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The Convergence of Mimetic Discretization for Rough Grids

J. M. HYMAN
Los Alamos National Laboratory
T-7, MS-B284, Los Alamos, NM 87545, U.S.A.

S. STEINBERG*
Department of Mathematics and Statistics, University of New Mexico
Albuquerque, NM 87131, U.S.A.
stanly@math.unm.edu

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Abstract—We prove that the mimetic finite-difference discretizations of Laplace's equation converges on rough logically-rectangular grids with convex cells. Mimetic discretizations for the invariant operators' divergence, gradient, and curl satisfy exact discrete analogs of many of the important theorems of vector calculus. The mimetic discretization of the Laplacian is given by the composition of the discrete divergence and gradient. We first construct a mimetic discretization on a single cell by geometrically constructing inner products for discrete scalar and vector fields, then constructing a finite-volume discrete divergence, and then constructing a discrete gradient that is consistent with the discrete divergence theorem. This construction is then extended to the global grid. We demonstrate the convergence for the two-dimensional Laplace equation with Dirichlet boundary conditions on grids with a lower bound on the angles in the cell corners and an upper bound on the cell aspect ratios. The best convergence rate to be expected is first order, which is what we prove. The techniques developed apply to far more general initial boundary-value problems.

Keywords—Mimetic discretization, Convergence, Finite-volume method.

1. INTRODUCTION

We prove that the solutions of mimetic finite-difference discretizations of Laplace's equation, and their gradients, converge at first order to the solution and gradient of the continuum problem in logically-rectangular grids with convex cells. Numerical examples indicate that the method is first-order convergent on rough grids and second-order convergent on smooth grids.

Mimetic finite-difference methods, derived by the support-operators method [1], create discretizations of the gradient, curl, and divergence that satisfy many properties of the continuum

*Author to whom all correspondence should be addressed.

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operators, for example, that the discrete curl of the discrete gradient is zero, that the discrete gradient and divergence exactly satisfy a discrete analog of the divergence theorem, that if the curl of a discrete vector field is zero, then it is a discrete gradient of a discrete scalar field, etc. [2–8].

The study [9] showed that on nice problems, the mimetic finite-difference method was of comparable accuracy to other methods, and on difficult problems, the mimetic method was superior to the mixed finite-element method. Extensions to higher-order methods can be found in [10–12]. Mimetic finite-difference methods have been applied in two-dimensional diffusion [9,13–15], three-dimensional diffusion [3,16], Maxwell's equations [17,18], hydrodynamics [19,20], and flow through porous media [21]. These successes motivate the need to prove the convergence properties of the method on rough grids. The proof techniques presented in this paper can be extended to general grids with convex cells (with triangles, quadrilaterals, and more complex polygonal cells), to problems with variable—even discontinuous—and anisotropic material properties, to general Robin or mixed boundary conditions, to higher dimensions, and other coordinate systems.

The derived mimetic discretization, on nonuniform grids, are not given by local operators. However, in [14,16], it was observed that by adding some auxiliary discrete scalar variables, the support-operators method could be viewed as local, with some computational advantages. In this paper, we extend this idea to a complete development of the mimetic ideas from a local, that is, single cell point of view. This is done by having discrete scalar variables located at the cell center and at the centers of the cell edges, while the normal component of vector fields are located only at the centers of cell edges. Then one can retain the scalar values at the cell edge center and use the solution methods in [14,16], or these values can be eliminated and the usual support-operators method recovered.

In fact, the development of mimetic finite-difference methods for a general quadrilateral cell in two dimensions is algebraically complex, so we begin all discussions with the one-dimensional case, as this adds clarity, and all calculations can be done explicitly. Many of the formulas in this paper were derived using a computer algebra system. As these systems do not support abstract vector computation, their use for our problems is not straightforward.

Most of the early work on the support-operators method was done by Samarskii *et al.* [22,23] and the term "support-operators method" comes from early translations of this work. The textbooks [24,25] have extensive references to the early literature, and proofs of convergence for finite-difference methods similar to the methods presented here. An alternate convergence proof for the mimetic finite-difference method, based on viewing mimetic discretization as a mixed finite-element method, is contained in [26]. The paper [27] contains proofs of higher-order convergence for similar finite-difference methods in grids whose elements are rectangular, while the paper [28] studies "almost" second-order convergence for variable coefficients and mixed boundary conditions in rectangular grids. The related 1-D natural finite-volume difference discretization introduced in [29] is shown to converge at second order on rough grids. Also, because we use a standard finite-volume discretization of the divergence, our method is a finite-volume method [30], and is closely related to the mixed finite-element method [31–35].

Methods where the solution of a discretization converge at a rate better than the order of the truncation error are called *supraconvergent* in [36]. Although the solutions of the mimetic finite-difference discretization of Laplace's equation are first-order convergent, the mimetic finite-difference discretization of the Laplacian has zero-order truncation error on irregular grids or on uniform grids near the boundary, even in one dimension, so the discretization is supraconvergent. However, both the divergence and gradient have truncation error of at least order one. In our analysis, we estimate the error in the solution of Laplace's equation in terms of the error in the divergence and gradient, as is natural in a mimetic formulation. The methods described in [25] and [24] can also have zero-order truncation error for the Laplacian on irregular grids. These authors rely on decompositions of the errors and special estimates for each part of the error to obtain their convergence results.

In this paper, we prove that the mimetic finite-difference method converges under modest assumptions on the grids. The proof is based on the mimetic properties of the discretization, and thus, will directly generalize to mimetic discretizations of other problems.

We prove our low-order convergence estimates without relying on the smoothness of the grids. We also lay some groundwork for later studying higher-order convergence on such grids. In fact, suppose that we have a smooth map of the unit square to a domain and then we build grids on the domain by mapping rectangular grids on the unit square to the domain [37], then as these grids are refined, the cells approach parallelograms given by the image of a small rectangle transformed by the Jacobian of the map generating the grid. Numerical studies have verified that the mimetic method is second-order accurate for smooth grids, which indicates that our discretization should have special properties for parallelograms. This observation motivates our expressing the error estimates so that we can see what happens when the grid cell is a parallelogram.

In Section 2, we develop the mimetic finite-difference method for a single quadrilateral cell. The first objective is to show that the scheme satisfies a summation by parts (discrete divergence) theorem. The proof of the mimetic properties requires that we define discrete inner products for both scalar and vector fields. In one and two dimensions, the inner product of discrete scalars is simply the product of the values at the cell centers times the size of the cell; the cell edge values of the scalar are not used in the inner product. The inner products of vectors are given by a geometric construction, which is not so simple for two-dimensional irregular quadrilateral cells. The divergence is discretized using standard finite-volume ideas and then the discrete analog of the divergence theorem is used to define the discrete gradient. In two dimensions, the values for the gradient are defined by solving a local system when the scalar edge values are retained, or a global system when the scalar edge values are eliminated. Finally, the inner products are bounded from above and below and then the errors in the divergence and gradient are estimated.

In Section 3, the results for a single cell are translated to a global logically rectangular grid. The discrete gradient satisfies a system of equations that is independent of the cell-edge values of the scalar field. This system is symmetric positive definite and strictly diagonally dominant.

In Section 4, we introduce the global formal operators that allow us to explicitly compute the discrete inner products for scalar and vector fields, the discrete divergence, and most importantly, the system of equations that define the gradient. This is useful for creating and analyzing solution algorithms, and provides a framework for proving a discrete Friedrichs-Poincaré inequality that is critical to the convergence proof.

In Section 5, we discretize the Laplace equation with Dirichlet boundary conditions using the discrete divergence and gradient defined in Section 3. We then introduce an abstract mimetic setting and use this to prove that the error in the solution of the boundary-value problem and its gradient are estimated by the truncation error in the divergence and gradient, observed in Section 3. Throughout the paper, all estimates are given in terms of a cell aspect ratio and the sine of the smallest angle in the grid. Therefore, all of our estimates are uniform and convergence is clear for families of grids where the aspect ratios are bounded above and the angles are bounded below.

2. THE LOCAL MIMETIC DISCRETIZATION

The central goal in the mimetic method is to find a discretization of the divergence and gradient operators that satisfy exactly a discrete analog of the divergence theorem. For a grid cell C, this theorem states that

$$\int_C \vec{\nabla} \cdot \vec{\mathbf{w}} \mathbf{u} \, dV + \int_C \vec{\mathbf{w}} \cdot \vec{\nabla} \mathbf{u} \, dV = \int_{\partial C} \mathbf{u} \vec{\mathbf{w}} \cdot \vec{n} \, dS, \tag{2.1}$$

where ∂C is the boundary of the cell, $\vec{n}$ is the outward normal to the boundary of the cell, dS is the surface differential on the boundary, dV is the volume differential, $\mathbf{u}$ and $\vec{\mathbf{w}}$ are, respectively, smooth scalar and vector fields defined on the closure of the cell, $\vec{\nabla}\cdot$ is the divergence, and $\vec{\nabla}$ is the gradient (see [1]).

If $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ are scalar fields and $\vec{\mathbf{w}}^{(1)}$ and $\vec{\mathbf{w}}^{(2)}$ are vector fields, then the natural cell-based continuum inner products for scalars and vectors are

$$\left\langle \mathbf{u}^{(1)}, \mathbf{u}^{(2)} \right\rangle = \int_C \mathbf{u}^{(1)}\mathbf{u}^{(2)}\, dV \qquad \text{and} \qquad \left\langle \vec{\mathbf{w}}^{(1)}, \vec{\mathbf{w}}^{(2)} \right\rangle = \int_C \mathbf{w}^{(1)} \cdot \mathbf{w}^{(2)}\, dV, \tag{2.2}$$

and the divergence theorem (2.1) can be written as

$$\left\langle \vec{\nabla} \cdot \vec{\mathbf{w}}, \mathbf{u} \right\rangle + \left\langle \vec{\mathbf{w}}, \vec{\nabla}\mathbf{u} \right\rangle = \int_{\partial C} \mathbf{u}\vec{\mathbf{w}} \cdot \vec{n}\, dS. \tag{2.3}$$

We first derive the one-dimensional discretization before studying the more complex two-dimensional case. In both cases, we define discrete analogs of the continuum inner products (2.2) and use the finite-volume method to discretize the divergence and the discrete divergence theorem to discretize the gradient.

2.1. One-Dimensional Discretization

A generic one-dimensional cell (shown in Figure 1) is given by its left and right end points $x_L < x_R$. We define $x_C = (x_L + x_R)/2$ to be the center of the cell and $\mathsf{L}_C = x_R - x_L$ to be the length of the cell.

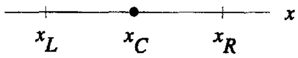


Figure 1. A typical cell $[x_L, x_R]$ in a one-dimensional grid with center $x_C = (x_L + x_R)/2$.

In one dimension, the divergence and the gradient are simply the first derivative and the divergence theorem is integration by parts for two scalar fields defined on the cell in Figure 1,

$$\int_{x_L}^{x_R} \mathbf{w}'(x)\mathbf{u}(x)\, dx + \int_{x_L}^{x_R} \mathbf{w}(x)\mathbf{u}'(x)\, dx = \mathbf{w}(x_R)\mathbf{u}(x_R) - \mathbf{w}(x_L)\mathbf{u}(x_L). \tag{2.4}$$

2.1.1. Discrete variables and their inner products

We introduce a discrete scalar field u that has the value u_C associated with the cell center, and two auxiliary values u_L and u_R associated with the cell end points, and also a discrete vector field w with values w_L and w_R that are associated with the cell end points. The values u_L and u_R are called *auxiliary* because they can be eliminated from the discretization except at the boundary of the domain.

If $u^{(1)}$ and $u^{(2)}$ are two discrete scalar fields and $w^{(1)}$ and $w^{(2)}$ are two vector fields defined on the cell, then their inner products are

$$\left\langle u^{(1)}, u^{(2)} \right\rangle_{\mathcal{S}} = u_C^{(1)} u_C^{(2)} \mathsf{L}_C, \qquad \left\langle w^{(1)}, w^{(2)} \right\rangle_{\mathcal{V}} = \frac{w_L^{(1)} w_L^{(2)} + w_R^{(1)} w_R^{(2)}}{2} \mathsf{L}_C, \tag{2.5}$$

where $\mathsf{L}_C = x_R - x_L$ is the length of the cell. The values u_L and u_R are not included in the inner product because they are eliminated in the global discretization.

The two discrete inner products correspond to midpoint and trapezoid integration rules, which are third-order accurate on a single cell. The upper and lower bounds on the inner products in one dimension are

$$\langle u, u \rangle = \mathsf{L}_C u_C^2, \qquad \langle w, w \rangle = \mathsf{L}_C \frac{w_L^2 + w_R^2}{2}. \tag{2.6}$$

2.1.2. The divergence

In (2.4), $\mathbf{w}'$ is the analog of the divergence $\vec{\nabla}\cdot$ of a vector field which produces a scalar field. So the discrete divergence of a discrete vector variable is a cell quantity

$$(\mathcal{D}w)_C = \frac{w_R - w_L}{\mathsf{L}_C}. \tag{2.7}$$

This definition is the standard finite-volume discretization.

2.1.3. Integration by parts and the gradient

In (2.4), $\mathbf{u}'$ is the analog of the gradient of a scalar field which produces a vector field. So the discrete gradient $\mathcal{G}$ of a discrete scalar field u has two values in a cell: $(\mathcal{G}u)_L$ and $(\mathcal{G}u)_R$. The natural discrete analogs of the terms in (2.4) are

$$\int_{x_L}^{x_R} \mathbf{w}'(x)\mathbf{u}(x)\,dx \mapsto \langle \mathcal{D}w, u\rangle_{\mathcal{S}} = (w_R - w_L)\,u_C; \tag{2.8}$$

$$\int_{x_L}^{x_R} \mathbf{w}(x)\mathbf{u}'(x)\,dx \mapsto \langle w, \mathcal{G}u\rangle_{\mathcal{V}} = \frac{w_L\,(\mathcal{G}u)_L + w_R\,(\mathcal{G}u)_R}{2}\mathsf{L}_C; \tag{2.9}$$

and

$$\mathbf{w}(x_R)\mathbf{u}(x_R) - \mathbf{w}(x_L)\mathbf{u}(x_L) \mapsto w_R u_R - w_L u_L. \tag{2.10}$$

The discrete analog of the integration by parts theorem is

$$(w_R - w_L)\,u_C + \frac{w_L\,(\mathcal{G}u)_L + w_R\,(\mathcal{G}u)_R}{2}\mathsf{L}_C = w_R u_R - w_L u_L. \tag{2.11}$$

For (2.11) to hold for all vector fields w, the gradient must be defined by

$$(\mathcal{G}u)_L = \frac{u_C - u_L}{\mathsf{L}_C/2}, \qquad (\mathcal{G}u)_R = \frac{u_R - u_C}{\mathsf{L}_C/2}. \tag{2.12}$$

We now have a mimetic discretization of the two inner products (2.5) and the analogs of the divergence and gradient which satisfy a discrete analog of the integration by parts theorem (2.4)

$$\langle w, \mathcal{G}u\rangle_{\mathcal{V}} + \langle \mathcal{D}w, u\rangle_{\mathcal{S}} = w_R u_R - w_L u_L. \tag{2.13}$$

2.1.4. The projections from the continuum

We analyze the accuracy of the mimetic discretization, by comparing the discrete values with the values of continuum fields projected onto the cell. The projection $\mathcal{P}_{\mathcal{S}}$ of a continuum field $\mathbf{u}$ are its values at the mid and end points of the cell

$$\mathrm{u}_\sigma = (\mathcal{P}_{\mathcal{S}}\mathbf{u})_\sigma = \mathbf{u}\,(x_\sigma), \qquad \sigma \in \{C, L, R\}, \tag{2.14}$$

while the projection $\mathcal{P}_{\mathcal{V}}$ of a vector field are only its values at the end points of the cell

$$\mathrm{w}_\sigma = (\mathcal{P}_{\mathcal{V}}\mathbf{w})_\sigma = \mathbf{w}(x_\sigma), \qquad \sigma \in \{L, R\}. \tag{2.15}$$

2.1.5. The truncation errors

The *truncation error* $\mathcal{T}_\mathcal{D}$ for the discrete *divergence* (2.7) is defined by

$$\mathcal{T}_\mathcal{D}(\mathbf{w}) = \mathcal{P}_\mathcal{S}\mathbf{w}' - \mathcal{D}\mathcal{P}_\mathcal{V}\mathbf{w}, \tag{2.16}$$

where $\mathbf{w}$ is any smooth field. A Taylor series expansion gives

$$|\mathcal{T}_\mathcal{D}(\mathbf{w})| = \left|\mathbf{w}'(x_C) - \frac{\mathbf{w}(x_R) - \mathbf{w}(x_L)}{x_R - x_L}\right| \leq C_3 \mathsf{L}_C^2, \tag{2.17}$$

where C_k is a numerical factor, independent of L_C, times the maximum of the k-derivative of the continuum fields over the cell.

The *truncation error* $\mathcal{T}_\mathcal{G}$ for the discrete *gradient* (2.12) is defined by

$$\mathcal{T}_\mathcal{G}(\mathbf{u}) = \mathcal{P}_\mathcal{V}\mathbf{u}' - \mathcal{G}\mathcal{P}_\mathcal{S}\mathbf{u}, \tag{2.18}$$

where $\mathbf{u}$ is any smooth field. A Taylor series expansion gives

$$|\mathcal{T}_\mathcal{G}(\mathbf{u})_L| = \left|\mathbf{u}'(x_L) - \frac{\mathbf{u}(x_C) - \mathbf{u}(x_L)}{x_C - x_L}\right| \leq C_2 \mathsf{L}_C, \tag{2.19}$$

with a similar estimate at the right end point.

The divergence given by (2.7) is second-order accurate (2.17) while the gradient given by (2.12) is first-order accurate (2.19).

2.2. Two-Dimensional Discretization

We begin with a detailed discussion of the geometry of a quadrilateral cell and a coordinate system in the cell based on bilinear interpolation. We define a local coordinate representation of all of the geometric formulas for the cell in terms of the vectors that are the sides of the cells. This formulation is independent of the coordinate system, and the resulting formulas are significantly simpler than if they were written in terms of a global coordinate system. Next, the inner product for vectors is given by an intuitive geometric construction. The discrete divergence is defined by a finite-volume approximation of the divergence theorem for a vector field, and the gradient is uniquely defined implicitly as the solution to a system of equations based on the divergence theorem (2.1).

The upper and lower bounds on the inner products play an important role in the convergence theory for the discrete operators. We introduce the projections of continuum fields onto the grid and estimate the accuracy of the discrete divergence and gradient. We define a compact geometric formulation for the truncation error for the divergence and then estimate its accuracy based on a Taylor series analysis. The truncation error in the gradient is derived from the system of equations that the gradient satisfies.

2.2.1. The quadrilateral

The two-dimensional cell shown in Figure 2 is determined by its four corners $P_i = (x_i, y_i)$, $0 \leq i \leq 3$. The sides of the cells are labeled by the letters D, R, U, and L (which stand for down, right, up, and left). We assume that the lengths of all of the sides of the cell are positive, that the angles in the corners of the cells are nonzero, and that the cell is convex. If the vectors $\vec{P}_i$ connect the origin to the point P_i, then vectors tangent to the edges are given by the edges

$$\vec{T}_D = \vec{P}_1 - \vec{P}_0, \qquad \vec{T}_R = \vec{P}_2 - \vec{P}_1, \qquad \vec{T}_U = \vec{P}_2 - \vec{P}_3, \qquad \vec{T}_L = \vec{P}_3 - \vec{P}_0, \tag{2.20}$$

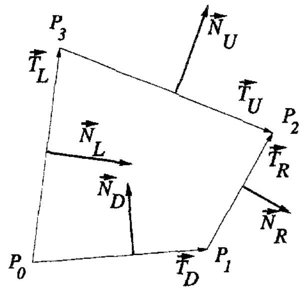


Figure 2. The cell edge tangent vectors $\vec{T}$ and normal vectors $\vec{N}$ in a generic two-dimensional grid cell satisfy the identities $\vec{T}_D + \vec{T}_R = \vec{T}_L + \vec{T}_U$ and $\vec{N}_D + \vec{N}_L = \vec{N}_R + \vec{N}_U$.

while vectors normal to the edges are given by

$$\vec{N}_D = \vec{k} \times \vec{T}_D, \qquad \vec{N}_R = -\vec{k} \times \vec{T}_R, \qquad \vec{N}_U = \vec{k} \times \vec{T}_U, \qquad \vec{N}_L = -\vec{k} \times \vec{T}_L, \tag{2.21}$$

where $\vec{k}$ is the unit vector normal to the coordinate plane. The directions of these tangents and normals are chosen to be those that are used in a global logically rectangular grid rather than, say, exterior normals. The lengths of side of the quadrilateral are given by

$$\mathsf{L}_\sigma = |T_\sigma| = |N_\sigma|, \qquad \sigma \in \{L, R, D, U\}. \tag{2.22}$$

NOTE. We define the formulas for all the important parameters of a cell in terms of tangent vectors. The formulas are more compact than if these are expressed in terms of the coordinates of the points, and importantly, the tangent vector representation formulas are clearly invariant under coordinate transformations. The tangent identity in Figure 2 tells us that these formulas are not unique; we can always eliminate one of the tangent vectors. This makes using a computer algebra system to find the formulas decidedly nontrivial, particularly, it is difficult to find the "simplest" representation of the formulas.

The bilinear transformation

$$\begin{aligned} \vec{P}(\xi,\eta) &= (1-\xi)(1-\eta)\vec{P}_0 + (1-\xi)\eta\vec{P}_3 + \xi(1-\eta)\vec{P}_1 + \xi\eta\vec{P}_2 \\ &= \vec{P}_0 + (1-\xi)\eta\vec{T}_L + \xi(1-\eta)\vec{T}_D + \xi\eta\left(\vec{T}_D + \vec{T}_R\right) \\ &= \vec{P}_0 + (1-\xi)\eta\vec{T}_L + \xi(1-\eta)\vec{T}_D + \xi\eta\left(\vec{T}_L + \vec{T}_U\right) \end{aligned} \tag{2.23}$$

maps the corners of the unit square $0 \le \xi,\ \eta \le 1$ to the corners of the cell shown in Figure 2. Because the cell is convex, the mapping from the unit square is onto the cell. Later, we will show that the Jacobian of this map is nonzero for convex cells with nonzero angles in the corners, so this mapping is one-to-one. Consequently, this bilinear map defines a coordinate system, called *logical* coordinates, for the quadrilateral.

PROPOSITION 2.1. *The bilinear map (2.23) is linear if and only if the quadrilateral is a parallelogram.*

PROOF. A quadrilateral is a parallelogram if and only if $\vec{T}_L = \vec{T}_R$ and $\vec{T}_D = \vec{T}_U$. We can rewrite (2.23) as

$$\vec{P}(\xi,\eta) = \vec{P}_0 + \eta\vec{T}_L + \xi\vec{T}_D + \xi\eta\left(\vec{T}_R - \vec{T}_L\right) = \vec{P}_0 + \eta\vec{T}_L + \xi\vec{T}_D + \xi\eta\left(\vec{T}_U - \vec{T}_D\right),$$

which makes the result clear.

