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Complete sets of metamorphoses: Twofold 4-cycle systems into twofold 6-cycle systems

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ABSTRACT

Let (X, C) denote a twofold k-cycle system with an even number of cycles. If these k-cycles can be paired together so that: (i) each pair contains a common edge; (ii) removal of the repeated common edge from each pair leaves a $(2k-2)$-cycle; (iii) all the repeated edges, once removed, can be rearranged exactly into a collection of further $(2k-2)$-cycles; then this is a *metamorphosis* of a twofold k-cycle system into a twofold $(2k-2)$-cycle system. The existence of such metamorphoses has been dealt with for the case of 3-cycles (Gionfriddo and Lindner, 2003) [3] and 4-cycles (Yazıcı, 2005) [7].

If a twofold k-cycle system (X, C) of order n exists, which has not just one but has k different metamorphoses, from k different pairings of its cycles, into twofold $(2k-2)$-cycle systems, such that the collection of all removed double edges from all k metamorphoses precisely covers $2K_n$, we call this a *complete set* of twofold paired k-cycle metamorphoses into twofold $(2k-2)$-cycle systems.

In this paper, we show that there exists a twofold 4-cycle system (X, C) of order n with a complete set of metamorphoses into twofold 6-cycle systems if and only if $n \equiv 0, 1, 9, 16 \pmod{24}$, $n \neq 9$.

1. Introduction

Let X be the vertex set of a complete graph K_n. A k-cycle is a graph with k vertices, $x_1, x_2, \ldots, x_k$, having k edges, $\{x_1, x_k\}$ and $\{x_i, x_{i+1}\}$ for $1 \leqslant i \leqslant k-1$. A k-cycle system (X, C) of order n is a collection C of k-cycles which partitions the edge-set of K_n.

If the edges of λK_n (this n-vertex graph has λ edges between each pair of distinct vertices) are partitioned into a collection C of k-cycles, then (X, C) is a λ-fold k-cycle system of order n.

Let (X, C) be a twofold k-cycle system of order n with an even number of cycles. If these k-cycles can be paired together so that:

(i) each pair of cycles contains a common edge;
(ii) removal of these common (double) edges leaves a $(2k-2)$-cycle;
(iii) the collection of removed double edges can be rearranged into further $(2k-2)$-cycles;

then we refer to this as a *metamorphosis* of a paired twofold k-cycle system into a twofold $(2k-2)$-cycle system.

Clearly, for such a metamorphosis of this kind to exist, some necessary requirements are that:

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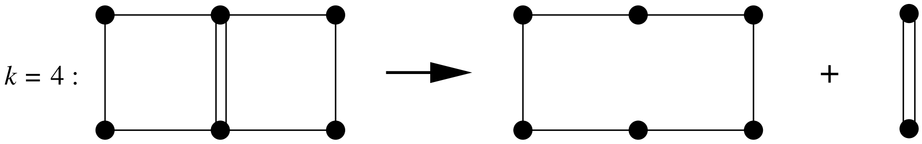

Fig. 1. Illustrating the case $k = 4$; 4-cycle pairs to 6-cycles.

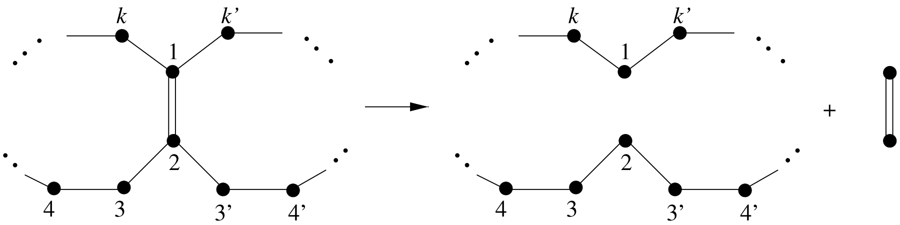

Fig. 2. Illustrating the general k-cycle case; pairs of k-cycles to $(2k - 2)$-cycles.

(a) the pairs of k-cycles with a common edge contain altogether $2k - 2$ distinct vertices, so only the end points of the common edge lie in both cycles, and all other vertices are distinct;
(b) the number of k-cycles, besides being even for the pairing of cycles, must be 0 (mod $(2k-2)$), in order that the repeated edges, upon removal, can be formed into further $2k - 2$-cycles (see Figs. 1 and 2).

We use the notation $P_kM_{2k-2}(n)$ to denote such a metamorphosis. Despite the fact that the subscript $2k-2$ is somewhat redundant, it is included here as a reminder.

Considerable work has been done on other so-called metamorphosis problems in the context of graph decompositions. The first paper in this area was by Lindner and Street [5] in 2000, and to date at least 22 papers have now appeared on the topic. A formal definition of a metamorphosis for an arbitrary G-design appears in [2].

For $k = 3$, such a $P_3M_4(n)$ metamorphosis was given by Gionfriddo and Lindner [3] (for appropriate orders n), and for $k = 4$, a $P_4M_6(n)$ metamorphosis was given by Yazıcı [7].

Henceforth, we restrict our attention to the case $k = 4$, although other values of k could be treated similarly. The following treatment for the case $k = 3$ is to appear in [4].

The expected *spectrum* or set of orders n for which a $P_4M_6(n)$ could exist is 0, 1, 9 or 16 (mod 24). This follows easily because the number of 4-cycles must be even (to form pairs) and must be 0 (mod 3) (so that the number of removed double edges is 0 (mod 6)); thus $n(n-1)/24$ must be an integer.

