Precise asymptotics for the first moment of the error variance estimator in linear models

Ke-Ang Fu*, Wei-Dong Liu, Li-Xin Zhang

Department of Mathematics, Zhejiang University, Hangzhou 310027, China

Received 9 June 2006; received in revised form 16 March 2007; accepted 2 July 2007

Abstract

Let \( \sigma^2 \) be the unknown error variance of a linear model and let \( \hat{\sigma}^2 \) be the estimator of \( \sigma^2 \) based on the residual sum of squares. In this work, we show the precise asymptotics in the law of the logarithm for the first moment of the error variance estimator.© 2007 Elsevier Ltd. All rights reserved.

Keywords: Precise asymptotics; Linear model; Moment convergence; Error variance estimator; The law of the logarithm

1. Introduction and main result

Consider the linear model

\[ Y_i = X_i' \beta + e_i, \quad i = 1, 2, \ldots, n, \]

where \( \beta \) is a \( q \)-dimensional unknown parametric vector, and \( \{e_i\} \) is a sequence of i.i.d. trial errors with \( Ee_1 = 0 \) and \( 0 < \sigma^2 = Ee_1^2 < \infty \). By ordinary least squares and the characteristic of linear models, the estimator of \( \sigma^2 \) always takes the following form:

\[
\hat{\sigma}^2_n = \frac{1}{n - \gamma} \left\{ \sum_{i=1}^{n} e_i^2 - \sum_{j=1}^{\gamma} \left( \sum_{i=1}^{n} a_{nj} e_i \right)^2 \right\}, \tag{1.1}
\]

where \( \gamma = \gamma_n \) is the rank of the design matrix \( X_n = (X_1, \ldots, X_n) \) satisfying \( \gamma_n \leq q \) and \( \{a_{nl}\} \) is a sequence of real numbers satisfying

\[
\sum_{i=1}^{n} a_{nl} a_{nm} = \begin{cases} 1, & l = m, \\ 0, & l \neq m, \end{cases} \tag{1.2}
\]

and \( X_n' (X_n X_n')^{-1} X_n = (a_{nl})' \left( \begin{smallmatrix} I_{\gamma} \\ 0 \end{smallmatrix} \right) (a_{nl}) \), where \( I_{\gamma} \) is a \( \gamma \times \gamma \) identity matrix. The limit properties of the error variance estimator have been widely discussed, and we refer the reader to the literature [1,2,7,9] and references therein.

* Project supported by the National Natural Science Foundation of China (Nos. 10671176 & 10771192).
* Corresponding author.
E-mail address: fukeang@hotmail.com (K.-A. Fu).

© 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2007.07.018
It is well known that, for i.i.d. random variables, Chow [3] discussed the complete moment convergence, and got the following result.

**Theorem A.** Let \( \{X, X_n, n \geq 1\} \) be a sequence of i.i.d. random variables with \( E X = 0 \), and set \( S_n = \sum_{i=1}^{n} X_i, n \geq 1 \). Assume \( p \geq 1, \alpha > 1/2, p\alpha > 1 \) and \( E(|X|^p) < \infty \). Then for any \( \varepsilon > 0 \),

\[
\sum_{n=1}^{\infty} n^{p\alpha^{-2} - 2} E\{\max_{j \leq n} |S_j| - \varepsilon n^\alpha\} < \infty.
\]

Recently, Jiang and Zhang [5] established the following precise rates in the law of the logarithm for the moment convergence of i.i.d. random variables by using the strong approximation method.

**Theorem B.** Let \( \{X, X_n, n \geq 1\} \) be a sequence of i.i.d. random variables with \( E X = 0, E X^2 = \sigma^2 < \infty \) and \( E(|X|^2 / (\log |X|)^\gamma) < \infty \). Set \( S_n = \sum_{k=1}^{n} X_k, n \geq 1 \). Then for \( r > 1 \), we have

\[
\lim_{\varepsilon \searrow 0} \frac{1}{\log(\varepsilon^2 + (r - 1))} \sum_{n=1}^{\infty} n^{r - 2 - 1/2} E\{\|S_n\| - \sigma \varepsilon / \sqrt{2n \log n}\} = \frac{\sigma}{(r - 1) \sqrt{2\pi}}.
\]

Inspired by Chow [3] and Jiang and Zhang [5], here we study the precise asymptotics in the law of the logarithm for the first moment of the error variance estimator. Our main result reads as follows.

