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JOURNAL OF DIFFERENTIAL EQUATIONS **39**, 151–185 (1981)

Spectral Analysis of a Nonself-Adjoint Differential Operator*

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Received December 19, 1979; revised June 11, 1980

1. INTRODUCTION

Let p be a measurable complex-valued function on $\mathbb{R}$ such that for some positive number η, which will be fixed throughout this paper,

$$\int_{-\infty}^{\infty} e^{2\eta|x|}\,|p(x)|\,dx < \infty. \tag{1.1}$$

Consider the differential operator

$$l = -d^2/dx^2 + p(x) \tag{1.2}$$

and the unbounded operator L on $L^2(\mathbb{R})$ defined by $Lf = lf$ on the domain

$$\mathrm{Dom}(L) = \{f \in L^2(\mathbb{R}) \cap C^1(\mathbb{R}) \colon f' \text{ is absolutely continuous on every bounded interval, and } lf \in L^2(\mathbb{R})\}.$$

It is known (cf. Kemp [12], Krall [13]) that L is a closed operator, and its adjoint L^* is given by

$$L^*f = l^*f \qquad \text{on } \mathrm{Dom}(L^*) = \{f \in L^2(\mathbb{R}) \colon \bar{f} \in \mathrm{Dom}(L)\}$$

where $l^* = -d^2/dx^2 + \overline{p(x)}$. In this paper we shall study the theory of the eigenfunction expansion associated to the operator L. The case where p is real-valued, i.e., where L is self-adjoint, is classical, and we shall be mainly concerned with the phenomena which are peculiar to the case $\operatorname{Im} p \neq 0$.

Problems of this sort arise in a variety of situations in physics involving energy dissipation. Indeed, our original motivation for this work came from the study of dielectric waveguides with heat loss. (In this situation p represents a dielectric coefficient, and the equation $lf = \lambda f\ [\lambda \in \mathbb{C}]$ arises from Maxwell's equations by separation of variables: cf. McKenna [16].) Operators of the form (1.2), with complex p, have also turned up in recent

* Research partially supported by the National Science Foundation.

0022-0396/81/020151–35$02.00/0

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work of Adler and Moser [1] and Deift and Trubowitz [7] on inverse scattering theory and the Korteweg–deVries equation.

In [9] we studied the analogous problem on a half-line, that is, the eigenfunction expansion associated to the operator (1.2) on $L^2(0, \infty)$, subject to a boundary condition at 0. In both that situation and the present one, it is not hard to obtain a formal eigenfunction expansion for any $f \in L^2$, but the expansion may diverge because of the possible presence of singular points in the continuous spectrum—the so-called "spectral singularities." In both cases, the remedy we propose is to introduce a rather subtle convergence factor into the expansion formula for f so that the modified expression converges, and so that f is obtained as the limit in the L^2 norm of this expression as the convergence factor is made to disappear. We also use this technique to construct a bounded functional calculus for L. In broad outline, the arguments in this paper are similar to those in [9], but in detail they are rather different and frequently more involved—partly because the present situation is intrinsically more complicated, and partly because we are less frequently able to rely on known results. We refer the reader to [9] for a fuller discussion of the motivation for the ideas used here, as well as an explication of these ideas in a simpler situation.

Aspects of the spectral theory of the operator L have previously been studied by a number of authors, including Benzinger [3], Blashchak [4, 5], Funtakov [10], Huige [11], Kemp [12], Krall [13], and Stone [19]. Of these, only Blashchak [5] makes a serious attempt to come to grips with the spectral singularities. He obtains an expansion formula which converges in a norm weaker than the L^2 norm, similar to the formula of Ljance [15] for the analogous problem on a half-line; he also shows that L is a generalized spectral operator in the sense of Ljance [14] and is a spectral operator provided there are no spectral singularities. Our present expansion formula (Theorem 2) is superior to that of Blashchak in that it converges in the L^2 norm and is more convenient for applications, and the results on the spectrality of L follow easily from our functional calculus (Theorem 4).

Also of interest is the inverse problem of recovering the potential p from suitable scattering data of L, which provides a method for constructing L's with prescribed eigenvalues and spectral singularities. This problem has been solved by Blashchak [6]. (The reader should beware of mistakes in the English version of [6]. In particular, on page 791, column 1, line 14, "real" should be "nonreal.")

We take the work of Kemp [12] as our starting point. Kemp expresses his results in a way which exhibits the symmetry between l and l^*. We find it more convenient to express everything in terms of the eigenfunctions of l. However, since these are merely the complex conjugates of the eigenfunctions of l^*, it is a trivial matter to translate one formulation into the other.

The following notational conventions will be in force for the remainder of the paper: (1) If λ is a complex number which is not real and positive, $\sqrt{\lambda}$ will denote the square root of λ with positive imaginary part. (2) The Lebesgue spaces $L^p(\mathbb{R})$ $(1 \leqslant p \leqslant \infty)$ will be denoted simply by L^p. (3) We use the classical notation "l.i.m." for "limit in the L^2 norm."

2. Preliminaries

Since the potential $p(x)$ is small near $x = \pm\infty$, for $\operatorname{Im} s \geqslant 0$ there is a solution y_1 of $ly = s^2 y$ which is asymptotic to e^{isx} as $x \to +\infty$, and another solution y_2 which is asymptotic to e^{-isx} as $x \to -\infty$. These solutions can be conveniently expressed in the following form:

PROPOSITION 1 (Agranovich and Marchenko [2]). *Let η be as in* (1.1). *There exist kernels $k_+(x, t)$ and $k_-(x, t)$ defined, respectively, for $0 \leqslant x \leqslant t < \infty$ and $-\infty < t \leqslant x \leqslant 0$ with the following properties*:

(a) *k_+ and k_- are absolutely continuous in each variable.*

(b) *For some $C > 0$,*

$$|k_\pm(x, t)| \leqslant C e^{-\eta|x+t|},$$

$$|\partial k_\pm/\partial x\,(x, t)| \leqslant C(|p((x+t)/2)| + e^{-\eta|x+t|}).$$

(c) *If $\operatorname{Im} s > -\eta$, the functions $y_1(x, s)$ and $y_2(x, s)$ defined, respectively, for $x \geqslant 0$ and $x \leqslant 0$ by*

$$y_1(x, s) = e^{isx} + \int_x^\infty k_+(x, t)\, e^{ist}\, dt, \tag{2.1}$$

$$y_2(x, s) = e^{-isx} + \int_{-\infty}^x k_-(x, t)\, e^{-ist}\, dt \tag{2.2}$$

satisfy $ly = s^2 y$.

Although formulas (2.1) and (2.2) are valid only for $x \geqslant 0$ and $x \leqslant 0$, respectively, y_1 and y_2 can, of course, be uniquely continued as solutions of $ly = s^2 y$ to $-\infty < x < \infty$. Proposition 3 below, together with (2.1) and (2.2), will provide a means of computing y_1 and y_2 on the other half-lines.

For any differentiable functions f, g on $\mathbb{R}$, we shall denote their Wronskian $fg' - f'g$ by $w(f, g)$. Since the operator l has no first order term, if f and g

are solutions of $ly = s^2 y$ the Wronskian $w(f, g)$ will be constant as a function of x. The following Wronskians will appear repeatedly in the sequel:

$$U_1(s) = w(y_1(\cdot, s), y_1(\cdot, -s)),$$
$$U_2(s) = w(y_2(\cdot, s), y_2(\cdot, -s)),$$
$$V(s) = w(y_1(\cdot, s), y_2(\cdot, -s)),$$
$$W(s) = w(y_1(\cdot, s), y_2(\cdot, s)).$$

U_1, U_2, and V are defined for $|\operatorname{Im} s| < \eta$, while W is defined for $\operatorname{Im} s > -\eta$. We summarize the facts we shall need about these functions in the following proposition.

PROPOSITION 2.

(a) *If* $|\operatorname{Im} s| < \eta$, $U_1(s) = -U_2(s) = -2is$.

(b) *For any* $\delta < \eta$, $V(s)$ *is bounded in the strip* $|\operatorname{Im} s| \leqslant \delta$.

(c) *For any* $\delta < \eta$, $W(s) + 2is$ *is bounded in the half-plane* $\operatorname{Im} s > -\delta$.

(d) *If* $|\operatorname{Im} s| < \eta$, $U_1(s)\, U_2(s) + V(s)\, V(-s) - W(s)\, W(-s) = 0$.

Proof. (a) From (2.1), for fixed s we have

$$y_1(x, s) = e^{isx}(1 + o(1)),$$
$$\partial y_1/\partial x\,(x, s) = e^{isx}(is + o(1)) \text{ as } x \to +\infty.$$

Therefore $U_1(s) = -2is + o(1)$ as $x \to +\infty$, and since $U_1(s)$ is independent of x, $U_1(s) = -2is$. Similarly, $U_2(s) = 2is$.

(b) From the variation of parameters formula it follows that y_1 and y_2 satisfy the integral equations

$$y_1(x, s) = e^{isx} + \int_x^\infty s^{-1}(\sin s(\xi - x))\, p(\xi)\, y_1(\xi, s)\, d\xi,$$

$$y_2(x, s) = e^{-isx} + \int_{-\infty}^x s^{-1}(\sin s(x - \xi))\, p(\xi)\, y_2(\xi, s)\, d\xi.$$

If we substitute these expressions and their derivatives into the definition of $V(s)$ and set $x = 0$, we find that

$$V(s) = \int_0^\infty e^{is\xi} p(\xi)\, y_1(\xi, s)\, d\xi + \int_{-\infty}^0 e^{is\xi} p(\xi)\, y_2(\xi, -s)\, d\xi$$
$$+ \left[\int_0^\infty s^{-1}(\sin s\xi)\, p(\xi)\, y_1(\xi, s)\, d\xi\right] \times$$

$$\times \left[\int_{-\infty}^{0} (\cos s\xi)\, p(\xi)\, y_2(\xi, -s)\, d\xi \right]$$

$$- \left[\int_{0}^{\infty} (\cos s\xi)\, p(\xi)\, y_1(\xi, s)\, d\xi \right]$$

$$\times \left[\int_{-\infty}^{0} s^{-1}(\sin s\xi)\, p(\xi)\, y_2(\xi, -s)\, d\xi \right].$$

But from Proposition 1 we see that for $\xi \geqslant 0$,

$$|y_1(\xi, s)| \leqslant e^{-(\operatorname{Im} s)\xi} + C \int_{\xi}^{\infty} e^{-\eta(\xi+t)} e^{-(\operatorname{Im} s)t}\, dt \leqslant C' e^{-(\operatorname{Im} s)\xi}$$

and likewise for $\xi \leqslant 0$,

$$|y_2(\xi, s)| \leqslant C' e^{(\operatorname{Im} s)\xi}.$$

The boundedness of $V(s)$ for $|\operatorname{Im} s| \leqslant \delta < \eta$ then follows from these estimates together with (1.1).

(c) is derived in the same way as (b).

(d) follows by a simple calculation from the definitions of the Wronskians.

From Proposition 2(a) it follows that for $|\operatorname{Im} s| < \eta$, $s \neq 0$, each of the pairs $y_1(\cdot, s)$, $y_1(\cdot, -s)$ and $y_2(\cdot, s)$, $y_2(\cdot, -s)$ is a basis for the solutions of $ly = s^2 y$. It is easy to find the expressions for members of one pair as linear combinations of members of the other. Here is the result.

PROPOSITION 3. *If* $|\operatorname{Im} s| < \eta$, $s \neq 0$,

$$\begin{aligned} y_1(x, s) &= \frac{V(s)}{U_2(s)}\, y_2(x, s) - \frac{W(s)}{U_2(s)}\, y_2(x, -s) \\ &= \frac{V(s)}{2is}\, y_2(x, s) - \frac{W(s)}{2is}\, y_2(x, -s), \end{aligned} \tag{2.3}$$

$$\begin{aligned} y_2(x, s) &= \frac{-V(-s)}{U_1(s)}\, y_1(x, s) + \frac{W(s)}{U_1(s)}\, y_1(x, -s) \\ &= \frac{V(-s)}{2is}\, y_1(x, s) - \frac{W(s)}{2is}\, y_1(x, -s). \end{aligned} \tag{2.4}$$

At this point we identify the spectrum of the Hilbert space operator L, which we shall denote by spec(L).