In any one $P_4M_6(n)$, the number of doubled edges removed and formed into 6-cycles is $n(n-1)/8 = \frac{1}{4}\binom{n}{2}$. So our aim in this paper is to construct, for each admissible order n, *one* twofold 4-cycle system of order n where $n \equiv 0, 1, 9, 16 \pmod{24}$, and perform *four* different metamorphoses, into twofold 6-cycle systems, on its pairs of 4-cycles sharing a common edge, so that the collection of all *four* lots of double edges precisely covers $2K_n$. So in the complete set of four metamorphoses, their sets of double edges must be disjoint and each will cover one quarter of the edges of $2K_n$.

For this to be possible, it is straightforward to see that the twofold 4-cycle system must be *super-simple*, that is, every pair of cycles has at most two vertices in common. This is because *every* edge is paired in one of the four metamorphoses, and if two 4-cycles shared three vertices, then they would share (at least) one edge, so pairing these 4-cycles at the common edge would not result in a 6-cycle with 6 distinct vertices upon removal of the double edge.

We shall call such a set of four metamorphoses a *complete set* of twofold paired 4-cycle metamorphoses into 6-cycles, or CP_4M_6. Any such system of order n will be denoted $CP_4M_6(n)$, while such a system on a different (non-complete) graph such as $K_{6,8}$ will be denoted by $CP_4M_6(K_{6,8})$. We also warn the reader that in [6] the term "complete" has been used with a different meaning from our use here of "complete set".

We also remark that the most difficult part of this problem is finding the so-called "small" cases for the recursive construction in Section 3 to work. Also we note that the same problem for a $CP_kM_{(2k-2)}$ involves k sets of metamorphoses.

The following example illustrates the case of order 25.

Example 1.1. A $CP_4M_6(25)$.

With vertex set $\mathbb{Z}_{25}$, we use the twofold 4-cycle system given by the following six starter cycles, modulo 25. (The cycle (a_1, a_2, a_3, a_4) is said to be a starter cycle in $\mathbb{Z}_n$ if all the cycles $\{(a_1+i, a_2+i, a_3+i, a_4+i) \mid i \in \mathbb{Z}_n\}$, addition modulo n, are taken. The differences are then said to be $\min\{|a_j - a_{j+1}|, |a_{j+1} - a_j|\}, j \in \{1, 2, 3, 4\}, j+1 = 1$ if $j = 4$.)

(0, 1, 3, 7), (0, 1, 19, 23),
(0, 3, 23, 9), (0, 3, 19, 5),
(0, 6, 21, 8), (0, 6, 23, 10).

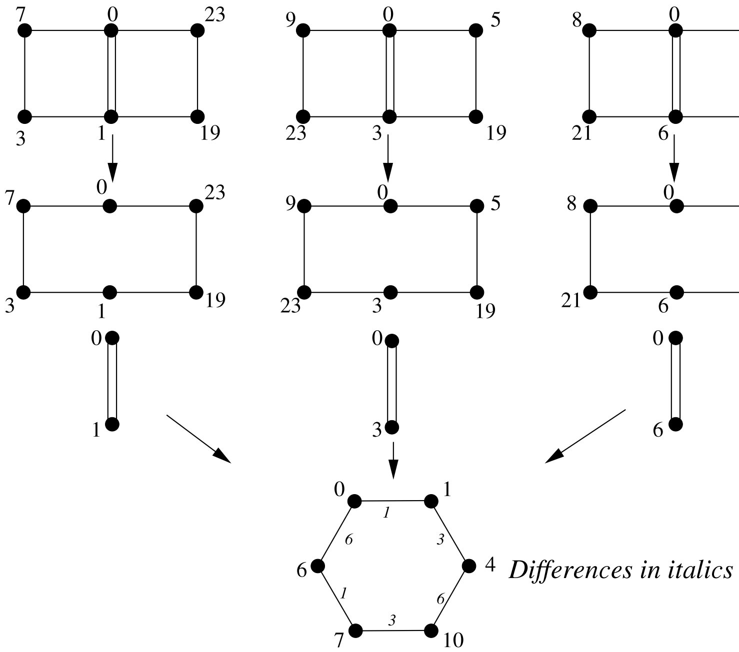

Fig. 3. Case (A), order 25; all modulo 25.

Note that these starter cycles use the differences, respectively:

1, 2, 4, 7; 1, 2, 4, 7;
3, 5, 9, 11; 3, 5, 9, 11;
6, 8, 10, 12; 6, 8, 10, 12.

There are four different ways to pair cycles giving a metamorphosis into a twofold 6-cycle system each time. We list these four ways as (A), (B), (C) and (D), and illustrate (A) pictorially in Fig. 3.

(A) Pair the 4-cycles as follows:

(0, 1, 3, 7) and (0, 1, 19, 23);
(0, 3, 23, 9) and (0, 3, 19, 5);
(0, 6, 21, 8) and (0, 6, 23, 10) (all mod 25).

Then the doubled edges (which use differences 1, 3, 6) form a starter 6-cycle (0, 1, 4, 10, 7, 6) (mod 25), and the remaining 6-cycles from the paired 4-cycles are (0, 7, 3, 1, 19, 23), (0, 9, 23, 3, 19, 5), (0, 8, 21, 6, 23, 10) (mod 25).

(B) This time we use differences 2, 5, 8 for the double edge, so we pair as follows (mod 25):

(0, 2, 6, 24), (0, 2, 3, 21); (0, 5, 2, 11), (0, 5, 19, 3); (0, 8, 21, 6), (0, 8, 2, 12).

Removal of the double edges gives the starter 6-cycle (0, 2, 7, 15, 10, 8) (mod 25).