**Theorem 1.1.** Suppose \( E\|e_1\| = 0, 0 < \sigma^2 = Ee_1^2 < \infty \) and \( Ee_1^4 < \infty \), and set \( \nu = \text{Var}(e_1^2) \). Then, for \( d > 0 \) and \( 1/2 < b + 1/d < 1 \), we have

\[
\lim_{\varepsilon \to 0} \frac{\nu^{2b + (2/d) - 1}}{\nu^{1/2} \nu^{d/2}} E\{\|n^{1/2} \nu^{1/2} \nu^{d/2} - \sigma^2\| - \varepsilon (\log n)^{d/2}\} = \frac{d E\|N\|^{2b+2/d}}{(bd + 1)(2bd + 2 - d)},
\]

where \( N \) is the standard normal random variable.

**Remark 1.1.** Here we consider the moment convergence rates, which extend the results of Lu and Qiu [7]. Also we consider the general result for the logarithm while in many other references (cf. Jiang and Zhang [5]) they only consider the special case \( d = 1 \).

**Remark 1.2.** As we know, to obtain such results, one way is using strong approximation (cf. Jiang and Zhang [5]), but this method is not applicable here. Another way is using the Berry–Esseen inequality (cf. Li [6]), and we do not take this approach either.

### 2. Proof of Theorem 1.1

In this section, for \( M > 4 \) and \( 0 < \varepsilon < 1/4 \), we set

\[
c(\varepsilon) = \exp(M / \varepsilon^{2/d}).
\]

Without loss of generality, assume \( \nu = 1 \) and \( a_{ij} = a_{ij} \) for the same \( i \) in the sequel, and in the sequel, let \( C \) denote a positive constant whose value possibly varies from place to place and the notation \([x]\) means the largest integer \( \leq x \). The proof of Theorem 1.1 is based on the following four propositions.

**Proposition 2.1.** For \( d > 0 \) and \( b + 1/d > 1/2 \), we have

\[
\lim_{\varepsilon \to 0} \frac{\nu^{2b + (2/d) - 1}}{\nu^{1/2} \nu^{d/2}} E\{|N| - \varepsilon (\log n)^{d/2}\} = \frac{d E\|N\|^{2b+2/d}}{(bd + 1)(2bd + 2 - d)},
\]

where \( N \) is defined in Theorem 1.1.

**Proof.** By applying Theorem A.1 from Appendix, we can get that
\[
\lim_{\varepsilon \searrow 0} \varepsilon^{2b+(2/d)-1} \sum_{n \leq \varepsilon(n)} \frac{(\log n)^{bd-d/2}}{n} \left| E\{ |N| - \varepsilon (\log n)^{d/2} \} \right| = 0. \quad (2.2)
\]

**Proof.** Set \( \Delta_n = \sup_{x \in \mathbb{R}} |P(|N| \geq x) - P(n^{1/2} |\hat{\sigma}_n^2 - \sigma^2| \geq x)| \). Then, from Theorem A.2, it follows that \( \Delta_n \to 0 \) as \( n \to \infty \). Notice that

\[
\lim_{\varepsilon \searrow 0} \varepsilon^{2b+(2/d)-1} \sum_{n \leq \varepsilon(n)} \frac{(\log n)^{bd-d/2}}{n} \left| E\{ |N| - \varepsilon (\log n)^{d/2} \} \right| = 0. \quad (2.2)
\]