If $\vec{r} = (x, y)$, then the center of the cell and the centers of the faces of the cell are given by

$$\begin{gathered} \vec{r}_C \equiv \vec{P}\left(\frac{1}{2}, \frac{1}{2}\right), \qquad \vec{r}_D \equiv \vec{P}\left(\frac{1}{2}, 0\right), \qquad \vec{r}_U \equiv \vec{P}\left(\frac{1}{2}, 1\right), \\ \vec{r}_L \equiv \vec{P}\left(0, \frac{1}{2}\right), \qquad \vec{r}_R \equiv \vec{P}\left(1, \frac{1}{2}\right), \end{gathered} \tag{2.24}$$

and then

$$\vec{r}_D = \vec{P}_0 + \frac{\vec{T}_D}{2}, \qquad \vec{r}_U = \vec{P}_0 + \vec{T}_L + \frac{\vec{T}_U}{2}, \qquad \vec{r}_L = \vec{P}_0 + \frac{\vec{T}_L}{2}, \qquad \vec{r}_R = \vec{P}_0 + \vec{T}_D + \frac{\vec{T}_R}{2}, \tag{2.25}$$

and

$$\vec{r}_C = \frac{\vec{P}_0 + \vec{P}_1 + \vec{P}_2 + \vec{P}_3}{4} = \vec{P}_0 + \frac{\vec{T}_D}{2} + \frac{\vec{T}_L + \vec{T}_R}{4} = \vec{P}_0 + \frac{\vec{T}_L}{2} + \frac{\vec{T}_D + \vec{T}_U}{4}. \tag{2.26}$$

It will simplify the formulas to define the vectors relative to the center of the cell

$$\begin{aligned} \vec{r}_U - \vec{r}_C &= +\frac{1}{4}\left(\vec{T}_L + \vec{T}_R\right), \qquad & \vec{r}_D - \vec{r}_C &= -\frac{1}{4}\left(\vec{T}_L + \vec{T}_R\right), \\ \vec{r}_R - \vec{r}_C &= +\frac{1}{4}\left(\vec{T}_D + \vec{T}_U\right), \qquad & \vec{r}_L - \vec{r}_C &= -\frac{1}{4}\left(\vec{T}_D + \vec{T}_U\right). \end{aligned} \tag{2.27}$$

The coordinate lines in the unit square are given by (ξ, η), $0 \le \xi \le 1$, with η fixed in $[0,1]$, and (ξ, η), $0 \le \eta \le 1$, with ξ fixed in $[0,1]$. The image of these coordinate lines under the bilinear map (2.23) are the logical coordinate lines in the quadrilateral. Tangents to these coordinate lines are given by

$$\begin{aligned} \vec{T}_\xi(\eta) &= \frac{\partial}{\partial \xi}\vec{P}(\xi, \eta) = (1-\eta)\vec{T}_D + \eta \vec{T}_U, \\ \vec{T}_\eta(\xi) &= \frac{\partial}{\partial \eta}\vec{P}(\xi, \eta) = (1-\xi)\vec{T}_L + \xi \vec{T}_R. \end{aligned} \tag{2.28}$$

The normal vectors pointing in the direction of increasing ξ and η are given by

$$\begin{aligned} \vec{N}_\xi(\xi) &= -\vec{k} \times \vec{T}_\eta(\xi) = (1-\xi)\vec{N}_L + \xi \vec{N}_R, \\ \vec{N}_\eta(\eta) &= +\vec{k} \times \vec{T}_\xi(\eta) = (1-\eta)\vec{N}_D + \eta \vec{N}_U, \end{aligned} \tag{2.29}$$

where $\vec{k}$ is the unit normal to the x-y-plane. The Jacobian for the bilinear map is given by

$$J(\xi, \eta) = \vec{k} \cdot \vec{T}_\xi(\eta) \times \vec{T}_\eta(\xi) = \vec{k} \cdot \vec{T}_D \times \vec{T}_L + \xi \vec{k} \cdot \vec{T}_D \times \vec{T}_U + \eta \vec{k} \cdot \vec{T}_R \times \vec{T}_L. \tag{2.30}$$

The last two terms in the Jacobian are zero for a parallelogram. That is, the Jacobian for a parallelogram is constant. The area of the cell is given by integrating the Jacobian

$$\begin{aligned} \mathsf{A}_C &= \vec{k} \cdot \vec{T}_D \times \vec{T}_L + \frac{1}{2}\vec{k} \cdot \vec{T}_D \times \vec{T}_U + \frac{1}{2}\vec{k} \cdot \vec{T}_R \times \vec{T}_L \\ &= J\left(\frac{1}{2}, \frac{1}{2}\right) \\ &= \frac{1}{4}\vec{k} \cdot \left(\vec{T}_D + \vec{T}_U\right) \times \left(\vec{T}_L + \vec{T}_R\right). \end{aligned} \tag{2.31}$$

The areas of the triangles defined by two adjoining sides of the quadrilateral play an important role in the definition of the discrete inner product for vectors. These areas can be written in terms of the cross products of tangents (being careful of the order)

$$\begin{aligned} \mathsf{A}_{D,L} &= \frac{\vec{k} \cdot \vec{T}_D \times \vec{T}_L}{2}, \qquad & \mathsf{A}_{D,R} &= \frac{\vec{k} \cdot \vec{T}_D \times \vec{T}_R}{2}, \\ \mathsf{A}_{U,R} &= \frac{\vec{k} \cdot \vec{T}_U \times \vec{T}_R}{2}, \qquad & \mathsf{A}_{U,L} &= \frac{\vec{k} \cdot \vec{T}_U \times \vec{T}_L}{2}, \end{aligned} \tag{2.32}$$

and then the area of the cell can also be written

$$\mathsf{A}_C = \mathsf{A}_{D,L} + \mathsf{A}_{U,R} = \mathsf{A}_{D,R} + \mathsf{A}_{U,L}. \tag{2.33}$$

We can complete the list of formulas given in (2.32)

$$\vec{k} \cdot \vec{T}_D \times \vec{T}_U = 2\left(\mathsf{A}_{D,R} - \mathsf{A}_{D,L}\right) = 2\left(\mathsf{A}_{U,R} - \mathsf{A}_{U,L}\right), \tag{2.34}$$

$$\vec{k} \cdot \vec{T}_L \times \vec{T}_R = 2\left(\mathsf{A}_{D,L} - \mathsf{A}_{U,L}\right) = 2\left(\mathsf{A}_{D,R} - \mathsf{A}_{U,R}\right). \tag{2.35}$$

A little algebra gives

$$J(\xi,\eta) = 2\left(\mathsf{A}_{D,L}(1-\xi)(1-\eta) + \mathsf{A}_{U,L}(1-\xi)\eta + \mathsf{A}_{D,R}\xi(1-\eta) + \mathsf{A}_{U,R}\xi\eta\right). \tag{2.36}$$

If the areas of the corner triangles are positive then, as the Jacobian is a convex combination of these areas, the Jacobian is positive.

The dot product of tangents and normals can be computed from the formula for the triple scalar product

$$\vec{T}_\sigma \cdot \left(\vec{k} \times \vec{T}_\tau\right) = -\vec{k} \cdot \left(\vec{T}_\sigma \times \vec{T}_\tau\right) \tag{2.37}$$

and the definitions of the normal vectors (2.21). This allows us to write the formulas in (2.32) in several different ways in terms of normal and tangent vectors. The angles in the corners of the quadrilateral shown in Figure 2 are given by

$$\begin{aligned} \sin\left(\theta_{D,L}\right) &= \frac{2\mathsf{A}_{D,L}}{\mathsf{L}_D \mathsf{L}_L}, & \sin\left(\theta_{D,R}\right) &= \frac{2\mathsf{A}_{D,R}}{\mathsf{L}_D \mathsf{L}_R}, \\ \sin\left(\theta_{U,R}\right) &= \frac{2\mathsf{A}_{U,R}}{\mathsf{L}_U \mathsf{L}_R}, & \sin\left(\theta_{U,L}\right) &= \frac{2\mathsf{A}_{U,L}}{\mathsf{L}_U \mathsf{L}_L}. \end{aligned} \tag{2.38}$$

The error estimates will be a function of the size of the cell, so we define

$$\mathsf{L}_{\min} = \min\left\{\mathsf{L}_L, \mathsf{L}_R, \mathsf{L}_D, \mathsf{L}_U\right\}, \qquad \mathsf{L}_{\max} = \max\left\{\mathsf{L}_L, \mathsf{L}_R, \mathsf{L}_D, \mathsf{L}_U\right\}. \tag{2.39}$$

We define ρ to measure the aspect ratio of the quadrilateral

$$\rho = \frac{\mathsf{L}_{\max}}{\mathsf{L}_{\min}}. \tag{2.40}$$

The distortion of the cell can be measured by

$$\beta = \min\left\{\sin\left(\theta_{D,L}\right), \sin\left(\theta_{D,R}\right), \sin\left(\theta_{U,R}\right), \sin\left(\theta_{U,L}\right)\right\}. \tag{2.41}$$

PROPOSITION 2.2.

$$0 < \beta \leq 1, \qquad 1 \leq \rho < \infty. \tag{2.42}$$

PROOF. The assumption that the angles are not zero gives $\beta > 0$. In an orthogonal grid, $\beta = 1$. The assumption that the sides of the cell are positive implies that ρ is finite.

2.2.2. The discrete scalar and vector variables

In the cell shown in Figure 2, for a discrete scalar variable u, we define the cell value u_C and the four auxiliary edge values u_D, u_R, u_U, and u_L. For a discrete vector variable w, we define the four normal flux variables w_D, w_R, w_U, and w_L. As in one dimension, the scalar variables on the cell edges are called *auxiliary* because, except on the boundary of the domain, they can be eliminated from the global formulation of the discretization [14,16].

2.2.3. The discrete inner products

The values of u_D, u_R, u_U, and u_L do not explicitly appear in the inner product for scalars or the global formulation of the discretization. So a natural cell-based inner product for two discrete scalar fields $u^{(1)}$ and $u^{(2)}$ is

$$\left\langle u^{(1)}, u^{(2)} \right\rangle_{\mathcal{S}} = u_C^{(1)} u_C^{(2)} \mathsf{A}_C, \tag{2.43}$$

where A_C is the area of the cell.

An inner product for discrete vector fields, called the *vertex* inner product, is defined using a geometric construction [38]. At the corner P_0 between the L and D sides in Figure 2, a vector $\vec{\mathbf{w}}$ can be represented in terms of the tangent vectors. Let

$$w_D = \frac{\vec{\mathbf{w}} \cdot \vec{N}_D}{\mathsf{L}_D}, \qquad w_L = \frac{\vec{\mathbf{w}} \cdot \vec{N}_L}{\mathsf{L}_L}, \tag{2.44}$$

and then

$$\vec{\mathbf{w}} = \frac{w_D \mathsf{L}_D}{\vec{T}_L \cdot \vec{N}_D} \vec{T}_L + \frac{w_L \mathsf{L}_L}{\vec{T}_D \cdot \vec{N}_L} \vec{T}_D = \frac{w_D \mathsf{L}_D}{2\mathsf{A}_{D,L}} \vec{T}_L + \frac{w_L \mathsf{L}_L}{2\mathsf{A}_{D,L}} \vec{T}_D. \tag{2.45}$$

Here, we have used (2.37), (2.21), and (2.32) to see that

$$\vec{T}_D \cdot \vec{N}_L = \vec{T}_L \cdot \vec{N}_D = \vec{T}_L \cdot \left(\vec{k} \times \vec{T}_D\right) = -\vec{k} \cdot \left(\vec{T}_L \times \vec{T}_D\right) = \vec{k} \cdot \left(\vec{T}_D \times \vec{T}_L\right) = 2\mathsf{A}_{D,L}. \tag{2.46}$$

We can view (2.45) as a way of interpolating the normal components w_D and w_L to a constant vector field on the triangle with vertex at the corner 0.

If two vectors $w^{(1)}$ and $w^{(2)}$ are represented by (2.45), then their Cartesian inner product is given by

$$\begin{aligned} \left\langle w^{(1)}, w^{(2)} \right\rangle_{D,L} &= \vec{\mathbf{w}}^{(1)} \cdot \vec{\mathbf{w}}^{(2)} \\ &= \frac{\mathsf{L}_D^2 \mathsf{L}_L^2 \left(w_D^{(1)} w_D^{(2)} + w_L^{(1)} w_L^{(2)}\right) + \mathsf{L}_D \mathsf{L}_L \vec{T}_L \cdot \vec{T}_D \left(w_L^{(1)} w_D^{(2)} + w_D^{(1)} w_L^{(2)}\right)}{4\mathsf{A}_{D,L}^2}. \end{aligned} \tag{2.47}$$

This can be repeated for each corner of a cell and the weighted average of the four values used to define a bilinear form.

A natural choice for the weights is the area of the triangle near the corner divided by the area of the quadrilateral. By (2.33), the sum of the weights is 2, so dividing by this and multiplying by the area defines a vector bilinear form

$$\begin{aligned} \left\langle w^{(1)}, w^{(2)} \right\rangle_{\mathcal{V}} = & \frac{\mathsf{L}_D \mathsf{L}_L \left(\vec{T}_L \cdot \vec{T}_D \left(w_L^{(1)} w_D^{(2)} + w_D^{(1)} w_L^{(2)}\right) + \mathsf{L}_D \mathsf{L}_L \left(w_D^{(1)} w_D^{(2)} + w_L^{(1)} w_L^{(2)}\right)\right)}{8\mathsf{A}_{D,L}} \\ & + \frac{\mathsf{L}_D \mathsf{L}_R \left(\vec{T}_L \cdot \vec{T}_R \left(w_R^{(1)} w_D^{(2)} + w_D^{(1)} w_R^{(2)}\right) + \mathsf{L}_D \mathsf{L}_R \left(w_D^{(1)} w_D^{(2)} + w_R^{(1)} w_R^{(2)}\right)\right)}{8\mathsf{A}_{D,R}} \\ & + \frac{\mathsf{L}_L \mathsf{L}_U \left(\vec{T}_U \cdot \vec{T}_L \left(w_U^{(1)} w_L^{(2)} + w_L^{(1)} w_U^{(2)}\right) + \mathsf{L}_L \mathsf{L}_U \left(w_L^{(1)} w_L^{(2)} + w_U^{(1)} w_U^{(2)}\right)\right)}{8\mathsf{A}_{U,L}} \\ & + \frac{\mathsf{L}_R \mathsf{L}_U \left(\vec{T}_U \cdot \vec{T}_R \left(w_U^{(1)} w_R^{(2)} + w_R^{(1)} w_U^{(2)}\right) + \mathsf{L}_R \mathsf{L}_U \left(w_R^{(1)} w_R^{(2)} + w_U^{(1)} w_U^{(2)}\right)\right)}{8\mathsf{A}_{U,R}}. \end{aligned} \tag{2.48}$$

PROPOSITION 2.3. *The bilinear forms $\langle u^{(1)}, u^{(2)} \rangle_{\mathcal{S}}$ and $\langle w^{(1)}, w^{(2)} \rangle_{\mathcal{V}}$ are symmetric and positive definite, and thus, inner products.*

PROOF. The properties of the scalar form are obvious. The properties of the vector form follow from the observation that each corner inner product is an inner product on the normal vectors to the sides adjacent to the corner, and the bilinear form is a positive convex combination of the corner inner products.

2.2.4. The divergence

The finite-volume discrete divergence

$$(\mathcal{D}w)_C = \frac{\mathsf{L}_R w_R - \mathsf{L}_L w_L + \mathsf{L}_U w_U - \mathsf{L}_D w_D}{\mathsf{A}_C} \tag{2.49}$$

is a natural choice given the way variables are defined in a cell. The signs of the terms are determined by noting that w_R and w_U use outer normals while w_D and w_L use inner normals.

2.2.5. The gradient

The gradient $\mathcal{G}u$ of a discrete scalar field u is a discrete vector field given by the normal vector components $(\mathcal{G}u)_D$, $(\mathcal{G}u)_R$, $(\mathcal{G}u)_U$, and $(\mathcal{G}u)_L$. The natural analog of the divergence theorem (2.1) is the discrete divergence theorem

$$\langle \mathcal{D}w, u\rangle_S + \langle w, \mathcal{G}u\rangle_{\mathcal{V}} = \mathsf{L}_U u_U w_U - \mathsf{L}_D u_D w_D + \mathsf{L}_R u_R w_R - \mathsf{L}_L u_L w_L, \tag{2.50}$$

where $\langle\cdot,\cdot\rangle_S$ is given in (2.43), $\langle\cdot,\cdot\rangle_{\mathcal{V}}$ is given in (2.48), the divergence $\mathcal{D}$ is given by (2.49), and the gradient $\mathcal{G}$ is to be found. The signs of the terms for the boundary integral are determined by noting that divergence theorem (2.1) uses the outward normal while w_D and w_L use the inner normal.

PROPOSITION 2.4. *Formula (2.50) uniquely determines the gradient $\mathcal{G}$.*

PROOF. Because $\langle\cdot,\cdot\rangle_{\mathcal{V}}$ is an inner product, and we want (2.50) to hold for all discrete vector fields w, there must be a unique $\mathcal{G}u$ satisfying (2.50).

2.2.6. Estimates for the inner products

THEOREM 2.5. *For β defined in (2.41) and $\mathsf{L}_{\min}$ and $\mathsf{L}_{\max}$ defined in (2.39), the scalar inner product (2.43) satisfies the estimate*

$$\beta \mathsf{L}_{\min}^2 u_C^2 \le \langle u, u\rangle_S \le \mathsf{L}_{\max}^2 u_C^2, \tag{2.51}$$

while the vector inner product (2.48) satisfies the estimate

$$\frac{\beta^2}{4}\mathsf{L}_{\min}^2\left(w_D^2 + w_R^2 + w_U^2 + w_L^2\right) \le \langle w, w\rangle_{\mathcal{V}} \le \frac{1}{\beta}\mathsf{L}_{\max}^2\left(w_D^2 + w_R^2 + w_U^2 + w_L^2\right). \tag{2.52}$$

PROOF. The estimate for the scalar inner product is clear. For the vector inner product, the proof is based on the well-known inequality

$$(1 - |\alpha|)\left(x^2 + y^2\right) \le x^2 + 2\alpha xy + y^2 \le (1 + |\alpha|)\left(x^2 + y^2\right). \tag{2.53}$$

We also need to know that if $\sin(\theta)$ satisfies (2.41), then

$$|\cos(\theta)| \le 1 - \frac{\beta^2}{2}. \tag{2.54}$$

From the definition of the corner inner product (2.47), we have

$$\begin{aligned} \mathsf{A}_{D,L}\langle w, w\rangle_{D,L} &= \frac{\mathsf{L}_D^2 \mathsf{L}_L^2}{4\mathsf{A}_{D,L}}\left(w_D^2 + w_L^2 + 2\frac{\vec{T}_L\cdot\vec{T}_D}{\mathsf{L}_D\mathsf{L}_L} w_L w_D\right), \\ &= \frac{\mathsf{L}_L\mathsf{L}_D}{2\sin\left(\theta_{D,L}\right)}\left(w_D^2 + w_L^2 + 2\cos\left(\theta_{D,L}\right) w_L w_D\right). \end{aligned} \tag{2.55}$$

From (2.39), (2.41), and (2.54), we have

$$\begin{aligned} \mathsf{L}_{\min} \le \mathsf{L}_\sigma \le \mathsf{L}_{\max}, &\qquad \sigma \in \{L, R, D, U\}, \\ 1 \le \frac{1}{\sin(\theta_{D,L})} \le \frac{1}{\beta}, & \\ \frac{\beta^2}{2} \le 1 - |\cos(\theta)|, &\qquad 1 + |\cos(\theta)| \le 2, \end{aligned}$$

so that

$$\mathsf{L}_{\min}^2 \frac{\beta^2}{4}\left(w_D^2 + w_L^2\right) \le \mathsf{A}_{D,L}\langle w, w\rangle_{D,L} \le \mathsf{L}_{\max}^2 \frac{1}{\beta}\left(w_D^2 + w_L^2\right). \tag{2.56}$$

This estimate holds for any corner, so we can sum these four estimates and divide by 2 to get the desired estimate.

2.2.7. Projections

To assess the accuracy of the discretization, we project continuum fields to the grid. If $\mathbf{u}$ is a smooth scalar field and $\vec{\mathbf{w}}$ is a smooth vector field, then their projections are

$$\mathbf{u}_\sigma = (\mathcal{P}_S \mathbf{u})_\sigma = \mathbf{u}(\vec{r}_\sigma), \qquad \sigma \in \{C, L, R, D, U\}, \tag{2.57}$$

$$\vec{\mathbf{w}}_\sigma = (\mathcal{P}_S \vec{\mathbf{w}})_\sigma = \vec{\mathbf{w}}(\vec{r}_\sigma), \qquad \sigma \in \{L, R, D, U\}. \tag{2.58}$$

2.2.8. The accuracy of the divergence

The truncation error for the divergence is

$$\mathcal{T}_\mathcal{D}(\vec{\mathbf{w}}) = \mathcal{P}_S \vec{\nabla}\cdot\vec{\mathbf{w}} - \mathcal{D}\mathcal{P}_\mathcal{V}\vec{\mathbf{w}}, \tag{2.59}$$

where $\vec{\mathbf{w}}$ is any smooth vector field.

THEOREM 2.6. *The truncation error for the divergence satisfies the estimate*

$$|\mathcal{T}_\mathcal{D}(\vec{\mathbf{w}})| \le \frac{\rho}{\beta} C_3 \left\{\mathsf{L}_D^2 + \mathsf{L}_R^2 + \mathsf{L}_U^2 + \mathsf{L}_L^2\right\} + \frac{\rho}{\beta} C_2 \left\{\left|\vec{T}_R - \vec{T}_L\right| + \left|\vec{T}_U - \vec{T}_D\right|\right\}, \tag{2.60}$$

where β is defined in (2.41), ρ is defined in (2.40), and C_k is a numerical constant times the maximum of the absolute values of all of the k^{th} derivatives of the components of $\vec{\mathbf{w}}$ over the cell. If the quadrilateral is a parallelogram, then the last term in the estimate is zero and the truncation error for the divergence is second order.

PROOF. We will estimate the truncation error for the divergence in the logical coordinates given in (2.23). For any vector field

$$\vec{\mathbf{w}} = \vec{\mathbf{w}}(\xi, \eta) = \vec{\mathbf{w}}(x(\xi,\eta), y(\xi,\eta)), \tag{2.61}$$

a chain-rule computation gives

$$\vec{\nabla}\cdot\vec{\mathbf{w}}(\xi,\eta) = \frac{\vec{N}_\xi(\xi)\cdot\frac{\partial\vec{\mathbf{w}}}{\partial\xi} + \vec{N}_\eta(\eta)\cdot\frac{\partial\vec{\mathbf{w}}}{\partial\eta}}{J(\xi,\eta)}. \tag{2.62}$$

From this, we see that the projection of the divergence is

$$\begin{aligned} \left(\mathcal{P}_S\vec{\nabla}\cdot\vec{\mathbf{w}}\right)_C &= \vec{\nabla}\cdot\vec{\mathbf{w}}\left(\frac{1}{2},\frac{1}{2}\right) \\ &= \frac{\left(\left(\vec{N}_L + \vec{N}_R\right)/2\right)\cdot\frac{\partial\vec{\mathbf{w}}}{\partial\xi}(1/2,1/2) + \left(\left(\vec{N}_D + \vec{N}_U\right)/2\right)\cdot\frac{\partial\vec{\mathbf{w}}}{\partial\eta}(1/2,1/2)}{\mathsf{A}_C}, \end{aligned} \tag{2.63}$$

while the discrete divergence of the projection of the vector field is

$$(\mathcal{D}\mathcal{P}_{\mathcal{V}}\vec{\mathbf{w}})_C = \frac{+\vec{N}_R \cdot \vec{\mathbf{w}}(1,1/2) - \vec{N}_L \cdot \vec{\mathbf{w}}(0,1/2) + \vec{N}_U \cdot \vec{\mathbf{w}}(1/2,1) - \vec{N}_D \cdot \vec{\mathbf{w}}(1/2,0)}{\mathsf{A}_C}. \tag{2.64}$$

We write the truncation error as

$$\mathcal{T}_{\mathcal{D}}(\vec{\mathbf{w}}) = \frac{\mathcal{T}_{D,U} + \mathcal{T}_{L,R}}{\mathsf{A}_C}, \tag{2.65}$$

where

$$\mathcal{T}_{L,R} = \frac{\vec{N}_L + \vec{N}_R}{2} \cdot \frac{\partial \vec{\mathbf{w}}}{\partial \xi}\left(\frac{1}{2},\frac{1}{2}\right) - \left(\vec{N}_R \vec{\mathbf{w}}\left(1,\frac{1}{2}\right) - \vec{N}_L \vec{\mathbf{w}}\left(0,\frac{1}{2}\right)\right) \tag{2.66}$$

and

$$\mathcal{T}_{D,U} = \frac{\vec{N}_D + \vec{N}_U}{2} \cdot \frac{\partial \vec{\mathbf{w}}}{\partial \eta}\left(\frac{1}{2},\frac{1}{2}\right) - \left(\vec{N}_U \vec{\mathbf{w}}\left(\frac{1}{2},1\right) - \vec{N}_D \vec{\mathbf{w}}\left(\frac{1}{2},0\right)\right). \tag{2.67}$$

These terms can be rewritten as

$$\begin{aligned}\mathcal{T}_{L,R} = {} & \frac{\vec{N}_L + \vec{N}_R}{2} \cdot \left(\frac{\partial \vec{\mathbf{w}}}{\partial \xi}\left(\frac{1}{2},\frac{1}{2}\right) - \left(\vec{\mathbf{w}}\left(1,\frac{1}{2}\right) - \vec{\mathbf{w}}\left(0,\frac{1}{2}\right)\right)\right) \\ & + \frac{\vec{N}_L - \vec{N}_R}{2} \cdot \left(\vec{\mathbf{w}}\left(1,\frac{1}{2}\right) - 2\vec{\mathbf{w}}\left(\frac{1}{2},\frac{1}{2}\right) + \vec{\mathbf{w}}\left(0,\frac{1}{2}\right)\right) \\ & + \left(\vec{N}_L - \vec{N}_R\right) \cdot \vec{\mathbf{w}}\left(\frac{1}{2},\frac{1}{2}\right),\end{aligned} \tag{2.68}$$

and

$$\begin{aligned}\mathcal{T}_{D,U} = {} & \frac{\vec{N}_D + \vec{N}_U}{2} \cdot \left(\frac{\partial \vec{\mathbf{w}}}{\partial \eta}\left(\frac{1}{2},\frac{1}{2}\right) - \left(\vec{\mathbf{w}}\left(\frac{1}{2},1\right) - \vec{\mathbf{w}}\left(\frac{1}{2},0\right)\right)\right) \\ & + \frac{\vec{N}_D - \vec{N}_U}{2} \cdot \left(\vec{\mathbf{w}}\left(\frac{1}{2},1\right) - 2\vec{\mathbf{w}}\left(\frac{1}{2},\frac{1}{2}\right) + \vec{\mathbf{w}}\left(\frac{1}{2},0\right)\right) \\ & + \left(\vec{N}_D - \vec{N}_U\right) \cdot \vec{\mathbf{w}}\left(\frac{1}{2},\frac{1}{2}\right).\end{aligned} \tag{2.69}$$

When we form the sum $\mathcal{T}_{D,U} + \mathcal{T}_{L,R}$, the last terms in the above expressions will add to zero by the identities given in Figure 2, so they do not need to be estimated. We estimate the remaining terms by using Taylor's theorem in the cell coordinates (ξ, η) and the chain rule to transform logical derivatives to spatial derivatives.