PROPOSITION 4 (Kemp [12]). *The discrete spectrum of L consists of those numbers $\lambda = s^2$ such that* $\operatorname{Im} s > 0$ *and* $W(s) = 0$, *and the eigenfunctions of L with eigenvalue s^2 are the scalar multiples of $y_1(\cdot, s)$ (or equivalently of $y_2(\cdot, s)$, since the condition $W(s) = 0$ means that $y_1(\cdot, s)$ and $y_2(\cdot, s)$ are proportional). The continuous spectrum of L consists of the half-line $[0, \infty)$, and the residual spectrum of L is empty.*

We note that W is an analytic function in the half-plane $\operatorname{Im} s > -\eta$. Moreover, from Proposition 2(c) it follows that $W(s) \neq 0$ when $|s|$ is large and $\operatorname{Im} s \geqslant 0$. Therefore, W has only finitely many zeros in the half-plane $\operatorname{Im} s > 0$. We denote them by $s_1, \ldots, s_M$, and we denote their multiplicities by $\alpha_1, \ldots, \alpha_M$. We further set $\lambda_m = s_m^2$ for $1 \leqslant m \leqslant M$, so that

$$\operatorname{spec}(L) = \{\lambda_1, \ldots, \lambda_M\} \cup [0, \infty).$$

If the multiplicity α_m of s_m is greater than 1, we must consider not only the eigenspace $\{cy_1(\cdot, s_m) : c \in \mathbb{C}\}$ but the "generalized eigenspace" consisting of the linear span of the functions

$$(d/d\lambda)^k y_j(\cdot, \sqrt{\lambda})|_{\lambda = \lambda_m} \qquad (0 \leqslant k \leqslant \alpha_m - 1,\ 1 \leqslant j \leqslant 2).$$

For convenience of notation, for $\lambda \notin [0, \infty)$ we set

$$Y_j(x, \lambda) = y_j(x, \sqrt{\lambda}), \qquad Y_j^{(k)}(x, \lambda) = (d/d\lambda)^k Y_j(x, \lambda).$$

(Recall that $\operatorname{Im} \sqrt{\lambda} > 0$ by convention.)

PROPOSITION 5. (a) *The linear span of $\{Y_1^{(k)}(\cdot, \lambda_m) : 0 \leqslant k \leqslant \alpha_m - 1\}$ is the same as the linear span of $\{Y_2^{(k)}(\cdot, \lambda_m) : 0 \leqslant k \leqslant \alpha_m - 1\}$.*

(b) *The functions $Y_j^{(k)}(\cdot, \lambda_m)$ $(0 \leqslant k \leqslant \alpha_m - 1,\ 1 \leqslant m \leqslant M,\ 1 \leqslant j \leqslant 2)$ are all in L^2.*

(c) $(l - \lambda_m)\, Y_j^{(k)}(\cdot, \lambda_m) = k Y_j^{(k-1)}(\cdot, \lambda_m)$ *for all j, k, m.*

Proof. (a) Let $Z(\cdot, \lambda)$ be a solution of $lZ = \lambda Z$, defined for λ in some (perhaps disconnected) neighborhood U of $\{\lambda_1, \ldots, \lambda_m\}$, which depends analytically on λ and is linearly independent of $Y_1(\cdot, \lambda)$ in U. Then $Y_2(\cdot, \lambda)$ is a linear combination of $Y_1(\cdot, \lambda)$ and $Z(\cdot, \lambda)$ in U; more precisely, as in Proposition 3 we have

$$Y_2(\lambda) = \frac{[w(Y_2, Z)](\lambda)\, Y_1(x, \lambda) + W(\sqrt{\lambda})\, Z(x, \lambda)}{[w(Y_1, Z)](\lambda)}. \tag{2.5}$$

Since $W(\sqrt{\lambda})$ vanishes to order α_m at $\lambda = \lambda_m$, from (2.5) it is clear that $Y_2^{(k)}(\cdot, \lambda_m)$ is a linear combination of $Y_1^{(0)}(\cdot, \lambda_m), \ldots, Y_1^{(k)}(\cdot, \lambda_m)$ for $k < \alpha_m$.

(b) From (2.1) and (2.2) it is easy to see that for any $k \geqslant 0$, $Y_1^{(k)}(x, \lambda_m)$ decays exponentially as $x \to +\infty$, whereas $Y_2^{(k)}(x, \lambda_m)$ decays exponentially as $x \to -\infty$. Thus by (a), if $k < \alpha_m$, $Y_1^{(k)}(x, \lambda_m)$ and $Y_2^{(k)}(x, \lambda_m)$ decay exponentially as $x \to \pm\infty$, and in particular these functions are in L^2.

(c) Note that $(l - \lambda_m) Y_j(\cdot, \lambda_m) = (\lambda - \lambda_m) Y_j(\cdot, \lambda)$. On both sides of this equation, expand $Y_j(\cdot, \lambda)$ in its Taylor series about $\lambda = \lambda_m$; comparison of coefficients yields the desired result.

The function W may also have finitely many zeros on the real axis; if so, the squares of these zeros, which lie in the continuous spectrum of L, are the *spectral singularities* of L. Actually, it is more appropriate to consider the zeros of $W(s)/s$ (that is, 0 should be counted as a spectral singularity only if $W(0) = W'(0) = 0$). Thus, we denote the real zeros of $W(s)/s$ by $t_1, \ldots, t_N$ and their multiplicities by $\beta_1, \ldots, \beta_N$. Also we denote the real zeros of $W(s)\, W(-s)/s^2$ by $u_1, \ldots, u_{N'}$ and their multiplicities by $\gamma_1, \ldots, \gamma_{N'}$. We note that

$$\{u_1, \ldots, u_{N'}\} = \{t_1, \ldots, t_N\} \cup \{-t_1, \ldots, -t_N\},$$

and that if $u_{n'} = \pm t_n$ then $\gamma_{n'} \geqslant \beta_n$, with equality if and only if $W(\mp t_n)/(\mp t_n) \neq 0$. The meaning of all these quantities will become clearer later on; in particular, cf. the remark following the proof of Theorem 2.

Next, we introduce some useful linear operators on L^2.

(1) $\mathscr{F}$ will denote the Fourier transform:

$$\mathscr{F} f = \underset{\xi \to \infty}{\text{l.i.m.}} \int_{-\xi}^{\xi} f(x) \exp[ix(\cdot)]\, dx.$$

The adjoint of $\mathscr{F}$ is then given by $\mathscr{F}^* f(s) = \mathscr{F} f(-s)$.

(2) If ϕ is a measurable complex-valued function on $\mathbb{R}$, $\phi(S)$ will denote the operation of multiplication by ϕ:

$$[\phi(S) f](s) = \phi(s) f(s).$$

If $\phi \in L^\infty$ then $\phi(S)$ is a bounded operator on L^2. If $\phi \notin L^\infty$, we may regard $\phi(S)$ either as an unbounded operator on L^2 with domain $\{f \in L^2 : \phi f \in L^2\}$ or as a mapping from all of L^2 to the space of a.e.-defined measurable functions on $\mathbb{R}$, whichever is appropriate. We note the following corollary of the Lebesgue dominated convergence theorem, which we shall use without comment later on: *If $\{\phi_n\}$ is a bounded sequence in L^∞ and $\phi_n \to \phi$ a.e., then $\phi_n(S) \to \phi(S)$ strongly.*

(3) We set $P_+ = \chi_{(0,\infty)}(S)$ and $P_- = \chi_{(-\infty,0)}(S)$, where χ_E denotes the characteristic function of E. Thus P_+ and P_- are the canonical projections

of L^2 onto the subspaces of functions supported on the positive and negative half-lines.

(4) We combine the kernels k_+ and k_- of Proposition 1 into a single kernel k as follows:

$$k(x,t)=\begin{cases} k_+(x,t) & \text{if } 0\leqslant x\leqslant t<\infty, \\ k_-(x,t) & \text{if } -\infty<t\leqslant x\leqslant 0, \\ 0 & \text{otherwise.}\end{cases}$$

We then define integral operators K and K' by

$$Kf(t)=\int_{-\infty}^{\infty} k(x,t)f(x)\,dx, \qquad K'f(x)=\int_{-\infty}^{\infty} k(x,t)f(t)\,dt.$$

We note that because of the estimates on $k_\pm$ in Proposition 1, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |k(x,t)|^2 dx\,dt<\infty,$$

so that K and K' are Hilbert–Schmidt operators on L^2.

PROPOSITION 6. *$I+K$ is invertible. In fact, there is a kernel $\tilde{k}(x,t)$ supported in the set where $-\infty<t\leqslant x\leqslant 0$ or $0\leqslant x\leqslant t<\infty$ and satisfying $|\tilde{k}(x,t)|\leqslant C'e^{-\eta|x+t|}$ such that $(I+K)^{-1}=I+\tilde{K}$, where $\tilde{K}f(t)=\int_{-\infty}^{\infty}\tilde{k}(x,t)f(x)\,dx$.*

Proof. Notice that for any $f\in L^2$

$$P_+Kf(t)=\int_0^t k_+(x,t)f(x)\,dx \qquad \text{if } t\geqslant 0, =0 \text{ if } t<0.$$

Hence P_+K is a Volterra integral operator with L^2 kernel. By a well-known argument (cf. Tricomi [20]), the estimate $|k_+(x,t)|\leqslant Ce^{-\eta(x+t)}$ implies that the iterated kernels k_+^n defined recursively by

$$k_+^1=k_+,\ k_+^n(x,t)=\int_x^t k_+(x,\xi)\,k_+^{n-1}(\xi,t)\,d\xi \qquad (0\leqslant x\leqslant t<\infty)$$

satisfy

$$|k_+^n(x,t)|\leqslant\frac{C^n}{(2\eta)^{n-1}\sqrt{(n-2)!}}e^{-\eta(x+t)} \qquad (n\geqslant 2),$$

so that the series $\sum_1^\infty (-1)^n k_+^n(x,t)$ converges absolutely and its sum $\tilde{k}_+(x,t)$ satisfies $|\tilde{k}_+(x,t)| \leqslant C' e^{-\eta(x+t)}$. Moreover, the operator $\tilde{K}_+$ defined by

$$\tilde{K}_+ f(t) = \int_0^t \tilde{k}_+(x,t) f(x)\,dx \qquad \text{if } t \geqslant 0, =0 \text{ if } t<0,$$

satisfies $I + \tilde{K}_+ = (I + P_+ K)^{-1}$. Likewise, $(I + P_- K)^{-1} = I + \tilde{K}_-$ where $\tilde{K}_-$ has similar properties. But then, since

$$I + K = I + (P_+ + P_-)K + KP_+ P_- K = (I + P_+ K)(I + P_- K),$$

$I + K$ is invertible and

$$(I+K)^{-1} = (I + \tilde{K}_-)(I + \tilde{K}_+) = I + \tilde{K}_+ + \tilde{K}_-.$$

This completes the proof.

If $f \in L^2$, the "Fourier data" of f with respect to the operator L consist of the functions $y_1 f$, $y_2 f$ on $\mathbb{R}$ defined by

$$y_j f(s) = \int_{-\infty}^{\infty} f(x)\, y_j(x,s)\,dx \tag{2.6}$$

together with the numbers

$$Y_j^{(k)} f(\lambda_m) = \int_{-\infty}^{\infty} f(x)\, Y_j^{(k)}(x, \lambda_m)\,dx \quad (0 \leqslant k \leqslant \alpha_m - 1,\ 1 \leqslant m \leqslant M). \tag{2.7}$$

Both (2.6) and (2.7) require further comment. We first discuss (2.7).

By Proposition 5(b), the integrals in (2.7) all converge absolutely. If we were to replace λ_m in (2.7) by some other $\lambda \in \mathbb{C}$, these integrals would (in general) diverge, since then $Y_j^{(k)}(\cdot, \lambda) \notin L^2$. Nonetheless, it is convenient to treat the numbers $Y_j^{(k)} f(\lambda_m)$ as if they were the derivatives at λ_m of an analytic function $Y_j f(\lambda)$. To be more specific, these quantities will typically occur in expressions of the form

$$[\phi \cdot Y_j f]^{(k)}(\lambda_m) \qquad (0 \leqslant k \leqslant \alpha_m - 1) \tag{2.8}$$

where ϕ is an analytic function in some neighborhood of λ_m, and (2.8), by definition, denotes

$$\sum_{i=0}^{k} \binom{k}{i} \phi^{(k-i)}(\lambda_m)\, Y_j^{(i)} f(\lambda_m).$$

(See also the discussion of "functions on the spectrum of L" in Section 4.)

