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Book Review

Review of Totally Nonnegative Matrices by Shaun M. Fallat and Charles R. Johnson, Princeton University Press, Princeton and Oxford (2011), xv + 248 pp., Princeton Series in Applied Mathematics, ISBN 978-0-691-12157-4 cloth ☆

The monograph is intended as a self-contained development of the most fundamental properties of the totally nonnegative (abbreviated henceforth by TN) and totally positive (TP) matrices, the class of matrices which have *all* their minors nonnegative and positive, respectively. These matrices arise in a remarkable variety of ways within mathematics and in many areas to which mathematics is applied, see the listing in Section 0.2 of the book under review.

For a long time, the only monographs on these matrices have been the books by Gantmacher and Krein [3] and Karlin [5]; both appeared in the sixties. As an obstacle, the book by Gantmacher and Krein was for a long time available only in Russian and German, with an English translation appearing in 2002.[1] Ando collected 1984 or earlier in the survey paper [1] most of the properties of TN matrices known at that time. Another source of information was the collection of 23 papers [4] presented at a workshop on total positivity held in 1994 in Jaca, Spain. A considerable amount of research has been done since then. So the time was ripe for an update and extension of the preceding monographs.

In 2010, the monograph by Pinkus [8] was published.[2] He prefers the terms *totally positive* and *strictly totally positive* (which is the terminology used in [5]) instead of *totally nonnegative* and *totally positive* (used in [3]). Since the book under review was published only about one year later it is natural to compare both books. Purely quantitatively, the book by Pinkus is about three quarters of the extent of the book by Fallat and Johnson.

The book by Fallat and Johnson has 11 chapters. As in the book by Pinkus, nearly all chapters start with an introduction into the theme of the chapter, often followed by a section in which the necessary definitions, notations, and terms used within this chapter are given. The Introduction in Chapter 0 provides the basic definitions and notation as well as many examples of TN matrices; here tridiagonal matrices are discussed in detail. Applications of the TN matrices are presented, ranging from TP kernels to B-splines. Special emphasis is put on the strong relation to Pólya frequency functions which has produced numerous important results and associated applications.

☆ A working list of errata can be found under http://www.math.uregina.ca/∼sfallat/Research/tnbook.html.

[1] Besides an English translation which appeared as document AEC-tr-4481 (physics) of the Office of Technical Documentation, Department of Commerce, Washington, DC, in April 1961.

[2] See the review of this book in this journal, vol. 433 (2010), pp. 1052–1053.

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Chapter 1 provides the elementary and fundamental properties of TN matrices along with some other useful background material from matrix theory. Chapter 2 is devoted to the important and useful tool of bidiagonal factorization. Many of the results on TN matrices can be derived from the fact that any TN matrix admits a factorization into TN bidiagonal matrices, e.g., from this factorization follow the fact that the TN matrices are the topological closure of the TP matrices and a test for an entrywise nonnegative matrix for being TN/TP as well as a procedure for the construction of TN/TP matrices. The bidiagonal factorization is put into historical perspective, viz. it can be traced back to a paper by Anne Whitney dated from 1952 [9] in which a lemma is proven that may be used to deduce the existence of such a factorization. The bidiagonal factorization is now recognized as the fundamental parametrization of TN matrices. In this chapter, the representation of a bidiagonal factorization in terms of planar diagrams is also introduced, which is employed in the following chapters to prove many results.

Chapter 3 answers the question of the way in which a given matrix can be efficiently tested for being TN or TP. In the next chapter the key results concerning the sign variation diminution of TN and TP matrices are developed, a property which is important for application in Computer Aided Geometric Design, e.g., a linear transformation which is associated with a TP matrix cannot increase the number of the sign changes in a vector.