(C) Using the differences 4, 9, 10 for the doubled edges, we pair as follows (mod 25):

(0, 4, 22, 23), (0, 4, 6, 7); (0, 9, 23, 3), (0, 9, 6, 11); (0, 10, 4, 12), (0, 10, 23, 6).

The starter 6-cycle from the removed double edges is (0, 4, 13, 23, 14, 10) (mod 25).

(D) Using the differences 7, 11, 12 for the doubled edges, we pair as follows (mod 25):

(0, 7, 3, 1), (0, 7, 6, 4); (0, 11, 2, 5), (0, 11, 6, 9); (0, 12, 4, 10), (0, 12, 2, 8).

The starter 6-cycle from the removed double edges is (0, 7, 18, 5, 19, 12).

It is clear that the doubled edges, from (A), (B), (C) and (D) above, precisely cover $2K_{25}$ because they involved the differences 1, 3, 6; 2, 5, 8; 4, 9, 10; 7, 11, 12; each twice. So this is an example of a complete set of twofold paired 4-cycle metamorphoses into 6-cycles, of order 25. □

2. Some necessary examples

We begin this section with two examples of complete sets of twofold paired 4-cycle metamorphoses into 6-cycles, for two bipartite graphs. These examples use small latin squares in their construction.

Example 2.1. A $\mathrm{CP}_4\mathrm{M}_6(K_{6,6})$.

Let the vertex set of $K_{6,6}$ be $\{0_a, 1_a, 2_a, 0_c, 1_c, 2_c\} \cup \{0_b, 1_b, 2_b, 0_d, 1_d, 2_d\}$, and take any 3×3 latin square on the symbols $\{0, 1, 2\}$. For each symbol s in cell (x, y) of the latin square, we take the 4-cycles

$$(x_a, y_b, x_c, s_d), \qquad (x_a, y_b, (x+1)_c, (s+2)_d),$$

where $x + 1$ and $s + 2$ are calculated modulo 3. This gives two 4-cycles for each of the 9 symbols in the latin square; so as required, the total number of 4-cycles is 18. Now the four different pairings for the four metamorphoses are as follows.

(A) Take the pairs to be edges between vertices $\{0_a, 1_a, 2_a\}$ and $\{0_b, 1_b, 2_b\}$; in other words all edges between points with subscripts a and b. It is immediate from the form of the 4-cycles that this covers all 9 edges between these points. Moreover, this graph of the repeated edges is $2K_{3,3}$, which has an easy decomposition into 6-cycles, thus providing a metamorphosis in this case (A).
(B) This time the 4-cycles are paired with all edges between points with subscripts a and d. Again the repeated edges precisely cover $2K_{3,3}$, yielding 6-cycles from the repeated edges.
(C) This time the 4-cycles are paired with all edges between points with subscripts c and b; the rest follows as above.
(D) This time the 4-cycles are paired with all edges between points with subscripts c and d; the rest follows as above.

The result is a $\mathrm{CP}_4\mathrm{M}_6(K_{6,6})$. □

Example 2.2. A $\mathrm{CP}_4\mathrm{M}_6(K_{6,8})$.

In this case, we take the vertex set $\{0_a, 1_a, 2_a, 0_c, 1_c, 2_c\} \cup \{0_b, 1_b, 2_b, 3_b, 0_d, 1_d, 2_d, 3_d\}$. We take a partial latin square such as the following.

*	2	1	0
0	*	2	1
1	0	*	2
2	1	0	*

Now for each symbol s in an occupied cell (x, y) of this partial latin square, we take the 4-cycles

$$(s_a, x_b, s_c, y_d), \qquad (s_a, x_b, (s+1)_c, (y+1)_d),$$

where the addition $s + 1$ is taken mod 3, while $y + 1$ is taken mod 4.

As in the previous example, the four different edge-pairings, giving four different metamorphoses, are as follows:

(A) Pair edges s_a, x_b for all $0 \leqslant s \leqslant 2$ and $0 \leqslant x \leqslant 3$; this gives 12 double edges. These form a copy of $2K_{3,4}$; its 24 edges easily decompose into four 6-cycles.
(B) Pair edges with end points a and d.
(C) Pair edges with end points c and b.
(D) Pair edges with end points c and d.

The result is a $\mathrm{CP}_4\mathrm{M}_6(K_{6,8})$. □

Example 2.3. A $\mathrm{CP}_4\mathrm{M}_6(16)$.

Let the vertex set of K_{16} be $\{i_j \mid i = 0, 1, 2, 3, 4,\ j = 1, 2, 3\} \cup \{\infty\}$. We work modulo 5, and take the following 12 starters modulo 5 (with subscripts and ∞ fixed).

$$\begin{array}{llll}
(0_1, 1_1, 0_2, \infty), & (0_2, 2_1, 0_3, \infty), & (0_3, 1_1, 3_1, \infty), & (0_1, 1_1, 3_1, 2_2),\\
(0_1, 0_2, 3_1, 0_3), & (0_1, 0_2, 4_1, 1_3), & (0_1, 3_2, 1_2, 0_3), & (0_1, 4_3, 1_2, 1_3),\\
(0_2, 1_2, 3_2, 3_3), & (0_2, 1_2, 2_3, 4_1), & (0_2, 2_3, 3_3, 1_3), & (1_2, 3_3, 1_3, 0_3).
\end{array}$$

The four pairings to give metamorphoses into 6-cycle systems are as follows.
(A) Take the first edge in each of the above 4-cycles, and pair according to this edge. This may require cycling modulo 5 to find the pair; for instance the second listed cycle is $(0_2, 2_1, 0_3, \infty)$, and this pairs with the cycle $(2_1, 0_2, 3_2, 2_3)$, obtained (mod 5) from $(0_1, 3_2, 1_2, 0_3)$ (which is the seventh cycle listed above).