We now turn to (2.6). If $s \in \mathbb{R}$, the functions $y_j(\cdot, s)$ are not in L^2, so the integral in (2.6) will usually not converge absolutely. However, if $f \in L^2$ has compact support,

$$\int_0^\infty f(x)\, y_1(x,\cdot)\, dx = \mathscr{F} P_+(I+K)f,$$

$$\int_{-\infty}^0 f(x)\, y_2(x,\cdot)\, dx = \mathscr{F}^* P_-(I+K)f.$$

(This follows easily from (2.1), (2.2) and the Fubini theorem.) Hence, in view of (2.3) and (2.4),

$$y_1 f = \left[\mathscr{F} P_+ + \frac{V(S)}{2iS}\mathscr{F}^* P_- - \frac{W(S)}{2iS}\mathscr{F} P_-\right](I+K)f, \tag{2.9}$$

$$y_2 f = \left[\mathscr{F}^* P_- + \frac{V(-S)}{2iS}\mathscr{F} P_+ - \frac{W(S)}{2iS}\mathscr{F}^* P_+\right](I+K)f. \tag{2.10}$$

The expressions on the right of (2.9) and (2.10) make sense for any $f \in L^2$ and define $y_1 f$ and $y_2 f$ almost everywhere as measurable functions on $\mathbb{R}$. Hence we take (2.9) and (2.10) as the definition of $y_1 f$ for arbitrary $f \in L^2$.

By Proposition 2(b, c), the functions $V(s)/s$ and $W(s)/s$ are bounded on $\mathbb{R}$ except perhaps near $s = 0$, so $y_1 f$ and $y_2 f$ are square-integrable on $\mathbb{R}\backslash(-\varepsilon, \varepsilon)$ for any $\varepsilon > 0$. Moreover, notice that $V(0) = W(0)$, so that $|V(s) - W(s)| = O(|s|)$ and $|V(-s) - W(s)| = O(|s|)$ as $s \to 0$. It follows that $y_1 f$ and $y_2 f$ are in L^2 provided that either $W(0) = 0$ or the functions $g_\pm = \mathscr{F} P_\pm(I+K)f$ are differentiable at 0, since the latter case

$$|\mathscr{F} P_\pm(I+K)f(s) - \mathscr{F}^* P_\pm(I+K)f(s)| = |g_\pm(s) - g_\pm(-s)| = O(|s|).$$

Similar considerations lead to the following result, which will be used later.

Proposition 7. *If $g \in L^1$, the integrals $\int_0^\infty g(s)\, y_j(x,s)\, ds$ $(j = 1, 2)$ converge absolutely. If $g \in L^2$ and $S^{-1}W(S)\, P_+ g \in L^2$ (which implies that $S^{-1}V(\pm S)\, P_+ g \in L^2$), the L^2 limits*

$$\theta_j = \operatorname*{l.i.m.}_{\xi \to \infty} \int_0^\xi g(s)\, y_j(\cdot, s)\, ds \qquad (j = 1, 2)$$

exist, and

$$\theta_1 = (I+K')\left[P_+\mathscr{F} + P_-\mathscr{F}^* \frac{V(S)}{2iS} - P_-\mathscr{F} \frac{W(S)}{2iS}\right] P_+ g,$$

$$\theta_2 = (I+K')\left[P_-\mathscr{F}^* + P_+\mathscr{F} \frac{V(-S)}{2iS} - P_+\mathscr{F}^* \frac{W(s)}{2iS}\right] P_+ g.$$

3. Expansion Theorems

We first prove an expansion formula which requires no convergence factors but is valid only for functions in a certain dense subspace X of L^2.

Let Ξ be the set of all $g \in L^2$ with compact support such that $\mathscr{F}P_+ g$ and $\mathscr{F}P_- g$ vanish to order (at least) γ_n at u_n, $1 \leqslant n \leqslant N'$. (The latter condition makes sense because Fourier transforms of functions with compact support are restrictions to the real line of entire analytic functions.) We note that since the set $\{u_1, ..., u_{N'}\}$ is symmetric about zero, and since $\gamma_n = \gamma_{n'}$ when $u_n = -u_{n'}$, the condition $g \in \Xi$ implies that $\mathscr{F}^*P_+ g$ and $\mathscr{F}^*P_- g$ also vanish to order γ_n at u_n, $1 \leqslant n \leqslant N'$. We then define

$$X = \{f \in L^2 : (I + K)f \in \Xi\}.$$

Proposition 8. *X is dense in L^2.*

Proof. By Proposition 6, it suffices to show that Ξ is dense in L^2. We shall show that if $\phi \in L^2$ is orthogonal to Ξ then $\phi = 0$.

First, notice that if $g \in L^2$ vanishes outside the interval $[0, a]$ $(a \geqslant 0)$, then $g \in \Xi$ if and only if

$$\int_0^a g(x)\, x^j e^{iu_n x}\, dx = 0 \qquad (0 \leqslant j \leqslant \gamma_n - 1,\ 1 \leqslant n \leqslant N').$$

Thus if ϕ is orthogonal to Ξ, for $x \in [0, a]$ we must have

$$\phi(x) = \sum_{n=1}^{N'} \sum_{j=0}^{\gamma_n - 1} c_{jn} x^j e^{iu_n x} \tag{3.1}$$

for some constants c_{jn}. Since the functions $x^j e^{iu_n x}$ are linearly independent on every interval (being independent solutions of

$$\prod_{n=1}^{N'} \left(\frac{d}{dx} - u_n\right)^{\gamma_n} \phi = 0),$$

the coefficients c_{jn} must be independent of a, so the representation (3.1) of ϕ holds on $[0, \infty)$. But no function of the form (3.1) is in $L^2(0, \infty)$ except the zero function, so $\phi = 0$ on $[0, \infty)$. Likewise $\phi = 0$ on $(-\infty, 0]$.

Proposition 9. *If $f \in X$, then*

(a) *$|f(x)| = O(e^{-\eta|x|})$ as $|x| \to \infty$; in particular, $f \in L^1$;*

(b) *the functions $y_1 f$, $y_2 f$ extend analytically to the strip $|\mathrm{Im}\, s| < \eta$ and vanish to order γ_n at u_n, $1 \leqslant n \leqslant N'$.*

Proof. Since $f = (I+K)^{-1}g$ where g has compact support, (a) follows from Proposition 6. Moreover, since $\mathscr{F}P_\pm g$ and $\mathscr{F}^*P_\pm g$ are entire functions, (b) follows from formulas (2.9) and (2.10). (As we remarked before, the apparent singularity at $s=0$ in these formulas is removable.)

The first step in deriving the expansion theorem is to study the resolvent operator $R_z = (L - zI)^{-1}$ for $z \notin \operatorname{spec}(L)$. For $\operatorname{Im} s > -\eta$ and $x, \xi \in \mathbb{R}$ we set

$$h(x, \xi; s) = \begin{cases} y_1(x,s)\, y_2(\xi,s)/W(s) & \text{if } \xi \leqslant x, \\ y_1(\xi,s)\, y_2(x,s)/W(s) & \text{if } \xi \geqslant x, \end{cases}$$

and for $z \notin \operatorname{spec}(L)$ we set

$$r(x, \xi; z) = h(x, \xi; \sqrt{z}) \qquad (\operatorname{Im} \sqrt{z} > 0).$$

PROPOSITION 10 (Kemp [12]). *If $z \notin \operatorname{spec}(L)$, R_z is given by*

$$R_z f(x) = \int_{-\infty}^{\infty} r(x, \xi; z) f(\xi)\, d\xi.$$

PROPOSITION 11 (Kemp [12]). *If $|\operatorname{Im} s| < \eta$,*

$$sh(x, \xi; s) - sh(x, \xi; -s) = \frac{2is^2}{W(s)\, W(-s)} \sum_{j=1}^{2} y_j(x,s)\, y_j(\xi, -s).$$

PROPOSITION 12 (Kemp [12]). *If $z \notin \operatorname{spec}(L)$ and $0 < \varepsilon < \operatorname{Im} \sqrt{z}$, $\varepsilon < \min\{\operatorname{Im} s_m : 1 \leqslant m \leqslant M\}$, then*

$$\begin{aligned} r(x, \xi; z) = {} & \frac{1}{\pi i} \int_{\operatorname{Im} s = \varepsilon} \frac{sh(x, \xi; s)}{s^2 - z}\, ds \\ & - \sum_{m=1}^{M} \operatorname{Res}_{\lambda = \lambda_m} \frac{r(x, \xi; \lambda)}{\lambda - z}. \end{aligned} \tag{3.2}$$

We wish to rewrite the right-hand side of (3.2) in a more convenient form. First,

$$\operatorname{Res}_{\lambda=\lambda_m} \frac{r(x,\xi;\lambda)}{\lambda - z} = \begin{cases} \operatorname{Res}_{\lambda=\lambda_m} \dfrac{Y_1(x,\lambda)\, Y_2(\xi,\lambda)}{(\lambda - z)\, W(\sqrt{\lambda})} & \text{if } \xi \leqslant x, \\[2ex] \operatorname{Res}_{\lambda=\lambda_m} \dfrac{Y_1(\xi,\lambda)\, Y_2(x,\lambda)}{(\lambda - z)\, W(\sqrt{\lambda})} & \text{if } \xi \geqslant x. \end{cases}$$

But if we use the expression (2.5) for $Y_2(x, \lambda)$, we see that

$$\frac{Y_1(x,\lambda)\, Y_2(\xi,\lambda) - Y_1(\xi,\lambda)\, Y_2(x,\lambda)}{(\lambda - z)\, W(\sqrt{\lambda})} = \frac{Y_1(x,\lambda)\, Z(\xi,\lambda) - Y_1(\xi,\lambda)\, Z(x,\lambda)}{(\lambda - z)[w(Y_1, Z)](\lambda)},$$

which is analytic at $\lambda = \lambda_m$. Hence

$$\operatorname{Res}_{\lambda=\lambda_m} \frac{r(x, \xi; \lambda)}{\lambda - z} = \operatorname{Res}_{\lambda=\lambda_m} \frac{Y_1(x, \lambda)\, Y_2(\xi, \lambda)}{(\lambda - z)\, W(\sqrt{\lambda})}$$

for all x, ξ. Thus, if we set

$$B_m(\lambda) = \frac{-(\lambda - \lambda_m)^{\alpha_m}}{(\alpha_m - 1)!\, W(\sqrt{\lambda})},$$

the usual formula for residues yields

$$-\operatorname{Res}_{\lambda=\lambda_m} \frac{r(x, \xi; \lambda)}{\lambda - z} = \left(\frac{d}{d\lambda}\right)^{\alpha_m - 1} [B_m(\lambda)\, Y_1(x, \lambda)\, Y_2(\xi, \lambda)/(\lambda - z)]|_{\lambda=\lambda_m}. \tag{3.3}$$

Next, we tackle the integral in (3.2). Let δ be a positive number such that $\delta < \eta$, $\delta < \operatorname{Im} \sqrt{z}$, $\delta < \min\{\operatorname{Im} s_m : 1 \leqslant m \leqslant M\}$, and $\delta < \frac{1}{2}\min\{|u_n - u_{n'}| : 1 \leqslant n < n' \leqslant N'\}$. Let Γ'_δ be the contour consisting of the real line, oriented from $-\infty$ to $+\infty$, with the intervals $|s - u_n| < \delta$ replaced by the semicircles of radius δ in the upper half-plane about the points u_n, $1 \leqslant n \leqslant N'$. Then clearly

$$\frac{1}{\pi i} \int_{\operatorname{Im} s = \epsilon} \frac{sh(x, \xi; s)}{s^2 - z}\, ds = \frac{1}{\pi i} \int_{\Gamma'_\delta} \frac{sh(x, \xi; s)}{s^2 - z}\, ds.$$

Let us suppose for the moment that $0 \notin \{u_1, \ldots, u_{N'}\}$. Let Γ''_δ be the contour consisting of the real line, oriented from $-\infty$ to $+\infty$, with the intervals $|s - u_n| < \delta$ $(1 \leqslant n \leqslant N')$ replaced by the semicircles of radius δ about u_n in the upper or lower half-plane according as $u_n > 0$ or $u_n < 0$. Then

$$\frac{1}{\pi i} \int_{\Gamma'_\delta} \frac{sh(x, \xi; s)}{s^2 - z}\, ds = \frac{1}{\pi i} \int_{\Gamma''_\delta} \frac{sh(x, \xi; s)}{s^2 - z}\, ds + 2 \sum_{\substack{u_n < 0 \\ W(u_n) = 0}} \operatorname{Res}_{s=u_n} \frac{sh(x, \xi; s)}{s^2 - z}.$$

The contour Γ''_δ is symmetric about the origin; hence, if we denote by Γ_δ the portion of Γ''_δ lying in the right half-plane, by Proposition 11, we have

$$\begin{aligned} \frac{1}{\pi i} \int_{\Gamma''_\delta} \frac{sh(x, \xi; s)}{s^2 - z}\, ds &= \frac{1}{\pi i} \int_{\Gamma_\delta} \frac{sh(x, \xi; s) - sh(x, \xi; -s)}{s^2 - z}\, ds \\ &= \frac{2}{\pi} \int_{\Gamma_\delta} \sum_{j=1}^{2} \frac{s^2 y_j(x, s)\, y_j(x, -s)}{(s^2 - z)\, W(s)\, W(-s)}\, ds. \end{aligned}$$