If $\mathbf{f}$ is a smooth scalar field of a single variable, then Taylor's theorem with remainder gives

$$\begin{aligned}\mathbf{f}(1) - \mathbf{f}(0) - \mathbf{f}'\left(\frac{1}{2}\right) &= \frac{1}{24}\mathbf{f}'''(\xi_1), \qquad 0 < \xi_1 < 1, \\ \mathbf{f}(1) - 2\mathbf{f}\left(\frac{1}{2}\right) + \mathbf{f}(0) &= \frac{1}{4}\mathbf{f}''(\xi_2), \qquad 0 < \xi_1 < 1.\end{aligned} \tag{2.70}$$

For any smooth scalar field $\mathbf{u}$,

$$\frac{\partial \mathbf{u}}{\partial \xi} = \vec{T}_\xi(\eta) \cdot \vec{\nabla}\mathbf{u}, \tag{2.71}$$

where $\vec{\nabla}$ is the gradient with respect to the spatial variables. Because the tangent vector is independent of ξ,

$$\frac{\partial^2 \mathbf{u}}{\partial \xi^2} = \vec{T}_\xi(\eta) H(u) \vec{T}_\xi(\eta), \tag{2.72}$$

where H, the Hessian, is the matrix of second derivatives of $\mathbf{u}$, with similar expressions for the higher derivatives.

From (2.28), we have

$$\left|\vec{T}_\xi(\eta)\right| \leq \mathsf{L}_D + \mathsf{L}_U. \tag{2.73}$$

Consequently, the logical derivatives can be estimated by

$$\left|\frac{\partial^k \mathbf{u}}{\partial \xi^k}\right| \leq C_k \left(\mathsf{L}_D + \mathsf{L}_U\right)^k, \qquad \left|\frac{\partial^k \mathbf{u}}{\partial \eta^k}\right| \leq C_k \left(\mathsf{L}_L + \mathsf{L}_R\right)^k, \tag{2.74}$$

where C_k is some numerical constant times the maximum of the absolute values of all of the k^{th} derivatives of $\mathbf{u}$ over the cell.

To estimate an expression like

$$\vec{\mathbf{w}}\left(1,\frac{1}{2}\right) - \vec{\mathbf{w}}\left(0,\frac{1}{2}\right) - \frac{\partial \vec{\mathbf{w}}}{\partial \xi}\left(\frac{1}{2},\frac{1}{2}\right), \tag{2.75}$$

we apply Taylor's theorem in the ξ variable to each component of $\vec{\mathbf{w}}$ and then apply the first estimate in (2.74). Thus,

$$\begin{aligned}
\left|\vec{\mathbf{w}}\left(1,\frac{1}{2}\right) - \vec{\mathbf{w}}\left(0,\frac{1}{2}\right) - \frac{\partial \vec{\mathbf{w}}}{\partial \xi}\left(\frac{1}{2},\frac{1}{2}\right)\right| &\leq C_3 \left(\mathsf{L}_D + \mathsf{L}_U\right)^3, \\
\left|\vec{\mathbf{w}}\left(\frac{1}{2},1\right) - \vec{\mathbf{w}}\left(\frac{1}{2},0\right) - \frac{\partial \vec{\mathbf{w}}}{\partial \eta}\left(\frac{1}{2},\frac{1}{2}\right)\right| &\leq C_3 \left(\mathsf{L}_L + \mathsf{L}_R\right)^3, \\
\left|\vec{\mathbf{w}}\left(1,\frac{1}{2}\right) - 2\vec{\mathbf{w}}\left(\frac{1}{2},\frac{1}{2}\right) + \vec{\mathbf{w}}\left(0,\frac{1}{2}\right)\right| &\leq C_2 \left(\mathsf{L}_D + \mathsf{L}_U\right)^2, \\
\left|\vec{\mathbf{w}}\left(\frac{1}{2},1\right) - 2\vec{\mathbf{w}}\left(\frac{1}{2},\frac{1}{2}\right) + \vec{\mathbf{w}}\left(\frac{1}{2},0\right)\right| &\leq C_2 \left(\mathsf{L}_L + \mathsf{L}_R\right)^2.
\end{aligned} \tag{2.76}$$

These terms are divided by the area A_C in the discrete divergence, therefore we need an estimate of the form

$$\frac{\left(\mathsf{L}_L + \mathsf{L}_R\right)^2}{\mathsf{A}_C} \leq 2\frac{\mathsf{L}_L^2 + \mathsf{L}_R^2}{\mathsf{A}_C} \leq \frac{2\mathsf{L}_L^2}{\mathsf{A}_{D,L}} + \frac{2\mathsf{L}_R^2}{\mathsf{A}_{U,R}} = \frac{4\mathsf{L}_L}{\mathsf{L}_D \sin\left(\theta_{D,L}\right)} + \frac{4\mathsf{L}_R}{\mathsf{L}_U \sin\left(\theta_{U,R}\right)} \leq 4\frac{\rho}{\beta}, \tag{2.77}$$

and similarly

$$\frac{\left(\mathsf{L}_D + \mathsf{L}_U\right)^2}{\mathsf{A}_C} \leq 4\frac{\rho}{\beta}, \tag{2.78}$$

where β is defined in (2.41) and ρ is defined in (2.40).

Combining these estimates gives

$$\begin{aligned}
\left|\frac{\mathcal{T}_{D,U} + \mathcal{T}_{L,R}}{\mathsf{A}_C}\right| \leq \left|\vec{N}_D + \vec{N}_U\right| \frac{\rho}{\beta} C_3(\mathsf{L}_D + \mathsf{L}_U) + \left|\vec{N}_L + \vec{N}_R\right| \frac{\rho}{\beta} C_3(\mathsf{L}_L + \mathsf{L}_R) \\
+ \left|\vec{N}_D - \vec{N}_U\right| C_2 + \left|\vec{N}_L - \vec{N}_R\right| C_2.
\end{aligned} \tag{2.79}$$

Noting that

$$\begin{aligned}
\left|\vec{N}_D + \vec{N}_U\right| &\leq \mathsf{L}_D + \mathsf{L}_U, & \left|\vec{N}_L + \vec{N}_R\right| &\leq \mathsf{L}_L + \mathsf{L}_R, \\
\left|\vec{N}_D - \vec{N}_U\right| &= \left|\vec{T}_D - \vec{T}_U\right|, & \left|\vec{N}_L - \vec{N}_R\right| &= \left|\vec{T}_L - \vec{T}_R\right|,
\end{aligned} \tag{2.80}$$

gives the required estimate (2.60).

COROLLARY 2.7. *Under the assumptions of Theorem 2.6, the divergence is first-order accurate,*

$$|\mathcal{T}_\mathcal{D}(\vec{w})| \le \frac{\rho}{\beta}\left(C_2 \mathsf{L}_{\max} + C_3 \mathsf{L}_{\max}^2\right), \tag{2.81}$$

in any convex cell with positive sides and second-order accurate,

$$|\mathcal{T}_\mathcal{D}(\vec{w})| \le \frac{\rho}{\beta} C_3 \mathsf{L}_{\max}^2, \tag{2.82}$$

when the cell is a parallelogram.

PROOF. From Theorem 2.6, the first-order accuracy follows from the trivial estimates

$$\left|\vec{T}_R - \vec{T}_L\right| \le \mathsf{L}_R + \mathsf{L}_L, \qquad \left|\vec{T}_U - \vec{T}_D\right| \le \mathsf{L}_U + \mathsf{L}_D, \tag{2.83}$$

while in a parallelogram $\vec{T}_R = \vec{T}_L$ and $\vec{T}_U = \vec{T}_D$.

2.2.9. The accuracy of the gradient

Because of the complexity of the vector inner product, the discrete divergence theorem (2.50) does not provide us with a simple formula for $\mathcal{G}$. It is possible, using a computer algebra system to find the formula for the gradient in terms of the coordinates of the corners of the quadrilateral, but the resulting expressions are so large that they are computationally useless and difficult to analyze. We will take an indirect route to analyze the accuracy of the gradient.

The vector inner product (2.48) can be written in the form

$$\left\langle w^{(1)}, w^{(2)} \right\rangle_\mathcal{V} = \sum_{\sigma,\tau \in \{L,R,D,U\}} \mathsf{L}_\sigma \mathsf{L}_\tau \mathbf{M}_{\sigma,\tau} w_\sigma^{(1)} w_\tau^{(2)}, \tag{2.84}$$

where $\mathbf{M}_{\sigma,\tau} = \mathbf{M}_{\tau,\sigma}$ and

$$\begin{gathered}
\mathbf{M}_{D,D} = \frac{\mathsf{L}_L^2}{8\mathsf{A}_{D,L}} + \frac{\mathsf{L}_R^2}{8\mathsf{A}_{D,R}}, \qquad \mathbf{M}_{D,R} = \frac{\vec{T}_D \cdot \vec{T}_R}{8\mathsf{A}_{D,R}}, \qquad \mathbf{M}_{D,L} = \frac{\vec{T}_D \cdot \vec{T}_L}{8\mathsf{A}_{D,L}}, \qquad \mathbf{M}_{D,U} = 0, \\
\mathbf{M}_{R,R} = \frac{\mathsf{L}_D^2}{8\mathsf{A}_{D,R}} + \frac{\mathsf{L}_U^2}{8\mathsf{A}_{U,R}}, \qquad \mathbf{M}_{R,L} = 0, \qquad \mathbf{M}_{R,U} = \frac{\vec{T}_U \cdot \vec{T}_R}{8\mathsf{A}_{U,R}}, \\
\mathbf{M}_{L,L} = \frac{\mathsf{L}_D^2}{8\mathsf{A}_{D,L}} + \frac{\mathsf{L}_U^2}{8\mathsf{A}_{U,L}}, \qquad \mathbf{M}_{L,U} = \frac{\vec{T}_U \cdot \vec{T}_L}{8\mathsf{A}_{U,L}}, \\
\mathbf{M}_{U,U} = \frac{\mathsf{L}_L^2}{8\mathsf{A}_{U,L}} + \frac{\mathsf{L}_R^2}{8\mathsf{A}_{U,R}}.
\end{gathered} \tag{2.85}$$

The discrete divergence theorem (2.50) holds for all discrete vector fields w, so collecting the coefficients of the components of w, and then multiplying each component by an appropriate L_σ, gives the following system of determining equations for the gradient:

$$\begin{aligned}
u_C - u_D &= \mathsf{L}_D \mathcal{G}_D \mathbf{M}_{D,D} + \mathsf{L}_R \mathcal{G}_R \mathbf{M}_{R,D} + \mathsf{L}_U \mathcal{G}_U \mathbf{M}_{U,D} + \mathsf{L}_L \mathcal{G}_L \mathbf{M}_{L,D}, \\
u_R - u_C &= \mathsf{L}_D \mathcal{G}_D \mathbf{M}_{D,R} + \mathsf{L}_R \mathcal{G}_R \mathbf{M}_{R,R} + \mathsf{L}_U \mathcal{G}_U \mathbf{M}_{U,R} + \mathsf{L}_L \mathcal{G}_L \mathbf{M}_{L,R}, \\
u_U - u_C &= \mathsf{L}_D \mathcal{G}_D \mathbf{M}_{D,U} + \mathsf{L}_R \mathcal{G}_R \mathbf{M}_{R,U} + \mathsf{L}_U \mathcal{G}_U \mathbf{M}_{U,U} + \mathsf{L}_L \mathcal{G}_L \mathbf{M}_{L,U}, \\
u_C - u_L &= \mathsf{L}_D \mathcal{G}_D \mathbf{M}_{D,L} + \mathsf{L}_R \mathcal{G}_R \mathbf{M}_{R,L} + \mathsf{L}_U \mathcal{G}_U \mathbf{M}_{U,L} + \mathsf{L}_L \mathcal{G}_L \mathbf{M}_{L,L},
\end{aligned} \tag{2.86}$$

where we have written $\mathcal{G}_\sigma$ for $(\mathcal{G}u)_\sigma$. The gradient is the unique solution of this system of

Our next goal is to estimate the accuracy of the gradient defined by system (2.86). The truncation error $\mathcal{T}_\mathcal{G}$ for the gradient is defined by

$$\mathcal{T}_\mathcal{G}(\mathbf{u}) = \mathcal{P}_\mathcal{V}\vec{\nabla}\mathbf{u} - \mathcal{G}\mathcal{P}_\mathcal{S}\mathbf{u}, \tag{2.87}$$

where $\mathbf{u}$ is any smooth scalar field. We first observe that the truncation error, with the abbreviations $\mathcal{T}_\sigma = \mathcal{T}_\mathcal{G}(\mathbf{u})_\sigma$, satisfies the system of equation

$$\begin{aligned}
\mathsf{L}_D\mathcal{T}_D\mathbf{M}_{D,D} + \mathsf{L}_R\mathcal{T}_R\mathbf{M}_{R,D} + \mathsf{L}_U\mathcal{T}_U\mathbf{M}_{U,D} + \mathsf{L}_L\mathcal{T}_L\mathbf{M}_{L,D} &= R_D,\\
\mathsf{L}_D\mathcal{T}_D\mathbf{M}_{D,R} + \mathsf{L}_R\mathcal{T}_R\mathbf{M}_{R,R} + \mathsf{L}_U\mathcal{T}_U\mathbf{M}_{U,R} + \mathsf{L}_L\mathcal{T}_L\mathbf{M}_{L,R} &= R_R,\\
\mathsf{L}_D\mathcal{T}_D\mathbf{M}_{D,U} + \mathsf{L}_R\mathcal{T}_R\mathbf{M}_{R,U} + \mathsf{L}_U\mathcal{T}_U\mathbf{M}_{U,U} + \mathsf{L}_L\mathcal{T}_L\mathbf{M}_{L,U} &= R_U,\\
\mathsf{L}_D\mathcal{T}_D\mathbf{M}_{D,L} + \mathsf{L}_R\mathcal{T}_R\mathbf{M}_{R,L} + \mathsf{L}_U\mathcal{T}_U\mathbf{M}_{U,L} + \mathsf{L}_L\mathcal{T}_L\mathbf{M}_{L,L} &= R_L,
\end{aligned} \tag{2.88}$$

where

$$\begin{aligned}
R_D = \vec{N}_D \cdot \vec{\nabla}\mathbf{u}(\vec{r}_D)\mathbf{M}_{D,D} + \vec{N}_R \cdot \vec{\nabla}\mathbf{u}(\vec{r}_R)\mathbf{M}_{R,D} + \vec{N}_U \cdot \vec{\nabla}\mathbf{u}(\vec{r}_U)\mathbf{M}_{U,D}\\
+\vec{N}_L \cdot \vec{\nabla}\mathbf{u}(\vec{r}_L)\mathbf{M}_{L,D} - (\mathbf{u}_C - \mathbf{u}_D),
\end{aligned} \tag{2.89}$$

with similar formulas for R_R, R_U, and R_L. The symmetry of these equations allows us to consider only one of the R formulas in detail.

We first show that the truncation error is estimated by R_σ, $\sigma \in \{L, R, D, U\}$, and then that the Rs are small.

THEOREM 2.8. *The truncation error for the gradient (2.87) satisfies*

$$\mathcal{T}_D^2 + \mathcal{T}_R^2 + \mathcal{T}_U^2 + \mathcal{T}_L^2 \le \frac{16\rho^2}{\beta^4\mathsf{L}_{\min}^2}\left(R_D^2 + R_R^2 + R_U^2 + R_L^2\right), \tag{2.90}$$

where $\mathsf{L}_{\min}$ and $\mathsf{L}_{\max}$ is defined in (2.39), β is defined in (2.41), and ρ is defined in (2.40).

PROOF. We first multiply the equations in (2.88) by $\mathsf{L}_D\mathcal{T}_D$, $\mathsf{L}_R\mathcal{T}_R$, $\mathsf{L}_U\mathcal{T}_U$, and $\mathsf{L}_L\mathcal{T}_L$ to obtain

$$\begin{aligned}
\langle\mathcal{T},\mathcal{T}\rangle_\mathcal{V} &= R_D\mathsf{L}_D\mathcal{T}_D + R_R\mathsf{L}_R\mathcal{T}_R + R_U\mathsf{L}_U\mathcal{T}_U + R_L\mathsf{L}_L\mathcal{T}_L\\
&\le \left(R_D^2\mathsf{L}_D^2 + R_R^2\mathsf{L}_R^2 + R_U^2\mathsf{L}_U^2 + R_L^2\mathsf{L}_L^2\right)^{1/2}\left(\mathcal{T}_D^2 + \mathcal{T}_R^2 + \mathcal{T}_U^2 + \mathcal{T}_L^2\right)^{1/2}.
\end{aligned} \tag{2.91}$$

From (2.52), we have

$$\frac{\beta^2}{4}\mathsf{L}_{\min}^2\left(\mathcal{T}_D^2 + \mathcal{T}_R^2 + \mathcal{T}_U^2 + \mathcal{T}_L^2\right) \le \langle\mathcal{T},\mathcal{T}\rangle_\mathcal{V}. \tag{2.92}$$

These two estimates give

$$\mathcal{T}_D^2 + \mathcal{T}_R^2 + \mathcal{T}_U^2 + \mathcal{T}_L^2 \le \frac{16}{\beta^4\mathsf{L}_{\min}^4}\left(R_D^2\mathsf{L}_D^2 + R_R^2\mathsf{L}_R^2 + R_U^2\mathsf{L}_U^2 + R_L^2\mathsf{L}_L^2\right). \tag{2.93}$$

Using the fact that $\mathsf{L}_\sigma \le \mathsf{L}_{\max}$ and the ρ defined in (2.40) gives the estimate.

THEOREM 2.9. *The R_σ satisfy*

$$\max\{|R_D|, |R_R|, |R_U|, |R_L|\} \le C_2\frac{1}{\beta}\mathsf{L}_{\max}^2, \tag{2.94}$$

where β is defined in (2.41), $\mathsf{L}_{\min}$ and $\mathsf{L}_{\max}$ are defined in (2.39), and C_2 is a numerical constant times the maximum of the absolute values of second derivatives of $\mathbf{u}$ over the cell.

PROOF. We can use that fact that $\mathbf{M}_{U,D} = \mathbf{M}_{D,U} = \mathbf{M}_{R,L} = \mathbf{M}_{L,R} = 0$ to rearrange (2.89)

$$\begin{aligned} R_D &= \left(\mathbf{M}_{D,D}\vec{N}_D + \mathbf{M}_{R,D}\vec{N}_R + \mathbf{M}_{U,D}\vec{N}_U + \mathbf{M}_{L,D}\vec{N}_L\right) \cdot \vec{\nabla}\mathbf{u}(\vec{r}_D) \\ &\quad + \mathbf{M}_{R,D}\vec{N}_R \cdot \left(\vec{\nabla}\mathbf{u}(\vec{r}_R) - \vec{\nabla}\mathbf{u}(\vec{r}_D)\right) \\ &\quad + \mathbf{M}_{U,D}\vec{N}_U \cdot \left(\vec{\nabla}\mathbf{u}(\vec{r}_U) - \vec{\nabla}\mathbf{u}(\vec{r}_D)\right) \\ &\quad + \mathbf{M}_{L,D}\vec{N}_L \cdot \left(\vec{\nabla}\mathbf{u}(\vec{r}_L) - \vec{\nabla}\mathbf{u}(\vec{r}_D)\right) - (\mathbf{u}_C - \mathbf{u}_D) \\ &= \frac{1}{4}\left(\vec{T}_L + \vec{T}_R\right) \cdot \vec{\nabla}\mathbf{u}(\vec{r}_D) - (\mathbf{u}_C - \mathbf{u}_D) \\ &\quad + \mathbf{M}_{R,D}\vec{N}_R \cdot \left(\vec{\nabla}\mathbf{u}(\vec{r}_R) - \vec{\nabla}\mathbf{u}(\vec{r}_D)\right) \\ &\quad + \mathbf{M}_{L,D}\vec{N}_L \cdot \left(\vec{\nabla}\mathbf{u}(\vec{r}_L) - \vec{\nabla}\mathbf{u}(\vec{r}_D)\right). \end{aligned} \tag{2.95}$$

We now have the R_D written as the sum of two types of terms, and each can be estimated using Taylor series. For example, (2.27) gives

$$\begin{aligned} \left|\frac{1}{4}\left(\vec{T}_L + \vec{T}_R\right) \cdot \vec{\nabla}\mathbf{u}(\vec{r}_D) - (\mathbf{u}_C - \mathbf{u}_D)\right| &= \left|\mathbf{u}(\vec{r}_C) - \mathbf{u}(\vec{r}_D) - (\vec{r}_C - \vec{r}_D) \cdot \vec{\nabla}\mathbf{u}(\vec{r}_D)\right| \\ &\le C_2 (\mathsf{L}_L + \mathsf{L}_R)^2 \le C_2 \mathsf{L}_{\max}^2, \end{aligned} \tag{2.96}$$

where C_2 is a numerical constant times the maximum of the absolute values of second derivatives of $\mathbf{u}$. An example of the second type of terms is

$$\begin{aligned} \mathbf{M}_{R,D}\vec{N}_R \cdot \left(\vec{\nabla}\mathbf{u}(\vec{r}_R) - \vec{\nabla}\mathbf{u}(\vec{r}_D)\right) &= \frac{\vec{T}_D \cdot \vec{T}_R}{8\mathsf{A}_{D,R}} \vec{N}_R \cdot \left(\vec{\nabla}\mathbf{u}(\vec{r}_R) - \vec{\nabla}\mathbf{u}(\vec{r}_D)\right) \\ &= \frac{\cos(\theta_{D,R})}{4\sin(\theta_{D,R})} \vec{N}_R \cdot \left(\vec{\nabla}\mathbf{u}(\vec{r}_R) - \vec{\nabla}\mathbf{u}(\vec{r}_D)\right). \end{aligned} \tag{2.97}$$

So

$$\left|\mathbf{M}_{R,D}\vec{N}_R \cdot \left(\vec{\nabla}\mathbf{u}(\vec{r}_R) - \vec{\nabla}\mathbf{u}(\vec{r}_D)\right)\right| \le \frac{1}{\beta}\mathsf{L}_R C_2 |\vec{r}_R - \vec{r}_D| \le \frac{1}{\beta} C_2 \mathsf{L}_{\max}^2, \tag{2.98}$$

where C_2 is a numerical factor times the maximum of the absolute values of the second-derivative of $\mathbf{u}$. All other terms can be estimated in the same way, which gives the result.

We can now estimate of the accuracy of the gradient.

COROLLARY 2.10. *Under the assumptions of Theorems 2.8 and 2.9, the gradient is first-order accurate,*

$$\left|(\mathcal{T}_{\mathcal{G}}\mathbf{u})_\sigma\right| \le C_2 \frac{\rho^2}{\beta^3} \mathsf{L}_{\max}, \qquad \sigma \in \{L, R, D, U\}. \tag{2.99}$$

In this section, for a quadrilateral cell, we have introduced inner products for discrete scalar and vector fields and then defined a divergence and gradient that satisfy a summation by part theorem. We have given an upper and lower bound on the inner products and shown that the divergence and gradient are first-order accurate in cells that are convex and have side with positive length.

3. THE GLOBAL MIMETIC DISCRETIZATION

Global mimetic finite-difference discretizations for the divergence and gradient have been described in detail [1,9,13,15]. In this section, we give a concise description of how to use the local discretization described in the previous section to create a global discretization on *logically rectangular* grids in one and two dimensions. The global divergence and gradient satisfy a summation by parts formula and satisfy the same error estimate for global grids as they do on a single cell. An important result in this section is that the global gradient does not depend on the auxiliary scalar values. In two dimensions, we show that the gradient is defined by a symmetric diagonally-dominant system of linear equations.

3.1. One-Dimensional Discretization

We begin by describing a one-dimensional general grid in a finite interval $[a, b]$ along with a recipe for translating single cell quantities to globally indexed quantities. The global inner products are simply the sum of the cell inner products, while the global divergence is the same as the local divergence. To preserve the mimetic properties, the global gradient must be determined by solving a system of linear equations and does not depend on the auxiliary values of the scalar field except those on the boundary of the region.

3.1.1. A general grid on the region

The *nodes* of the grid are given by the points x_i for $0 \le i \le I$, with $x_0 = a < b = x_I$. The *cells* of the grid are labeled with $i + 1/2$ for $0 \le i \le I - 1$, with the nodes defining this cell given by x_i, and x_{i+1}. The *center* of this cell is $x_{i+1/2} = (x_i + x_{i+1})/2$ and the *length* of this cell is $\mathsf{L}_{i+1/2} = x_{i+1} - x_i$, $0 \le i \le I - 1$. The length of the region is given by

$$\mathsf{L} = x_I - x_0 = \sum_{i=0}^{I-1} \mathsf{L}_{i+1/2}. \tag{3.1}$$

All of the positions in the $i + 1/2$ cell are given by

$$i + \frac{1}{2} + \frac{k}{2}, \qquad |k| \le 1, \tag{3.2}$$

and so these points can be labeled with $k \in \{-1, 0, 1\}$. The nodes are given by the condition that $|k| = 1$. Most formulas for Section 2.1 can be translated to this global setting by the replacements $x_L \mapsto x_i$, $x_C \mapsto x_{i+1/2}$, $x_R \mapsto x_{i+1}$, and so forth.

