, 2$. Furthermore, $\mathbf{Y}' \mathbf{G}_1 \mathbf{Y}$ is independent of $\mathbf{Y}' \mathbf{G}_2 \mathbf{Y}$. ∎

The expression for $\mathbf{V}$ given in (3.1) represents a complete solution to the problem of characterizing the general p.d. IDP covariance matrix of the form $\mathbf{W} = \mathbf{V} \otimes \Sigma$ such that the quadratic forms $\mathbf{Y}' \mathbf{G}_i \mathbf{Y}$, $i = 1, 2$, are distributed as independent noncentral Wishart random matrices. The following theorem presents the second of the two main results of this paper.

THEOREM 3. *Let $\mathbf{Y} \sim \text{MN}(\mathbf{XB}, \mathbf{V} \otimes \Sigma)$ where $\mathbf{V} \in \mathbb{R}_n^{>}$ and $\Sigma \in \mathbb{R}_p^{>}$. The distribution of the test statistic $f(\mathbf{Q}, \mathbf{R})$ is identical to the distribution of $f(\mathbf{Q}, \mathbf{R})$ assuming the model $\mathbf{Y} \sim \text{MN}(\mathbf{XB}, \mathbf{I} \otimes \Sigma)$ if and only if*

$$\mathbf{V} = (\mathbf{I}_n - \mathbf{P} + \mathbf{P}_0) + (\mathbf{P} - \mathbf{P}_0) \mathbf{H} + \mathbf{H}'(\mathbf{P} - \mathbf{P}_0) \qquad (3.6)$$

where $\mathbf{P}$ and $\mathbf{P}_0$ are defined in (1.4) and (1.5), respectively, $\mathbf{H} = \mathbf{Z}[(\mathbf{I}_n - \mathbf{P} + \mathbf{P}_0) + \frac{1}{2} \mathbf{Z}'(\mathbf{P} - \mathbf{P}_0)]$, and $\mathbf{Z}$ is free to vary over the set $\{\mathbf{Z} \in \mathbb{R}_{n \times n}: \mathcal{N}(\mathbf{Z}') \cap \mathcal{N}(\mathbf{I}_n - \mathbf{P} + \mathbf{P}_0) = \{\mathbf{0}\}$ and $\mathcal{R}(\mathbf{I}_n - \mathbf{P} + \mathbf{P}_0) \cap \mathcal{R}[\mathbf{Z}'(\mathbf{P} - \mathbf{P}_0)] = \{\mathbf{0}\}\}$.

Proof. The proof follows directly from Theorem 2 if we let $\mathbf{G}_1 = \mathbf{I}_n - \mathbf{P}$ and $\mathbf{G}_2 = \mathbf{P}_0$. ∎

We note that the sufficiency portion of the last theorem is statistically relevant but mathematically trivial. On the other hand, the necessity

portion of the above theorem is statistically irrelevant but mathematically interesting.

We now give an example to demonstrate the form of an IDP covariance matrix for the multivariate general linear model. Consider the model (1.1) with the following parameterization. Let $n = 4$, $p = 2$, $q = 3$

$$\Sigma = \begin{bmatrix} 4 & 0.1 \\ 0.1 & 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{X} = \begin{bmatrix} 1 & 2 & 9 \\ 1 & 5 & 13 \\ 1 & 5 & 7 \\ 1 & 10 & 17 \end{bmatrix}.$$

Also, let

$$\mathbf{Z} = \begin{bmatrix} 3.25 & 3.25 & 3.75 & 4.75 \\ 3.25 & 3.25 & 3.75 & 4.75 \\ 3.25 & 3.25 & 3.75 & 4.75 \\ 3.25 & 3.25 & 3.75 & 4.75 \end{bmatrix}$$

in $\mathbf{H}$, where $\mathbf{H}$ is defined in Theorem 3, and let $\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

From expression (3.4) one possible IDP covariance structure for testing $H_0: \mathbf{CB} = \mathbf{0}$ vs $H_a: \mathbf{CB} \neq \mathbf{0}$ is

$$\mathbf{W} = \mathbf{V} \otimes \Sigma = \begin{bmatrix} 230.0 & 5.75 & 226.0 & 5.65 & 228.0 & 5.7 & 232.0 & 5.8 \\ 5.75 & 172.5 & 5.65 & 169.5 & 5.7 & 171.0 & 5.8 & 174.0 \\ 226.0 & 5.65 & 230.0 & 5.75 & 228.0 & 5.7 & 232.0 & 5.8 \\ 5.65 & 169.5 & 5.75 & 172.5 & 5.7 & 171.0 & 5.8 & 174.0 \\ 228.0 & 5.7 & 228.0 & 5.7 & 234.0 & 5.85 & 234.0 & 5.85 \\ 5.7 & 171.0 & 5.7 & 171.0 & 5.8 & 175.5 & 5.8 & 175.5 \\ 232.0 & 5.8 & 232.0 & 5.8 & 234.0 & 5.85 & 242.0 & 6.05 \\ 5.8 & 174.0 & 5.8 & 174.0 & 5.85 & 175.5 & 6.05 & 181.5 \end{bmatrix}.$$