On the other hand, we can evaluate the residues at $s = u_n$ in the same way as we evaluated the residues in (3.2), ending up with a formila similar to (3.3). All we need is that

$$2\,\mathrm{Res}_{s=u_n} \frac{sh(x,\xi;s)}{s^2-z} = \sum_{j,k=0}^{\gamma_n-1} c_{jkn}(z)\, y_1^{(j)}(x,u_n)\, y_2^{(k)}(\xi,u_n),$$

where superscripts (j) and (k) denote differentiation with respect to s. Combining these formulas with (3.3), we have

$$\begin{aligned} r(x,\xi;z) &= \frac{2}{\pi}\int_{\Gamma_\delta} \sum_{j=1}^{2} \frac{s^2 y_j(x,s)\, y_j(\xi,-s)}{(s^2-z)\, W(s)\, W(-s)}\, ds \\ &\quad + \sum_{m=1}^{M} \left(\frac{d}{d\lambda}\right)^{\alpha_m-1} [B_m(\lambda)\, Y_1(x,\lambda)\, Y_2(\xi,\lambda)/(\lambda-z)]_{\lambda=\lambda_m} \\ &\quad + \sum_{\substack{W(u_n)=0\\ u_n<0}} \sum_{j,k=0}^{\gamma_n-1} c_{jkn}(z)\, y_1^{(j)}(x,u_n)\, y_2^{(k)}(\xi,u_n). \end{aligned} \tag{3.4}$$

If $0 \in \{u_1,\dots,u_{N'}\}$, (3.4) remains valid provided that two modifications are made. First, Γ_δ should be understood to be the half-line $[\delta,\infty)$ with the intervals $|s-u_n|<\delta$ $(u_n>0)$ replaced by the semicircles of radius δ in the upper half-plane about the points u_n. Second, an extra term must be added, namely,

$$\frac{1}{\pi i}\int_{C_\delta} \frac{sh(x,\xi;s)}{s^2-z}\, ds$$

where C_δ is the semicircle $|s|=\delta$, $\operatorname{Im} s>0$ (oriented from left to right). If we denote by C_δ' the lower semicircle $|s|=\delta$, $\operatorname{Im} s<0$, oriented from left to right, and by γ the multiplicity associated to $0\in\{u_1,\dots,u_{N'}\}$, we obtain as above:

$$\begin{aligned} &\frac{1}{\pi i}\int_{C_\delta} \frac{sh(x,\xi;s)}{s^2-z}\, ds \\ &\quad = \frac{1}{2\pi i}\left[\int_{C_\delta} + \int_{C_\delta'}\right] + \frac{1}{2\pi i}\left[\int_{C_\delta} - \int_{C_\delta'}\right] \\ &\quad = \frac{1}{2\pi i}\int_{C_\delta} \frac{sh(x,\xi;s)-sh(x,\xi;-s)}{(s^2-z)}\, ds + \mathrm{Res}_{s=0} \frac{sh(x,\xi;s)}{s^2-z} \\ &\quad = \frac{1}{\pi}\int_{C_\delta} \sum_{j=1}^{2} \frac{s^2 y_j(x,s)\, y_j(\xi,-s)}{(s^2-z)\, W(s)\, W(-s)}\, ds \\ &\qquad + \sum_{j,k=0}^{\gamma-1} c_{jk}(z)\, y_1^{(j)}(x,0)\, y_2^{(k)}(\xi,0). \end{aligned}$$

PROPOSITION 13. *If* $f \in X$ *and* $z \notin \operatorname{spec}(L)$,

$$R_z f(x) = \frac{2}{\pi} \int_0^{\infty} \sum_{j=1}^{2} \frac{s^2 y_j f(-s)\, y_j(x, s)}{(s^2 - z)\, W(s)\, W(-s)}\, ds + \sum_{m=1}^{M} [((\cdot) - z)^{-1} \cdot B_m \cdot Y_2 f \cdot Y_1(x, \cdot)]^{(\alpha_m - 1)}(\lambda_m), \quad (3.5)$$

where the integral converges absolutely for all $x \in \mathbb{R}$.

Proof. We combine Proposition 10 with formula (3.4) (with appropriate modifications if $0 \in \{u_1, ..., u_{N'}\}$). Since $f \in L^1$ and

$$\int_{-\infty}^{\infty} |s^2 - z|^{-1}\, ds < \infty, \qquad \sup_{x, \xi \in \mathbb{R}, s \in \Gamma_\delta} \left| \frac{s^2 y_j(x, s)\, y_j(\xi, -s)}{W(s)\, W(-s)} \right| < \infty$$

(by Propositions 1, 2(c), and 3), we can interchange the order of integration of obtain

$$\begin{aligned} R_z f(x) = {} & \frac{2}{\pi} \int_{\Gamma_\delta} \sum_{j=1}^{2} \frac{s^2 y_j f(-s)\, y_j(x, s)}{(s^2 - z)\, W(s)\, W(-s)}\, ds \\ & + \sum_{m=1}^{M} [((\cdot) - z)^{-1} \cdot B_m \cdot Y_2 f \cdot Y_1(x, \cdot)]^{(\alpha_m - 1)}(\lambda_m) \\ & + \sum_{\substack{u_n < 0 \\ W(u_n) = 0}} \sum_{j,k=0}^{\gamma_n - 1} C_{jkn}(z)\, y_1^{(j)}(x, u_n)(y_2 f)^{(k)}(u_n). \end{aligned}$$

But by Proposition 9, $y_j f(s)$, and hence also $y_j f(-s)$, vanishes to order γ_n at $s = u_n$, so all terms in the last sum vanish and we can let $\delta \to 0$ to transform the integral over Γ_δ to an integral over $[0, \infty)$. The absolute convergence of this integral is clear since $\int |s^2 - z|^{-1}\, ds < \infty$ and the product of the remaining factors in the integrand is bounded. If $0 \in \{u_1, ..., u_{N'}\}$, the extra terms also vanish as $\delta \to 0$, so we are done.

We can now state the expansion theorem for functions in X. If $f \in X$, then by Proposition 7 and 9, for $0 < \xi < \infty$ the integral

$$T_\xi f(x) = \frac{2}{\pi} \int_0^{\xi} \sum_{j=1}^{2} \frac{s^2 y_j f(-s)\, y_j(x, s)}{W(s)\, W(-s)}\, ds$$

converges absolutely, and the limit

$$Tf = \operatorname*{l.i.m.}_{\xi \to \infty} T_\xi f$$

exists. Also, we set

$$Bf(x) = \sum_{m=1}^{M} [B_m \cdot Y_2 f \cdot Y_1(x, \cdot)]^{(\alpha_m - 1)}(\lambda_m). \tag{3.6}$$

THEOREM 1. *If $f \in X$ then $f = Tf + Bf$.*

In our proof of this result we have borrowed some ideas from Ljance [14]. First, we need two lemmas.

LEMMA 1. *If $g \in L^2$ and $z \notin \operatorname{spec}(L)$, then for $1 \leqslant j \leqslant 2$, $0 \leqslant k \leqslant \alpha_m - 1$, $1 \leqslant m \leqslant M$, $y_j(R_z g)(s) = y_j g(s)/(s^2 - z)$ and $Y_j^{(k)}(R_z g)(\lambda_m) = [((\cdot) - z)^{-1} \cdot Y_j g]^{(k)}(\lambda_m)$.*

Proof. Let $h = R_z g$. If g has compact support, it is clear from Proposition 10 and the definition of $r(x, \xi; z)$ that h decays exponentially at $\pm\infty$. Thus

$$\begin{aligned} y_j g(s) &= \int_{-\infty}^{\infty} (l - z)\, h(x)\, y_j(x, s)\, dx \\ &= \int_{-\infty}^{\infty} h(x)(l - z)\, y_j(x, s)\, dx = (s^2 - z)\, y_j h(s). \end{aligned}$$

The same relation then holds for any $g \in L^2$ by a simple limiting argument using (2.9) and (2.10). Likewise, by Proposition 5(c) we have

$$Y_j^{(k)} g(\lambda_m) = (\lambda_m - z)\, Y_j^{(k)} h(\lambda_m) + k Y_j^{(k-1)} h(\lambda_m).$$

On solving recursively for $Y_j^{(k)} h(\lambda_m)$, we obtain the desired result.

LEMMA 2. *If $f \in X$ and $h \in L^2$, then*

$$\int_{-\infty}^{\infty} Tf(x)\, h(x)\, dx = \frac{2}{\pi} \sum_{j=1}^{2} \int_0^{\infty} \frac{s^2 y_j f(-s)\, y_j h(s)}{W(s)\, W(-s)}\, ds. \tag{3.7}$$

Proof. If $W(0) = 0$, the map $h \to y_j h$ is bounded on L^2, and

$$\frac{S^2}{W(S)\, W(-S)}\, y_j f(-(\cdot)) \in L^2$$

by Propositions 2(c), 9(b). If $W(0) \neq 0$, the map

$$h \to \frac{S}{|S| + 1}\, y_j h$$

is bounded on L^2, and

$$\frac{S(|S|+1)}{W(S)\,W(-S)}\,y_j f(-(\cdot)) \in L^2.$$

In either case, for fixed $f \in X$, the integral on the right of (3.7) defines a bounded linear functional on $h \in L^2$. The same is clearly true of the integral on the left of (3.7), so it suffices to show that the two integrals are equal when h lies in the dense subspace $L^2 \cap L^1$. In this case, however, we have

$$\begin{aligned}\int_{-\infty}^{\infty} Tf(x)\,h(x)\,dx &= \lim_{\xi\to\infty} \int_{-\infty}^{\infty} T_\xi f(x)\,h(x)\,dx \\ &= \lim_{\xi\to\infty} \int_{-\infty}^{\infty} \int_0^{\xi} \frac{2}{\pi} \sum_{j=1}^{2} \frac{s^2 y_j f(-s)\,y_j(x,s)}{W(s)\,W(-s)}\,h(x)\,ds\,dx,\end{aligned}$$

and since $y_j(x, s)$ is bounded on $\mathbb{R} \times \mathbb{R}$, by Proposition 9(b) we can interchange the order of integration to obtain (3.7).

Proof of Theorem 1. First, from the definition of r it is clear that $r(x, \xi; z) = r(\xi, x; z)$. This implies that for any $f, g \in L^2$,

$$\int_{-\infty}^{\infty} f(x)\,R_z g(x)\,dx = \int_{-\infty}^{\infty} R_z f(x)\,g(x)\,dx. \tag{3.8}$$

It also implies that we can interchange x and ξ in (3.4), and hence that we can replace $y_j f(-s)\,y_j(x, s)$ by $y_j(x, -s)\,y_j f(s)$ and $Y_2 f \cdot Y_1(x, \cdot)$ by $Y_1 f \cdot Y_2(x, \cdot)$ in (3.5).

Now, fix $z \notin \operatorname{spec}(L)$, and suppose $f \in X$ and $g \in X$. By the preceding remarks,

$$\begin{aligned}&\int_{-\infty}^{\infty} f(x)\,R_z g(x)\,dx \\ &\quad= \int_{-\infty}^{\infty} f(x)\,\frac{2}{\pi} \int_0^{\infty} \sum_{j=1}^{2} \frac{s^2 y_j g(s)\,y_j(x,-s)}{(s^2-z)\,W(s)\,W(-s)}\,ds\,dx \\ &\qquad+ \int_{-\infty}^{\infty} f(x) \sum_{m=1}^{M} [((\cdot)-z)^{-1} \cdot B_m \cdot Y_1 g \cdot Y_2(x,\cdot)]^{(\alpha_m-1)}(\lambda_m)\,dx \\ &\quad= \frac{2}{\pi} \int_0^{\infty} \sum_{j=1}^{2} \frac{s^2 y_j g(s)\,y_j f(-s)}{(s^2-z)\,W(s)\,W(-s)}\,ds \\ &\qquad+ \sum_{m=1}^{M} [(\cdot)-z)^{-1} \cdot B_m \cdot Y_1 g \cdot Y_2 f]^{(\alpha_m-1)}(\lambda_m).\end{aligned}$$

(The interchange of integrals is easily justified since $f \in L^1$.) On the other hand, by Lemmas 1 and 2,

$$\int_{-\infty}^{\infty} Tf(x)\, R_z\, g(x) = \frac{2}{\pi} \int_0^{\infty} \sum_{j=1}^{2} \frac{s^2 y_j\, g(s)\, y_j f(-s)}{(s^2 - z)\, W(s)\, W(-s)}\, ds$$

and

$$\int_{-\infty}^{\infty} Bf(x)\, R_z\, g(x)\, dx = \sum_{m=1}^{M} [((\cdot) - z)^{-1} \cdot B_m \cdot Y_1 g \cdot Y_2 f]^{(\alpha_m - 1)}(\lambda_m).$$

Hence, by (3.8),

$$\int_{-\infty}^{\infty} R_z f(x)\, g(x)\, dx = \int_{-\infty}^{\infty} f(x)\, R_z\, g(x)\, dx = \int_{-\infty}^{\infty} Tf(x)\, R_z\, g(x)\, dx$$
$$+ \int_{-\infty}^{\infty} Bf(x)\, R_z\, g(x)\, dx = \int_{-\infty}^{\infty} R_z(Tf + Bf)(x)\, g(x)\, dx.$$

Since X is dense in L^2, it follows that $R_z f = R_z(Tf + Bf)$, and since R_z is injective, $f = Tf + Bf$. The proof is complete.