We define

$$\mathsf{L}_{\min} = \min_{0 \le i \le I-1} \mathsf{L}_{i+1/2}, \qquad \mathsf{L}_{\max} = \max_{0 \le i \le I-1} \mathsf{L}_{i+1/2}, \qquad \rho = \frac{\mathsf{L}_{\max}}{\mathsf{L}_{\min}}. \tag{3.3}$$

3.1.2. Scalar and vector fields on the grid

As in the local discretization, we introduce discrete scalar and vector fields on the global grid by giving their values for each cell. The space $\mathcal{H}_S$ of scalar-valued fields have a value for each cell

$$u_{i+1/2}, \qquad 0 \le i \le I - 1, \tag{3.4}$$

while the auxiliary values of scalar fields are given by

$$u_i, \qquad 0 \le i \le I. \tag{3.5}$$

The space of vector-valued fields $\mathcal{H}_V$ have a value for each node

$$w_i, \qquad 0 \le i \le I. \tag{3.6}$$

3.1.3. Grid inner products

The mimetic finite-difference method satisfies the discrete integration by parts theorem. Therefore, a discrete inner product must be defined on the spaces $\mathcal{H}_S$ and $\mathcal{H}_V$ [1]. So if $u^{(1)}, u^{(2)} \in \mathcal{H}_S$, then their inner product is given by the sum over all cells of the cell inner product for scalars defined in (2.5)

$$\left\langle u^{(1)}, u^{(2)} \right\rangle_S = \sum_{i=0}^{I-1} \left\langle u^{(1)}, u^{(2)} \right\rangle_{i+1/2} = \sum_{i=0}^{I-1} u^{(1)}_{i+1/2} u^{(2)}_{i+1/2} \mathsf{L}_{i+1/2}, \tag{3.7}$$

where $\mathsf{L}_{i+1/2} = x_{i+1} - x_i$.

If $w^{(1)}, w^{(2)} \in \mathcal{H}_\mathcal{V}$, then their inner product is given by the sum over all cells of the cell inner product for vectors defined in (2.5)

$$\left\langle w^{(1)}, w^{(2)} \right\rangle_\mathcal{V} = \sum_{i=0}^{I-1} \left\langle w^{(1)}, w^{(2)} \right\rangle_{i+1/2} = \sum_{i=0}^{I-1} \frac{w_i^{(1)} w_i^{(2)} + w_{i+1}^{(1)} w_{i+1}^{(2)}}{2} \mathsf{L}_{i+1/2}. \tag{3.8}$$

The norms associated with these inner products are $\|u\|_\mathcal{S}^2 = \langle u, u \rangle_\mathcal{S}$ and $\|w\|_\mathcal{V}^2 = \langle w, w \rangle_\mathcal{V}$. Because the one-dimensional cell scalar and vector bilinear forms are inner products, we have the following.

PROPOSITION 3.1. *The two symmetric bilinear forms $\langle u^{(1)}, u^{(2)} \rangle_\mathcal{S}$ and $\langle w^{(2)}, w^{(2)} \rangle_\mathcal{V}$ are inner products, and consequently, the two quadratic forms $\|u\|_\mathcal{S}$ and $\|w\|_\mathcal{V}$ are norms.*

3.1.4. The divergence and gradient

From (2.7), we see that the divergence is given by

$$(\mathcal{D}w)_{i+1/2} = \frac{w_{i+1} - w_i}{\mathsf{L}_{i+1/2}}, \qquad 0 \le i \le I-1. \tag{3.9}$$

In defining the gradient, we assume the scalar and gradient fields are continuous across the edges of the cells. That is, at the interface between two cells, the value u_R from a cell on the left equals the value of u_L from the cell on the right, and the same holds for $(\mathcal{G}u)_R$ from the cell on the left and $(\mathcal{G}u)_L$ from the cell on the right.

The continuity condition on the gradient gives a system of equations for computing the cell values of u from the auxiliary values or the auxiliary values of u from the boundary values and the cell values. For example, formulas (2.12) give

$$(\mathcal{G}u)_i = \frac{2\left(u_{i+1/2} - u_i\right)}{\mathsf{L}_{i+1/2}}, \quad (\mathcal{G}u)_{i+1} = \frac{2\left(u_{i+1} - u_{i+1/2}\right)}{\mathsf{L}_{i+1/2}}, \qquad 1 \le i \le I-1. \tag{3.10}$$

Equations (3.10) can be solved in terms of either

(1) the gradients;
(2) the values of u_i; or
(3) the values of $u_{i+1/2}$.

In particular, we have

$$(\mathcal{G}u)_i = 2\frac{u_{i+1/2} - u_{i-1/2}}{\mathsf{L}_{i+1/2} + \mathsf{L}_{i-1/2}}, \qquad 1 \le i \le I-1, \tag{3.11}$$

with the gradients at $i = 0$ and $i = I$ being still given by the formulas in (3.10). The analog of the system of equations (3.10) in two dimensions cannot be solved explicitly for the gradients, but can be easily solved numerically.

3.1.5. Summation by parts

A crucial property of the mimetic discrete divergence and gradient operators is that they satisfy a summation by parts formula.

PROPOSITION 3.2. *For any $u \in \mathcal{H}_\mathcal{S}$ and $w \in \mathcal{H}_\mathcal{V}$,*

$$\langle \mathcal{D}w, u \rangle_\mathcal{S} + \langle w, \mathcal{G}u \rangle_\mathcal{V} = w_I u_I - w_0 u_0. \tag{3.12}$$

PROOF. Both global inner products in (3.12) are sums of the local cell inner products. By summing the cell summation by parts formula (2.13) over all cells, the boundary terms form a collapsing sum to give the result.

3.1.6. Accuracy

Let $\mathbf{u}$ and $\mathbf{w}$ be smooth fields defined on the domain, then the projection of $\mathbf{u}$ on cells is

$$\mathbf{u}_{i+1/2} = (\mathcal{P}_S\mathbf{u})_{i+1/2} = \mathbf{u}\left(x_{i+1/2}\right), \qquad 0 \le i \le i-1, \tag{3.13}$$

while the projection of $\mathbf{u}$ on the nodes is

$$\mathbf{u}_i = (\mathcal{P}_S\mathbf{u})_i = \mathbf{u}(x_i), \qquad 0 \le i \le I. \tag{3.14}$$

Similarly, the projection of $\mathbf{w}$ onto the nodes is

$$\mathbf{w}_i = (\mathcal{P}_\mathcal{V}\mathbf{w})_i = \mathbf{w}(x_i), \qquad 0 \le i \le I. \tag{3.15}$$

As in (2.16) and (2.18), the truncation errors for the divergence and gradient on any smooth field $\mathbf{u}$ and $\mathbf{w}$ defined on the grid are defined by

$$\mathcal{T}_\mathcal{D}(\mathbf{w}) = \mathcal{P}_S\mathbf{w}' - \mathcal{D}\mathcal{P}_\mathcal{V}\mathbf{w}, \qquad \mathcal{T}_\mathcal{G}(\mathbf{u}) = \mathcal{P}_\mathcal{V}\mathbf{u}' - \mathcal{G}\mathcal{P}_S\mathbf{u}. \tag{3.16}$$

The truncation error estimates given for each cell in (2.17) and (2.19) imply the following.

PROPOSITION 3.3. *The divergence given by (3.9) is second-order accurate, the gradient given by (3.11) is first-order accurate*

$$\max_{0\le i\le I-1} \left|\mathcal{T}_\mathcal{D}(\mathbf{w})_{i-1/2}\right| \le C_3 \mathsf{L}_{\max}^2, \qquad \max_{0\le i\le I} \left|\mathcal{T}_\mathcal{G}(\mathbf{u})_i\right| \le C_2 \mathsf{L}_{\max}. \tag{3.17}$$

Here C_k is a numerical constant times the maximum of the k-derivative of $\mathbf{w}$ or $\mathbf{u}$ over the domain and $\mathsf{L}_{\max}$ is defined in (3.3).

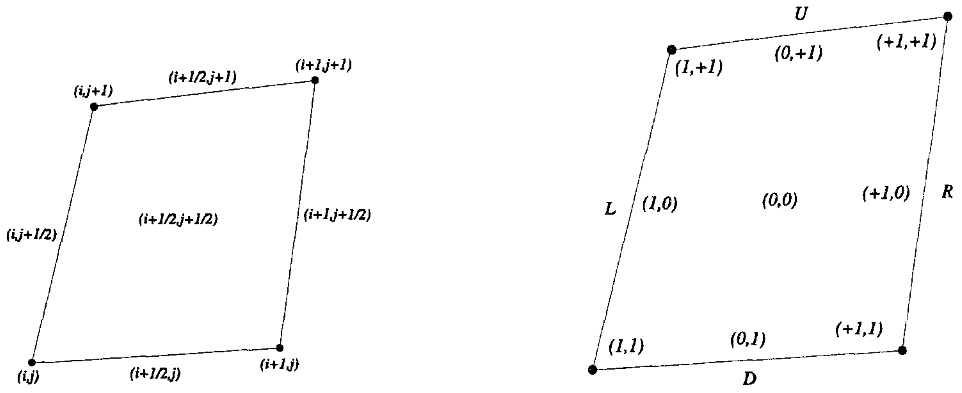
3.2. Two-Dimensional Discretization

The goal of this section is to create a global two-dimensional mimetic finite-difference discretization on a two-dimensional logically rectangular grid [5,6]. We first define the scalar and vector fields along with their global inner products. The divergence is the same as the local divergence, but global gradient is implicitly defined by a system of linear equations. This system is independent of the auxiliary values of the scalar field, and the system has excellent computational properties. Most importantly, the divergence and gradient satisfy a summation by parts formula. The local error estimates easily give error estimates on the global grid.

In two dimensions, the region Ω will be a polygon and we assume that the boundary of the grid and Ω are the same.

3.2.1. A general logically-rectangular grid on the region

A generic cell in the grid used for the mimetic discretization is displayed in Figure 3. Some of the geometric definitions come from viewing the cell as a two-dimensional projection of a three-dimensional cell. For example, the faces of a cell in three dimensions are the same as the edges of a cell in two dimensions and the nodes of a cell in one dimension. In two dimensions, the *nodes* of the grid are given by the points $(x_{i,j}, y_{i,j})$ for $0 \le i \le I$ and $0 \le i \le J$. For $0 \le i \le I-1$ and $0 \le i \le J-1$, the *cells* of the grid are labeled with $(i+1/2, j+1/2)$ and the nodes defining this cell are $(x_{i,j}, y_{i,j})$, $(x_{i+1,j}, y_{i+1,j})$, $(x_{i+1,j+1}, y_{i+1,j+1})$, $(x_{i,j+1}, y_{i,j+1})$, while the center of this cell is $(x_{i+1/2,j+1/2}, y_{i+1/2,j+1/2})$. There are two types of *edges* in the grid, those labeled with



(a) Global indexing. (b) Local indexing.

Figure 3. The global indexing for a typical cell and the local indexing for defining parts of the typical cell.

$(i+1/2, j)$ and given by the line joining the nodes (i,j) and $(i+1,j)$, and those labeled with $(i, j+1/2)$ and given by the line joining the nodes (i,j) and $(i,j+1)$.

The translation from the cell notation in Figure 2 to the global and local indexing given in the table in Figure 3 is clear from the figures. The local positions in the $(i+1/2, j+1/2)$ cell are given by

$$\left(i + \frac{k+1}{2}, j + \frac{l+1}{2}\right), \qquad |k| \le 1, \quad |l| \le 1,$$

and these points are labeled with (k,l). In particular, the nodes are given by the condition $|k| = |l| = 1$, while the edges are given by $|k| + |l| = 1$. The local coordinates will be used for defining the corner angles and the areas of the corner triangles, and for defining the elements of the bilinear form $\mathbf{B}$, and the elements of the matrices $\mathbf{M}$, $\mathbf{A}$, and $\mathbf{S}$ (see Appendix A).

The formulas for tangents (2.20) and the normals (2.21) become

$$\begin{aligned} \vec{T}_{i+1/2,j} &= \vec{P}_{i+1,j} - \vec{P}_{i,j}, & \vec{N}_{i+1/2,j} &= \vec{k} \times \vec{T}_{i+1/2,j}, \\ \vec{T}_{i+1,j+1/2} &= \vec{P}_{i+1,j+1} - \vec{P}_{i+1,j}, & \vec{N}_{i+1,j+1/2} &= -\vec{k} \times \vec{T}_{i+1,j+1/2}, \\ \vec{T}_{i+1/2,j+1} &= \vec{P}_{i+1,j+1} - \vec{P}_{i,j+1}, & \vec{N}_{i+1/2,j+1} &= \vec{k} \times \vec{T}_{i+1/2,j+1}, \\ \vec{T}_{i,j+1/2} &= \vec{P}_{i,j+1} - \vec{P}_{i,j}, & \vec{N}_{i,j+1/2} &= -\vec{k} \times \vec{T}_{i,j+1/2}, \end{aligned} \tag{3.18}$$

and the lengths of the cell sides are given by

$$\begin{aligned} \mathsf{L}_{i,j+1/2} &= \left|\vec{T}_{i,j+1/2}\right|, & 0 \le i \le I, \quad & 0 \le j \le J-1, \\ \mathsf{L}_{i+1/2,j} &= \left|\vec{T}_{i+1/2,j}\right|, & 0 \le i \le I-1, \quad & 0 \le j \le J. \end{aligned} \tag{3.19}$$

The formulas for the areas of triangles (2.32) and the angles in the corners (2.38) become

$$\begin{aligned} 2\mathsf{A}^{(-1,-1)}_{i+1/2,j+1/2} &= \vec{k} \cdot \vec{T}_{i+1/2,j} \times \vec{T}_{i,j+1/2}, \\ \sin\left(\theta^{(-1,-1)}_{i+1/2,j+1/2}\right) &= \frac{2\mathsf{A}^{(-1,-1)}_{i+1/2,j+1/2}}{\mathsf{L}_{i+1/2,j}\mathsf{L}_{i,j+1/2}}, \\ 2\mathsf{A}^{(-1,+1)}_{i+1/2,j+1/2} &= \vec{k} \cdot \vec{T}_{i+1/2,j} \times \vec{T}_{i+1,j+1/2}, \\ \sin\left(\theta^{(-1,+1)}_{i+1/2,j+1/2}\right) &= \frac{2\mathsf{A}^{(-1,+1)}_{i+1/2,j+1/2}}{\mathsf{L}_{i+1/2,j}\mathsf{L}_{i+1,j+1/2}}, \\ 2\mathsf{A}^{(+1,+1)}_{i+1/2,j+1/2} &= \vec{k} \cdot \vec{T}_{i+1/2,j+1} \times \vec{T}_{i+1,j+1/2}, \end{aligned} \tag{3.20}$$

$$\sin\left(\theta^{(+1,+1)}_{i+1/2,j+1/2}\right) = \frac{2\mathsf{A}^{(+1,+1)}_{i+1/2,j+1/2}}{\mathsf{L}_{i+1/2,j+1}\mathsf{L}_{i+1,j+1/2}},$$
$$2\mathsf{A}^{(+1,-1)}_{i+1/2,j+1/2} = \vec{k}\cdot\vec{T}_{i+1/2,j+1}\times\vec{T}_{i,j+1/2}. \quad (3.20)(\text{cont.})$$
$$\sin\left(\theta^{(+1,-1)}_{i+1/2,j+1/2}\right) = \frac{2\mathsf{A}^{(+1,-1)}_{i+1/2,j+1/2}}{\mathsf{L}_{i+1/2,j+1}\mathsf{L}_{i,j+1/2}}.$$

The area of the quadrilateral is then

$$\mathsf{A}_{i+1/2,j+1/2} = \mathsf{A}^{(-1,-1)}_{i+1/2,j+1/2} + \mathsf{A}^{(+1,+1)}_{i+1/2,j+1/2} = \mathsf{A}^{(-1,+1)}_{i+1/2,j+1/2} + \mathsf{A}^{(+1,-1)}_{i+1/2,j+1/2}, \quad (3.21)$$

and the area of the region is given by

$$A = \sum_{\substack{0\le i\le I-1\\ 0\le j\le J-1}} \mathsf{A}_{i+1/2,j+1/2}. \quad (3.22)$$

Using the same notation as in the single cell, we define

$$\mathsf{L}_{\min} = \min\left(\min_{\substack{0\le i\le I-1\\ 0\le j\le J}} \mathsf{L}_{i+1/2,j},\ \min_{\substack{0\le i\le I\\ 0\le j\le J-1}} \mathsf{L}_{i,j+1/2}\right),$$
$$\mathsf{L}_{\max} = \max\left(\max_{\substack{0\le i\le I-1\\ 0\le j\le J}} \mathsf{L}_{i+1/2,j},\ \max_{\substack{0\le i\le I\\ 0\le j\le J-1}} \mathsf{L}_{i,j+1/2}\right), \quad (3.23)$$

and then set

$$\rho = \frac{\mathsf{L}_{\max}}{\mathsf{L}_{\min}}. \quad (3.24)$$

We also define β to be the sine of the smallest angle in the grid

$$\beta = \min_{\substack{0\le i\le I-1\\ 0\le j\le J-1}} \min\Big\{\sin\left(\theta^{(-1,-1)}_{i+1/2,j+1/2}\right), \sin\left(\theta^{(-1,+1)}_{i+1/2,j+1/2}\right), \sin\left(\theta^{(+1,+1)}_{i+1/2,j+1/2}\right), \sin\left(\theta^{(+1,-1)}_{i+1/2,j+1/2}\right)\Big\}. \quad (3.25)$$

3.2.2. Scalar and vector fields on the grid

As in the local discretization, we introduce the space of scalar fields $\mathcal{H}_{\mathcal{S}}$ associated with the cells

$$u_{i+1/2,j+1/2}, \qquad 0\le i\le I-1, \quad 0\le j\le J-1. \quad (3.26)$$

Additionally, we use the auxiliary values of the scalar on the cell edges

$$u_{i,j+1/2}, \qquad 0\le i\le I, \qquad 0\le j\le J-1,$$
$$u_{i+1/2,j}, \qquad 0\le i\le I-1, \quad 0\le j\le J. \quad (3.27)$$

The space of vector fields $\mathcal{H}_{\mathcal{V}}$ associated with the edges as given by the w has the values

$$w_{i+1/2,j}, \qquad 0\le i\le I, \qquad 0\le j\le J-1;$$
$$w_{i,j+1/2}, \qquad 0\le i\le I-1, \quad 0\le j\le J. \quad (3.28)$$

3.2.3. Grid inner products

The global inner products of scalar and vector fields is the sum of the cell inner products given in (2.43) and (2.48). We define the global inner product for $u^{(1)}, u^{(2)} \in \mathcal{H}_S$ as the sum over all cells of the cell inner product (2.43)

$$\begin{aligned}\left\langle u^{(1)}, u^{(2)} \right\rangle_S &= \sum_{i=0}^{I-1} \sum_{j=0}^{J-1} \left\langle u^{(1)}, u^{(2)} \right\rangle_{i+1/2,j+1/2} \\ &= \sum_{i=0}^{I-1} \sum_{j=0}^{J-1} u^{(1)}_{i+1/2,j+1/2} u^{(2)}_{i+1/2,j+1/2} \mathsf{A}_{i+1/2,j+1/2}.\end{aligned} \tag{3.29}$$

If $w^{(1)}, w^{(2)} \in \mathcal{H}_\mathcal{V}$, then the global inner product is defined by the sum over all cells of the cell inner product (2.48) as written in (2.84)

$$\left\langle w^{(1)}, w^{(2)} \right\rangle_\mathcal{V} = \sum_{i=0}^{I-1} \sum_{j=0}^{J-1} \left\langle w^{(1)}, w^{(2)} \right\rangle_{i+1/2,j+1/2}, \tag{3.30}$$

where

$$\begin{aligned}\left\langle w^{(1)}, w^{(2)} \right\rangle_{i+1/2,j+1/2} &= \mathsf{A}_{i+1/2,j+1/2} \sum_{\substack{|k|+|l|=1 \\ |r|+|s|=1}} \mathbf{B}^{(k,l)(r,s)}_{i+1/2,j+1/2} \\ &\quad \times w^{(1)}_{i+1/2+k/2,j+1/2+l/2} w^{(2)}_{i+1/2+r/2,j+1/2+s/2}, \\ \mathbf{B}^{(k,l)(r,s)}_{i+1/2,j+1/2} &= \frac{\mathsf{L}_{i+1/2+k/2,j+1/2+l/2} \mathsf{L}_{i+1/2+r/2,j+1/2+s/2}}{\mathsf{A}_{i+1/2,j+1/2}} \\ &\quad \times \mathbf{M}^{(k,l)(r,s)}_{i+1/2,j+1/2},\end{aligned} \tag{3.31}$$

where the matrix $\mathbf{M}$ is described in (A.1) in Appendix A. Both $\mathbf{M}$ and $\mathbf{B}$ are symmetric in their upper indices. The norms associated with these inner products are $\|u\|_S = \langle u, u \rangle_S$ and $\|w\|_\mathcal{V} = \langle w, w \rangle_\mathcal{V}$. Because the two-dimensional cell scalar and vector bilinear forms are inner products, we have the following.

PROPOSITION 3.4. *The two bilinear forms $\langle u, v \rangle_S$ and $\langle v, w \rangle_\mathcal{V}$ are inner products, and consequently, the two quadratic forms $\|u\|_S$ and $\|w\|_\mathcal{V}$ are norms.*

3.2.4. The divergence and gradient

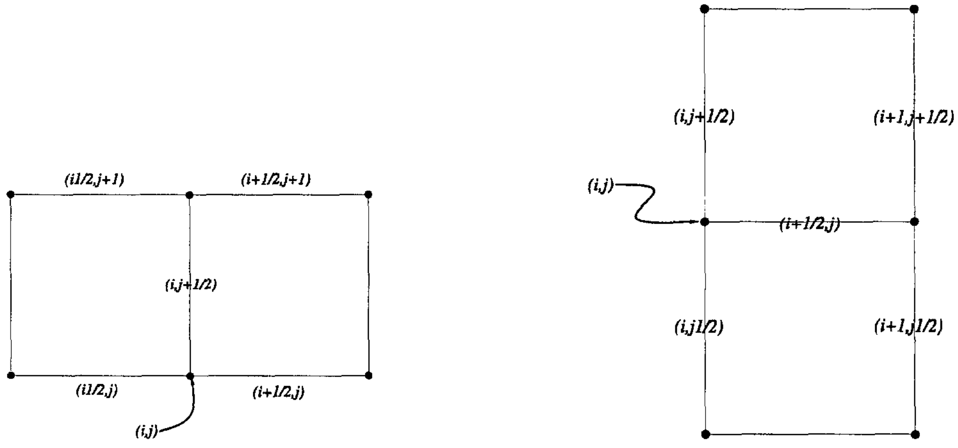
In the global grid notation, the discrete divergence (2.49) is given by

$$\begin{aligned}(\mathcal{D}w)_{i+1/2,j+1/2} &= \frac{\mathsf{L}_{i+1,j+1/2} w_{i+1,j+1/2} - \mathsf{L}_{i,j+1/2} w_{i,j+1/2}}{\mathsf{A}_{i+1/2,j+1/2}} \\ &+ \frac{\mathsf{L}_{i+1/2,j+1} w_{i+1/2,j+1} - \mathsf{L}_{i+1/2,j} w_{i+1/2,j}}{\mathsf{A}_{i+1/2,j+1/2}}, \qquad 0 \le i \le I-1, \quad 0 \le j \le J-1.\end{aligned} \tag{3.32}$$

Recall that the global gradient is required to satisfy the system of determining equations (2.86). These equations are written out in detail in (A.2) in Appendix A in a form that eliminates the auxiliary edge values of the scalar variable to produce as system of equations with the stencil or footprint illustrated in Figure 4.

PROPOSITION 3.5. *The gradient, with $\mathcal{G}_{i,j+1/2} = (\mathcal{G}u)_{i,j+1/2}$ and $\mathcal{G}_{i+1/2,j} = (\mathcal{G}u)_{i+1/2,j}$, satisfies a system of equations of the form*

$$\begin{aligned}\mathbf{A}^{(0,0)}_{i,j+1/2} \mathcal{G}_{i,j+1/2} &+ \sum_{|k|=|l|=1} \mathbf{A}^{(k,l)}_{i,j+1/2} \mathcal{G}_{i+k/2,j+1/2+l/2} \\ &= u_{i+1/2,j+1/2} - u_{i-1/2,j+1/2}, \\ 1 \le i \le I-1, &\qquad 0 \le j \le J-1;\end{aligned} \tag{3.33}$$



(a) Stencil for the gradient at $(i, j + 1/2)$. (b) Stencil for gradient at $(i + 1/2, j)$.