Thus, for \( P_n \), by applying Theorem A.3, we have
\[ e^{2b+(2/d)-1} \sum_{n \leq c(\varepsilon)} \frac{(\log n)^{bd}}{n} P_{n1} \leq e^{2b+(2/d)-1} \sum_{n \leq c(\varepsilon)} \frac{(\log n)^{bd}}{n} (\log n)^{-d/2} \Delta_n^{-1/2} \Delta_n \]
\[ = e^{2b+(2/d)-1} \sum_{n \leq c(\varepsilon)} \frac{(\log n)^{bd-d/2}}{n} \Delta_n^{1/2} \]
\[ \leq M^{bd+1-d/2} \left( \frac{1}{(\log(c(\varepsilon)))^{bd+1-d/2}} \sum_{n \leq c(\varepsilon)} \frac{(\log n)^{bd-d/2}}{n} \Delta_n^{1/2} \right) \rightarrow 0, \quad \text{as} \quad \varepsilon \searrow 0. \]

For \( P_{n2} \), by Markov’s inequality and Theorem A.3, we have
\[ e^{2b+(2/d)-1} \sum_{n \leq c(\varepsilon)} \frac{(\log n)^{bd}}{n} P_{n2} \leq C e^{2b+(2/d)-1} \sum_{n \leq c(\varepsilon)} \frac{(\log n)^{bd}}{n} \int_{(\log n)^{-d/2} \Delta_n^{-1/2}}^{\infty} \frac{1}{(x+\varepsilon)^2(\log n)^d} dx \]
\[ \leq e^{2b+(2/d)-1} \sum_{n \leq c(\varepsilon)} \frac{(\log n)^{bd-d/2}}{n} \Delta_n^{1/2} \rightarrow 0, \quad \text{as} \quad \varepsilon \searrow 0. \]

Hence (2.2) holds. \( \square \)

**Proposition 2.3.** For \( d > 0 \) and \( b + 1/d > 1/2 \), we have
\[ \lim_{M \to \infty} \lim_{\varepsilon \searrow 0} e^{2b+(2/d)-1} \sum_{n > c(\varepsilon)} \frac{(\log n)^{bd-d/2}}{n} E[|N| - \varepsilon(\log n)^{d/2}]_+ = 0. \] (2.3)

**Proof.** Note that
\[ e^{2b+(2/d)-1} \sum_{n > c(\varepsilon)} \frac{(\log n)^{bd-d/2}}{n} E[|N| - \varepsilon(\log n)^{d/2}]_+ \]
\[ = e^{2b+(2/d)-1} \int_{c(\varepsilon)}^{\infty} \frac{(\log y)^{bd-d/2}}{y} \int_{(\log y)^{d/2}}^{\infty} P(|N| \geq x) dx dy \]
\[ = \int_{M^{d/2}}^{\infty} \int_{c(\varepsilon)}^{\infty} \frac{(\log y)^{bd-d/2}}{y} \int_{(\log y)^{d/2}}^{\infty} P(|N| \geq x) dx dy \]
\[ = \int_{M^{d/2}}^{\infty} P(|N| \geq x) \int_{c(\varepsilon)}^{\infty} y^{2b+(2/d)-2} dx dy \]
\[ \leq C \int_{M^{d/2}}^{\infty} P(|N| \geq x) x^{2b+(2/d)-1} dx \rightarrow 0, \quad \text{as} \quad M \to \infty. \]

So (2.3) is proved now. \( \square \)

**Proposition 2.4.** For \( d > 0 \) and \( 1/2 < b + 1/d < 1 \), we have
\[ \lim_{M \to \infty} \lim_{\varepsilon \searrow 0} e^{2b+(2/d)-1} \sum_{n > c(\varepsilon)} \frac{(\log n)^{bd-d/2}}{n} E[n^{1/2} |\hat{\sigma}_n^2 - \sigma^2| - \varepsilon(\log n)^{d/2}]_+ = 0. \] (2.4)

**Proof.** By the representation of \( \hat{\sigma}_n^2 - (1.1) \), we have that
\[ n(\hat{\sigma}_n^2 - \sigma^2) = \frac{n}{n-\gamma} \sum_{i=1}^{n} (s_i^2 - \sigma^2) + \frac{n\gamma}{n-\gamma} \sigma^2 - \frac{n}{n-\gamma} \sum_{j=1}^{\gamma} \left( \sum_{i=1}^{n} a_{nj} e_i \right)^2. \]