We now show how to modify the expansion formula so that it is valid for all $f \in L^2$. First, we recall some definitions.

The Hardy space H^2 is the set of all analytic functions F on the upper half-plane such that

$$\sup_{t>0} \int_{-\infty}^{\infty} |F(s + it)|^2\, ds < \infty.$$

Functions in H^2 possess boundary values almost everywhere on the real line, which define elements of L^2. Thus we can identify H^2 with a subspace of L^2; under this identification, the Paley–Wiener characterization of H^2 [17] states that

$$H^2 = \mathscr{F} P_+(L^2) = \mathscr{F}^* P_-(L^2).$$

We denote by A^∞ the set of bounded analytic functions on the upper half-plane which extend analytically to some neighborhood of the real axis, and we note that if $F \in H^2$ and $G \in A^\infty$ then $FG \in H^2$.

As in [9], an *approximating family for* L is a collection $\{\phi_\epsilon\}_{\epsilon>0}$ of functions in $H^2 \cap A^\infty$ with the following properties:

(a) $\sup_{\epsilon>0} \sup_{\operatorname{Im} s>0} |\phi_\epsilon(s)| \leqslant 1$,

(b) $\phi_\epsilon(s)$ vanishes to order γ_n at $s = u_n$, $1 \leqslant n \leqslant N'$,

(c) $\lim_{\epsilon\to 0} \phi_\epsilon(s) = 1$ for almost every $s \in \mathbb{R}$.

Given an approximating family $\{\phi_\epsilon\}$ for L, we define the functions $y_1^\epsilon(x, s)$ and $y_2^\epsilon(x, s)$ for $x, s \in \mathbb{R}$ by

$$y_1^\epsilon(x, s) = \begin{cases} \phi_\epsilon(s)\, y_1(x, s) & \text{if} \quad x \geqslant 0, \\ (2is)^{-1}[\phi_\epsilon(s)\, V(s)\, y_2(x, s) & \\ \qquad - \phi_\epsilon(-s)\, W(s)\, y_2(x, -s)] & \text{if} \quad x < 0, \end{cases}$$

$$y_2^\epsilon(x, s) = \begin{cases} \phi_\epsilon(s)\, y_2(x, s) & \text{if} \quad x \leqslant 0, \\ (2is)^{-1}[\phi_\epsilon(s)\, V(-s)\, y_1(x, s) & \\ \qquad - \phi_\epsilon(-s)\, W(s)\, y_1(x, -s)] & \text{if} \quad x > 0. \end{cases}$$

Notice that $y_1^\epsilon(x, s)$ and $y_2^\epsilon(x, s)$ are solutions of $ly = s^2 y$ for $x > 0$ and $x < 0$, but are usually discontinuous at $x = 0$. Moreover, as $\varepsilon \to 0$, $y_j^\epsilon(x, s) \to y_j(x, s)$ uniformly in x, for almost every s, by Proposition 3.

If $f \in L^2$ has compact support, we set

$$y_j^\epsilon f(s) = \int_{-\infty}^{\infty} f(x)\, y_j^\epsilon(x, s)\, dx \qquad (j = 1, 2).$$

As before, we extend this definition to all $f \in L^2$ by setting

$$y_1^\epsilon f = \left[\phi_\epsilon(S) \mathscr{F} P_+ + \frac{\phi_\epsilon(S)\, V(S)}{2iS} \mathscr{F}^* P_- - \frac{\phi_\epsilon(-S)\, W(S)}{2iS} \mathscr{F} P_-\right] (I + K) f, \tag{3.9}$$

$$y_2^\epsilon f = \left[\phi_\epsilon(S) \mathscr{F}^* P_- + \frac{\phi_\epsilon(S)\, V(-S)}{2iS} \mathscr{F} P_+ - \frac{\phi_\epsilon(-S)\, W(S)}{2iS} \mathscr{F}^* P_+\right] (I + K) f. \tag{3.10}$$

What actually occurs in the expansion formula is not $y_j f(s)$ but $y_j f(-s)$: this reflection in the origin has the effect of interchanging $\mathscr{F}$ and $\mathscr{F}^*$ and replacing S by $-S$ in these formulas.

Here, then, is our main theorem.

THEOREM 2. *If $f \in L^2$, $x \in \mathbb{R}$, and $\varepsilon > 0$, the integral*

$$T^\epsilon f(x) = \frac{2}{\pi} \int_0^\infty \sum_{j=1}^{2} \frac{s^2 y_j^\epsilon f(-s)\, y_j(x, s)}{W(s)\, W(-s)}\, ds$$

converges absolutely. Moreover, $T^\epsilon f \in L^2$ *and*

$$f = \underset{\epsilon\to 0}{\text{l.i.m.}}\, T^\epsilon f + Bf \tag{3.11}$$

where Bf *is defined by* (3.6).

Proof. The proof is similar to the proof of Theorem 1 in [9], where some of the arguments are given in greater detail. The idea is as follows. One shows that the operators T^ϵ are bounded on L^2 and that $\text{l.i.m.}_{\epsilon\to 0} T^\epsilon f = Tf$ for all $f \in X$. By Theorem 1, then, (3.11) is true for $f \in X$. Next one shows that the operators T^ϵ are uniformly bounded and converge strongly to a bounded operator as $\varepsilon \to 0$. Since X is dense in L^2, a simple limiting argument then shows that (3.11) is true for all $f \in L^2$.

It is easily verified that because of the properties of the approximating family $\{\phi_\epsilon\}$, not only does the integral defining $T^\epsilon f(x)$ converge absolutely, but so do the integrals of the individual terms

$$\int_0^\infty \frac{s^2 y_j^\epsilon f(-s)\, y_j(x, s)}{W(s)\, W(-s)}\, ds \qquad (j = 1, 2).$$

(It is here that the condition $\phi_\epsilon \in H^2$ is used; details are left to the reader.) We now perform some computations with these integrals. First, using (3.9) and (3.10) and the remarks following them together with Proposition 7, we find that

$$T^\epsilon = \frac{2}{\pi}(I + K')A^\epsilon(I + K)$$

where

$$\begin{aligned} A^\epsilon = {} & \left[P_+\mathscr{F} + P_-\mathscr{F}^* \frac{V(S)}{2iS} - P_-\mathscr{F}\frac{W(S)}{2iS}\right] P_+ \frac{S^2}{W(S)\,W(-S)} \\ & \times \left[\phi_\epsilon(-S)\mathscr{F}^*P_+ - \frac{\phi_\epsilon(-S)\,V(-S)}{2iS}\mathscr{F}P_- + \frac{\phi_\epsilon(S)\,W(-S)}{2iS}\mathscr{F}^*P_-\right] \\ & + \left[P_-\mathscr{F}^* + P_+\mathscr{F}\frac{V(-S)}{2iS} - P_+\mathscr{F}^*\frac{W(S)}{2iS}\right] P_+ \frac{S^2}{W(S)\,W(-S)} \\ & \times \left[\phi_\epsilon(-S)\mathscr{F}P_- - \frac{\phi_\epsilon(-S)\,V(S)}{2iS}\mathscr{F}^*P_+ + \frac{\phi_\epsilon(S)\,W(-S)}{2iS}\mathscr{F}P_+\right]. \end{aligned} \tag{3.12}$$

We multiply this all out, using the fact that P_+ commutes with multiplication operators, and collect the terms beginning with P_+ and P_-, thus:

$$A^\epsilon = P_+A_+^\epsilon + P_-A_-^\epsilon$$

where

$$A_+^\epsilon = \mathscr{F}P_+ \frac{\phi_\epsilon(-S)\,S^2}{W(S)\,W(-S)} \mathscr{F}^*P_+ - \mathscr{F}P_+ \frac{\phi_\epsilon(-S)\,V(-S)S}{2iW(S)\,W(-S)} \mathscr{F}P_-$$
$$+ \mathscr{F}P_+ \frac{\phi_\epsilon(S)S}{2iW(S)} \mathscr{F}^*P_- + \mathscr{F}P_+ \frac{\phi_\epsilon(-S)\,V(-S)S}{2iW(S)\,W(-S)} \mathscr{F}P_-$$
$$+ \mathscr{F}P_+ \frac{\phi_\epsilon(-S)\,V(S)\,V(-S)}{4W(S)\,W(-S)} \mathscr{F}^*P_+ - \mathscr{F}P_+ \frac{\phi_\epsilon(S)\,V(-S)}{4W(S)} \mathscr{F}P_+$$
$$- \mathscr{F}^*P_+ \frac{\phi_\epsilon(-S)S}{2iW(-S)} \mathscr{F}P_- - \mathscr{F}^*P_+ \frac{\phi_\epsilon(-S)\,V(S)}{4W(-S)} \mathscr{F}^*P_+$$
$$+ \mathscr{F}^*P_+ \frac{\phi_\epsilon(S)}{4} \mathscr{F}P_+ \tag{3.13}$$

and A_-^ϵ is a similarly unattractive sum of nine terms. However, things are not really so bad. First, notice that the second and fourth terms on the right of (3.13) cancel out. Moreover, if we introduce the operator J defined by $Jg(s) = g(-s)$, and note that

$$J^2 = I, \qquad J\mathscr{F} = \mathscr{F}J = \mathscr{F}^*, \qquad J\mathscr{F}^* = \mathscr{F}^*J = \mathscr{F},$$
$$P_+J = JP_-, \qquad P_-J = JP_+,$$

and $J\psi(S) = \psi(-S)J$ for any function ψ, we see that the seventh term on the right of (3.13) is equal to

$$-\mathscr{F}^*P_+J^2 \frac{\phi_\epsilon(-S)S}{2iW(-S)} \mathscr{F}P_- = \mathscr{F}P_- \frac{\phi_\epsilon(S)S}{2iW(S)} \mathscr{F}^*P_-,$$

so that the third and seventh terms add up to

$$\mathscr{F} \frac{\phi_\epsilon(S)S}{2iW(S)} \mathscr{F}^*P_-.$$

Likewise, the sixth and eigth terms add up to

$$-\mathscr{F} \frac{\phi_\epsilon(S)\,V(-S)}{4W(S)} \mathscr{F}P_+.$$

Finallt, by Proposition 2(a), (d) the sum of the first and fifth terms is

$$\mathscr{F}P_+ \frac{\phi_\epsilon(-S)}{4} \mathscr{F}^*P_+,$$

which can be combined as above with the ninth term to yield

$$\mathscr{F}^* \frac{\phi_\epsilon(S)}{4} \mathscr{F} P_+ .$$

In short,

$$A_+^\epsilon = \mathscr{F}^* \frac{\phi_\epsilon(S)}{4} \mathscr{F} P_+ + \mathscr{F} \frac{\phi_\epsilon(S) S}{2iW(S)} \mathscr{F}^* P_- - \mathscr{F} \frac{\phi_\epsilon(S)\, V(-S)}{4W(S)} \mathscr{F} P_+ . \tag{3.14}$$

A similar calculation shows that

$$A_-^\epsilon = \mathscr{F} \frac{\phi_\epsilon(S)}{4} \mathscr{F}^* P_- + \mathscr{F}^* \frac{\phi_\epsilon(S) S}{2iW(S)} \mathscr{F} P_+ - \mathscr{F}^* \frac{\phi_\epsilon(S)\, V(S)}{4W(S)} \mathscr{F}^* P_- . \tag{3.15}$$

Some comments are in order at this point. In the first place, products and quotients of multiplication operators by functions are to be understood as multiplication by the corresponding product or quotient of functions; for example,

$$\frac{\phi_\epsilon(S) S}{2iW(S)} = \text{multiplication by } \frac{\phi_\epsilon(s) s}{2iW(s)}.$$

The reason for this bit of pedantry is that the entire product or quotient may be bounded even when the individual factors are not. Indeed, the whole point of introducing the ϕ_ϵ's as we have done is that *every individual term in the expression* (3.13) *for* A_+^ϵ *and in the subsequent calculations is a bounded operator on* L^2, *and likewise for* A_-^ϵ. This is easily verified by using the properties of the approximating family $\{\phi_\epsilon\}$, Proposition 2(b), (c), and the fact that $V(0) = W(0)$. Hence we can combine terms fearlessly to arrive at the simplified expressions (3.14) and (3.15). Moreover, although the original formula (3.12) for A^ϵ contains some unbounded multiplication operators, it certainly displays A^ϵ as a densely defined operator on L^2 which agrees with the bounded operator $P_+ A_+^\epsilon + P_- A_-^\epsilon$ on its domain of definition. Therefore A^ϵ, and hence also T^ϵ, is bounded.