Figure 4. The stencil or footprint of the finite-difference equations defining the discrete gradient in (3.33) and (3.34).

$$
\begin{aligned}
&\mathbf{A}^{(0,0)}_{i+1/2,j}\mathcal{G}_{i+1/2,j} + \sum_{|k|=|l|=1} \mathbf{A}^{(k,l)}_{i+1/2,j}\mathcal{G}_{i+1/2+k/2,j+l/2} \\
&\quad = u_{i+1/2,j+1/2} - u_{i+1/2,j-1/2}, \\
&\quad 0 \le i \le I-1, \qquad 1 \le j \le J-1;
\end{aligned} \tag{3.34}
$$

$$
\begin{aligned}
&\mathbf{A}^{(0,0)}_{0,j+1/2}\mathcal{G}_{0,j+1/2} + \sum_{|l|=1} \mathbf{A}^{(+1,l)}_{0,j+1/2}\mathcal{G}_{1/2,j+1/2+l/2} = u_{1/2,j+1/2} - u_{0,j+1/2}, \\
&\qquad 0 \le j \le J-1; \\
&\mathbf{A}^{(0,0)}_{I,j+1/2}\mathcal{G}_{I,j+1/2} + \sum_{|l|=1} \mathbf{A}^{(-1,l)}_{I,j+1/2}\mathcal{G}_{I-1/2,j+1/2+l/2} = u_{I,j+1/2} - u_{I-1/2,j+1/2}, \\
&\qquad 0 \le j \le J-1; \\
&\mathbf{A}^{(0,0)}_{i+1/2,0}\mathcal{G}_{i+1/2,0} + \sum_{|k|=1} \mathbf{A}^{(k,+1)}_{i+1/2,0}\mathcal{G}_{i+1/2+k/2,1/2} = u_{i+1/2,1/2} - u_{i+1/2,0}, \\
&\qquad 0 \le i \le I-1; \\
&\mathbf{A}^{(0,0)}_{i+1/2,J}\mathcal{G}_{i+1/2,J} + \sum_{|k|=1} \mathbf{A}^{(k,-1)}_{i+1/2,J}\mathcal{G}_{i+1/2+k/2,J-1/2} = u_{i+1/2,J} - u_{i+1/2,J-1/2}, \\
&\qquad 0 \le i \le I-1.
\end{aligned} \tag{3.35}
$$

The explicit formulas for matrix $\mathbf{A}$ *and a proof that it is diagonally dominant are given in Appendix A.*

3.2.5. Summation by parts

The two-dimensional divergence and gradient satisfy a summation by parts formula obtained by summing the one-cell summation by parts formula (2.50) over all the cells.

PROPOSITION 3.6. *For any* $u \in \mathcal{H}_S$ *and* $w \in \mathcal{H}_\mathcal{V}$

$$
\begin{aligned}
\langle \mathcal{D}w, u\rangle_S + \langle w, \mathcal{G}u\rangle_\mathcal{V} = &- \sum_{i=0,I-1} \mathsf{L}_{i+1/2,0} w_{i+1/2,0} u_{i+1/2,0} \\
&+ \sum_{i=0,I-1} \mathsf{L}_{i+1/2,J} w_{i+1/2,J} u_{i+1/2,J}
\end{aligned} \tag{3.36}
$$

$$- \sum_{j=0,J-1} \mathsf{L}_{0,j+1/2} w_{0,j+1/2} u_{0,j+1/2} + \sum_{j=0,J-1} \mathsf{L}_{I,j+1/2} w_{I,j+1/2} u_{I,j+1/2}. \tag{3.36 (cont.)}$$

PROOF. Both inner products in (3.36) are sums of cells inner products. If we sum the cell summation by parts formula (2.50) over all cells, then the cell boundary terms form collapsing sums to give the result.

3.2.6. Accuracy

The projection of the smooth scalar field $\mathbf{u}$ on cells is

$$\mathbf{u}_{i+1/2,j+1/2} = (\mathcal{P}_{\mathcal{S}}\mathbf{u})_{i+1/2,j+1/2} = \mathbf{u}\left(x_{i+1/2,j+1/2}, y_{i+1/2,j+1/2}\right), \quad 0 \le i \le I-1, \quad 0 \le j \le J-1, \tag{3.37}$$

and the projection of $\mathbf{u}$ onto edges is

$$\begin{aligned} \mathbf{u}_{i,j+1/2} &= (\mathcal{P}_{\mathcal{S}}\mathbf{u})_{i,j+1/2} = \mathbf{u}(x_{i,j+1/2}, y_{i,j+1/2}), \quad 0 \le i \le I, \quad 0 \le j \le J-1, \\ \mathbf{u}_{i+1/2,j} &= (\mathcal{P}_{\mathcal{S}}\mathbf{u})_{i+1/2,j} = \mathbf{u}(x_{i+1/2,j}, y_{i+1/2,j}), \quad 0 \le i \le I-1, \quad 0 \le j \le J. \end{aligned} \tag{3.38}$$

Similarly, the projection of a smooth continuum vector field $\vec{\mathbf{w}}$ onto the edges is

$$\begin{aligned} \vec{\mathbf{w}}_{i,j+1/2} &= (\mathcal{P}_{\mathcal{V}}\vec{\mathbf{w}})_{i,j+1/2} = \vec{\mathbf{w}}\left(x_{i,j+1/2}, y_{i,j+1/2}\right), \quad 0 \le i \le I, \quad 0 \le j \le J-1, \\ \vec{\mathbf{w}}_{i+1/2,j} &= (\mathcal{P}_{\mathcal{V}}\vec{\mathbf{w}})_{i+1/2,j} = \vec{\mathbf{w}}\left(x_{i+1/2,j}, y_{i+1/2,j}\right), \quad 0 \le i \le I-1, \quad 0 \le j \le J. \end{aligned} \tag{3.39}$$

The truncation error for the divergence and gradient are defined as in (2.59) and (2.87)

$$\mathcal{T}_{\mathcal{D}}(\vec{\mathbf{w}}) = \mathcal{P}_{\mathcal{S}}\vec{\nabla} \cdot \vec{\mathbf{w}} - \mathcal{D}\mathcal{P}_{\mathcal{V}}\vec{\mathbf{w}}, \qquad \mathcal{T}_{\mathcal{G}}(\mathbf{u}) = \mathcal{P}_{\mathcal{V}}\vec{\nabla}\mathbf{u} - \mathcal{G}\mathcal{P}_{\mathcal{S}}\mathbf{u}, \tag{3.40}$$

where again $\mathbf{u}$ and $\vec{\mathbf{w}}$ are any smooth fields defined on the region Ω.

From (2.60) and (2.99) come the following.

PROPOSITION 3.7. *The divergence $\mathcal{D}$ given by (2.49) and the gradient $\mathcal{G}$ given by Corollary 3.5 are first-order accurate*

$$\begin{aligned} \max_{\substack{0\le i\le I-1 \\ 0\le j\le J-1}} \left|\mathcal{T}_{\mathcal{D}}(\vec{\mathbf{w}})_{i+1/2,j_{1/2}}\right| &\le \frac{\rho}{\beta}\left(C_2 + C_3 \mathsf{L}_{\max}\right)\mathsf{L}_{\max}, \\ \max\left\{ \max_{\substack{0\le i\le I \\ 0\le j\le J-1}} \left|\mathcal{T}_{\mathcal{G}}(\mathbf{u})_{i,j+1/2}\right|, \max_{\substack{0\le i\le I-1 \\ 0\le j\le J}} \left|\mathcal{T}_{\mathcal{G}}(\mathbf{u})_{i+1/2,j}\right| \right\} &\le C_2 \frac{\rho^2}{\beta^3}\mathsf{L}_{\max}, \end{aligned} \tag{3.41}$$

where $\mathsf{L}_{\max}$ is defined in (3.23), β is defined in (3.25), ρ is defined in (3.24), and C_k is a numerical constant times the maximum of the absolute values of k^{th} derivatives of $\mathbf{u}$ *or* $\mathbf{w}$ *over the cell.*

4. THE FORMAL MIMETIC THEORY

We now introduce *formal* inner products on the spaces of discrete scalar and vector fields along with the formal discrete divergence and gradient operators. We use these tools to define the discrete *natural* divergence and gradient operators introduced in the previous section. This viewpoint facilitates both the programming and analysis of mimetic finite-difference methods. At the end of this section, we prove natural Friedrichs-Poincaré inequality for the discrete natural operators, critical in the proof of convergence, in one and two dimensions.

4.1. One-Dimensional Formal Operators

If $u^{(1)}, u^{(2)} \in \mathcal{H}_S$ and $w^{(1)}, w^{(2)} \in \mathcal{H}_\mathcal{V}$, then their formal inner products are

$$\left\langle\left\langle u^{(1)}, u^{(2)}\right\rangle\right\rangle_S = \sum_{i=0}^{I-1} u^{(1)}_{i+1/2} u^{(2)}_{i+1/2}, \qquad \left\langle\left\langle w^{(1)}, w^{(2)}\right\rangle\right\rangle_\mathcal{V} = \sum_{i=0}^{I} w^{(1)}_i w^{(2)}_i. \tag{4.1}$$

The associated norms are $|||u|||_S^2 = \langle\langle u, u\rangle\rangle_S$ and $|||u|||_\mathcal{V}^2 = \langle\langle u, u\rangle\rangle_\mathcal{V}$.

To represent the natural inner products and the divergence and gradient in terms of formal objects, we begin by noting the following.

PROPOSITION 4.1. *Because the formal and natural bilinear forms are inner products on the same space, there exist positive-definite symmetric operators $\mathcal{M}$ and $\mathcal{S}$ such that*

$$\left\langle u^{(1)}, u^{(2)}\right\rangle_S = \left\langle\left\langle u^{(1)}, \mathcal{M}u^{(2)}\right\rangle\right\rangle_S, \qquad \left\langle w^{(1)}, w^{(2)}\right\rangle_\mathcal{V} = \left\langle\left\langle w^{(1)}, \mathcal{S}w^{(2)}\right\rangle\right\rangle_\mathcal{V}. \tag{4.2}$$

The definition of the scalar inner product (3.7) implies that

$$(\mathcal{M}u)_{i+1/2} = \mathsf{L}_{i+1/2} u_{i+1/2}, \qquad 0 \le i \le I-1, \tag{4.3}$$

while the definition of the vector inner product (3.8) gives

$$(\mathcal{S}w)_0 = \frac{\mathsf{L}_{1/2}}{2} w_0, \qquad (\mathcal{S}w)_i = \frac{\mathsf{L}_{i-1/2} + \mathsf{L}_{i-1/2}}{2} w_i, \qquad (\mathcal{S}w)_I = \frac{\mathsf{L}_{I-1/2}}{2} w_I, \qquad 1 \le i \le I-1. \tag{4.4}$$

Consequently, we have the estimates

$$\mathsf{L}_{\min}|||u|||_S^2 \le \langle\langle u, \mathcal{M}u\rangle\rangle_S \le \mathsf{L}_{\max}|||u|||_S^2, \qquad \frac{\mathsf{L}_{\min}}{2}|||w|||_\mathcal{V}^2 \le \langle\langle w, \mathcal{S}w\rangle\rangle_\mathcal{V} \le \mathsf{L}_{\max}|||w|||_\mathcal{V}^2, \tag{4.5}$$

where $\mathsf{L}_{\min}$ and $\mathsf{L}_{\max}$ are defined in (3.3). Applying Theorem 3 from [39, p. 201] bounds the norms of the $\mathcal{M}$ and $\mathcal{S}$ operators

$$\mathsf{L}_{\min} \le |||\mathcal{M}|||_S \le \mathsf{L}_{\max}, \qquad \frac{\mathsf{L}_{\min}}{2} \le |||\mathcal{S}|||_\mathcal{V} \le \mathsf{L}_{\max}. \tag{4.6}$$

4.1.1. The formal divergence and gradient

The formal divergence $\mathbf{D}$ and the formal gradient $\mathbf{G}$ operators are defined by

$$(\mathbf{D}w)_{i+1/2} = w_{i+1} - w_i, \qquad 0 \le i \le I-1, \tag{4.7}$$

$$(\mathbf{G}u)_0 = u_{1/2} - u_0, \quad (\mathbf{G}u)_I = u_I - u_{I-1/2}, \quad (\mathbf{G}u)_i = u_{i+1/2} - u_{i-1/2}, \qquad 1 \le i \le I-1. \tag{4.8}$$

PROPOSITION 4.2. *For any $u \in \mathcal{H}_S$ and $w \in \mathcal{H}_\mathcal{V}$, the formal divergence and gradient satisfy a summation by parts formula*

$$\langle\langle \mathbf{D}w, u\rangle\rangle_S + \langle\langle w, \mathbf{G}u\rangle\rangle_\mathcal{V} = w_I u_I - w_0 u_0. \tag{4.9}$$

PROOF. We start with an analog of the product rule

$$(w_{i+1} - w_i)\, u_{i+1/2} + w_i \left(u_{i+1/2} - u_{i-1/2}\right) = w_{i+1}u_{i+1/2} - w_i u_{i-1/2}, \tag{4.10}$$

and sum this to get

$$\sum_{i=1}^{I-1} (w_{i+1} - w_i)\, u_{i+1/2} + \sum_{i=1}^{I-1} w_i \left(u_{i+1/2} - u_{i-1/2}\right) = w_I u_{I-1/2} - w_1 u_{1/2}. \tag{4.11}$$

Adding one more term to the first sum gives

$$\sum_{i=0}^{I-1} (w_{i+1} - w_i)\, u_{i+1/2} + \sum_{i=1}^{I-1} w_i \left(u_{i+1/2} - u_{i-1/2}\right) = w_I u_{I-1/2} - w_0 u_{1/2}. \tag{4.12}$$

Now adding

$$w_0 \left(u_{1/2} - u_0\right) + w_I \left(u_I - u_{I-1/2}\right) \tag{4.13}$$

to both sides of the equation gives the result.

PROPOSITION 4.3. *The formal divergence and gradient can be represented as*

$$\mathcal{D} = \mathcal{M}^{-1}\mathbf{D}, \qquad \mathcal{G} = \mathcal{S}^{-1}\mathbf{G}. \tag{4.14}$$

PROOF. The formula for the natural $\mathcal{D}$ follows from (3.9) and definition (4.3) of the $\mathcal{M}$ operator. The formula for the natural gradient follows from formula (3.11) and definition (4.4).

To motivate the formula for the natural gradient in two dimensions, compare the natural summation by parts formula (3.12) with the formal summation by parts formula (4.9) and note that

$$\langle \mathcal{D}w, u\rangle_{\mathcal{S}} + \langle w, \mathcal{G}u\rangle_{\mathcal{V}} = \langle\langle \mathbf{D}w, u\rangle\rangle_{\mathcal{S}} + \langle\langle w, \mathbf{G}u\rangle\rangle_{\mathcal{V}}. \tag{4.15}$$

We also know from (4.2) that

$$\begin{aligned}\langle \mathcal{D}w, u\rangle_{\mathcal{S}} &= \left\langle \mathcal{M}^{-1}\mathbf{D}w, u\right\rangle_{\mathcal{S}} = \left\langle\left\langle \mathcal{M}^{-1}\mathbf{D}w, \mathcal{M}u\right\rangle\right\rangle_{\mathcal{S}} = \langle\langle \mathbf{D}w, u\rangle\rangle_{\mathcal{S}}\,,\\ \langle w, \mathcal{G}u\rangle_{\mathcal{S}} &= \left\langle w, \mathcal{S}^{-1}\mathbf{G}u\right\rangle_{\mathcal{S}} = \left\langle\left\langle w, \mathcal{S}\mathcal{S}^{-1}\mathbf{G}u\right\rangle\right\rangle_{\mathcal{S}} = \langle\langle w, \mathbf{G}u\rangle\rangle_{\mathcal{S}}\,.\end{aligned} \tag{4.16}$$

These equations are strong consistency requirements between the natural and formal summation by parts formulas and the formulas for the natural gradient and divergence and their formal counterparts.

4.2. Two-Dimensional Formal Operators

The formal inner product of $u^{(1)}, u^{(2)} \in \mathcal{H}_{\mathcal{S}}$ is

$$\left\langle\left\langle u^{(1)}, u^{(2)}\right\rangle\right\rangle_{\mathcal{S}} = \sum_{i=0}^{I-1}\sum_{j=0}^{J-1} u^{(1)}_{i+1/2,j+1/2}u^{(2)}_{i+1/2,j+1/2}, \tag{4.17}$$

while the formal inner product of $w^{(1)}, w^{(2)} \in \mathcal{H}_{\mathcal{V}}$ is

$$\left\langle\left\langle w^{(1)}, w^{(2)}\right\rangle\right\rangle_{\mathcal{V}} = \sum_{i=0}^{I-1}\sum_{j=0}^{J} w^{(1)}_{i+1/2,j}w^{(2)}_{i+1/2,j} + \sum_{i=0}^{I}\sum_{j=0}^{J-1} w^{(1)}_{i,j+1/2}w^{(2)}_{i,j+1/2}. \tag{4.18}$$

The associated norms are $|||u|||^2_{\mathcal{S}} = \langle\langle u, u\rangle\rangle_{\mathcal{S}}$ and $|||u|||^2_{\mathcal{V}} = \langle\langle u, u\rangle\rangle_{\mathcal{V}}$.

PROPOSITION 4.4. *There exist positive-definite symmetric operators $\mathcal{M}$ and $\mathcal{S}$ such that*

$$\left\langle u^{(1)}, u^{(2)}\right\rangle_{\mathcal{S}} = \left\langle\left\langle u^{(1)}, \mathcal{M}u^{(2)}\right\rangle\right\rangle_{\mathcal{S}}, \qquad \left\langle w^{(1)}, w^{(2)}\right\rangle_{\mathcal{V}} = \left\langle\left\langle w^{(1)}, \mathcal{S}w^{(2)}\right\rangle\right\rangle_{\mathcal{V}}. \tag{4.19}$$

PROOF. Because the formal and natural bilinear forms are inner products on the same space, such operators must exist.

PROPOSITION 4.5. *The operator $\mathcal{M}$ is given by*

$$(\mathcal{M}u)_{i+1/2,j+1/2} = \mathsf{A}_{i+1/2,j+1/2}u_{i+1/2,j+1/2}, \qquad 0 \le i \le I-1, \quad 0 \le j \le J-1. \tag{4.20}$$

In the interior, the operator $\mathcal{S}$ is given by

$$\begin{aligned}(\mathcal{S}w)_{i,j+1/2} &= \mathbf{S}^{(0,0)}_{i,j+1/2}w_{i,j+1/2} + \sum_{|k|=|l|=1} \mathbf{S}^{(k,l)}_{i,j+1/2}w_{i+k/2,j+1/2+l/2},\\ &1 \le i \le I-1, \quad 0 \le j \le J-1,\end{aligned} \tag{4.21}$$

$$\begin{aligned}(\mathcal{S}w)_{i+1/2,j} &= \mathbf{S}^{(0,0)}_{i+1/2,j}w_{i+1/2,j} + \sum_{|k|=|l|=1} \mathbf{S}^{(k,l)}_{i+1/2,j}w_{i+1/2+k/2,j+l/2},\\ &0 \le i \le I-1, \quad 1 \le j \le J-1,\end{aligned} \tag{4.22}$$

while on the boundary the operator $\mathcal{S}$ is given by

$$(\mathcal{S}w)_{0,j+1/2} = \mathbf{S}^{(0,0)}_{0,j+1/2} w_{0,j+1/2} + \sum_{|l|=1} \mathbf{S}^{(+1,l)}_{0,j+1/2} w_{1/2,j+1/2+l/2}, \qquad 0 \le j \le J-1,$$

$$(\mathcal{S}w)_{I,j+1/2} = \mathbf{S}^{(0,0)}_{I,j+1/2} w_{I,j+1/2} + \sum_{|l|=1} \mathbf{S}^{(-1,l)}_{I,j+1/2} w_{I-1/2,j+1/2+l/2}, \qquad 0 \le j \le J-1,$$

$$(\mathcal{S}w)_{i+1/2,0} = \mathbf{S}^{(0,0)}_{i+1/2,0} w_{i+1/2,0} + \sum_{|k|=1} \mathbf{S}^{(k,+1)}_{i+1/2,0} w_{i+1/2+k/2,1/2}, \qquad 0 \le i \le I-1,$$

$$(\mathcal{S}w)_{i+1/2,J} = \mathbf{S}^{(0,0)}_{i+1/2,J} w_{i+1/2,J} + \sum_{|k|=1} \mathbf{S}^{(k,-1)}_{i+1/2,J} w_{i+1/2+k/2,J-1/2}, \qquad 0 \le i \le I-1.$$

The formula for $\mathcal{M}$ follows from (3.29), while formulas for the **S** *matrix are given in Section A.3 in Appendix A.*

We will also need the operator $\mathcal{L} : \mathcal{H}_\mathcal{V} \to \mathcal{H}_\mathcal{V}$ defined by

$$\begin{aligned} (\mathcal{L}w)_{i,j+1/2} &= \mathsf{L}_{i,j+1/2} w_{i,j+1/2}, \qquad 0 \le i \le I, \qquad 0 \le j \le J-1, \\ (\mathcal{L}w)_{i+1/2,j} &= \mathsf{L}_{i+1/2,j} w_{i+1/2,j}, \qquad 0 \le i \le I-1, \quad 0 \le j \le J. \end{aligned} \tag{4.23}$$

PROPOSITION 4.6. *The operators $\mathcal{L}$ and $\mathcal{M}$ are diagonal matrices with positive entries, while the operator $\mathcal{S}$ has a five-band symmetric matrix with positive entries. All three matrices are positive definite.*

PROOF. Because $\mathcal{M}$ and $\mathcal{S}$ give inner products (see (4.19)), their matrices must be symmetric and positive definite. The formulas for the matrices, (4.20), (4.23), and (4.22), define the band structure and also show that they are symmetric. We estimate the positive definiteness of these operators in the next proposition.

PROPOSITION 4.7. *The operators $\mathcal{M}$, $\mathcal{S}$, and $\mathcal{L}$ satisfy*

$$\begin{aligned} \mathsf{L}^2_{\min}\beta |||u|||^2_\mathcal{S} &\le \langle\langle u, \mathcal{M}u\rangle\rangle_\mathcal{S} \le \mathsf{L}^2_{\max} |||u|||^2_\mathcal{S}, \\ \frac{\beta^2}{4}\mathsf{L}^2_{\min} |||w|||^2_\mathcal{S} &\le \langle\langle w, \mathcal{S}w\rangle\rangle_\mathcal{V} \le \frac{2}{\beta}\mathsf{L}^2_{\max} |||w|||^2_\mathcal{V}, \\ \mathsf{L}_{\min} |||w|||^2_\mathcal{V} &\le \langle\langle w, \mathcal{L}w\rangle\rangle_\mathcal{V} \le \mathsf{L}_{\max} |||w|||^2_\mathcal{V}, \end{aligned} \tag{4.24}$$

where $\mathsf{L}_{\min}$ and $\mathsf{L}_{\max}$ are defined in (3.23), and β is defined in (3.25) as the sine of the smallest angle. These estimates imply the operators are positive definite.

PROOF. The estimate for $\mathcal{M}$ follows from (4.20) and the estimate for $\mathcal{L}$ follows from (4.23). For $\mathcal{S}$, we sum estimate (2.52) over all of the cells, then except at the boundaries, each value of w is counted twice, so the lower estimate remains unchanged but the upper estimate must be multiplied by 2.

COROLLARY 4.8.

$$\begin{aligned} \mathsf{L}^2_{\min}\beta &\le |||\mathcal{M}|||_\mathcal{S} \le \mathsf{L}^2_{\max}, \\ \frac{\beta^2}{4}\mathsf{L}^2_{\min} &\le |||\mathcal{S}|||_\mathcal{V} \le \frac{2}{\beta}\mathsf{L}^2_{\max}, \\ \mathsf{L}_{\min} &\le |||\mathcal{L}|||_\mathcal{V} \le \mathsf{L}_{\max}. \end{aligned} \tag{4.25}$$

PROOF. This follows from Theorem 3 of [39, p. 201].

4.2.1. The formal divergence and gradient

The formal divergence is defined by

$$(\mathbf{D}w)_{i+1/2,j+1/2} = w_{i+1,j+1/2} - w_{i,j+1/2} + w_{i+1/2,j+1} + w_{i+1/2,j}, \tag{4.26}$$

where $0 \le i \le I-1$, $0 \le j \le J-1$.

The definition of the formal gradient has two parts: for $0 \le j \le J-1$,

$$\begin{aligned}(\mathbf{G}u)_{0,j+1/2} &= u_{1/2,j+1/2} - u_{0,j+1/2},\\ (\mathbf{G}u)_{i,j+1/2} &= u_{i+1/2,j+1/2} - u_{i-1/2,j+1/2}, \qquad 1 \le i \le I-1,\\ (\mathbf{G}u)_{I,j+1/2} &= u_{I,j+1/2} - u_{I-1/2,j+1/2},\end{aligned} \tag{4.27}$$

and for $0 \le i \le I-1$,

$$\begin{aligned}(\mathbf{G}u)_{i+1/2,0} &= u_{i+1/2,1/2} - u_{i+1/2,0},\\ (\mathbf{G}u)_{i,j+1/2} &= u_{i+1/2,j+1/2} - u_{i+1/2,j-1/2}, \qquad 1 \le j \le J-1,\\ (\mathbf{G}u)_{i+1/2,J} &= u_{i+1/2,J} - u_{i+1/2,J-1/2}.\end{aligned} \tag{4.28}$$

The formal divergence and gradient satisfy a summation by part formula in two dimensions.

PROPOSITION 4.9. *For any $u \in \mathcal{H}_S$ and $w \in \mathcal{H}_V$,*

$$\begin{aligned}\langle\langle \mathbf{D}w, u\rangle\rangle_S + \langle\langle w, \mathbf{G}u\rangle\rangle_V = &- \sum_{i=0,I-1} w_{i+1/2,0}u_{i+1/2,0}\\ &+ \sum_{i=0,I-1} w_{i+1/2,J}u_{i+1/2,J}\\ &- \sum_{j=0,J-1} w_{0,j+1/2}u_{0,j+1/2}\\ &+ \sum_{j=0,J-1} w_{I,j+1/2}u_{I,j+1/2}.\end{aligned} \tag{4.29}$$

PROOF. This follows from two parameterized applications of the one-dimensional proof.