When paired according to the first listed edges in the cycles, these doubled edges decompose to give the following ten 6-cycles:

$$\begin{array}{lll}
(1_1, 0_1, 0_2, 4_2, 1_3, 2_1), & (0_1, 3_2, 4_2, 4_1, 2_2, 4_3), & (1_1, 1_2, 0_2, 2_1, 1_3, 4_2),\\
(1_1, 1_2, 0_2, 4_2, 3_2, 0_3), & (0_1, 4_1, 3_1, 2_1, 2_2, 4_3), & (1_1, 0_3, 3_2, 2_2, 4_1, 4_2),\\
(2_2, 2_1, 0_2, 2_3, 3_1, 1_2), & (0_1, 1_1, 2_1, 3_1, 2_3, 0_2), & (0_1, 4_1, 3_3, 1_2, 3_1, 3_2),\\
(3_1, 3_2, 2_2, 1_2, 3_3, 4_1). & &
\end{array}$$

(B) This time the edges taken in the above 12 starter cycles, as repeated edges for this second pairing, are, respectively:

$$\begin{array}{llll} 0_2\infty, & 0_2\infty, & 1_13_1, & 1_13_1, \\ 3_10_3, & 4_11_3, & 1_20_3, & 1_24_3, \\ 0_23_3, & 1_22_3, & 0_21_3, & 1_20_3. \end{array}$$

These repeated edges decompose to give the following ten 6-cycles:

$$\begin{array}{lll} (0_1, 2_1, 4_1, 1_3, 3_2, 2_3), & (1_2, 2_3, 4_2, 0_3, 2_2, \infty), & (1_1, 4_1, 2_1, 4_3, 0_2, 3_3), \\ (0_1, 3_1, 1_1, 3_3, 4_2, 2_3), & (2_2, 1_3, 4_1, 1_1, 3_1, 0_3), & (0_2, 1_3, 2_2, 3_3, 4_2, \infty), \\ (0_3, 3_1, 0_1, 2_1, 4_3, 1_2), & (0_2, 1_3, 3_2, \infty, 2_2, 3_3), & (0_2, \infty, 1_2, 2_3, 3_2, 4_3), \\ (1_2, 4_3, 3_2, \infty, 4_2, 0_3). & & \end{array}$$

(C) This time the repeated edges taken in the above 12 starter cycles are, respectively:

$$\begin{array}{llll} 0_1\infty, & 2_10_3, & 3_1\infty, & 0_12_2, \\ 3_10_2, & 0_11_3, & 1_23_2, & 0_11_3, \\ 1_23_2, & 4_12_3, & 2_33_3, & 0_31_3. \end{array}$$

These repeated edges decompose to give the following ten 6-cycles:

$$\begin{array}{lll} (0_1, 1_3, 2_3, 1_1, 4_3, 3_3), & (\infty, 3_1, 0_2, 2_2, 4_2, 2_1), & (2_1, 0_3, 1_3, 3_1, 4_3, 3_3), \\ (1_1, 2_3, 3_3, 2_1, 0_3, 4_3), & (1_1, 3_2, 1_2, 4_2, 2_1, \infty), & (1_1, \infty, 0_1, 2_2, 0_2, 3_2), \\ (0_1, 2_2, 4_2, 1_2, 4_1, \infty), & (0_1, 3_3, 2_3, 4_1, 0_3, 1_3), & (3_1, 1_3, 2_3, 4_1, 0_3, 4_3), \\ (0_2, 3_1, \infty, 4_1, 1_2, 3_2). & & \end{array}$$

(D) Finally the repeated edges in the above 12 starter cycles are, respectively:

$$\begin{array}{llll} 1_10_2, & 0_3\infty, & 0_3\infty, & 3_12_2, \\ 0_10_3, & 4_10_2, & 0_10_3, & 1_21_3, \\ 3_23_3, & 4_10_2, & 1_33_3, & 1_33_3. \end{array}$$

These repeated edges decompose to give the following ten 6-cycles:

$$\begin{array}{lll} (0_1, 1_2, 2_1, 3_2, 3_3, 0_3), & (0_1, 4_2, 3_1, 3_3, \infty, 0_3), & (0_1, 1_2, 1_3, \infty, 4_3, 4_2), \\ (2_1, 1_2, 1_3, 3_3, 0_3, 2_3), & (3_1, 2_2, 2_3, 4_3, 1_3, 3_3), & (1_1, 1_3, 4_3, 4_2, 3_1, 2_2), \\ (1_1, 0_2, 4_1, 4_3, \infty, 1_3), & (1_1, 0_2, 0_3, \infty, 2_3, 2_2), & (2_1, 2_3, 0_3, 0_2, 4_1, 3_2), \\ (4_1, 3_2, 3_3, \infty, 2_3, 4_3). & & \end{array}$$

This completes the decomposition. □

Example 2.4. A $CP_4M_6(24)$.