Next, suppose $f \in X$. We can then perform the same calculations as above, but without the ϕ's, to see that

$$Tf = \frac{2}{\pi} (I + K')[P_+ A_+ + P_- A_-](I + K) f,$$

where A_+ and A_- are given by (3.14) and (3.15) with $\phi_\epsilon(S)$ replaced by I. Here the nice properties of the space X replace those of the functions ϕ_ϵ in the verification that the terms in this equation and in the calculations leading to it all make sense. Since $\phi_\epsilon(S) \to I$ strongly as $\varepsilon \to 0$, it follows that

$$\operatorname*{l.i.m.}_{\epsilon \to 0} T^\epsilon f = Tf \qquad \text{for} \quad f \in X.$$

Finally, we shall use (3.14) and (3.15) to show that the operators $P_+A_+^\epsilon$ and $P_-A_-^\epsilon$ are bounded uniformly in ε and converge strongly to bounded operators on L^2 as $\varepsilon \to 0$. The same is then true of the operators T^ϵ, which is what we need to finish the proof.

Let $\psi_1(s)$ be the sum of the singular parts of the Laurent expansions of the function $s/2iW(s)$ about its poles on the real axis. ψ_1 is thus a rational function which vanishes at infinity, such that $\psi_1(s) - s/2iW(s)$ is bounded on $\mathbb{R}$ and $\psi_1\phi_\epsilon \in A^\infty$ for $\varepsilon > 0$. Likewise, let $\psi_2(s)$ (resp. $\psi_3(s)$) be the sum of the singular parts of the Laurent expansions of the function $V(-s)/4W(s)$ (resp. $V(s)/4W(s)$) about its poles on the real axis. As in [9], the essential point is that the operators

$$P_+\mathscr{F}\psi_1(S)\,\phi_\epsilon(S)\mathscr{F}^*P_-, \quad P_+\mathscr{F}\psi_2(S)\,\phi_\epsilon(S)\mathscr{F}P_+,$$
$$P_-\mathscr{F}^*\psi_1(S)\,\phi_\epsilon(S)\mathscr{F}P_+, \quad P_-\mathscr{F}^*\psi_3(S)\,\phi_\epsilon(S)\mathscr{F}^*P_-$$

are all identically zero. This follows from the Paley–Wiener theorem: if $g \in L^2$, then (for example) $\mathscr{F}^*P_-\, g \in H^2$, so also

$$h = \psi_1(S)\,\phi_\epsilon(S)\mathscr{F}^*P_-\, g \in H^2.$$

But then $\mathscr{F}h$ is supported on $(-\infty, 0]$, so $P_+\mathscr{F}h = 0$. Likewise for the other operators. It therefore follows easily that $P_+A_+^\epsilon$ and $P_-A_-^\epsilon$ are bounded uniformly in ε and converge strongly as $\varepsilon \to 0$ to

$$\frac{\pi}{2}P_+ + P_+\mathscr{F}\left[\frac{S}{2iW(S)} - \psi_1(S)\right]\mathscr{F}^*P_-$$
$$- P_+\mathscr{F}\left[\frac{V(-S)}{4W(S)} - \psi_2(S)\right]\mathscr{F}P_+$$

and

$$\frac{\pi}{2}P_- + P_-\mathscr{F}^*\left[\frac{S}{2iW(S)} - \psi_1(S)\right]\mathscr{F}P_+$$
$$- P_-\mathscr{F}^*\left[\frac{V(S)}{4W(S)} - \psi_3(S)\right]\mathscr{F}^*P_-,$$

respectively. (We have used the fact that $\mathscr{F}\mathscr{F}^* = \mathscr{F}^*\mathscr{F} = 2\pi I$.) This completes the proof.

Remark. Although the expansion formula (3.11) involves the singularities of $s^2/W(s)\,W(-s)$, in (3.14) and (3.15) only the singularities of $s/W(s)$ appear (since $V(0) = W(0)$). Thus, it is necessary to assume that $\phi_\epsilon(s)$ vanishes to order γ_n at $s = u_n$, $1 \leqslant n \leqslant N'$, in order to arrive at (3.14) and (3.15); but as far as the boundedness and convergence as $\varepsilon \to 0$ of these operators is concerned, the weaker condition that $\phi_\epsilon(s)$ should vanish to order β_n at $s = t_n$, $1 \leqslant n \leqslant N$, would suffice. It is also the points t_n and their multiplicities β_n that will be significant in the construction of a functional calculus in the next section.

As a corollary of Theorem 2, we obtain the "Parseval equation" associated to the operator L.

THEOREM 3. *For any $f, g \in L^2$,*

$$\int_{-\infty}^{\infty} f(x)\,g(x)\,dx = \lim_{\epsilon\to 0} \frac{2}{\pi} \int_0^\infty \sum_{j=1}^{2} \frac{s^2 y_j^\epsilon f(-s)\, y_j\, g(s)}{W(s)\,W(-s)}\,ds + \sum_{m=1}^{M} [B_m \cdot Y_1 f \cdot Y_2 g]^{(\alpha_m - 1)}(\lambda_m).$$

Proof. By Theorem 2 we have

$$\int_{-\infty}^{\infty} f(x)\,g(x)\,dx = \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} T^\epsilon f(x)\,g(x)\,dx + \int_{-\infty}^{\infty} Bf(x)\,g(x)\,dx.$$

But

$$\int_{-\infty}^{\infty} T^\epsilon f(x)\,g(x)\,dx = \frac{2}{\pi} \int_{-\infty}^{\infty} \sum_{j=1}^{2} \frac{s^2 y_j^\epsilon f(-s)\, y_j\, g(s)}{W(s)\,W(-s)}\,ds$$

and

$$\int_{-\infty}^{\infty} Bf(x)\,g(x)\,dx = \sum_{m=1}^{M} [B_m \cdot Y_2 f \cdot Y_1 g]^{(\alpha_m - 1)}(\lambda_m).$$

The second of these equations is obvious, while the first is proved in the same way as (3.7).

4. Functional Calculus

As a first step in the development of a functional calculus for L, we consider rational functions of L, which can be treated in an elementary way.

Let $\mathscr{Q}$ be the algebra of all rational functions on $\mathbb{C}$ which vanish at infinity and have no poles on spec(L), and let $\mathscr{B}(L^2)$ be the Banach algebra of all bounded operators on L^2 (with the norm topology). By the partial fraction decomposition, each $Q \in \mathscr{Q}$ can be written uniquely in the form

$$Q(\lambda) = \sum_{k=1}^{K} \sum_{j=1}^{\mu_k} c_{jk}(\lambda - z_k)^{-j} \tag{4.1}$$

where $z_1, ..., z_K$ are the poles of Q and $\mu_1, ..., \mu_K$ are their multiplicities. If $Q \in \mathscr{Q}$ is given by (4.1), we define $Q_*(L) \in \mathscr{B}(L^2)$ by

$$Q_*(L) = \sum_{k=1}^{K} \sum_{j=1}^{\mu_k} c_{jk} R_{z_k}^j$$

where $R_z = (L - zI)^{-1}$.

PROPOSITION 14. *The mapping $Q \to Q_*(L)$ is an algebra homomorphism from $\mathscr{Q}$ to $\mathscr{B}(L^2)$.*

This result is not hard to verify directly, the only nontrivial point being multiplicativity. Alternatively, one may invoke the analytic functional calculus for closed operators (cf. Dunford and Schwartz [8]).

PROPOSITION 15. *If $Q \in \mathscr{Q}$ and $f \in X$, then*

$$\begin{aligned} Q_*(L) f(x) = {} & \frac{2}{\pi} \int_0^\infty \sum_{j=1}^{2} \frac{Q(s^2)\, s^2 y_j f(-s)\, y_j(x, s)}{W(s)\, W(-s)}\, ds \\ & + \sum_{m=1}^{M} [Q \cdot B_m \cdot Y_2 f \cdot Y_1(x, \cdot)]^{(\alpha_m - 1)}(\lambda_m). \end{aligned}$$

Proof. It suffices to verify this when Q is of the form $Q(\lambda) = (\lambda - z)^{-k}$, so that $Q_*(L) = R_z^k$. However, we have the well-known identity

$$R_z^k = \frac{1}{(k-1)!} \left(\frac{d}{dz}\right)^{k-1} R_z,$$

so the result follows by differentiating the formula (3.5) for $R_z f(x)$ $(k - 1)$ times with respect to z, noting that the differentiated integrals are again absolutely convergent.

We now proceed to the main theme of this section. As in [9], we define a *function on the spectrum of L* to be a mapping which assigns to each $\lambda \in [0, \infty)$ a complex number $G(\lambda)$ and to each eigenvalue λ_m of L an α_m-

tuple of complex numbers $(G^{(0)}(\lambda_m),\ldots, G^{(\alpha_m-1)}(\lambda_m))$, such that the function $\lambda \to G(\lambda)$ on $[0, \infty)$ is measurable. We consider two functions G, H on the spectrum of L to be equal if $G(\lambda) = H(\lambda)$ for almost every $\lambda \in [0, \infty)$ and $G^{(j)}(\lambda_m) = H^{(j)}(\lambda_m)$ for all j, m. The set of functions on the spectrum of L forms a commutative algebra, with the vector operations defined pointwise and multiplication defined by

$$G \cdot H(\lambda) = G(\lambda)\, H(\lambda) \qquad \text{for} \quad \lambda \in [0, \infty),$$

$$(G \cdot H)^{(j)}(\lambda_m) = \sum_{k=0}^{j} \binom{j}{k} G^{(k)}(\lambda_m)\, H^{(j-k)}(\lambda_m).$$

If ϕ is an analytic function on a neighborhood of spec(L) in $\mathbb{C}$ (for example, if $\phi \in \mathscr{Q}$), ϕ canonically determines a function G_ϕ on the spectrum of L by the equations

$$G_\phi(\lambda) = \phi(\lambda) \qquad \text{for} \quad \lambda \in [0, \infty),$$

$$G_\phi^{(j)}(\lambda_m) = (d^j\phi/d\lambda^j)(\lambda_m).$$

We shall always identify G_ϕ with ϕ. In general, functions G on the spectrum of L are subject to the convention introduced earlier for the quantities $Y_k^{(j)}f(\lambda_m)$: the numbers $G^{(j)}(\lambda_m)$ are to be treated as if they were the derivatives of an analytic function $G(\lambda)$ at λ_m.

For any function G on the spectrum of L, we define the function $\tilde{G}$ on $\mathbb{R}$ by $\tilde{G}(s) = G(s^2)$.

If $s_0 \in \mathbb{R}$ and k is a positive integer, we denote by $D^k(s_0)$ the set of all functions $\phi\colon \mathbb{R} \to \mathbb{C}$ for which there exist complex constants $a_0,\ldots, a_{k-1}$ such that

$$\left|\phi(s) - \sum_{j=0}^{k-1} \frac{a_j}{j!}(s - s_0)^j\right| = O(|s - s_0|^k) \qquad \text{for } s \text{ near } s_0.$$

Clearly, if ϕ is $k-1$ times continuously differentiable near s_0 and $\phi^{(k-1)}$ satisfies a Lipschitz condition at s_0 then $\phi \in D^k(s_0)$. Moreover, for any $\phi \in D^k(s_0)$ the numbers $a_0,\ldots, a_{k-1}$ are obviously uniquely determined. They are called the *de la Vallée Poussin derivatives* of ϕ at s_0, and we shall denote them by $\phi^{(0)}(s_0),\ldots, \phi^{(k-1)}(s_0)$.