Note the inequality \(|x + y + z - \varepsilon| \leq |x - (\varepsilon/3)| + |y - (\varepsilon/3)| + |z - (\varepsilon/3)|\), and the equality \( e^{2b+(2/d)-1} = z^{2b+(2/d)-1}(\varepsilon/3)^{2b+(2/d)-1} \), and hence it suffices to show that
\[
\lim_{M \to \infty} \lim_{\varepsilon \searrow 0} \varepsilon^{2b+(2/d)-1} \sum_{n > c(\varepsilon)} \frac{(\log n)^{bd-d/2}}{n^{3/2}} E \left\{ \left| \sum_{i=1}^{n} (e_i^2 - \sigma^2) \right| - \varepsilon n^{1/2} (\log n)^{d/2} \right\} \geq 0, \tag{2.5}
\]

\[
\lim_{M \to \infty} \lim_{\varepsilon \searrow 0} \varepsilon^{2b+(2/d)-1} \sum_{n > c(\varepsilon)} \frac{(\log n)^{bd-d/2}}{n^{3/2}} E \{ \gamma^2 - \varepsilon n^{1/2} (\log n)^{d/2} \} = 0 \tag{2.6}
\]

and

\[
\lim_{M \to \infty} \lim_{\varepsilon \searrow 0} \varepsilon^{2b+(2/d)-1} \sum_{n > c(\varepsilon)} \frac{(\log n)^{bd-d/2}}{n^{3/2}} E \left\{ \left| \sum_{i=1}^{n} a_{ni} e_i \right|^2 - \varepsilon n^{1/2} (\log n)^{d/2} \right\} \geq 0. \tag{2.7}
\]

For (2.5), set \( S_n = \sum_{i=1}^{n} (e_i^2 - \sigma^2) \). Thus \( S_n \) are partial sums of i.i.d. random variables with mean zero and finite variance. Then we have that

\[
\varepsilon^{2b+2/d-1} \sum_{n > c(\varepsilon)} \frac{(\log n)^{bd-d/2}}{n^{3/2}} E \{ |S_n| - \varepsilon n^{1/2} (\log n)^{d/2} \} \geq 0.
\]

\[
\geq \varepsilon^{2b+2/d-1} \sum_{n > c(\varepsilon)} \frac{(\log n)^{bd-d/2}}{n^{3/2}} \int_{\varepsilon n^{1/2} (\log n)^{d/2}}^{\infty} P(|S_n| \geq x) \, dx
\]

\[
\leq C \varepsilon^{2b+2/d-1} \sum_{n > c(\varepsilon)} \frac{(\log n)^{bd-d}}{n^{1/2}} \int_{\varepsilon n^{1/2} (\log n)^{d/2}}^{\infty} x^{-2} \, dx
\]

\[
\leq C \varepsilon^{2b+2/d-2} \frac{(\log \varepsilon)^{bd-d+1}}{n} = CM^{bd-d+1} \to 0, \quad \text{as } M \to \infty.
\]

For (2.6), this is trivially true. Since \( \varepsilon n^{1/2} (\log n)^{d/2} \geq \varepsilon M^{(2s/2)^d} \to \infty \) as \( n > c(\varepsilon) \) and \( \varepsilon \searrow 0 \), and \( \gamma^2 \) is a constant, it is easily seen that \( E \{ \gamma^2 - \varepsilon n^{1/2} (\log n)^{d/2} \} \geq 0 \) as \( \varepsilon \) is small enough, and hence (2.6) follows.