Let the vertex set of K_{24} be $\{i_j \mid i \in \mathbb{Z}_3,\ 1 \leqslant j \leqslant 8\}$. The twofold 4-cycle system we use has 46 starters modulo 3 (subscripts are fixed).

$$\begin{array}{llll} (0_4, 2_5, 0_5, 0_8), & (0_7, 1_3, 1_6, 1_8), & (0_3, 1_8, 0_5, 2_8), & (0_1, 2_2, 0_2, 1_8), \\ (0_4, 1_4, 0_6, 2_7), & (0_3, 0_8, 2_5, 2_4), & (0_3, 1_3, 1_1, 2_5), & (1_5, 2_1, 1_6, 2_7), \\ (0_7, 2_7, 2_8, 1_1), & (0_2, 2_1, 1_6, 0_5), & (1_7, 2_7, 2_2, 0_5), & (0_2, 1_3, 0_6, 2_7), \\ (0_1, 2_2, 1_5, 1_7), & (0_2, 1_4, 2_6, 2_7), & (0_3, 0_8, 1_6, 1_7), & (2_5, 2_3, 0_5, 2_7), \\ (0_1, 1_1, 1_6, 2_7), & (0_6, 2_1, 1_8, 1_6), & (2_4, 2_2, 2_3, 0_7), & (1_2, 2_3, 0_4, 0_6), \\ (0_5, 2_6, 2_1, 2_7), & (1_3, 2_3, 1_6, 1_4), & (0_1, 2_3, 0_6, 0_5), & (0_4, 2_4, 1_8, 1_1), \\ (0_8, 2_8, 0_7, 0_4), & (2_3, 2_5, 0_2, 1_5), & (2_3, 2_7, 2_8, 0_2), & (0_6, 0_2, 0_8, 1_2), \\ (0_2, 0_6, 1_4, 1_7), & (2_4, 0_5, 2_3, 1_1), & (0_1, 1_1, 1_4, 1_5), & (0_2, 1_4, 0_3, 2_4), \\ (1_1, 2_6, 2_3, 1_8), & (1_5, 0_6, 2_8, 1_6), & (0_1, 0_2, 0_8, 0_5), & (0_4, 2_5, 1_5, 1_2), \\ (0_8, 2_3, 0_6, 2_2), & (0_1, 0_2, 0_7, 2_4), & (1_5, 2_1, 2_4, 0_8), & (2_5, 1_4, 2_6, 0_6), \\ (0_3, 2_7, 1_8, 0_4), & (1_7, 1_3, 1_1, 2_4), & (0_1, 2_3, 2_2, 1_3), & (0_2, 0_4, 2_8, 1_6), \\ (0_1, 1_2, 0_2, 1_7), & (1_8, 2_8, 0_1, 0_7). & & \end{array}$$

The above starter cycles are ordered so that the first edge in each cycle is repeated and used in metamorphosis (A), and the second edge in (B) (see Example 2.2 for instance). But the repeated edges for (C) and (D) are mixed, so we explicitly list these below.

The doubled edges in each case (A), (B), (C) and (D) can be rearranged into 23 6-cycles (*not* cycled), as follows:

6-*cycles from*(A) :	6-*cycles from*(B) :
$(0_1, 0_2, 0_4, 1_4, 2_5, 1_6)$,	$(0_1, 0_3, 1_6, 0_8, 1_4, 0_6)$,
$(1_1, 2_1, 2_2, 0_3, 0_5, 2_6)$,	$(2_4, 2_1, 2_3, 2_2, 2_8, 0_6)$,
$(2_1, 1_3, 0_2, 1_4, 0_4, 1_5)$,	$(1_4, 1_1, 1_3, 2_4, 0_6, 0_3)$,
$(2_4, 2_2, 1_1, 0_2, 1_3, 1_5)$,	$(1_4, 2_6, 1_3, 1_2, 1_7, 0_8)$,
$(0_4, 1_5, 0_6, 0_2, 0_1, 2_2)$,	$(1_5, 0_5, 1_2, 0_2, 0_7, 2_8)$,
$(0_3, 2_7, 2_3, 1_7, 0_7, 1_3)$,	$(1_1, 0_6, 1_3, 2_4, 2_1, 1_6)$,
$(1_3, 0_7, 2_7, 1_7, 2_3, 2_8)$,	$(2_4, 1_6, 0_3, 0_2, 1_2, 1_8)$,
$(1_3, 2_3, 1_2, 2_1, 0_6, 1_5)$,	$(0_5, 2_3, 0_6, 0_3, 0_2, 2_5)$,
$(1_2, 2_4, 0_5, 1_4, 2_5, 1_6)$,	$(1_4, 1_1, 0_8, 2_5, 1_5, 0_3)$,
$(0_4, 2_5, 0_1, 1_2, 1_4, 2_4)$,	$(0_1, 2_6, 1_8, 1_2, 2_2, 2_8)$,
$(1_4, 1_2, 2_4, 2_2, 1_1, 0_5)$,	$(0_1, 0_4, 2_3, 2_2, 1_5, 2_8)$,
$(0_3, 0_7, 2_7, 2_3, 0_8, 1_8)$,	$(0_5, 1_5, 0_3, 2_6, 2_1, 1_8)$,
$(2_1, 1_2, 2_3, 2_8, 1_8, 1_3)$,	$(1_1, 1_3, 1_2, 2_2, 0_2, 0_8)$,
$(0_3, 2_7, 1_7, 1_3, 1_8, 0_8)$,	$(2_5, 0_2, 0_7, 0_8, 1_6, 1_3)$,
$(0_3, 0_7, 1_7, 1_3, 2_8, 0_8)$,	$(2_1, 2_3, 0_5, 1_2, 1_7, 1_8)$,
$(0_1, 2_1, 0_2, 1_1, 1_2, 1_6)$,	$(2_2, 1_5, 2_5, 0_5, 1_8, 2_7)$,
$(1_4, 0_2, 0_6, 2_1, 1_5, 2_4)$,	$(2_5, 1_3, 2_6, 1_8, 1_7, 0_8)$,
$(0_4, 2_2, 0_1, 1_1, 0_5, 2_4)$,	$(0_1, 0_3, 2_6, 0_4, 2_8, 0_6)$,
$(2_3, 0_3, 1_1, 1_2, 0_1, 2_5)$,	$(0_2, 0_8, 0_7, 2_8, 2_7, 2_2)$,
$(0_1, 2_1, 0_2, 0_4, 2_5, 2_3)$,	$(0_4, 1_6, 2_4, 1_8, 2_7, 2_8)$,
$(0_3, 1_3, 2_3, 0_8, 2_8, 1_8)$,	$(1_1, 0_6, 1_4, 2_6, 2_3, 1_6)$,
$(2_3, 0_1, 1_1, 2_6, 2_2, 0_3)$,	$(0_1, 0_4, 2_3, 1_6, 2_1, 2_6)$,
$(2_6, 2_2, 2_1, 1_1, 0_3, 0_5)$,	$(1_3, 1_6, 0_4, 2_6, 2_3, 0_6)$.