We define Φ to be the set of all functions G on the spectrum of L such that $\tilde{G}$ is bounded and agrees almost everywhere with a function $\phi \in \bigcap_{n=1}^{N} D^{\beta_n}(t_n)$ (in which case we shall set $\tilde{G}^{(j)}(t_n) = \phi^{(j)}(t_n)$ for $0 \leqslant j \leqslant \beta_n - 1$, $1 \leqslant n \leqslant N$). Notice that $\mathscr{Q} \subset \Phi$. We define a norm on Φ by

$$\|G\|_\Phi = \operatorname*{ess.\,sup}_{\lambda > 0} |G(\lambda)| + \sum_{n=1}^{N} \sum_{j=0}^{\beta_n - 1} |\tilde{G}^{(j)}(t_n)|$$
$$+ \sum_{n=1}^{N} \operatorname*{ess.\,sup}_{|s-t_n| \leqslant 1} \left| \tilde{G}(s) - \sum_{j=0}^{\beta_n - 1} \frac{\tilde{G}^{(j)}(t_n)}{j!} (s - t_n)^j \right| |s - t_n|^{-\beta_n}$$
$$+ \sum_{m=1}^{M} \sum_{j=0}^{\alpha_m - 1} |G^{(j)}(\lambda_m)|.$$

PROPOSITION 16. *Φ is a commutative Banach algebra with identity.*

The verification of this fact is entirely straightforward and is left to the reader. (The identity element 1 of Φ is of course defined by the analytic function $\phi(\lambda) \equiv 1$.) One remark is in order: with our definition of the norm on Φ, we have $\|GH\|_\Phi \leqslant C\|G\|_\Phi \|H\|_\Phi$ for some $C \geqslant 1$ (independent of G and H) and $\|1\|_\Phi = 1 + N + M$. Normally one requires of a Banach algebra with identity that $\|xy\| \leqslant \|x\| \|y\|$ and $\|1\| = 1$. This point is of no importance to us, but it may be remedied in the usual fashion: replace $\| \; \|_\Phi$ by the equivalent norm

$$\|G\|'_\Phi = \sup_{H \in \Phi, H \neq 0} \|GH\|_\Phi / \|H\|_\Phi.$$

In addition to the norm topology on Φ, we shall need another notion of convergence in Φ which, for want of a better name, we shall call σ-convergence. Namely, we say that a sequence $\{G_k\}$ in Φ σ-converges to $G \in \Phi$ (symbolically: $G_k \to_\sigma G$) if

(a) $G_k(\lambda) \to G(\lambda)$ for a.e. $\lambda \in [0, \infty)$,

(b) $G_k^{(j)}(\lambda_m) \to G^{(j)}(\lambda_m)$ for $0 \leqslant j \leqslant \alpha_m - 1$, $1 \leqslant m \leqslant M$,

(c) $\tilde{G}_k^{(j)}(t_n) \to \tilde{G}^{(j)}(t_n)$ for $0 \leqslant j \leqslant \beta_n - 1$, $1 \leqslant n \leqslant N$,

(d) the functions $\tilde{G}_k(s)$ and

$$\left[\tilde{G}_k(s) - \sum_{j=0}^{\beta_n - 1} \frac{\tilde{G}_k^{(j)}(t_n)}{j!} (s - t_n)^j \right] (s - t_n)^{-\beta_n} \qquad (1 \leqslant n \leqslant N)$$

are essentially bounded on $\mathbb{R}$, the bound being uniform in k. We note that conditions (a), (c), and (d) are fulfilled if $\tilde{G}_k$ and $\tilde{G}$ are C^∞, $\tilde{G}_k^{(j)}$ is bounded uniformly in k for each j, and $\tilde{G}_k^{(j)} \to \tilde{G}^{(j)}$ uniformly on compact sets for all j.

Remark. The algebra Φ properly includes the algebra denoted by Φ in [9]. Theorem 3 and Proposition 6 of [9] can be sharpened by using the present algebra Φ and the notion of σ-convergence.

If $\mathscr{A}$ is a subset of Φ, we denote by $[\mathscr{A}]_\sigma$ the set of all limits of σ-convergent sequences in $\mathscr{A}$. Since σ-convergence is not defined by a metric on Φ, $[\mathscr{A}]_\sigma$ itself will generally not be closed under σ-convergence.

PROPOSITION 17. $\Phi = [[[\mathscr{Q}]_\sigma]_\sigma]_\sigma$.

For the proof we shall need the following well-known facts about approximations to the identity on $\mathbb{R}$, proofs of which can be found, for example, in Stein [18].

LEMMA 3. *Let ϕ be a nonnegative C^∞ function on $\mathbb{R}$ such that*

$$\int_{-\infty}^{\infty} \phi(s)\, ds = 1, \qquad \int_{-\infty}^{\infty} [\sup_{|t| \geqslant |s|} \phi(t)]\, ds < \infty,$$

and $\phi^{(j)} \in L^1$ for all j. For $k = 1, 2, 3,\ldots$, set $\phi_k(s) = k\phi(ks)$, and for $f \in L^\infty$, set

$$f_k(s) = f * \phi_k(s) = \int_{-\infty}^{\infty} f(t)\, \phi_k(s-t)\, dt.$$

Then f_k is C^∞,

$$\|f_k^{(j)}\|_{L^\infty} \leqslant \|\phi_k^{(j)}\|_{L^1} \|f\|_{L^\infty} \qquad \text{for all } j, k,$$

and $f_k \to f$ almost everywhere as $k \to \infty$. If also f is C^∞ and the derivatives of f are all bounded, then for all $j \geqslant 0$, $\|f_k^{(j)}\|_{L^\infty} \leqslant \|f^{(j)}\|_{L^\infty}$ and $f_k^{(j)} \to f^{(j)}$ uniformly on compact sets as $k \to \infty$.

We shall also need the following fact, whose proof is a matter of simple linear algebra: cf. the proof of Proposition 5 in [9].

LEMMA 4. *Let Φ_0 be the set of all $G \in \Phi$ such that $G^{(j)}(\lambda_m) = 0$ for $0 \leqslant j \leqslant \alpha_m - 1$, $1 \leqslant m \leqslant M$ and $\tilde{G}^{(j)}(t_n) = 0$ for $0 \leqslant j \leqslant \beta_n - 1$, $1 \leqslant n \leqslant N$. Then for each $G \in \Phi$ there exist $G_0 \in \Phi_0$ and $Q \in \mathscr{Q}$ such that $G = G_0 + Q$.*

Proof of Proposition 17. By Lemma 4, it suffices to show that $\Phi_0 \subset [[[\mathscr{Q}]_\sigma]_\sigma]_\sigma$. The proof will be accomplished in three steps.

Step 1. Suppose $G \in \Phi_0$. We claim that there is a sequence $\{G_k\} \subset \Phi$ which σ-converges to G, such that $\tilde{G}_k$ is C^∞, $G_k^{(j)}$ is bounded for all j, k, and $G_k^{(j)}(\lambda_m) = 0$ for all j, k, m.

Choose a C^∞ function ϕ on $\mathbb{R}$ such that $\phi(s) = \phi(-s) \geqslant 0$ for all s, $\phi(s) = 0$ for $|s| \geqslant 1$, and $\int \phi(s)\, ds = 1$. (Thus ϕ satisfies the hypotheses of Lemma 3.) Set $\phi_k(s) = k\phi(ks)$ and $\tilde{G}_k = \tilde{G} * \phi_k$, and note that since $\tilde{G}$ and ϕ_k are even functions, so is $\tilde{G}_k$. Then define G_k on the spectrum of L by $G_k(\lambda) = \tilde{G}_k(\sqrt{\lambda})$ for $\lambda \geqslant 0$ and $G_k^{(j)}(\lambda_m) = 0$ for all j, m. By Lemma 3, $\tilde{G}_k$ is C^∞, the derivatives of $\tilde{G}_k$ are bounded, $\|\tilde{G}_k\|_{L^\infty} \leqslant \|\tilde{G}\|_{L^\infty}$, and $\tilde{G}_k \to \tilde{G}$ a.e.

Next, we examine the derivatives $\tilde{G}_k^{(j)}(t_n)$. Fix an n; by making the change of variables $s \to s - t_n$ we may assume that $t_n = 0$, and we set $\beta = \beta_n$. Since $\tilde{G}_k^{(j)} = \tilde{G} * \phi_k^{(j)}$ and $\phi_k^{(j)}(s) = k^{j+1}\phi^{(j)}(ks)$, we have

$$\begin{aligned}
|\tilde{G}_k^{(j)}(0)| &= \left| k^{j+1} \int_{-1/k}^{1/k} \phi^{(j)}(-kt)\, \tilde{G}(t)\, dt \right| \\
&\leqslant k^{j+1} \int_{-1/k}^{1/k} \|\phi^{(j)}\|_{L^\infty} \|G\|_\Phi\, |t|^\beta\, dt \\
&\leqslant Ck^{j-\beta} \to 0 \qquad \text{as } k \to \infty.
\end{aligned}$$

Moreover

$$\begin{aligned}
&\tilde{G}_k(s) - \sum_{j=0}^{\beta-1} \frac{G_k^{(j)}(0)}{j!} s^j \\
&\quad = \int_{-\infty}^{\infty} k \left[\phi(ks - kt) - \sum_{j=0}^{\beta-1} \frac{\phi^{(j)}(-kt)}{j!} (ks)^j \right] \tilde{G}(t)\, dt. \qquad (4.2)
\end{aligned}$$

The integrand on the right is zero unless $|t - s| < k^{-1}$ or $|t| < k^{-1}$, and by Taylor's theorem the expression in square brackets is bounded by a constant, independent of k, times $|ks|^\beta$. We distinguish two cases: either $|s| \leqslant 2k^{-1}$ or $|s| > 2k^{-1}$. In the former case the intervals $|t - s| < k^{-1}$ and $|t| < k^{-1}$ are both contained in the interval $|t| < 3k^{-1}$, and (4.2) is bounded by

$$\int_{-3/k}^{3/k} k[C\,|ks|^\beta]\, \|G\|_\Phi\, |t|^\beta\, dt \leqslant C'\, |s|^\beta,$$

with C' independent of k. In the latter case the intervals $|t - s| < k^{-1}$ and $|t| < k^{-1}$ are disjoint. The integral over $|t| < k^{-1}$ is bounded by $C'\,|s|^\beta$ as above. Also, when $|t - s| < k^{-1}$ we have $\phi^{(j)}(-kt) = 0$, so the integral over $|t - s| < k^{-1}$ is bounded by

$$\begin{aligned}
&\int_{s-(1/k)}^{s+(1/k)} k\phi(ks - kt)\, \|G\|_\Phi\, |t|^\beta\, dt \\
&\qquad \leqslant \|G\|_\Phi\, (|s| + (1/k))^\beta \int_{s-(1/k)}^{s+(1/k)} k\phi(ks - kt)\, dt \\
&\qquad \leqslant \|G\|_\Phi\, (3\,|s|/2)^\beta \int_{-1}^{1} \phi(u)\, du = \|G\|_\Phi\, (3/2)^\beta\, |s|^\beta.
\end{aligned}$$

This completes the proof that $G_k \to_\sigma G$.

Step 2. Suppose that $H \in \Phi$ is such that $\tilde{H}$ is C^∞, $\tilde{H}^{(j)}$ is bounded for every j, and $H^{(j)}(\lambda_m) = 0$ for all j, m. Let $\psi(s) = [\pi(s^2 + 1)]^{-1}$. Then ψ satisfies the hypotheses of Lemma 3, and $\psi(s) = \psi(-s)$. As in Step 1, set

$\psi_k(s) = k\psi(ks)$, $\tilde{H}_k = \tilde{H} * \psi_k$, and then define H_k on the spectrum of L by $H_k(\lambda) = \tilde{H}_k(\sqrt{\lambda})$ for $\lambda \in [0, \infty)$ and $H_k^{(j)}(\lambda_m) = 0$ for all j, m. By Lemma 3, we see that $H_k \to_\sigma H$.