In Chapter 5 the eigen-structure of square TN matrices is explored. Since TN matrices are entrywise nonnegative, the Perron–Frobenius theory applies. But by the special structure of the TN matrices much more can be said about their eigenvalues (they are nonnegative) and the sign patterns of the entries of the eigenvectors. The eigenvalues of an oscillatory matrix,[3] which is a TN matrix with a TP integral power, are all positive and distinct. Other spectral properties investigated in this chapter are: the interlacing of the eigenvalues of a TN matrix and those of some special principal submatrices, majorization between the eigenvalues and the diagonal elements, eigenvalue inequalities for products of TN matrices, as well as some inverse eigenvalue problems. In Chapter 6 determinantal inequalities for TN matrices are presented. Herein it is shown that some classical determinantal inequalities due to Hadamard, Fischer, and Koteljanskiĭ are members of a class of general multiplicative principal minor inequalities.

The remaining four chapters cover a wide range of specialized topics: the distribution of rank deficient submatrices within a TN matrix (Chapter 7), the Hadamard (i.e., the entrywise) product of TN matrices (Chapter 8), various aspects of matrix completion problems associated with TN matrices (Chapter 9). Of special interest is Section 9.5 in which the question is partially answered which entries of a TN/TP matrix may be increased or decreased without losing the property of being a TN/TP matrix. The results confirm the practical experience of ones own work with TN matrices. Often a small perturbation of some entries in a TN matrix leads to a matrix which is not TN (in contrast to other classes of matrices like the *M* matrices). As a useful result, the maximum allowable perturbations of some single entries of a TN/TP matrix are now quantified. The book is concluded with a brief review of a number of subtopics connected with TN matrices, including powers and roots, subdirect sums, and Perron complements of TN matrices as well as TP/TN polynomial matrices.

The book provides a bibliography of 20 pages length of papers which are cited in the text or others of potential interest to readers (compared to only six pages in Pinkus' book). The bibliography is largely complete. Among the references I am missing are the book [7] on the relation of TN matrices to shape preserving representations in Computer Aided Geometric Design (with a reference in Section 0.2, see below), a paper by R.A. Brualdi and S. Kirkland [2] on TN (0,1)-matrices which contains the result of Theorem 1.6.9, and papers by K.R. Goodearl, S. Launois, and T.H. Lenagan which relate TN matrices with quantum matrices and matrix Poisson varieties, e.g. [6]. However, these papers appeared during the final stages of writing the book, or even later. A comprehensive list of symbols used in the text (of five pages length) at the end of the book facilitates the reading.

A book on TN matrices cannot cover all aspects of the theory and applications. So, in the book under review some are only surveyed, e.g., the relation of TN matrices to shape preserving representations in Computer Aided Geometric Design, e.g. [7]. The book "takes a core, matrix theoretic perspective

[3] Such a matrix is called an *oscillation matrix* in Pinkus' book.

common to all sources of interest in the subject" (p. xiii). As such, numerical issues, e.g., accurate computations with TN matrices are beyond its scope. However, a more algorithmic description of the Neville algorithm and a detailed presentation of its properties (which are somewhat scattered in the text) would have been helpful for those who want to apply this useful tool to practical problems.

The book by Fallat and Johnson is clearly written and very well organized. The proofs are given in necessary detail. Many parts of the book heavily reflect the authors own research, and as a result the presentation is always at the frontline of current research. A good example is Chapter 7, in which the distribution of ranks among submatrices of a TN matrix is explored, which is less free than in a general matrix. It seems that the main results of this chapter are not published by the authors prior to the appearance of the book.

Comparing it with Pinkus' book I would recommend Pinkus' book to a reader who is mainly interested in the classical theory and the historical aspects; as such it is largely self-contained and perfectly suited as a classroom text (although the important topic of bidiagonalization is not made an integral part of the theoretical development). The book by Fallat and Johnson is of greatest value and indispensable for an active researcher in the field of TN matrices and related fields and will certainly be the future reference book for TN matrices, as the books *Matrix Analysis* and *Topics in Matrix Analysis* by one of the authors (C.R.J.) and R.A. Horn became for matrix theory.

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