The edges for the metamorphosis and pairing for (C) are (mod 3):

0_40_8,	1_61_8,	0_32_8,	0_21_8,	0_42_7,	2_52_4,	0_32_5,	1_52_7,
0_71_1,	1_60_5,	1_70_5,	0_22_7,	1_51_7,	0_22_7,	0_31_7,	2_52_7,
0_12_7,	1_81_6,	2_30_7,	0_40_6,	2_12_7,	1_61_4,	0_10_5,	0_41_1,
0_80_4,	2_31_5,	2_80_2,	0_81_2,	0_21_7,	2_31_1,	1_41_5,	0_22_4,
2_31_8,	2_81_6,	0_10_5,	0_41_2,	0_82_2,	0_12_4,	2_40_8,	2_50_6,
1_80_4,	1_72_4,	0_11_3,	2_81_6,	0_21_7,	0_10_7.		

The edges for the metamorphosis and pairing for (D) are (mod 3):

0_80_5,	1_80_7,	2_80_5,	1_80_1,	2_70_6,	2_40_3,	2_51_1,	2_71_6,
1_12_8,	0_50_2,	0_52_2,	2_70_6,	1_70_1,	2_72_6,	1_71_6,	2_70_5,
2_71_6,	1_60_6,	0_72_4,	0_61_2,	2_70_5,	1_41_3,	0_50_6,	1_11_8,
0_40_7,	1_50_2,	0_22_3,	1_20_6,	1_71_4,	1_12_4,	1_50_1,	2_40_3,
1_81_1,	1_61_5,	0_50_8,	1_21_5,	2_20_6,	2_40_7,	0_81_5,	0_62_6,
0_40_3,	2_41_1,	1_32_2,	1_60_2,	1_70_1,	0_71_8.		

6-*cycles from*(C) :	6-*cycles from*(D) :
$(1_1, 2_3, 0_7, 2_2, 1_8, 0_4)$,	$(2_1, 0_5, 2_2, 0_6, 0_7, 0_4)$,
$(1_4, 1_8, 2_3, 0_7, 1_2, 2_8)$,	$(2_3, 0_2, 0_5, 2_8, 1_1, 2_4)$,
$(0_4, 0_8, 2_4, 2_5, 2_7, 1_2)$,	$(2_7, 1_4, 1_3, 2_2, 0_6, 0_5)$,
$(1_1, 2_3, 1_5, 2_6, 2_4, 1_7)$,	$(1_7, 0_4, 1_3, 2_2, 2_5, 2_8)$,
$(2_2, 1_4, 1_8, 1_6, 0_5, 1_7)$,	$(0_1, 1_5, 1_2, 0_6, 0_5, 0_8)$,
$(1_2, 2_8, 0_3, 2_5, 0_6, 0_8)$,	$(0_4, 0_3, 2_4, 2_7, 2_6, 0_7)$,
$(1_3, 0_5, 1_6, 1_8, 0_4, 0_8)$,	$(2_2, 0_5, 0_8, 1_5, 0_2, 1_6)$,
$(0_4, 1_1, 1_5, 2_3, 1_8, 0_6)$,	$(2_1, 0_8, 2_7, 2_6, 2_5, 2_8)$,
$(1_4, 2_8, 0_2, 2_4, 1_7, 1_5)$,	$(1_5, 0_7, 2_1, 2_8, 1_1, 1_8)$,
$(0_5, 0_1, 2_7, 1_2, 0_8, 1_3)$,	$(1_1, 2_7, 1_6, 2_6, 2_5, 1_8)$,
$(0_1, 2_4, 2_8, 1_6, 1_4, 0_7)$,	$(0_1, 1_4, 2_3, 2_4, 0_7, 1_8)$,
$(1_4, 0_7, 1_1, 1_5, 1_7, 2_1)$,	$(1_2, 0_3, 2_4, 2_7, 1_6, 2_6)$,
$(2_1, 1_4, 1_5, 2_7, 0_2, 1_7)$,	$(1_1, 2_5, 1_7, 2_6, 0_6, 2_7)$,
$(2_5, 2_1, 2_7, 0_1, 0_5, 0_7)$,	$(1_5, 0_1, 1_7, 2_6, 0_6, 1_6)$,
$(1_3, 2_7, 0_4, 0_5, 0_7, 0_1)$,	$(1_2, 0_3, 0_4, 1_7, 1_6, 1_5)$,
$(2_1, 0_3, 2_8, 2_4, 0_2, 2_7)$,	$(0_7, 1_6, 1_7, 2_5, 1_8, 1_5)$,
$(0_7, 1_1, 1_7, 0_3, 2_1, 2_5)$,	$(0_1, 1_7, 0_6, 1_6, 0_7, 1_8)$,