Now we begin to deal with (2.7). Define \( T_n = \sum_{i=1}^{n} a_{ni} e_i \), and then by the Chebyshev’s inequality and the orthogonality in (1.2), we have that

\[
P(|T_n| \geq \sqrt{x}) \leq x^{-2} E T_n^4 \leq C x^{-2} \left( (E e_1^4) \sum_{i=1}^{n} a_{ni}^4 + (E e_1^2)^2 \sum_{i \neq j} a_{ni}^2 a_{nj}^2 \right).
\]

Notice that \( \sum_{i=1}^{n} a_{ni}^4 \leq \sum_{i=1}^{n} a_{ni}^2 = 1 \) and \( \sum_{i \neq j} a_{ni}^2 a_{nj}^2 \leq (\sum_{i=1}^{n} a_{ni}^2)^2 = 1 \). Therefore,

\[
\varepsilon^{2b+(2/d)-1} \sum_{n > c(\varepsilon)} \frac{(\log n)^{bd-d/2}}{n^{3/2}} E \{ |T_n|^2 - \varepsilon n^{1/2} (\log n)^{d/2} \} \geq 0.
\]

\[
\lim_{M \to \infty} \lim_{\varepsilon \searrow 0} \varepsilon^{2b+(2/d)-1} \sum_{n > c(\varepsilon)} \frac{(\log n)^{bd-d/2}}{n^{3/2}} E \{ |T_n|^2 - \varepsilon n^{1/2} (\log n)^{d/2} \} = 0, \quad \text{as } \varepsilon \searrow 0. \quad \Box
\]
Now we turn to the proof of Theorem 1.1:

**Proof.** From Propositions 2.1–2.4, Theorem 1.1 immediately follows. □

Acknowledgements

The authors thank the referees for pointing out some errors in a previous version, as well as for several comments that have led to improvements in this work.

Appendix

**Theorem A.1** (Lemma 2.4 of Huang and Zhang [4]). For \( n \geq 1 \), let \( \alpha_n(\varepsilon) > 0, \beta_n(\varepsilon) > 0 \) and \( f(\varepsilon) > 0 \) satisfy

\[
\alpha_n(\varepsilon) \sim \beta_n(\varepsilon), \quad \text{as } n \to \infty \text{ and } \varepsilon \to \varepsilon_0,
\]

and

\[
f(\varepsilon)\beta_n(\varepsilon) \to 0, \quad \text{as } \varepsilon \to \varepsilon_0, \forall n \geq 1.
\]

Then

\[
\limsup_{\varepsilon \to \varepsilon_0} \liminf_{\varepsilon \to \varepsilon_0} f(\varepsilon) \sum_{n=1}^{\infty} \alpha_n(\varepsilon) = \limsup_{\varepsilon \to \varepsilon_0} \liminf_{\varepsilon \to \varepsilon_0} f(\varepsilon) \sum_{n=1}^{\infty} \beta_n(\varepsilon).
\]

**Theorem A.2** (Theorem 1 of Chen [2]). Suppose that \( Ee_1 = 0, 0 < \sigma^2 = Ee_1^2 < \infty \) and \( Ee_1^4 < \infty \), and set \( \nu = \text{Var}(e_1^2) \). Then we have

\[
n^{1/2} \nu^{-1/2} (\hat{\sigma}_n^2 - \sigma^2) \overset{d}{\to} N,
\]

where \( \overset{d}{\to} \) and \( N \) denote convergence in distribution and the standard normal random variable, respectively.

**Theorem A.3** (Lemma 3.2.3 of Stout [8, p. 120]). Let \( \{a_{ni}\} \) be a matrix of real numbers and \( \{x_i\} \) a sequence of real numbers. Let \( x_i \to x \) as \( i \to \infty \). Then

\[
\sum_{i=1}^{\infty} |a_{ni}| \leq M < \infty \quad \text{for all } n \geq 1,
\]

\[
\sum_{i=1}^{\infty} a_{ni} \to 1 \quad \text{as } n \to \infty
\]

and

\[
a_{ni} \to 0 \quad \text{as } n \to \infty \text{ for each } i \geq 1
\]

imply that

\[
\sum_{i=1}^{\infty} a_{ni}x_i \to x \quad \text{as } n \to \infty.
\]

References