Step 3. Suppose $F \in \Phi$ satisfies $F^{(j)}(\lambda_m) = 0$ for all j, m, and $\tilde{F} = \tilde{H} * \psi_A$ where H, ψ are as in Step 2 and A is a fixed integer which is sufficiently large that $(t \pm i/A)^2$ is not an eigenvalue of L for any $t \in \mathbb{R}$. We shall construct a sequence $\{Q_k\}$ in $\mathscr{Q}$ which σ-converges to F, and this will complete the proof. First, we set

$$\tilde{P}_k(s) = \sum_{j=-k^2}^{k^2} \frac{A\tilde{H}(j/k)}{k\pi[(As - Aj/k)^2 + 1]}.$$

Thus $\tilde{P}_k(s)$ is a Riemann sum for the integral

$$\int_{-\infty}^{\infty} \frac{A\tilde{H}(t)\,dt}{\pi[(As - At)^2 + 1]}\,dt = \tilde{F}(s),$$

and $\tilde{P}_k^{(j)}(s)$ is the corresponding Riemann sum for the integral defining $\tilde{F}^{(j)}(s)$, for all j. It is thus easy to see that for all $j \geqslant 0$, $\tilde{P}_k^{(j)}$ is bounded uniformly in k and converges to $\tilde{F}^{(j)}$ uniformly on compact sets. Moreover, $\tilde{P}_k$ is an even rational function of s, and hence is actually a rational function of s^2, which we denote by P_k: thus $P_k(s^2) = \tilde{P}_k(s)$. Finally, we set

$$Q_k(\lambda) = P_k(\lambda) \prod_{m=1}^{M} \left(\frac{\lambda - \lambda_m}{\lambda - \lambda_m + i/k}\right)^{\alpha_m}.$$

Q_k is a rational function which vanishes at infinity and has poles at $(j/k + i/A)^2(-k^2 \leqslant j \leqslant k^2)$ and at $\lambda_m - i/k$ $(1 \leqslant m \leqslant M)$, so that $Q_k \in \mathscr{Q}$, at least for k sufficiently large. Also, $Q_k^{(j)}(\lambda_m) = 0$ for $0 \leqslant j \leqslant \lambda_m - 1$ and $1 \leqslant m \leqslant M$, and for $j \geqslant 0$, $\tilde{Q}_k^{(j)}$ is bounded on $\mathbb{R}$ uniformly in k and converges to $\tilde{F}^{(j)}$ uniformly on compact sets. Thus $Q_k \to_\sigma F$, and we are done.

We now come to our final major result.

THEOREM 4. (a) *Fix an approximating family* $\{\phi_\epsilon\}$ *for* L. *If* $G \in \Phi$, $f \in L^2$, *and* $\varepsilon > 0$, *the integral*

$$G(L)^\epsilon f(x) = \frac{2}{\pi} \int_0^\infty \sum_{j=1}^{2} \frac{G(s^2)\, s^2 y_j^\epsilon f(-s)\, y_j(x, s)}{W(s)\, W(-s)}\, ds$$

converges absolutely for all $x \in \mathbb{R}$ *and defines a function in* L^2. *Moreover,* $\text{l.i.m.}_{\epsilon \to 0}\, G(L)^\epsilon f$ *exists.*

(b) *For $G \in \Phi$ and $f \in L^2$, set*

$$G(L)f = \operatorname*{l.i.m.}_{\epsilon \to 0} G(L)^\epsilon f + \sum_{m=1}^{M} [G \cdot B_m \cdot Y_2 f \cdot Y_1(\cdot, \cdot)]^{(\alpha_m - 1)}(\lambda_m).$$

Then $G(L) \in \mathscr{B}(L^2)$ and $G(L)$ is independent of the choice of approximating family for L.

(c) *The correspondence $G \to G(L)$ is a Banach algebra homomorphism from Φ to $\mathscr{B}(L^2)$.*

(d) *If $\{G_i\}$ is a sequence in Φ which σ-converges to $G \in \Phi$, then $G_i(L)$ converges strongly to $G(L)$.*

(e) *If $G \in \mathscr{Q}$ then $G(L) = G_*(L)$.*

Proof. (Cf. the proof of Theorem 3 in [9].) The absolute convergence of $G(L)^\epsilon f(x)$ follows immediately from the absolute convergence of $T^\epsilon f(x)$ (Theorem 2). To prove the remaining assertions in (a) and (b), we repeat the calculations in the proof of Theorem 2, inserting a factor of $G(S^2)$ into each term, and we find that

$$\begin{aligned} G(L)^\epsilon = \frac{2}{\pi}(I + K') \Bigg[& P_+ \mathscr{F}^* \frac{G(S^2)\,\phi_\epsilon(S)}{4} \mathscr{F} P_+ \\ & + P_+ \mathscr{F} \frac{G(S^2)\,\phi_\epsilon(S) S}{2iW(S)} \mathscr{F}^* P_- - P_+ \mathscr{F} \frac{G(S^2)\,\phi_\epsilon(S)\, V(-S)}{4W(S)} \mathscr{F} P_+ \\ & + P_- \mathscr{F} \frac{G(S^2)\,\phi_\epsilon(S)}{4} \mathscr{F}^* P_- + P_- \mathscr{F} \frac{G(S^2)\,\phi_\epsilon(S) S}{2iW(S)} \mathscr{F} P_+ \\ & - P_- \mathscr{F}^* \frac{G(S^2)\,\phi_\epsilon(S)\, V(S)}{4W(S)} \mathscr{F}^* P_- \Bigg] (I + K). \end{aligned} \tag{4.3}$$

This shows that $G(L)^\epsilon \in \mathscr{B}(L^2)$. As for the behavior of $G(L)^\epsilon$ as $\varepsilon \to 0$, the first and fourth terms inside the brackets in (4.3) clearly remain bounded and converge strongly to

$$P_+ \mathscr{F}^* G(S^2) \mathscr{F} P_+/4 \qquad \text{and} \qquad P_- \mathscr{F} G(S^2) \mathscr{F}^* P_-/4.$$

Next, let ψ_1 be as in the proof of Theorem 2. The second term inside the brackets in (4.3) is then equal to

$$\begin{aligned} & P_+ \mathscr{F} G(S^2)\,\phi_\epsilon(S) \left[\frac{S}{2iW(S)} - \psi_1(S) \right] \mathscr{F}^* P_- \\ & \quad + P_+ \mathscr{F} G(S^2)\,\phi_\epsilon(S)\,\psi_1(S) \mathscr{F}^* P_- \end{aligned} \tag{4.4}$$

The first term of (4.4) converges strongly as $\varepsilon \to 0$ to the corresponding operator with $\phi_\epsilon(S)$ omitted. As in [9], we handle the second term by means of the following elementary fact:

Let $\mathscr{R}$ be the space of rational functions of the form $R(s) = P(s)(s+i)^{-r}$ where $r = (\sum_{n=1}^{N} \beta_n) - 1$ and P is a polynomial of degree at most r. Then for each $G \in \Phi$ there is a unique $R_G \in \mathscr{R}$ such that $R_G^{(j)}(t_n) = \tilde{G}^{(j)}(t_n)$ for $0 \leqslant j \leqslant \beta_n - 1$, $1 \leqslant n \leqslant N$.

Setting $H_G = \tilde{G} - R_G$, so that $G(s^2) = H_G(s) + R_G(s)$, we rewrite the second term of (4.4) as

$$P_+ \mathscr{F} H_G(S)\, \phi_\epsilon(S)\, \psi_1(S) \mathscr{F} P_- + P_+ \mathscr{F} R_G(S)\, \phi_\epsilon(S)\, \psi_1(S) \mathscr{F} P_- . \quad (4.5)$$

The second term of (4.5) vanishes, by the Paley–Wiener argument used in the proof of Theorem 2. Since $|H_G(s)| = O(|s - t_n|^{\beta_n})$ for s near t_n, the first term of (4.5) converges strongly as $\varepsilon \to 0$ to the corresponding operator with $\phi_\epsilon(S)$ omitted. The remaining three terms inside the brackets in (4.3) are handled in exactly the same way. Thus (a) and (b) are proved.

The correspondence $G \to G(L)$ is clearly linear. We now show that it is bounded from Φ to $\mathscr{B}(L^2)$. Indeed, if we trace through the preceding calculations, we find that for some $C_1 > 0$,

$$\|G(L)\|_{\mathscr{B}(L^2)} \leqslant C_1 \left[\operatorname*{ess.sup}_{\lambda \geqslant 0} |G(\lambda)| + \sum_{m=1}^{M} \sum_{j=0}^{\alpha_m - 1} |G^{(j)}(\lambda_m)| \right.$$
$$\left. + \sum_{k=1}^{3} \operatorname*{ess.sup}_{s \in \mathbb{R}} |H_G(s)\, \psi_k(s)| \right]$$

where the ψ_k's are as in the proof of Theorem 2. Since the ψ_k's are linear combinations of terms of the form $(s - t_n)^{-j}$ with $1 \leqslant j \leqslant \beta_n$, $1 \leqslant n \leqslant N$, we need only show that for some $C_2 > 0$,

$$\operatorname*{ess.sup}_{s \in \mathbb{R}} |H_G(s)|\, |s - t_n|^{-j} \leqslant C_2 \|G\|_\Phi \qquad (1 \leqslant j \leqslant \beta_n,\ 1 \leqslant n \leqslant N). \quad (4.6)$$

To see this, we note that there exists $C_3 > 0$ such that for $0 \leqslant k \leqslant \beta_n$,

$$\sup_{s \in \mathbb{R}} |R_G^{(k)}(s)| \leqslant C_3 \sum_{n=1}^{N} \sum_{j=0}^{\beta_n - 1} |R_G^{(j)}(t_n)|$$
$$= C_3 \sum_{n=1}^{N} \sum_{j=0}^{\beta_n - 1} |\tilde{G}^{(j)}(t_n)| \leqslant C_3 \|G\|_\Phi, \quad (4.7)$$

since $\mathscr{R}$ is finite-dimensional and the expression on the right of the first inequality in (4.7) defines a norm on $\mathscr{R}$. Taking $k=0$, we obtain

$$\operatorname*{ess.\,sup}_{s\in\mathbb{R}} |H_G(s)| \leqslant \operatorname*{ess.\,sup}_{s\in\mathbb{R}} (|R_G(s)| + |\tilde{G}(s)|) \leqslant (C_3+1)\,\|G\|_\Phi. \tag{4.8}$$

On the other hand,

$$\begin{aligned} &\operatorname*{ess.\,sup}_{|s-t_n|\leqslant 1} |H_G(s)|\,|s-t_n|^{-\beta_n} \\ &\leqslant \operatorname*{ess.\,sup}_{|s-t_n|\leqslant 1} \left| R_G(s) - \sum_{j=0}^{\beta_n-1} \frac{\tilde{G}^{(j)}(t_n)\,(s-t_n)^j}{j!} \right| |s-t_n|^{-\beta_n} \\ &\quad + \operatorname*{ess.\,sup}_{|s-t_n|\leqslant 1} \left| \sum_{j=0}^{\beta_n-1} \frac{\tilde{G}^{(j)}(t_n)}{j!}(s-t_n)^j - \tilde{G}(s) \right| |s-t_n|^{-\beta_n} \\ &\leqslant (C_3/\beta_n! + 1)\,\|G\|_\Phi, \end{aligned} \tag{4.9}$$

where we have estimated the term involving R_G by using (4.7) together with Taylor's theorem. The estimates (4.8) and (4.9) together imply (4.6).

We now prove (d). If $G_i \to_\sigma G$, the estimate (4.7) shows that $R_{G_i}^{(k)} \to R_G^{(k)}$ uniformly for $0 \leqslant k \leqslant \beta_n$. It follows, as above, that $H_{G_i} \to H_G$ a.e. and that the functions $H_{G_i}(s)(s-t_n)^{-j}$ $(1 \leqslant j \leqslant \beta_n,\ 1 \leqslant n \leqslant N)$ are bounded uniformly in i. Using the arguments in the proof of (a) and (b), it is then easy to see that $G_i(L) \to G(L)$ strongly.

Next, observe that if $f \in X$, $\text{l.i.m.}_{\epsilon\to 0}\, G(L)^\epsilon f$ can be calculated directly from (4.3) without any further machinations: it is equal to the right-hand side of (4.3) with $\phi_\epsilon(S)$ omitted throughout, applied to f. However, the same calculations that lead to (4.3) show that the latter quantity is equal to

$$\frac{2}{\pi}\int_0^\infty \sum_{j=1}^{2} \frac{G(s^2)\, s^2 y_j f(-s)\, y_j(\cdot, s)}{W(s)\, W(-s)}\, ds.$$

Thus if $G \in \mathscr{Q}$, it follows from Proposition 15 that $G(L)f = G_*(L)f$ for $f \in X$. Since $G(L)$ and $G_*(L)$ are bounded, and X is dense in L^2, (e) follows immediately.

It remains to show that $(G\cdot H)(L) = G(L)\,H(L)$ for any $G, H \in \Phi$. However, the map $(G, H) \to G\cdot H$ from $\Phi \times \Phi$ to Φ is easily seen to be continuous with respect to σ-convergence, while composition of operators in $\mathscr{B}(L^2)$ is continuous with respect to strong convergence. (Separate continuity, which is easier to prove, is sufficient.) The multiplicativity of the correspondence $G \to G(L)$ therefore follows from (d), (e), and Propositions 14 and 17. The proof is complete.

ACKNOWLEDGMENTS

The results in this paper were obtained while I enjoyed the hospitality of the Institute for Advanced Study and the University of Utah. I am indebted to J. McKenna for encouraging me to work on these problems in the first place, and to E. Stein and C. Fefferman for some helpful suggestions concerning the material in Section 4.

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