One can readily see that this IDP covariance matrix differs considerably from the i.i.d. covariance structure (1.2). In particular, it allows for non-zero covariance among observation vectors and for heteroscedastic marginal covariance matrices.

4. DISCUSSION

The covariance structure $\mathbf{W} = \mathbf{V} \otimes \Sigma$, where $\mathbf{V}$ is given in (3.6) and $\Sigma \in \mathbb{R}_p^>$, yields a set of IDP p.d. covariance matrices for the test statistics $f(\mathbf{Q}, \mathbf{R})$ for testing the linear hypothesis (1.3), provided $\mathbf{W}$ is of the general

form (1.8). Our IDP p.d. general covariance structure differs from most of the other results concerning IDP dependency structures for the linear model. The difference is that we give explicit restrictions on the general form for $\mathbf{V}$ which insures that all IDP covariance matrices are p.d. Note that for the univariate case where $\Sigma = \sigma^2$, the general IDP covariance structure presented in this paper reduces to a form of the general IDP dependency structure of Jeyaratnam [4].

The existence of IDP covariance structures for the multivariate regression model has two notable consequences for the application of multivariate regression analysis. First, the existence of IDP dependency structures implies that the error terms do not have to be independent in order for the distribution of the usual statistics for testing the linear hypothesis $H_0 : \mathbf{CB} = \mathbf{0}$ vs $H_a : \mathbf{CB} \neq \mathbf{0}$ to hold. This result implies that, under the normality assumption, there is at least some degree of robustness against dependent error terms. Second, under the general IDP error structure given in (3.6), the marginal covariance matrices of the error terms need not be identical. Thus, a test statistic $f(\mathbf{Q}, \mathbf{R})$, although derived under the usual i.i.d. assumption, $\mathbf{W} = \mathbf{I}_n \otimes \Sigma$ possesses some degree of robustness against heteroscedastic error terms in conjunction with dependence of the error terms.

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REFERENCES

1. T. W. Anderson, "An Introduction to Multivariate Analysis," Wiley, New York, 1984.
2. J. K. Baksalary, Nonnegative definite and positive definite solutions to the matrix equation $\mathbf{AXA}^* = \mathbf{B}$, *Linear and Multilinear Algebra* **16** (1984), 133–139.
3. M. Ghosh and B. K. Sinha, On the robustness of least squares procedures in regression models, *J. Multivariate Anal.* **10** (1980), 332–342.
4. S. Jeyaratnam, A sufficient condition on the covariance matrix for F tests in linear models to be valid, *Biometrika* **69** (1982), 679–680.
5. C. G. Khatri, Study of F-tests under a dependent model, *Sankhyā Ser. A* **43** (1981), 107–110.
6. G. Marsaglia and G. P. H. Styan, When does rank$(\mathbf{A} + \mathbf{B}) = $ rank$(\mathbf{A}) + $ rank$(\mathbf{B})$?, *Canad. Math. Bul.* **15** (1972), 451–452.
7. G. Marsaglia and G. P. H. Styan, Equalities and inequalities for ranks of matrices, *Linear and Multilinear Algebra* **2** (1974), 269–292.
8. T. Mathew and P. Bhimasankaram, On the robustness of the LRT with respect to specification errors in a linear model, *Sankhyā Ser. A* **45** (1983), 212–225.

9. L. M. Meaux, D. M. Young, and J. W. Seaman, Jr., Multivariate regression analysis with dependent observations: Conditions for the invariance of the distribution of the Lawley–Hotelling test for model utility, *Statistica* **54** (1995), 139–150.

10. R. Pavur, A characterization of the covariance structure in which certain quadratic forms are independent and follow chi-square distributions, *Sankhyā Ser. A* **51** (1989), 382–389.

11. G. B. Tranquilli and B. Baldessari, Regression analysis with dependent observations: Conditions for the invariance of the distribution of the *F*-statistic, *Statistica* **47** (1988), 49–57.

12. C. S. Wong, J. Masaro, and T. Wang, Multivariate version of Cochran's theorem, *J. Multivariate Anal.* **39** (1991), 154–174.