THEOREM 4.10. *The natural divergence and gradient can be expressed in terms of the formal divergence and gradient*

$$\mathcal{D} = \mathcal{M}^{-1}\mathbf{D}\mathcal{L}, \qquad \mathcal{G} = \mathcal{S}^{-1}\mathcal{L}\mathbf{G}. \tag{4.30}$$

PROOF. The formula for the divergence follows from (3.32). For the gradient, we combine the natural (3.36) and formal (4.29) summation by parts to get

$$\langle\langle \mathbf{D}\mathcal{L}w, u\rangle\rangle_S + \langle\langle \mathcal{L}w, \mathbf{G}u\rangle\rangle_V = \langle \mathcal{D}w, u\rangle_S + \langle w, \mathcal{G}u\rangle_V. \tag{4.31}$$

But

$$\langle \mathcal{D}w, u\rangle_S = \left\langle \mathcal{M}^{-1}\mathbf{D}\mathcal{L}w, u\right\rangle_S = \left\langle\left\langle \mathcal{M}^{-1}\mathbf{D}\mathcal{L}w, \mathcal{M}u\right\rangle\right\rangle_S = \langle\langle \mathbf{D}\mathcal{L}w, u\rangle\rangle_S, \tag{4.32}$$

and so

$$\langle\langle \mathcal{L}w, \mathbf{G}u\rangle\rangle_V = \langle w, \mathcal{G}u\rangle_V = \langle\langle w, \mathcal{S}\mathcal{G}u\rangle\rangle_V. \tag{4.33}$$

This implies that $\mathcal{L}\mathbf{G} = \mathcal{S}\mathcal{G}$ or $\mathcal{G} = \mathcal{S}^{-1}\mathcal{L}\mathbf{G}$ as was required.

COROLLARY 4.11. *The matrix of $\mathcal{S}$ is diagonally dominant.*

PROOF. We first observe that (4.30) gives $\mathcal{L}^{-1}\mathcal{S}\mathcal{G} = \mathbf{G}$. Comparing this with (3.33) and (3.34), we see that the matrix for $\mathcal{L}^{-1}\mathcal{S}$ is given by the matrix $\mathbf{A}$ which is shown to be diagonally dominant in Section A.2.1. Multiplying a diagonally dominant matrix by a positive diagonal matrix does not change the measure of dominance given in (A.14).

4.3. Formal Friedrichs-Poincaré Inequalities

In the continuum, the Friedrichs-Poincaré inequalities estimate the value of a scalar field in terms of its gradient along with some side condition that takes care of constant fields. The discrete analogs of these inequalities involve estimating the value of discrete fields in terms of differences. There are many possible estimates, but the mean-square estimate is fundamental in the discrete case.

LEMMA 4.12. *Let a_n, $0 \le n \le N$ be a sequence of real numbers, then*

$$|a_N - a_0|^2 \le N \sum_{n=0}^{N-1} (a_{n+1} - a_n)^2 . \tag{4.34}$$

PROOF. A standard "collapsing sum" argument gives

$$a_N - a_0 = \sum_{n=0}^{N-1} a_{n+1} - a_n, \tag{4.35}$$

and the Cauchy-Schwartz inequality gives

$$|a_N - a_0|^2 \le \sum_{n=0}^{N-1} (a_{n+1} - a_n)^2 \sum_{n=0}^{N-1} 1^2, \tag{4.36}$$

which gives the result.

4.3.1. One-dimensional inequalities

PROPOSITION 4.13. *If $u \in \mathcal{H}_S$ and $u_0 = 0$, then the discrete Friedrichs-Poincaré inequality holds*

$$|||u|||_S \le I|||\mathbf{G}u|||_\mathcal{V}. \tag{4.37}$$

PROOF. Choosing $N = i$, $a_0 = u_0 = 0$, $a_i = u_{i-1/2}$, $1 \le i \le I$, $a_{I+1} = u_I$, and $(\mathbf{G}u)_i = u_{i+1/2} - u_{i-1/2}$ in (4.34) gives

$$u_{i-1/2}^2 \le i \sum_{k=0}^{i-1} (\mathbf{G}u)_k^2 \le I \sum_{k=0}^{I} (\mathbf{G}u)_k^2 , \tag{4.38}$$

which implies that

$$u_{i+1/2}^2 \le I \sum_{k=0}^{I} (\mathbf{G}u)_k^2 . \tag{4.39}$$

Summing this equation

$$|||u|||_S^2 = \sum_{i=0}^{I-1} u_{i+1/2}^2 \le I^2 \sum_{k=0}^{I} (\mathbf{G}u)_k^2 , \tag{4.40}$$

and substituting

$$|||\mathbf{G}u|||_\mathcal{V}^2 = \sum_{i=0}^{I} (\mathbf{G}u)_i^2 , \tag{4.41}$$

gives

$$|||u|||_S^2 \le I^2 |||\mathbf{G}u|||_\mathcal{V}^2, \tag{4.42}$$

which gives the result.

COROLLARY 4.14. *Under the assumptions of Proposition 4.13,*

$$|||u|||_S \le \frac{\mathsf{L}}{\mathsf{L}_{\min}} |||\mathbf{G}u|||_\mathcal{V}, \tag{4.43}$$

where $\mathsf{L} = x_I - x_0$ is the length of the interval.

We now convert the formal Friedrichs-Poincaré inequality (4.43) to a natural Friedrichs-Poincaré inequality.

THEOREM 4.15. *If u is zero on the boundary of the grid, then there exists a constant K independent of the grid spacing such that*

$$\|u\|_{\mathcal{S}} \leq K\|\mathcal{G}u\|_{\mathcal{V}}. \tag{4.44}$$

PROOF.

$$\begin{aligned}
\|u\|_{\mathcal{S}}^2 &= \langle u, u\rangle_{\mathcal{S}} \\
&= \langle\langle u, \mathcal{M}u\rangle\rangle_{\mathcal{S}} && \text{(by (4.2))} \\
&\leq \mathsf{L}_{\max}|||u|||_{\mathcal{S}}^2 && \text{(by (4.5))} \\
&\leq \mathsf{L}^2\frac{\mathsf{L}_{\max}}{\mathsf{L}_{\min}^2}|||\mathbf{G}u|||_{\mathcal{V}}^2 && \text{(by (4.43))} \\
&= \mathsf{L}^2\frac{\mathsf{L}_{\max}}{\mathsf{L}_{\min}^2}|||\mathcal{S}\mathcal{S}^{-1}\mathbf{G}u|||_{\mathcal{V}}^2 \\
&= \mathsf{L}^2\frac{\mathsf{L}_{\max}}{\mathsf{L}_{\min}^2}|||\mathcal{S}\mathcal{G}u|||_{\mathcal{V}}^2 && \text{(by (4.14))} \\
&\leq \mathsf{L}^2\frac{\mathsf{L}_{\max}^3}{\mathsf{L}_{\min}^2}|||\mathcal{G}u|||_{\mathcal{V}}^2 && \text{(by (4.6))} \\
&\leq 2\mathsf{L}^2\frac{\mathsf{L}_{\max}^3}{\mathsf{L}_{\min}^3}\langle\langle \mathcal{G}u, \mathcal{S}\mathcal{G}u\rangle\rangle_{\mathcal{V}} && \text{(by (4.5))} \\
&= 2\mathsf{L}^2\rho^3\|\mathcal{G}u\|_{\mathcal{V}}^2 && \text{(by (3.3))}.
\end{aligned}$$

Here L is the length of the interval and $\rho = \mathsf{L}_{\max}/\mathsf{L}_{\min}$ is finite by assumption, so we have the estimate.

4.3.2. Two-dimensional inequalities

PROPOSITION 4.16. *If $u \in \mathcal{H}_{\mathcal{S}}$, $u_{0,j+1/2} = 0$, $0 \leq j \leq J-1$, and $u_{i+1/2,0} = 0$, $0 \leq i \leq I-1$, then*

$$|||u|||_{\mathcal{S}} \leq \min(I, J)|||\mathbf{G}u|||_{\mathcal{V}}. \tag{4.45}$$

PROOF. In two dimensions

$$|||u|||_{\mathcal{S}}^2 = \sum_{i=0}^{I-1}\sum_{j=0}^{J-1} u_{i+1/2,j+1/2}^2, \tag{4.46}$$

while

$$|||\mathbf{G}u|||_{\mathcal{V}}^2 = \sum_{i=0}^{I}\sum_{j=0}^{J-1}(\mathbf{G}u)_{i,j+1/2}^2 + \sum_{i=0}^{I-1}\sum_{j=0}^{J}(\mathbf{G}u)_{i+1/2,j}^2. \tag{4.47}$$

We can rewrite (4.39) as

$$u_{i+1/2,j+1/2}^2 \leq I\sum_{k=0}^{I}(\mathbf{G}u)_{k,j+1/2}^2, \tag{4.48}$$

and then repeat the one-dimensional arguments to get

$$|||u|||_{\mathcal{S}}^2 \leq I^2|||\mathbf{G}u|||_{\mathcal{V}}^2. \tag{4.49}$$

Interchanging the roles of i and j gives

$$|||u|||_{\mathcal{S}}^2 \leq J^2|||\mathbf{G}u|||_{\mathcal{V}}^2. \tag{4.50}$$

These last two estimates imply the result.

COROLLARY 4.17. *Under the assumptions of Proposition 4.16,*

$$|||u|||_{\mathcal{S}} \leq \frac{\mathsf{P}}{2\mathsf{L}_{\min}} |||\mathbf{G}u|||_{\mathcal{V}}, \tag{4.51}$$

where P *is the perimeter of the domain in which the grid lies.*

PROOF. There are $2(I+J)$ boundary segments of the grid lying on the boundary of the grid, and so the sum of their lengths must be less than the perimeter of the domain, but the length of each segment is longer than $\mathsf{L}_{\min}$, so

$$2(I+J)\mathsf{L}_{\min} \leq \mathsf{P}. \tag{4.52}$$

However, $\min(I,J) \leq \max(I,J) \leq I+J$ so

$$\min(I,J) \leq \frac{\mathsf{P}}{2\mathsf{L}_{\min}}, \tag{4.53}$$

which when combined with (4.45) gives the result.

We now convert the formal Friedrichs-Poincaré inequality (4.51) to a natural Friedrichs-Poincaré inequality.

THEOREM 4.18. *If* u *is zero on the boundary of the grid, then there exists a constant* K *independent of the grid spacing such that*

$$\|u\|_{\mathcal{S}} \leq K \|\mathcal{G}u\|_{\mathcal{V}}. \tag{4.54}$$

PROOF. If P is the perimeter of the domain, then

$$\begin{aligned}
\|u\|_{\mathcal{S}}^2 &= \langle u, u\rangle_{\mathcal{S}} \\
&= \langle\langle u, \mathcal{M}u\rangle\rangle_{\mathcal{S}} && \text{(by (4.19))} \\
&\leq \mathsf{L}_{\max}^2 |||u|||_{\mathcal{S}}^2 && \text{(by (4.24))} \\
&\leq \frac{\mathsf{P}^2}{4}\frac{\mathsf{L}_{\max}^2}{\mathsf{L}_{\min}^2} |||\mathbf{G}u|||_{\mathcal{V}}^2 && \text{(by (4.51))} \\
&= \frac{\mathsf{P}^2}{4}\frac{\mathsf{L}_{\max}^2}{\mathsf{L}_{\min}^2} \left|\left|\left|\mathcal{L}^{-1}\mathcal{S}\mathcal{S}^{-1}\mathcal{L}\mathbf{G}u\right|\right|\right|_{\mathcal{V}}^2 \\
&= \frac{\mathsf{P}^2}{4}\frac{\mathsf{L}_{\max}^2}{\mathsf{L}_{\min}^2} \left|\left|\left|\mathcal{L}^{-1}\mathcal{S}\mathcal{G}u\right|\right|\right|_{\mathcal{V}}^2 && \text{(by (4.30))} \\
&\leq \frac{\mathsf{P}^2}{4}\frac{\mathsf{L}_{\max}^2}{\mathsf{L}_{\min}^4} |||\mathcal{S}\mathcal{G}u|||_{\mathcal{V}}^2 && \text{(by (4.25))} \\
&\leq \frac{\mathsf{P}^2}{\beta^2}\frac{\mathsf{L}_{\max}^6}{\mathsf{L}_{\min}^4} |||\mathcal{G}u|||_{\mathcal{V}}^2 && \text{(by (4.25))} \\
&\leq \frac{4\mathsf{P}^2}{\beta^4}\frac{\mathsf{L}_{\max}^6}{\mathsf{L}_{\min}^6} \langle\langle \mathcal{G}u, \mathcal{S}\mathcal{G}u\rangle\rangle_{\mathcal{V}} && \text{(by (4.24))} \\
&\leq \frac{4\mathsf{P}^2\rho^6}{\beta^4} \|\mathcal{G}u\|_{\mathcal{V}}^2 && \text{(by (3.24))},
\end{aligned}$$

where $\rho = \mathsf{L}_{\max}/\mathsf{L}_{\min}$ and β defined in (3.25) is the sine of the smallest angle are finite by assumption. So we have the estimate with $K = 2\mathsf{P}\rho^3/\beta^2$.

5. CONVERGENCE OF SOLUTIONS OF BOUNDARY-VALUE PROBLEMS

We discretize the one- and two-dimensional Dirichlet boundary-value problem for the Laplacian using the mimetic discretizations of the divergence and gradient and then estimate the error in the discrete solution and its gradient. We first estimate the error on a single grid. The requirement that the constants in the error estimates remain bounded as the grid is refined gives the modest smoothness conditions the grids need to satisfy to give first-order convergence of the solutions and their gradients.

We begin by introducing an *abstract* mimetic theory independent of the detailed structure of the grid, the details of the definition of the inner products, and the details of the definition of the divergence and gradient. The crucial point is that the discrete divergence and gradient satisfy an abstract summation by parts formula.

5.1. Abstract Mimetic Formulation

The abstract mimetic formulation assumes there are two real linear spaces of discrete fields. The space $\mathcal{H}_S$ is associated with values of scalar fields in cells, while $\mathcal{H}_V$ is associated with values of normal components of vector fields on cell boundaries. Each of these spaces is endowed with an inner product: $\langle\cdot,\cdot\rangle_S$ on $\mathcal{H}_S$ and $\langle\cdot,\cdot\rangle_V$ on $\mathcal{H}_V$, and associated norms $\|\cdot\|_S^2$ and $\|\cdot\|_V^2$.

There are also two linear operators, the discrete divergence $\mathcal{D}$ and the discrete gradient $\mathcal{G}$ such that

$$\mathcal{D}:\mathcal{H}_V\to\mathcal{H}_S,\qquad \mathcal{G}:\mathcal{H}_S\to\mathcal{H}_V, \tag{5.1}$$

and most importantly

$$\langle\mathcal{D}w,u\rangle_S+\langle w,\mathcal{G}u\rangle_V=0,\qquad \forall\, u\in\mathcal{H}_S,\quad \forall\, w\in\mathcal{H}_V. \tag{5.2}$$

The mimetic one- and two-dimensional discretization described in Section 5, with the assumption that the scalar fields are zero on the boundary of region Ω, both satisfy these assumptions (see (3.4), (3.6), (3.9), (3.11), and (3.12) for one dimension and (3.26), (3.28), (3.32), Proposition 3.5, and (3.36) for two dimensions).

The discrete Laplacian defined by

$$\mathcal{L}=-\mathcal{D}\mathcal{G}:\mathcal{H}_S\to\mathcal{H}_S \tag{5.3}$$

is positive because, using (5.2),

$$\langle\mathcal{L}u,u\rangle_S=-\langle\mathcal{D}\mathcal{G}u,u\rangle_S=\langle\mathcal{G}u,\mathcal{G}u\rangle_V\ge 0. \tag{5.4}$$

It is more important to show that the Laplacian is positive *definite*, as we will do below.

5.2. The Boundary-Value Problem

We consider the continuum boundary-value problem (BVP) for the smooth scalar field $\mathbf{u}$ defined on polygonal domain Ω that satisfies Laplace's equation

$$-\vec{\nabla}\cdot\vec{\nabla}\mathbf{u}=\mathbf{f}, \tag{5.5}$$

where $\mathbf{f}$ is some given smooth scalar field on the interior of Ω and $\mathbf{u}=0$ on the boundary of the domain $\partial\Omega$. We assume that the boundary of the grid is identical to the boundary of the domain. The discrete BVP is to find a discrete scalar field $u\in\mathcal{H}_S$ that satisfies

$$\mathcal{L}u=f, \tag{5.6}$$

where $f\in\mathcal{H}_S$ is a given discrete scalar field on the *interior* of the grid and $u=0$ for all grid points on the boundary of the grid.

5.3. Solvability of the Discrete Boundary-Value Problem

We assume that if $u \in \mathcal{H}_\mathcal{S}$ and $u = 0$ on the boundary of the grid, then there exists a constant $C > 0$ that only depends on Ω such that the discrete Friedrichs-Poincaré inequality

$$\|u\|_\mathcal{S} \leq C \|\mathcal{G}u\|_\mathcal{V} \tag{5.7}$$

bounds the discrete scalar field in terms of its gradient (see (4.44) and (4.54)).

THEOREM 5.1. *The discrete Laplacian with Dirichlet boundary conditions is positive definite. That is, if $u \in \mathcal{H}_\mathcal{S}$ and $u = 0$ on the boundary of the grid, then*

$$\langle \mathcal{L}u, u \rangle_\mathcal{S} \geq \frac{1}{C^2} \langle u, u \rangle_\mathcal{S}, \tag{5.8}$$

where C is defined in (5.7). Consequently the discrete boundary-value problem is uniquely solvable and the solution satisfies

$$\|u\|_\mathcal{S} \leq C \|f\|_\mathcal{S}. \tag{5.9}$$

PROOF. Formula (5.4) and the Friedrichs-Poincaré inequality (5.7) imply that

$$\langle \mathcal{L}u, u \rangle_\mathcal{S} = \langle \mathcal{G}u, \mathcal{G}u \rangle_\mathcal{V} \geq \frac{1}{C^2} \langle u, u \rangle_\mathcal{S}, \tag{5.10}$$

which gives the first part of the theorem. A positive matrix that is bounded below is invertible, so the discrete boundary-value problem is uniquely solvable. Setting $\mathcal{L}u = f$ in the first part of the theorem gives the second part.

5.4. Projections and Truncation Error

We also assume that there are two linear operators: $\mathcal{P}_\mathcal{S}$ that maps smooth scalar fields $\mathbf{u}$ defined on Ω and zero on $\partial\Omega$ to a discrete scalar field $\mathcal{P}_\mathcal{S}\mathbf{u} \in \mathcal{H}_\mathcal{S}$ with $\mathcal{P}_\mathcal{S}\mathbf{u} = 0$ on the boundary of the grid; and $\mathcal{P}_\mathcal{V}$ that maps smooth vector fields $\vec{\mathbf{w}}$ defined on Ω to a discrete scalar field $\mathcal{P}_\mathcal{V}\vec{\mathbf{w}} \in \mathcal{H}_\mathcal{V}$.

If the continuum divergence is given by $\vec{\nabla}\cdot$, then the truncation error of the discrete divergence $\mathcal{D} : \mathcal{H}_\mathcal{V} \to \mathcal{H}_\mathcal{S}$ is

$$\mathcal{T}_\mathcal{D}(\vec{\mathbf{w}}) = \mathcal{P}_\mathcal{S}\vec{\nabla} \cdot \vec{\mathbf{w}} - \mathcal{D}\mathcal{P}_\mathcal{V}\vec{\mathbf{w}}. \tag{5.11}$$

If the continuum gradient is given by $\vec{\nabla}$, then the truncation error of the discrete gradient $\mathcal{G} : \mathcal{H}_\mathcal{S} \to \mathcal{H}_\mathcal{V}$ is

$$\mathcal{T}_\mathcal{G}(\mathbf{u}) = \mathcal{P}_\mathcal{V}\vec{\nabla}\mathbf{u} - \mathcal{G}\mathcal{P}_\mathcal{S}\mathbf{u}. \tag{5.12}$$

If the continuum Laplacian is given by $\Delta = -\vec{\nabla} \cdot \vec{\nabla}$, then the discrete Laplacian $\mathcal{L} = -\mathcal{D}\mathcal{G} : \mathcal{H}_\mathcal{S} \to \mathcal{H}_\mathcal{S}$ has truncation error

$$\mathcal{T}_\mathcal{L}(\mathbf{u}) = \mathcal{P}_\mathcal{S}\Delta\mathbf{u} - \mathcal{L}\mathcal{P}_\mathcal{S}\mathbf{u}. \tag{5.13}$$

The truncation error for the Laplacian can be broken into two parts,

$$\mathcal{T}_\mathcal{L}(\mathbf{u}) = -\left(\mathcal{D}\mathcal{T}_\mathcal{G}(\mathbf{u}) + \mathcal{T}_\mathcal{D}\left(\vec{\nabla}\mathbf{u}\right)\right). \tag{5.14}$$

5.5. The Error in the Solution and Its Gradient

We want to compare the solution of the continuum and discrete boundary-value problems, so let the smooth continuum scalar field $\mathbf{u}$ satisfy (5.5), while u is a discrete field that satisfies the discrete boundary-value problem (5.6) where $f = \mathcal{P}_\mathcal{S}\mathbf{f}$. The error $\mathbf{e}$ compares the projection of the solution $\mathbf{u}$ of the continuum problem (5.5) to the solution u of the discrete problem (5.6),

$$\mathbf{e} = \mathcal{P}_\mathcal{S}\mathbf{u} - u, \tag{5.15}$$

while the error in the discrete gradient of the solution is

$$\mathbf{E} = \mathcal{P}_\mathcal{V}\vec{\nabla}\mathbf{u} - \mathcal{G}u. \tag{5.16}$$

We assume that $\mathbf{e} = 0$ on $\partial\Omega$ because both $\mathbf{u} = 0$ and $u = 0$ on $\partial\Omega$.

PROPOSITION 5.2. *The error satisfies the discrete boundary-value problem*

$$\mathcal{L}\mathbf{e} = -\mathcal{T}_{\mathcal{L}}(\mathbf{u}) \tag{5.17}$$

and $\mathbf{e}$ *is zero on the boundary of the grid.*

PROOF. By assumption $\mathbf{e} = 0$ on $\partial\omega$. Next,

$$\begin{aligned} \mathcal{L}e &= \mathcal{L}\left(\mathcal{P}_{\mathcal{S}}\mathbf{u} - u\right) && \text{(by (5.15))} \\ &= \mathcal{L}\mathcal{P}_{\mathcal{S}}\mathbf{u} - \mathcal{P}_{\mathcal{S}}\vec{\nabla}\cdot\vec{\nabla}\mathbf{u} + \mathcal{P}_{\mathcal{S}}\vec{\nabla}\cdot\vec{\nabla}\mathbf{u} - \mathcal{L}u && \\ &= -\mathcal{T}_{\mathcal{L}}(\mathbf{u}) + \mathcal{P}_{\mathcal{S}}\mathbf{f} - f && \text{(by (5.13), (5.5), and (5.6))} \\ &= -\mathcal{T}_{\mathcal{L}}(\mathbf{u}) && \text{(by the definition of } f) \end{aligned}$$

as was to be proved.

It is easy to verify that the error in the gradient can be decomposed into two parts.

PROPOSITION 5.3.

$$\mathbf{E} = \mathcal{T}_{\mathcal{G}}(\mathbf{u}) + \mathcal{G}\mathbf{e}. \tag{5.18}$$

We estimate the error in the solution and gradient of the solution in terms of the truncation errors for the divergence and gradient.