$(2_2, 0_7, 1_2, 0_4, 0_5, 1_7)$, $(1_1, 2_4, 0_7, 0_6, 1_2, 2_5)$,
$(2_5, 0_6, 1_8, 0_2, 1_7, 0_3)$, $(0_1, 1_4, 1_7, 0_6, 2_7, 0_8)$,
$(0_4, 2_7, 2_5, 2_4, 0_8, 0_6)$, $(0_2, 1_6, 2_2, 2_5, 1_2, 2_6)$,
$(1_3, 2_7, 1_5, 2_6, 2_4, 0_1)$, $(1_5, 0_2, 2_6, 0_7, 2_1, 0_8)$,
$(1_4, 2_2, 0_8, 2_6, 2_8, 1_6)$, $(2_1, 0_4, 1_3, 1_4, 2_7, 0_5)$,
$(0_8, 2_2, 1_8, 0_2, 2_8, 2_6)$, $(0_2, 0_5, 2_8, 1_7, 1_4, 2_3)$. □

Example 2.5. A $\mathrm{CP}_4\mathrm{M}_6(33)$.

Let the vertex set of K_{33} be $\{i_j \mid i \in \mathbb{Z}_{11},\ j = 1, 2, 3\}$. The following 24 starter cycles, modulo 11, will be our twofold 4-cycle system.

$(0_1, 1_1, 8_3, 5_3)$, $(0_1, 3_3, 1_3, 0_3)$, $(1_1, 5_1, 1_2, 4_2)$, $(1_1, 9_3, 4_3, 3_3)$,
$(2_1, 0_1, 5_2, 10_2)$, $(2_1, 4_1, 8_3, 7_1)$, $(2_1, 0_3, 2_3, 1_2)$, $(2_1, 1_3, 3_2, 1_2)$,
$(3_1, 1_2, 0_2, 3_3)$, $(3_1, 2_1, 9_2, 6_2)$, $(5_1, 5_2, 4_2, 0_1)$, $(6_1, 10_1, 3_3, 8_2)$,
$(8_1, 0_1, 6_2, 10_3)$, $(8_1, 6_3, 4_2, 9_2)$, $(9_1, 6_3, 4_2, 6_2)$, $(10_1, 7_1, 2_3, 1_2)$,
$(0_2, 0_1, 5_2, 0_3)$, $(0_2, 2_1, 8_2, 0_3)$, $(0_2, 5_3, 0_3, 10_1)$, $(0_3, 1_1, 7_3, 3_2)$,
$(0_3, 6_2, 10_2, 6_1)$, $(0_3, 8_1, 4_3, 7_3)$, $(1_3, 2_2, 0_3, 4_2)$, $(2_3, 3_2, 10_2, 6_3)$.

For each starter cycle above, say written in the order (1, 2, 3, 4), the edges {1, 2} are paired for the metamorphosis (A); the edges {2, 3} are paired for the metamorphosis (B); the edges {3, 4} are paired for the metamorphosis (C); and the edges {4, 1} are paired for the metamorphosis (D). This is possible because each difference, whether pure or mixed, appears either twice or not at all in each of these four positions.

The extra 6-cycles in each case, from the repeated edges, are as follows (mod 11, with subscripts fixed):

(A) $(0_1, 1_1, 3_1, 6_1, 2_1, 0_2)$, $(0_1, 1_1, 3_1, 6_1, 2_1, 0_2)$,
$(0_1, 3_3, 4_1, 2_3, 3_2, 8_3)$, $(0_1, 3_3, 4_1, 2_3, 3_2, 8_3)$.

(B) $(0_1, 5_2, 9_1, 4_2, 8_2, 6_3)$, $(0_1, 5_2, 9_1, 2_3, 0_3, 6_3)$,
$(0_1, 6_2, 7_2, 0_2, 2_3, 7_3)$, $(0_1, 4_3, 2_3, 4_2, 5_2, 7_3)$.

(C) $(0_1, 4_2, 1_2, 3_2, 8_2, 1_3)$, $(0_1, 4_2, 1_2, 3_2, 8_2, 1_3)$,
$(0_2, 1_3, 5_2, 8_3, 5_3, 6_3)$, $(0_2, 1_3, 5_2, 8_3, 5_3, 6_3)$.

(D) $(0_1, 5_1, 2_2, 1_1, 0_2, 0_3)$, $(0_1, 3_2, 1_1, 6_1, 5_2, 2_3)$,
$(0_1, 1_2, 4_1, 6_2, 3_1, 5_3)$, $(0_2, 0_3, 0_1, 5_3, 1_3, 8_3)$. □

3. The construction

3.1. *The case of order* 0 *(mod* 24*)*

Let the vertex set of K_{24x} be $\{(i, j) \mid 1 \leqslant i \leqslant 4x,\ 1 \leqslant j \leqslant 6\}$. (The reader may wish to visualise these $24x$ points as six layers with $4x$ points per layer.) On each set $\{(i, j) \mid 4a - 3 \leqslant i \leqslant 4a,\ 1 \leqslant j \leqslant 6\}$, for $a = 1, \ldots, x$, place a $\mathrm{CP}_4\mathrm{M}_6(24)$; see Example 2.4.

Then on each set of vertices $\{(i_1, j) \mid 1 \leqslant j \leqslant 6\} \cup \{(i_2, j) \mid 1 \leqslant j \leqslant 6\}$, for all integers i_1 and i_2 satisfying $1 \leqslant \left\lceil \frac{i_1}{4} \right\rceil < \left\lceil \frac{i_2}{4} \right\rceil \leqslant x$, place a $\mathrm{CP}_4\mathrm{M}_6(K_{6,6})$; see Example 2.1. In other words, i_1 and i_2 belong to different sets of $\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \ldots, \{4x - 3, 4x - 2, 4x - 1, 4x\}$.