THEOREM 5.4. *There exists a constant* K *independent of the grid size such that*

$$\|\mathbf{e}\|_{\mathcal{S}} + \|\mathbf{E}\|_{\mathcal{V}} \le K\left(\|\mathcal{T}_{\mathcal{G}}(\mathbf{u})\|_{\mathcal{V}} + \left\|\mathcal{T}_{\mathcal{D}}\left(\vec{\nabla}\mathbf{u}\right)\right\|_{\mathcal{S}}\right). \tag{5.19}$$

PROOF. From (5.18), we have

$$\|\mathbf{E}\|_{\mathcal{V}} \le \|\mathcal{T}_{\mathcal{G}}(\mathbf{u})\|_{\mathcal{V}} + \|\mathcal{G}\mathbf{e}\|_{\mathcal{V}}, \tag{5.20}$$

and so we need to estimate the second term on the right of this inequality. Equations (5.17) and (5.14) give

$$\mathcal{L}\mathbf{e} = \mathcal{D}\mathcal{T}_{\mathcal{G}}(\mathbf{u}) + \mathcal{T}_{\mathcal{D}}\left(\vec{\nabla}\mathbf{u}\right). \tag{5.21}$$

If we take the inner product (5.21) with $\mathbf{e}$ and apply (5.2) to two of the terms, then

$$\langle\mathcal{G}\mathbf{e},\mathcal{G}\mathbf{e}\rangle_{\mathcal{V}} = \langle\mathcal{G}\mathbf{e},\mathcal{T}_{\mathcal{G}}(\mathbf{u})\rangle_{\mathcal{V}} + \left\langle\mathbf{e},\mathcal{T}_{\mathcal{D}}\left(\vec{\nabla}\mathbf{u}\right)\right\rangle_{\mathcal{S}}. \tag{5.22}$$

This implies that

$$\|\mathcal{G}\mathbf{e}\|_{\mathcal{V}}^2 \le \|\mathcal{G}\mathbf{e}\|_{\mathcal{V}}\|\mathcal{T}_{\mathcal{G}}(\mathbf{u})\|_{\mathcal{V}} + \|\mathbf{e}\|_{\mathcal{S}}\left\|\mathcal{T}_{\mathcal{D}}\left(\vec{\nabla}\mathbf{u}\right)\right\|_{\mathcal{S}}. \tag{5.23}$$

The Friedrichs-Poincaré inequality (5.7) and division by the norm of the discrete gradient of the error gives

$$\|\mathcal{G}\mathbf{e}\|_{\mathcal{V}} \le \|\mathcal{T}_{\mathcal{G}}(\mathbf{u})\|_{\mathcal{V}} + C\left\|\mathcal{T}_{\mathcal{D}}\left(\vec{\nabla}\mathbf{u}\right)\right\|_{\mathcal{S}}. \tag{5.24}$$

Now, (5.20) gives

$$\|\mathbf{E}\|_{\mathcal{V}} \le 2\|\mathcal{T}_{\mathcal{G}}(\mathbf{u})\|_{\mathcal{V}} + C\left\|\mathcal{T}_{\mathcal{D}}\left(\vec{\nabla}\mathbf{u}\right)\right\|_{\mathcal{S}}, \tag{5.25}$$

and the Friedrichs-Poincaré inequality (5.7) gives

$$\|\mathbf{e}\|_{\mathcal{S}} \le C\|\mathcal{T}_{\mathcal{G}}(\mathbf{u})\|_{\mathcal{V}} + C^2\left\|\mathcal{T}_{\mathcal{D}}\left(\vec{\nabla}\mathbf{u}\right)\right\|_{\mathcal{S}}. \tag{5.26}$$

These last two inequalities imply the theorem with $K = \max\{2+C, C+C^2\}$.

5.6. Solution Error Estimates for the Boundary-Value Problems

THEOREM 5.5. *In one dimension, if the discrete boundary-value problem (5.6) is created using (3.9) to discretize the divergence and (3.11) to discretize the gradient, then there exists a constant $\tilde{K}$ independent of the grid such that*

$$\|\mathbf{e}\|_{\mathcal{S}} + \|\mathbf{E}\|_{\mathcal{V}} \le \tilde{K} \mathsf{L}_{\max}, \tag{5.27}$$

where the error in solution $\mathbf{e}$ *is defined in (5.15) and the error* $\mathbf{E}$ *in the gradient is defined in (5.16).*

PROOF. In the following, C_k is a numerical constant times the maximum of the k-derivative of $\mathbf{u}$ over the domain. We need to estimate the right-hand side of (5.19) and start by estimating $\|\mathcal{T}_{\mathcal{D}}(\vec{\nabla}\mathbf{u})\|_{\mathcal{S}}$.

From (3.7), we see that

$$\|u\|_{\mathcal{S}} \le \sqrt{\mathsf{L}} \max_{0 \le i \le I-1} \left| u_{i+1/2} \right|, \tag{5.28}$$

so that

$$\left\| \mathcal{T}_{\mathcal{D}}\left(\vec{\nabla}\mathbf{u}\right) \right\|_{\mathcal{S}} \le \sqrt{\mathsf{L}} \max_{0 \le i \le I-1} \left| \mathcal{T}_{\mathcal{D}}\left(\vec{\nabla}\mathbf{u}\right)_{i+1/2} \right|, \tag{5.29}$$

where L is the length of domain (3.1). From (3.17), with $\mathbf{w} = \vec{\nabla}\mathbf{u}$, we get

$$\max_{0 \le i \le I-1} \left| \mathcal{T}_{\mathcal{D}}\left(\vec{\nabla}\mathbf{u}\right)_{i-1/2} \right| \le C_4 \mathsf{L}_{\max}^2. \tag{5.30}$$

Next, we estimate $\|\mathcal{T}_{\mathcal{G}}(\mathbf{u})\|_{\mathcal{V}}$. From (3.8), we see that

$$\|w\|_{\mathcal{V}} \le \sqrt{\mathsf{L}} \max_{0 \le i \le I} \left| w_{i+1/2} \right|, \tag{5.31}$$

so that

$$\|\mathcal{T}_{\mathcal{G}}(\mathbf{u})\|_{\mathcal{V}} \le \sqrt{\mathsf{L}} \max_{0 \le i \le I} \left| \mathcal{T}_{\mathcal{G}}(\mathbf{u})_{i+1/2} \right|. \tag{5.32}$$

It follows from (3.17), that

$$\max_{0 \le i \le I} \left| \mathcal{T}_{\mathcal{G}}(\mathbf{u})_i \right| \le C_2 \mathsf{L}_{\max}. \tag{5.33}$$

Combining these estimates gives the result with

$$\tilde{K} \le K \sqrt{\mathsf{L}} \left(C_2 + C_4 \mathsf{L}_{\max} \right). \tag{5.34}$$

THEOREM 5.6. *In two dimensions, if the discrete boundary-value problem (5.6) is created using the discrete divergence (3.32) and the discrete gradient given by (3.33)–(3.35), then there exists a constant $\tilde{K}$ such that*

$$\|\mathbf{e}\|_{\mathcal{S}} + \|\mathbf{E}\|_{\mathcal{V}} \le \tilde{K} \mathsf{L}_{\max}. \tag{5.35}$$

PROOF. From (3.41), with $\mathbf{w} = \vec{\nabla}\mathbf{u}$, we get

$$\max_{\substack{0 \le i \le I-1 \\ 0 \le j \le J-1}} \left| \mathcal{T}_{\mathcal{D}}\left(\vec{\nabla}\mathbf{u}\right)_{i+1/2,j+1/2} \right| \le \frac{\rho}{\beta} \left(C_3 + C_4 \mathsf{L}_{\max} \right) \mathsf{L}_{\max}. \tag{5.36}$$

Also, from (3.29), we see that

$$\|u\|_{\mathcal{S}} \le \sqrt{\mathsf{A}} \max_{\substack{0 \le i \le I-1 \\ 0 \le j \le J-1}} \left| u_{i+1/2,j+1/2} \right|, \tag{5.37}$$

where the area of region A is defined in (3.22). Replacing u by $\mathcal{T}_{\mathcal{D}}(\vec{\nabla}\mathbf{u})$ gives

$$\left\|\mathcal{T}_{\mathcal{D}}\left(\vec{\nabla}\mathbf{u}\right)\right\|_{\mathcal{S}} \le (C_3 + C_4\mathsf{L}_{\max})\sqrt{\mathsf{A}}\frac{\rho}{\beta}\mathsf{L}_{\max}. \tag{5.38}$$

From (3.41), we have

$$\max\left\{\max_{\substack{0\le i\le I\\0\le j\le J-1}}\left|\mathcal{T}_{\mathcal{G}}(\mathbf{u})_{i,j+1/2}\right|, \max_{\substack{0\le i\le I-1\\0\le j\le J}}\left|\mathcal{T}_{\mathcal{G}}(\mathbf{u})_{i+1/2,j}\right|\right\} \le C_2\frac{\rho^2}{\beta^3}\mathsf{L}_{\max}. \tag{5.39}$$

Also,

$$\mathsf{A}_C \ge \mathsf{L}_{\min}^2\beta, \tag{5.40}$$

where β is defined in (2.41) and then from (2.52)

$$\langle w, w\rangle_{\mathcal{V}} \le 4\mathsf{A}_C\frac{\rho^2}{\beta^2}\max_{X\in\{D,R,U,L\}} w_X^2, \tag{5.41}$$

where ρ is defined in (3.3). Summing this over all cells gives

$$\|w\|_{\mathcal{V}} \le 4\sqrt{\mathsf{A}}\frac{\rho}{\beta}\max\left\{\max_{\substack{0\le i\le I-1\\0\le j\le J}}\left|w_{i+1/2,j}\right|, \max_{\substack{0\le i\le I\\0\le j\le J-1}}\left|w_{i,j+1/2}\right|\right\}. \tag{5.42}$$

Replacing w with $\mathcal{T}_{\mathcal{G}}(\mathbf{u})$, we obtain

$$\|\mathcal{T}_{\mathcal{G}}(\mathbf{u})\|_{\mathcal{V}} \le 2C_2\sqrt{\mathsf{A}}\frac{\rho^3}{\beta^4}\mathsf{L}_{\max}. \tag{5.43}$$

Combining these estimates gives the result with

$$\tilde{K} \le K\sqrt{\mathsf{A}}\left((C_3 + C_4\mathsf{L}_{\max})\frac{\rho}{\beta} + 2C_2\frac{\rho^3}{\beta^4}\right). \tag{5.44}$$

5.7. Grid Convergence

To study convergence, we will choose a family of grids where $\mathsf{L}_{\max}$ converges to zero as the number of grid points goes to infinity, and moreover, the grids do not degenerate.

ASSUMPTION 5.7. *In one dimension, there is a finite ρ_0 such that for every grid in the family, $\rho < \rho_0$, where $\rho = \mathsf{L}_{\max}/\mathsf{L}_{\min}$ (see (3.3)).*

ASSUMPTION 5.8. *In two dimensions, there exists finite ρ_0 and $\beta_0 > 0$ such that for every grid in the family, $\rho < \rho_0$, and $\beta > \beta_0$, where $\rho = \mathsf{L}_{\max}/\mathsf{L}_{\min}$ (see (3.24)) and β is defined in (3.25) as the sine of the smallest angle in the grid.*

THEOREM 5.9. *In one dimension, assume we have a sequence of grids containing n points, with $n \to \infty$ that satisfy Assumption 5.7 and, in addition, there exists a constant $c > 0$ such that $\mathsf{L}_{\max} \le c/n$. Then, the solutions the discrete boundary-value problem (5.6) created using (3.9) to discretize the divergence and (3.11) to discretize the gradient converge at first order to the solution of the continuum boundary-value problem (5.5).*

PROOF. This follows from the estimate in Theorem 5.5 and that Assumption 5.7 bounds ρ.

THEOREM 5.10. *In two dimensions, assume we have a sequence of grids containing n points, with $n \to \infty$, and that there exist a constant $k > 0$ so that $I \geq k\sqrt{n}$ and $J \geq k\sqrt{n}$. Furthermore, the grids satisfy Assumption 5.8 and there exists a constant $c > 0$ such that $\mathsf{L}_{\max} \leq c/\sqrt{n}$. Under these assumptions, the solutions the discrete boundary-value problem (5.6) created using (3.32) to discretize the divergence and the discrete gradient given by (3.33)–(3.35) converge at first order to the solution of the continuum boundary-value problem (5.5).*

PROOF. This follows from the estimate in Theorem 5.6 and that Assumption 5.8 bounds ρ and $1/\beta$.

Note that the error for the solution of the boundary-value problem is estimated in different norms for different grids, so we need to know that the norms converge as the grid is resolved. We will show that the inner products converge, which implies that the norms converge.

PROPOSITION 5.11. *In one dimension, if $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ are smooth scalar fields on $[a, b]$, then*

$$\left\langle \mathcal{P}_S \mathbf{u}^{(1)}, \mathcal{P}_S \mathbf{u}^{(2)} \right\rangle_S \to \int_a^b \mathbf{u}^{(1)}(x)\mathbf{u}^{(2)}(x)\, dx, \tag{5.45}$$

while if $\mathbf{w}^{(1)}$ and $\mathbf{w}^{(2)}$ are smooth vector fields on $[a, b]$, then

$$\left\langle \mathcal{P}_\mathcal{V} \mathbf{w}^{(1)}, \mathcal{P}_\mathcal{V} \mathbf{w}^{(2)} \right\rangle_\mathcal{V} \to \int_a^b \mathbf{w}^{(1)}(x)\mathbf{w}^{(2)}(x)\, dx, \tag{5.46}$$

for any family of grids satisfying Assumption 5.7, as $\mathsf{L}_{\max} \to 0$.

PROOF. The discrete inner products (3.7) and (3.8) correspond to midpoint and trapezoid integration rules which are known to be globally second-order accurate. This implies that the discrete inner products converge to the global inner products with a second-order convergence rate.

PROPOSITION 5.12. *In two dimensions, if $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ are smooth scalar fields on the region Ω, then*

$$\left\langle \mathcal{P}_S \mathbf{u}^{(1)}, \mathcal{P}_S \mathbf{u}^{(2)} \right\rangle_S \to \int_\Omega \mathbf{u}^{(1)}(x,y)\mathbf{u}^{(2)}(x,y)\, dx\, dy, \tag{5.47}$$

while if $\vec{\mathbf{w}}^{(1)}$ and $\vec{\mathbf{w}}^{(2)}$ are smooth vector fields on the region Ω, then

$$\left\langle \mathcal{P}_\mathcal{V} \vec{\mathbf{w}}^{(1)}, \mathcal{P}_\mathcal{V} \vec{\mathbf{w}}^{(2)} \right\rangle_\mathcal{V} \to \int_\Omega \vec{\mathbf{w}}^{(1)}(x,y) \cdot \vec{\mathbf{w}}^{(2)}(x,y)\, dx\, dy, \tag{5.48}$$

for any family of grids satisfying Assumptions 5.8, as $\mathsf{L}_{\max} \to 0$.

PROOF. The inner product (3.29) is a Riemann sum for the integral, and therefore, converges. Furthermore, the convergence is at least first order.

For the inner product of vector fields, we will estimate the "corner" inner products, such as (2.47), by the inner product of the projection of the vector fields at the center of a cell. First, if $\vec{\mathbf{w}}$ is a smooth vector field, then $\vec{\mathbf{w}}(r_C) - \vec{\mathbf{w}}(r_D)$ and $\vec{\mathbf{w}}(r_C) - \vec{\mathbf{w}}(r_L)$ are first order in $\mathsf{L}_{\max}$, and then consequently, so are

$$(\vec{\mathbf{w}}(r_C) - \vec{\mathbf{w}}(r_D)) \cdot \frac{\vec{N}_D}{\mathsf{L}_D}, \qquad (\vec{\mathbf{w}}(r_C) - \vec{\mathbf{w}}(r_L)) \cdot \frac{\vec{N}_L}{\mathsf{L}_L} \tag{5.49}$$

(see (2.45)) because we have taken dot products with unit vectors. We can write these expressions as

$$\vec{\mathbf{w}}(r_C) \cdot \frac{\vec{N}_D}{\mathsf{L}_D} - \vec{\mathbf{w}}_D, \qquad \vec{\mathbf{w}}(r_C) \cdot \frac{\vec{N}_L}{\mathsf{L}_L} - \vec{\mathbf{w}}_L. \tag{5.50}$$

If we substitute these expression into (2.45), we find that

$$\vec{\mathbf{w}}(r_C) - \frac{\vec{\mathbf{w}}_D \mathsf{L}_D}{2\mathsf{A}_{D,L}} \vec{T}_L + \frac{\vec{\mathbf{w}}_L \mathsf{L}_L}{2\mathsf{A}_{D,L}} \vec{T}_D \tag{5.51}$$

is also order one with a constant proportional to ρ^2/β. This implies that

$$\vec{\mathbf{w}}^{(1)}(r_C)\cdot\vec{\mathbf{w}}^{(2)}(r_C)-\left\langle \mathcal{P}_\mathcal{V}\vec{\mathbf{w}}^{(1)},\mathcal{P}_\mathcal{V}\vec{\mathbf{w}}^{(2)}\right\rangle_{D,L} \tag{5.52}$$

is first order. Now multiplying by $\mathsf{A}_{D,L}/2$ and summing over the four corners gives

$$\vec{\mathbf{w}}^{(1)}(r_C)\cdot\vec{\mathbf{w}}^{(2)}(r_C)\mathsf{A}_C-\left\langle \mathcal{P}_\mathcal{V}\vec{\mathbf{w}}^{(1)},\mathcal{P}_\mathcal{V}\vec{\mathbf{w}}^{(2)}\right\rangle_{\mathcal{V}} \tag{5.53}$$

is order one with a constant that is proportional to $\mathsf{A}_C\rho^2/\beta$. As the sum over all cells of the first part of this expression is a Riemann sum for the integral, we have the convergence at order one for families of grids that satisfy Assumption 5.8.

6. SUMMARY

We have defined mimetic discretizations for the divergence and gradient operators and used these to discretize the Laplacian with homogeneous Dirichlet boundary conditions in one and two dimensions. The one-dimensional problem is used to motivate the more complex two-dimensional case. The main result is that in two dimensions this discretization is first-order accurate in logically-rectangular grids with a bounds on the ratio of cell edge lengths and the size of the angles in the corners of the cells.

The discretization is first defined on a single quadrilateral cell, where we show that the discrete divergence and gradient satisfy a discrete analog of the divergence theorem and are first-order accurate. This discretization is then moved to a global logically rectangular grid where the same results hold. Next, special representations of these natural divergence and gradient operators are given in terms of formal operators. This provides explicit matrix representations for the natural divergence and gradient operators along with the inner products of scalar and vector fields. These matrices are combined to give a matrix representation of the Laplacian. Additionally, the formal structure is used to prove a discrete analog of Friedrichs-Poincaré inequality.

The truncation error for the Laplacian is typically order zero in rough grids, and thus, cannot be used to prove convergence. We introduce an abstract version of the mimetic discretization and prove that the error in the discrete solution and the discrete gradient of this error are bounded by the truncation error in the discrete divergence and gradient operators, which leads to the convergence result.

APPENDIX A
EXPLICIT FORMULAS FOR THE MATRICES M, A, AND S

In this appendix, we will list the explicit formulas for matrix $\mathbf{M}$ required in (3.30). We also derive the formulas matrix $\mathbf{A}$ used in the determining equations (3.33)–(3.35). Using these formulas, in Section A.2.1 we give a proof that $\mathbf{A}$ is diagonally dominant. Finally, we give formulas for the matrix $\mathbf{S}$ used to define the $\mathcal{S}$ operator (4.22).

A.1. Formulas for Matrix M

The vertex-based inner product (3.30) requires formulas for matrix $\mathbf{M}$. We will translate the formulas in Figure 3 by using $\mathbf{M}_{\sigma,\tau}\mapsto\mathbf{M}^{(a,b)(c,d)}$ where (a,b) gives side σ and (c,d) gives side τ. The symmetry of $\mathbf{M}$ in σ and τ gives $\mathbf{M}^{(a,b)(c,d)}=\mathbf{M}^{(c,d)(a,b)}$, so we only need the explicit formulas

$$\mathbf{M}^{(0,-1)(0,-1)}_{i+1/2,j+1/2}=\frac{\mathsf{L}^2_{i,j+1/2}}{8\mathsf{A}^{(-1,-1)}_{i+1/2,j+1/2}}+\frac{\mathsf{L}^2_{i+1,j+1/2}}{8\mathsf{A}^{(-1,+1)}_{i+1/2,j+1/2}}, \tag{A.1}$$

$$
\begin{aligned}
\mathbf{M}_{i+1/2,j+1/2}^{(0,-1)(+1,0)} &= \frac{\vec{T}_{i+1/2,j} \cdot \vec{T}_{i+1,j+1/2}}{8\mathsf{A}_{i+1/2,j+1/2}^{(-1,+1)}}, \\
\mathbf{M}_{i+1/2,j+1/2}^{(0,-1)(-1,0)} &= \frac{\vec{T}_{i+1/2,j} \cdot \vec{T}_{i,j+1/2}}{8\mathsf{A}_{i+1/2,j+1/2}^{(-1,-1)}}, \\
\mathbf{M}_{i+1/2,j+1/2}^{(0,-1)(0,+1)} &= 0, \\
\mathbf{M}_{i+1/2,j+1/2}^{(+1,0)(+1,0)} &= \frac{\mathsf{L}_{i+1/2,j}^2}{8\mathsf{A}_{i+1/2,j+1/2}^{(-1,+1)}} + \frac{\mathsf{L}_{i+1/2,j+1}^2}{8\mathsf{A}_{i+1/2,j+1/2}^{(+1,+1)}}, \\
\mathbf{M}_{i+1/2,j+1/2}^{(+1,0)(-1,0)} &= 0, \\
\mathbf{M}_{i+1/2,j+1/2}^{(+1,0)(0,+1)} &= \frac{\vec{T}_{i+1/2,j+1} \cdot \vec{T}_{i+1,j+1/2}}{8\mathsf{A}_{i+1/2,j+1/2}^{(+1,+1)}}, \\
\mathbf{M}_{i+1/2,j+1/2}^{(-1,0)(-1,0)} &= \frac{\mathsf{L}_{i+1/2,j}^2}{8\mathsf{A}_{i+1/2,j+1/2}^{(-1,-1)}} + \frac{\mathsf{L}_{i+1/2,j+1}^2}{8\mathsf{A}_{i+1/2,j+1/2}^{(+1,-1)}}, \\
\mathbf{M}_{i+1/2,j+1/2}^{(-1,0)(0,+1)} &= \frac{\vec{T}_{i+1/2,j+1} \cdot \vec{T}_{i,j+1/2}}{8\mathsf{A}_{i+1/2,j+1/2}^{(+1,-1)}}, \\
\mathbf{M}_{i+1/2,j+1/2}^{(0,+1)(0,+1)} &= \frac{\mathsf{L}_{i,j+1/2}^2}{8\mathsf{A}_{i+1/2,j+1/2}^{(+1,-1)}} + \frac{\mathsf{L}_{i+1,j+1/2}^2}{8\mathsf{A}_{i+1/2,j+1/2}^{(+1,+1)}}.
\end{aligned}
\tag{A.1)(cont.}
$$

Recall that the global gradient is required to satisfy the system of determining equations (2.86) for $0 \le i \le I-1$, $0 \le j \le J-1$,

$$
\begin{aligned}
u_{i+1/2,j+1/2} - u_{i+1/2,j} = {} & \mathsf{L}_{i+1/2,j}\mathcal{G}_{i+1/2,j}\mathbf{M}_{i+1/2,j+1/2}^{(0,-1)(0,-1)} \\
& + \mathsf{L}_{i+1,j+1/2}\mathcal{G}_{i+1,j+1/2}\mathbf{M}_{i+1/2,j+1/2}^{(+1,0)(0,-1)} \\
& + \mathsf{L}_{i+1/2,j+1}\mathcal{G}_{i+1/2,j+1}\mathbf{M}_{i+1/2,j+1/2}^{(0,+1)(0,-1)} \\
& + \mathsf{L}_{i,j+1/2}\mathcal{G}_{i,j+1/2}\mathbf{M}_{i+1/2,j+1/2}^{(-1,0)(0,-1)}, \\
u_{i+1,j+1/2} - u_{i+1/2,j+1/2} = {} & \mathsf{L}_{i+1/2,j}\mathcal{G}_{i+1/2,j}\mathbf{M}_{i+1/2,j+1/2}^{(0,-1)(+1,0)} \\
& + \mathsf{L}_{i+1,j+1/2}\mathcal{G}_{i+1,j+1/2}\mathbf{M}_{i+1/2,j+1/2}^{(+1,0)(+1,0)} \\
& + \mathsf{L}_{i+1/2,j+1}\mathcal{G}_{i+1/2,j+1}\mathbf{M}_{i+1/2,j+1/2}^{(0,+1)(+1,0)} \\
& + \mathsf{L}_{i,j+1/2}\mathcal{G}_{i,j+1/2}\mathbf{M}_{i+1/2,j+1/2}^{(-1,0)(+1,0)}, \\
u_{i+1/2,j+1} - u_{i+1/2,j+1/2} = {} & \mathsf{L}_{i+1/2,j}\mathcal{G}_{i+1/2,j}\mathbf{M}_{i+1/2,j+1/2}^{(0,-1)(0,+1)} \\
& + \mathsf{L}_{i+1,j+1/2}\mathcal{G}_{i+1,j+1/2}\mathbf{M}_{i+1/2,j+1/2}^{(+1,0)(0,+1)} \\
& + \mathsf{L}_{i+1/2,j+1}\mathcal{G}_{i+1/2,j+1}\mathbf{M}_{i+1/2,j+1/2}^{(0,+1)(0,+1)} \\
& + \mathsf{L}_{i,j+1/2}\mathcal{G}_{i,j+1/2}\mathbf{M}_{i+1/2,j+1/2}^{(-1,0)(0,+1)}, \\
u_{i+1/2,j+1/2} - u_{i,j+1/2} = {} & \mathsf{L}_{i+1/2,j}\mathcal{G}_{i+1/2,j}\mathbf{M}_{i+1/2,j+1/2}^{(0,-1)(-1,0)} \\
& + \mathsf{L}_{i+1,j+1/2}\mathcal{G}_{i+1,j+1/2}\mathbf{M}_{i+1/2,j+1/2}^{(+1,0)(-1,0)}
\end{aligned}
\tag{A.2}
$$

$$+ \mathsf{L}_{i+1/2,j+1}\mathcal{G}_{i+1/2,j+1}\mathbf{M}^{(0,+1)(-1,0)}_{i+1/2,j+1/2}$$
$$+ \mathsf{L}_{i,j+1/2}\mathcal{G}_{i,j+1/2}\mathbf{M}^{(-1,0)(-1,0)}_{i+1/2,j+1/2}. \qquad \text{(A.2)(cont.)}$$

These equations are normalized so that the inhomogeneous terms are differences of the scalar field so that it is easy to eliminate the edge scalar fields from the global equations. This formulation naturally simplifies to the standard discretization on orthogonal grids.

A.2. Formulas for Matrix A

We derive a system of equations for the gradient on the interior of the domain and then work on the boundaries. To eliminate the edge values of the scalar field, we need to combine two of the above equations to eliminate the scalar edge values. In fact, there are only two cases that need to be considered and they are illustrated in Figure 4. For the horizontal arrangement, we translate the second equation in (A.2) to the cell with center $(i-1/2, j+1/2)$ by replacing i by $i-1$,

$$\begin{aligned} u_{i,j+1/2} - u_{i-1/2,j+1/2} = {} & \mathsf{L}_{i-1/2,j}\mathcal{G}_{i-1/2,j}\mathbf{M}^{(0,-1)(+1,0)}_{i-1/2,j+1/2} \\ & + \mathsf{L}_{i,j+1/2}\mathcal{G}_{i,j+1/2}\mathbf{M}^{(+1,0)(+1,0)}_{i-1/2,j+1/2} \\ & + \mathsf{L}_{i-1/2,j+1}\mathcal{G}_{i-1/2,j+1}\mathbf{M}^{(0,+1)(+1,0)}_{i-1/2,j+1/2} \\ & + \mathsf{L}_{i-1,j+1/2}\mathcal{G}_{i-1,j+1/2}\mathbf{M}^{(-1,0)(+1,0)}_{i-1/2,j+1/2}, \end{aligned}$$

where $1 \le i \le I$, $0 \le j \le J-1$. If we add this equation to the fourth equation in (A.2) and take into account that

$$\mathbf{M}^{(+1,0)(-1,0)}_{i+1/2,j+1/2} = \mathbf{M}^{(-1,0)(+1,0)}_{i-1/2,j+1/2} = 0, \qquad \text{(A.3)}$$

then we get

$$\begin{aligned} u_{i+1/2,j+1/2} - u_{i-1/2,j+1/2} = {} & \left(\mathbf{M}^{(-1,0)(-1,0)}_{i+1/2,j+1/2} + \mathbf{M}^{(+1,0)(+1,0)}_{i-1/2,j+1/2}\right)\mathsf{L}_{i,j+1/2}\mathcal{G}_{i,j+1/2} \\ & + \mathbf{M}^{(0,-1)(-1,0)}_{i+1/2,j+1/2}\mathsf{L}_{i+1/2,j}\mathcal{G}_{i+1/2,j} \\ & + \mathbf{M}^{(0,+1)(-1,0)}_{i+1/2,j+1/2}\mathsf{L}_{i+1/2,j+1}\mathcal{G}_{i+1/2,j+1} \\ & + \mathbf{M}^{(0,-1)(+1,0)}_{i-1/2,j+1/2}\mathsf{L}_{i-1/2,j}\mathcal{G}_{i-1/2,j} \\ & + \mathbf{M}^{(0,+1)(+1,0)}_{i-1/2,j+1/2}\mathsf{L}_{i-1/2,j+1}\mathcal{G}_{i-1/2,j+1}, \end{aligned}$$

where $1 \le i \le I-1$, $0 \le j \le J-1$. These equations can be written in the form (3.33) where

$$\begin{aligned} \mathbf{A}^{(0,0)}_{i,j+1/2} &= \mathsf{L}_{i,j+1/2}\left(\mathbf{M}^{(-1,0)(-1,0)}_{i+1/2,j+1/2} + \mathbf{M}^{(+1,0)(+1,0)}_{i-1/2,j+1/2}\right) \\ &= \mathsf{L}_{i,j+1/2}\left(\frac{\mathsf{L}^2_{i+1/2,j}}{8\mathsf{A}^{(-1,-1)}_{i+1/2,j+1/2}} + \frac{\mathsf{L}^2_{i+1/2,j+1}}{8\mathsf{A}^{(+1,-1)}_{i+1/2,j+1/2}}\right. \\ &\qquad \left. + \frac{\mathsf{L}^2_{i-1/2,j}}{8\mathsf{A}^{(-1,(+1)}_{i-1/2,j+1/2}} + \frac{\mathsf{L}^2_{i-1/2,j+1}}{8\mathsf{A}^{(+1,+1)}_{i-1/2,j+1/2}}\right) \\ &= \frac{\mathsf{L}_{i+1/2,j}}{4\sin\left(\theta^{(-1,-1)}_{i+1/2,j+1/2}\right)} + \frac{\mathsf{L}_{i+1/2,j+1}}{4\sin\left(\theta^{(+1,-1)}_{i+1/2,j+1/2}\right)} \\ &\qquad + \frac{\mathsf{L}_{i-1/2,j}}{4\sin\left(\theta^{(-1,+1)}_{i-1/2,j+1/2}\right)} + \frac{\mathsf{L}_{i-1/2,j+1}}{4\sin\left(\theta^{(+1,+1)}_{i-1/2,j+1/2}\right)}, \end{aligned}$$

and

$$\begin{aligned}\mathbf{A}_{i,j+1/2}^{(+1,+1)} &= \mathsf{L}_{i+1/2,j+1}\mathbf{M}_{i+1/2,j+1/2}^{(0,+1)(-1,0)} = \mathsf{L}_{i+1/2,j+1}\frac{\vec{T}_{i+1/2,j+1}\cdot\vec{T}_{i,j+1/2}}{8\mathsf{A}_{i+1/2,j+1/2}^{(+1,-1)}}\\ &= \frac{\mathsf{L}_{i+1/2,j+1}\cos\left(\theta_{i+1/2,j+1/2}^{(+1,-1)}\right)}{4\sin\left(\theta_{i+1/2,j+1/2}^{(+1,-1)}\right)},\end{aligned} \tag{A.4}$$

$$\begin{aligned}\mathbf{A}_{i,j+1/2}^{(+1,-1)} &= \mathsf{L}_{i+1/2,j}\mathbf{M}_{i+1/2,j+1/2}^{(0,-1)(-1,0)} = \mathsf{L}_{i+1/2,j}\frac{\vec{T}_{i+1/2,j}\cdot\vec{T}_{i,j+1/2}}{8\mathsf{A}_{i+1/2,j+1/2}^{(-1,-1)}}\\ &= \frac{\mathsf{L}_{i+1/2,j}\cos\left(\theta_{i+1/2,j+1/2}^{(-1,-1)}\right)}{4\sin\left(\theta_{i+1/2,j+1/2}^{(-1,-1)}\right)},\end{aligned} \tag{A.5}$$

$$\begin{aligned}\mathbf{A}_{i,j+1/2}^{(-1,+1)} &= \mathsf{L}_{i-1/2,j+1}\mathbf{M}_{i-1/2,j+1/2}^{(0,+1)(+1,0)} = \mathsf{L}_{i-1/2,j+1}\frac{\vec{T}_{i-1/2,j+1}\cdot\vec{T}_{i,j+1/2}}{8\mathsf{A}_{i-1/2,j+1/2}^{(+1,+1)}}\\ &= \frac{\mathsf{L}_{i-1/2,j+1}\cos\left(\theta_{i-1/2,j+1/2}^{(+1,-1)}\right)}{4\sin\left(\theta_{i-1/2,j+1/2}^{(+1,-1)}\right)},\end{aligned} \tag{A.6}$$

$$\begin{aligned}\mathbf{A}_{i,j+1/2}^{(-1,-1)} &= \mathsf{L}_{i-1/2,j}\mathbf{M}_{i-1/2,j+1/2}^{(0,-1)(+1,0)} = \mathsf{L}_{i-1/2,j}\frac{\vec{T}_{i-1/2,j}\cdot\vec{T}_{i,j+1/2}}{8\mathsf{A}_{i-1/2,j+1/2}^{(-1,+1)}}\\ &= \frac{\mathsf{L}_{i-1/2,j}\cos\left(\theta_{i-1/2,j+1/2}^{(-1,+1)}\right)}{4\sin\left(\theta_{i-1/2,j+1/2}^{(-1,+1)}\right)}.\end{aligned} \tag{A.7}$$

Next, we consider the vertical arrangement illustrated in Figure 4 and translate the third equation in (A.2) to the cell with center $(i+1/2, j-1/2)$ by replacing j by $j-1$,

$$\begin{aligned}u_{i+1/2,j} - u_{i+1/2,j-1/2} &= \mathsf{L}_{i+1/2,j-1}\mathcal{G}_{i+1/2,j-1}\mathbf{M}_{i+1/2,j-1/2}^{(0,-1)(0,+1)}\\ &+\mathsf{L}_{i+1,j-1/2}\mathcal{G}_{i+1,j-1/2}\mathbf{M}_{i+1/2,j-1/2}^{(+1,0)(0,+1)}\\ &+\mathsf{L}_{i+1/2,j}\mathcal{G}_{i+1/2,j}\mathbf{M}_{i+1/2,j-1/2}^{(0,+1)(0,+1)}\\ &+\mathsf{L}_{i,j-1/2}\mathcal{G}_{i,j-1/2}\mathbf{M}_{i+1/2,j-1/2}^{(-1,0)(0,+1)}.\end{aligned}$$

If we add this equation to the first equation in (A.2) and take into account (A.3), then we get

$$\begin{aligned}u_{i+1/2,j+1/2} - u_{i+1/2,j-1/2} &= \left(\mathbf{M}_{i+1/2,j-1/2}^{(0,+1)(0,+1)} + \mathbf{M}_{i+1/2,j+1/2}^{(0,-1)(0,-1)}\right)\mathsf{L}_{i+1/2,j}\mathcal{G}_{i+1/2,j}\\ &+\mathbf{M}_{i+1/2,j-1/2}^{(+1,0)(0,+1)}\mathsf{L}_{i+1,j-1/2}\mathcal{G}_{i+1,j-1/2}\\ &+\mathbf{M}_{i+1/2,j-1/2}^{(-1,0)(0,+1)}\mathsf{L}_{i,j-1/2}\mathcal{G}_{i,j-1/2}\\ &+\mathbf{M}_{i+1/2,j+1/2}^{(+1,0)(0,-1)}\mathsf{L}_{i+1,j+1/2}\mathcal{G}_{i+1,j+1/2}\\ &+\mathbf{M}_{i+1/2,j+1/2}^{(-1,0)(0,-1)}\mathsf{L}_{i,j+1/2}\mathcal{G}_{i,j+1/2},\end{aligned}$$

where $1 \le i \le I-1$, $0 \le j \le J-1$. These equations can be written in the form (3.34), where

$$\begin{aligned}\mathbf{A}_{i+1/2,j}^{(0,0)} &= \mathsf{L}_{i+1/2,j}\left(\mathbf{M}_{i+1/2,j-1/2}^{(0,+1)(0,+1)} + \mathbf{M}_{i+1/2,j+1/2}^{(0,-1)(0,-1)}\right)\\ &= \mathsf{L}_{i+1/2,j}\left(\frac{\mathsf{L}_{i,j-1/2}^2}{8\mathsf{A}_{i+1/2,j-1/2}^{(+1,-1)}} + \frac{\mathsf{L}_{i+1,j-1/2}^2}{8\mathsf{A}_{i+1/2,j-1/2}^{(+1,+1)}}\right.\end{aligned} \tag{A.8}$$

$$+\frac{\mathsf{L}^2_{i,j+1/2}}{8\mathsf{A}^{(-1,-1)}_{i+1/2,j+1/2}}+\frac{\mathsf{L}^2_{i+1,j+1/2}}{8\mathsf{A}^{(-1,+1)}_{i+1/2,j+1/2}}\Bigg)$$

$$=\frac{\mathsf{L}_{i,j-1/2}}{4\sin\left(\theta^{(+1,-1)}_{i+1/2,j-1/2}\right)}+\frac{\mathsf{L}_{i+1,j-1/2}}{4\sin\left(\theta^{(+1,+1)}_{i+1/2,j-1/2}\right)}$$
$$+\frac{\mathsf{L}_{i,j+1/2}}{4\sin\left(\theta^{(-1,-1)}_{i+1/2,j+1/2}\right)}+\frac{\mathsf{L}_{i+1,j+1/2}}{4\sin\left(\theta^{(-1,+1)}_{i+1/2,j+1/2}\right)}, \tag{A.8)(cont.}$$

and

$$\mathbf{A}^{(+1,+1)}_{i+1/2,j}=\mathsf{L}_{i+1,j+1/2}\mathbf{M}^{(+1,0)(0,-1)}_{i+1/2,j+1/2}=\mathsf{L}_{i+1,j+1/2}\frac{\vec{T}_{i+1/2,j}\cdot\vec{T}_{i+1,j+1/2}}{8\mathsf{A}^{(-1,+1)}_{i+1/2,j+1/2}}$$
$$=\frac{\mathsf{L}_{i+1,j+1/2}\cos\left(\theta^{(-1,+1)}_{i+1/2,j+1/2}\right)}{4\sin\left(\theta^{(-1,+1)}_{i+1/2,j+1/2}\right)}, \tag{A.9}$$

$$\mathbf{A}^{(+1,-1)}_{i+1/2,j}=\mathsf{L}_{i+1,j-1/2}\mathbf{M}^{(+1,0)(0,+1)}_{i+1/2,j-1/2}=\mathsf{L}_{i+1,j-1/2}\frac{\vec{T}_{i+1/2,j}\cdot\vec{T}_{i+1,j-1/2}}{8\mathsf{A}^{(+1,+1)}_{i+1/2,j-1/2}}$$
$$=\frac{\mathsf{L}_{i+1,j-1/2}\cos\left(\theta^{(+1,+1)}_{i+1/2,j-1/2}\right)}{4\sin\left(\theta^{(+1,+1)}_{i+1/2,j-1/2}\right)}, \tag{A.10}$$

$$\mathbf{A}^{(-1,+1)}_{i+1/2,j}=\mathsf{L}_{i,j+1/2}\mathbf{M}^{(-1,0)(0,-1)}_{i+1/2,j+1/2}=\mathsf{L}_{i,j+1/2}\frac{\vec{T}_{i+1/2,j}\cdot\vec{T}_{i,j+1/2}}{8\mathsf{A}^{(-1,-1)}_{i+1/2,j+1/2}}$$
$$=\frac{\mathsf{L}_{i,j+1/2}\cos\left(\theta^{(-1,-1)}_{i+1/2,j+1/2}\right)}{4\sin\left(\theta^{(-1,-1)}_{i+1/2,j+1/2}\right)}, \tag{A.11}$$

$$\mathbf{A}^{(-1,-1)}_{i+1/2,j}=\mathsf{L}_{i,j-1/2}\mathbf{M}^{(-1,0)(0,+1)}_{i+1/2,j-1/2}=\mathsf{L}_{i,j-1/2}\frac{\vec{T}_{i+1/2,j}\cdot\vec{T}_{i,j-1/2}}{8\mathsf{A}^{(+1,-1)}_{i+1/2,j-1/2}}$$
$$=\frac{\mathsf{L}_{i,j-1/2}\cos\left(\theta^{(+1,-1)}_{i+1/2,j-1/2}\right)}{4\sin\left(\theta^{(+1,-1)}_{i+1/2,j-1/2}\right)}. \tag{A.12}$$

The equations on the boundary will retain the scalar values on the boundary

$$\begin{aligned}u_{1/2,j+1/2}-u_{0,j+1/2}&=\mathsf{L}_{1/2,j}\mathcal{G}_{1/2,j}\mathbf{M}^{(0,-1)(-1,0)}_{1/2,j+1/2}\\&+\mathsf{L}_{1/2,j+1}\mathcal{G}_{1/2,j+1}\mathbf{M}^{(0,+1)(-1,0)}_{1/2,j+1/2}\\&+\mathsf{L}_{0,j+1/2}\mathcal{G}_{0,j+1/2}\mathbf{M}^{(-1,0)(-1,0)}_{1/2,j+1/2},\end{aligned}$$

$$\begin{aligned}u_{I,j+1/2}-u_{I-1/2,j+1/2}&=\mathsf{L}_{I-1/2,j}\mathcal{G}_{I-1/2,j}\mathbf{M}^{(0,-1)(+1,0)}_{I-1/2,j+1/2}\\&+\mathsf{L}_{I,j+1/2}\mathcal{G}_{I,j+1/2}\mathbf{M}^{(+1,0)(+1,0)}_{I-1/2,j+1/2}\\&+\mathsf{L}_{I-1/2,j+1}\mathcal{G}_{I-1/2,j+1}\mathbf{M}^{(0,+1)(+1,0)}_{I-1/2,j+1/2},\end{aligned}$$

$$\begin{aligned}u_{i+1/2,1/2}-u_{i+1/2,0}&=\mathsf{L}_{i+1/2,0}\mathcal{G}_{i+1/2,0}\mathbf{M}^{(0,-1)(0,-1)}_{i+1/2,1/2}\\&+\mathsf{L}_{i+1,1/2}\mathcal{G}_{i+1,1/2}\mathbf{M}^{(+1,0)(0,-1)}_{i+1/2,1/2}\\&+\mathsf{L}_{i,1/2}\mathcal{G}_{i,1/2}\mathbf{M}^{(-1,0)(0,-1)}_{i+1/2,1/2},\end{aligned}$$

$$\begin{aligned} u_{i+1/2,J} - u_{i+1/2,J-1/2} = {} & \mathsf{L}_{i+1,J-1/2}\mathcal{G}_{i+1,J-1/2}\mathbf{M}^{(+1,0)(0,+1)}_{i+1/2,J-1/2} \\ & + \mathsf{L}_{i+1/2,J}\mathcal{G}_{i+1/2,J}\mathbf{M}^{(0,+1)(0,+1)}_{i+1/2,J-1/2} \\ & + \mathsf{L}_{i,J-1/2}\mathcal{G}_{i,J-1/2}\mathbf{M}^{(-1,0)(0,+1)}_{i+1/2,J-1/2}. \end{aligned}$$

These boundary equations can be written in the form (3.35) where the off-diagonal values of **A** are the same as above, while the diagonal values are "half" of the values given above,

$$\begin{aligned} \mathbf{A}^{(0,0)}_{0,j+1/2} &= \mathsf{L}_{0,j+1/2}\mathbf{M}^{(-1,0)(-1,0)}_{1/2,j+1/2} = \frac{\mathsf{L}_{1/2,j}}{4\sin\left(\theta^{(-1,-1)}_{1/2,j+1/2}\right)} + \frac{\mathsf{L}_{1/2,j+1}}{4\sin\left(\theta^{(+1,-1)}_{1/2,j+1/2}\right)}, \\ \mathbf{A}^{(0,0)}_{I,j+1/2} &= \mathsf{L}_{I,j+1/2}\mathbf{M}^{(+1,0)(+1,0)}_{I-1/2,j+1/2} = \frac{\mathsf{L}_{I-1/2,j}}{4\sin\left(\theta^{(-1,+1)}_{I-1/2,j+1/2}\right)} + \frac{\mathsf{L}_{I-1/2,j+1}}{4\sin\left(\theta^{(+1,+1)}_{I-1/2,j+1/2}\right)}, \\ \mathbf{A}^{(0,0)}_{i+1/2,0} &= \mathsf{L}_{i+1/2,0}\mathbf{M}^{(0,-1)(0,-1)}_{i+1/2,0+1/2} = \frac{\mathsf{L}_{i,1/2}}{4\sin\left(\theta^{(-1,-1)}_{i+1/2,1/2}\right)} + \frac{\mathsf{L}_{i+1,1/2}}{4\sin\left(\theta^{(-1,+1)}_{i+1/2,1/2}\right)}, \\ \mathbf{A}^{(0,0)}_{i+1/2,J} &= \mathsf{L}_{i+1/2,J}\mathbf{M}^{(0,+1)(0,+1)}_{i+1/2,J-1/2} = \frac{\mathsf{L}_{i,J-1/2}}{4\sin\left(\theta^{(+1,-1)}_{i+1/2,J-1/2}\right)} + \frac{\mathsf{L}_{i+1,J-1/2}}{4\sin\left(\theta^{(+1,+1)}_{i+1/2,J-1/2}\right)}. \end{aligned}$$

A.2.1. Diagonal dominance of matrix A

The diagonal dominance of the matrix **A** is measured by

$$R = \frac{\mathbf{A}^{(0,0)} - \mathbf{A}^{(+1,+1)} - \mathbf{A}^{(+1,-1)} - \mathbf{A}^{(-1,+1)} - \mathbf{A}^{(-1,-1)}}{\mathbf{A}^{(0,0)}}. \tag{A.14}$$

In the interior of the domain, the numerator of R can be broken into four terms, while on the boundary the expression can be decomposed into two terms, where we can estimate a typical term using the results used to prove (2.52),

$$\frac{\mathsf{L}_{i+1/2,j}\left(1-\cos\left(\theta^{(0,-1)(-1,0)}_{i+1/2,j+1/2}\right)\right)}{4\sin\left(\theta^{(0,-1)(-1,0)}_{i+1/2,j+1/2}\right)} \geq \frac{\beta \mathsf{L}_{\min}}{8}. \tag{A.15}$$

So in the interior,

$$\mathbf{A}^{(0,0)} - \mathbf{A}^{(+1,+1)} - \mathbf{A}^{(+1,-1)} - \mathbf{A}^{(-1,+1)} - \mathbf{A}^{(-1,-1)} \geq \frac{\beta \mathsf{L}_{\min}}{2} \tag{A.16}$$

and

$$\mathbf{A}^{(0,0)} \leq \mathsf{L}_{\max}, \tag{A.17}$$

so

$$R \geq \frac{\beta}{2\rho}. \tag{A.18}$$

The same estimate holds on the boundary.

A.3. Formulas for Matrix S

The formulas for the matrix **S** for the $\mathcal{S}$ operator given in (4.22) can be computed in the same way as the formulas for the the matrix **A** computed above. The diagonal elements of $\mathcal{S}$ in the interior are given by

$$\begin{aligned} \mathbf{S}^{(0,0)}_{i,j+1/2} &= \mathbf{B}^{(-1,0)(-1,0)}_{i+1/2,j+1/2} + \mathbf{B}^{(+1,0)(+1,0)}_{i-1/2,j+1/2}, \qquad 1 \leq i \leq I-1, \quad 0 \leq j \leq J-1, \\ \mathbf{S}^{(0,0)}_{i+1/2,j} &= \mathbf{B}^{(0,+1)(0,+1)}_{i+1/2,j-1/2} + \mathbf{B}^{(0,-1)(0,-1)}_{i+1/2,j+1/2}, \qquad 0 \leq i \leq I-1, \quad 1 \leq j \leq J-1, \end{aligned}$$

and on the boundary, the diagonal elements of S are given by

$$\begin{aligned}
\mathbf{S}^{(0,0)}_{0,j+1/2} &= \mathbf{B}^{(-1,0)(-1,0)}_{1/2,j+1/2}, & 0 \le j \le J-1,\\
\mathbf{S}^{(0,0)}_{I,j+1/2} &= \mathbf{B}^{(+1,0)(+1,0)}_{I-1/2,j+1/2}, & 0 \le j \le J-1,\\
\mathbf{S}^{(0,0)}_{i+1/2,0} &= \mathbf{B}^{(0,-1)(0,-1)}_{i+1/2,0+1/2}, & 0 \le i \le I-1,\\
\mathbf{S}^{(0,0)}_{i+1/2,J} &= \mathbf{B}^{(0,+1)(0,+1)}_{i+1/2,J-1/2}, & 0 \le i \le I-1.
\end{aligned}$$

The off-diagonal terms of S are given by

$$\begin{aligned}
\mathbf{S}^{(+1,+1)}_{i,j+1/2} &= \mathbf{B}^{(0,+1)(-1,0)}_{i+1/2,j+1/2}, & 0 \le i \le I-1, && 0 \le j \le J-1,\\
\mathbf{S}^{(+1,-1)}_{i,j+1/2} &= \mathbf{B}^{(0,-1)(-1,0)}_{i+1/2,j+1/2}, & 0 \le i \le I-1, && 0 \le j \le J-1,\\
\mathbf{S}^{(-1,+1)}_{i,j+1/2} &= \mathbf{B}^{(0,+1)(+1,0)}_{i-1/2,j+1/2}, & 1 \le i \le I, && 0 \le j \le J-1,\\
\mathbf{S}^{(-1,-1)}_{i,j+1/2} &= \mathbf{B}^{(0,-1)(+1,0)}_{i-1/2,j+1/2}, & 1 \le i \le I, && 0 \le j \le J-1,
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{S}^{(+1,+1)}_{i+1/2,j} &= \mathbf{B}^{(+1,0)(0,-1)}_{i+1/2,j+1/2}, & 0 \le i \le I-1, && 0 \le j \le J-1,\\
\mathbf{S}^{(+1,-1)}_{i+1/2,j} &= \mathbf{B}^{(+1,0)(0,+1)}_{i+1/2,j-1/2}, & 0 \le i \le I-1, && 1 \le j \le J,\\
\mathbf{S}^{(-1,+1)}_{i+1/2,j} &= \mathbf{B}^{(-1,0)(0,-1)}_{i+1/2,j+1/2}, & 0 \le i \le I-1, && 0 \le j \le J-1,\\
\mathbf{S}^{(-1,-1)}_{i+1/2,j} &= \mathbf{B}^{(-1,0)(0,+1)}_{i+1/2,j-1/2}, & 0 \le i \le I-1, && 1 \le j \le J.
\end{aligned}$$

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