3.2. *The case of order* 1 *(mod* 24*)*

Let the vertex set of K_{24x+1} be $\{(i, j) \mid 1 \leqslant i \leqslant 4x,\ 1 \leqslant j \leqslant 6\} \cup \{\infty\}$. On each set of vertices $\{(i, j) \mid 4a - 3 \leqslant i \leqslant 4a,\ 1 \leqslant j \leqslant 6\} \cup \{\infty\}$, for $a = 1, \ldots, x$, place a $\mathrm{CP}_4\mathrm{M}_6(25)$; see Example 1.1. Then (as in the case 0 (mod 24)) on each set of vertices $\{(i_1, j) \mid 1 \leqslant j \leqslant 6\} \cup \{(i_2, j) \mid 1 \leqslant j \leqslant 6\}$, for all integers i_1 and i_2 satisfying $1 \leqslant \left\lceil \frac{i_1}{4} \right\rceil < \left\lceil \frac{i_2}{4} \right\rceil \leqslant x$, place a $\mathrm{CP}_4\mathrm{M}_6(K_{6,6})$; see Example 2.1.

3.3. *The case of order* 9 *(mod* 24*)*

A computer search has shown that no complete twofold paired 4-cycle metamorphosis into 6-cycles of order 9 exists. So the smallest case in this congruence class is order 33.

We take the vertex set of K_{24x+9} to be

$$\{\infty\} \cup \{a_i \mid 1 \leqslant i \leqslant 32\} \cup \{(i, j) \mid 1 \leqslant i \leqslant 24,\ 1 \leqslant j \leqslant x - 1\}.$$

On the 33 vertices $\{a_i \mid 1 \leqslant i \leqslant 32\} \cup \{\infty\}$, place a copy of Example 2.5. On the 25 vertices $\{(i, j) \mid 1 \leqslant i \leqslant 24\} \cup \{\infty\}$, for each $j = 1, 2, \ldots, x - 1$, place a $\mathrm{CP_4M_6}(25)$; see Example 1.1.

Then we use $\mathrm{CP_4M_6}(K_{6,6})$ and $\mathrm{CP_4M_6}(K_{6,8})$; see Examples 2.1 and 2.2. We can partition $\{a_i \mid 1 \leqslant i \leqslant 32\}$ into four parts of size 8, and each set $\{(i, j) \mid 1 \leqslant i \leqslant 24\}$, for $j = 1, 2, \ldots, x - 1$, into four parts of size 6. So we use $16(x - 1)$ copies of a $\mathrm{CP_4M_6}(K_{6,8})$ (Example 2.2) and $16\binom{x-1}{2}$ copies of a $\mathrm{CP_4M_6}(K_{6,6})$ (Example 2.1).

3.4. The case of order 16 *(mod* 24*)*

Let the vertex set of K_{24x+16} be $\{a_i \mid 1 \leqslant i \leqslant 16\} \cup \{(i, j) \mid 1 \leqslant i \leqslant 24,\ 1 \leqslant j \leqslant x\}$.

Then we use a $\mathrm{CP_4M_6}(16)$ (Example 2.3) on the vertex set $\{a_i \mid 1 \leqslant i \leqslant 16\}$, and x copies of a $\mathrm{CP_4M_6}(24)$ (Example 2.4) on the vertex sets $\{(i, j) \mid 1 \leqslant i \leqslant 24\}$ for $j = 1, 2, \ldots, x$.

Then, partitioning the vertex set $\{a_i \mid 1 \leqslant i \leqslant 16\}$ into two sets of eight, and each vertex set $\{(i, j) \mid 1 \leqslant i \leqslant 24\}$, for $j = 1, 2, \ldots, x$, into four, we use copies of a $\mathrm{CP_4M_6}(K_{6,8})$ (Example 2.2) and a $\mathrm{CP_4M_6}(K_{6,6})$ (Example 2.1); $8x$ copies of the former and $16\binom{x}{2}$ of the latter.

4. Concluding comments

We have now shown the following.

Theorem 4.1. *For all orders $n \equiv 0, 1, 9, 16$ (modulo 24), apart from order 9 , there exists a twofold 4-cycle decomposition of $2K_n$ which has four separate pairings to give metamorphoses into 6-cycle systems, such that the collection of 6-cycles formed from the repeated edges in the pairs of 4-cycles in all four metamorphoses themselves form a decomposition of $2K_n$.*

In terms of our notation above, this theorem states that for all $n \equiv 0, 1, 9, 16$ (modulo 24), $n \neq 9$, there exists a $\mathrm{CP_4M_6}(n)$.

As remarked above in the introduction, the twofold 4-cycle system used each time must be super-simple in order that there can exist four different pairings of the 4-cycles (each of which yields 6-cycles upon removal of the paired (double) edge). The expected spectrum of super-simple twofold 4-cycle systems is 0 or 1 (mod 4), and necessarily orders 4 and 5 are impossible. Our search for order 9 above also showed that there is no super-simple twofold 4-cycle system of order 9. However, there is one of order 8, and we have found super-simple twofold 4-cycle systems of all other admissible orders; a note regarding this has recently appeared [1].

If we require the double edges in the above four pairings to exactly cover $2K_n$ but if we drop the requirement that these double edges can be formed into 6-cycles, then the expected spectrum increases to 0 or 1 (mod 8). This is because in this case we no longer require the number of double edges removed to be a multiple of 3; we only require the number of 4-cycles to be even. This presents another open problem of interest.

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