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Asymptotic and Oscillatory Behavior of Third-Order Differential Equations of a Certain Type*

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1. INTRODUCTION

Let us consider the system

$$\bar{\mathbf{X}}' = A(t)\,\bar{\mathbf{X}}, \tag{1}$$

where $\bar{\mathbf{X}} = (x_1, x_2, x_3)$ and $A(t)$ is a 3×3 matrix given by

$$A(t) = \begin{bmatrix} 0 & 1 & 0 \\ \alpha & 0 & 1 \\ \beta & \gamma & 0 \end{bmatrix}_{(t)}. \tag{2}$$

The coefficients α, β, γ are real valued bounded functions defined on the ray $[0, \infty)$ such that α and γ are C^1 and β is continuous.

System (1) is equivalent to the equation

$$y''' = (\alpha + \gamma)\,y' + (\alpha' + \beta)\,y, \tag{3}$$

in the sense that the correspondence $y \to \bar{\mathbf{X}} = (y, y', y'' - \alpha y)$ establishes an isomorphism between the solution space of (3) and that of (1).

We denote by S the solution space of (1). For each $\bar{\mathbf{X}} \in S$ we define $V_{\bar{\mathbf{X}}}\colon [0, \infty) \to \mathbb{R}$ by

$$V_{\bar{\mathbf{X}}}(t) = x_2^2 - 2x_1x_3 + (\gamma(t) - \alpha(t))\,x_1^2. \tag{4}$$

We write $l(\bar{\mathbf{X}}) = \lim_{t\to\infty} V_{\bar{\mathbf{X}}}(t)$ when this limit exists. The following sets will be important in this work:

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$$S_0 = \{\bar{\mathbf{X}} \in S \mid \lim_{t\to\infty} \| \bar{\mathbf{X}}(t)\| = 0\},$$

$$S_+ = \{\bar{\mathbf{X}} \in S \mid V_{\bar{\mathbf{X}}}(t) \geqslant 0, \forall t \geqslant 0\},$$

$$S_- = \{\bar{\mathbf{X}} \in S \mid V_{\bar{\mathbf{X}}}(t) \leqslant 0, \forall t \geqslant 0\},$$

$$S_2 = \{\bar{\mathbf{X}} \in S \mid \| \bar{\mathbf{X}} \| \in L^2\}.^1$$

In Section 2 we will derive some propositions concerning S_0, S_2, S_+, S_- and the asymptotic behavior of the solutions of (1). From these propositions Theorems 1 and 2 (see Section 2) will be proved. In Section 3 we will apply these theorems to the equation

$$y''' = py' + qy, \tag{5}$$

to obtain Theorem 3 of this paper. Theorem 3 was motivated by Theorems 2.3 and 3.3 of [2]. We point out that as a difference with [2] we do not impose any sign restriction on p, and the condition q bounded has been weakened (see Theorem 3 below). (We note that p and q of (5) are the negatives of those in [2].) In Section 4 we derive some oscillatory properties of (5).

In the sequel we will need the following lemmas which we establish here without proof:

LEMMA 1. *In Equation* (1) *let us assume that* α, β, γ *are continuous and bounded for* $t \geqslant 0$; *then if* $\bar{\mathbf{X}} = (x_1, x_2, x_3) \in S$ *is such that* $x_1(t)$ *is bounded on* $[0, \infty)$, *then* $x_2(t)$ *and* $x_3(t)$ *are also bounded on* $[0, \infty)$.

LEMMA 2. *Under the assumptions of Lemma* 1, *if now* $x_1 \in L^2$ *then* $x_2 \in L^2$ *and* $x_3 \in L^2$.

LEMMA 3. *Let* $f \in C^1[0, \infty)$. *If* $f \in L^2$ *and* f' *is bounded then* $f(t) \to 0$ *as* $t \to \infty$.

Lemma 3 is elementary. For a proof of Lemmas 1 and 2 see, for instance, [1].

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We begin this section with the following:

PROPOSITION 1. *If* $(\alpha - \gamma)' + 2\beta \geqslant k > 0$ ($\leqslant -k < 0$), $k \in \mathbb{R}$, *then* $V_{\bar{\mathbf{X}}}(t)$ *is a strictly decreasing* (*strictly increasing*) *function of* $t \in [0, \infty)$ *for each nontrivial* $\bar{\mathbf{X}} \in S$. *From here* $l(\bar{\mathbf{X}})$ *exists and* $-\infty \leqslant l(\bar{\mathbf{X}}) < +\infty$ ($-\infty < l(\bar{\mathbf{X}}) \leqslant +\infty$).

[1] In what follows $f \in L^p$, $p \geqslant 1$ means $f \in L^p[0, \infty)$, $p \geqslant 1$.

Proof. It follows from the fact that for each $\bar{\mathbf{X}} \in S$ we have

$$V_{\bar{\mathbf{X}}}(t) - V_{\bar{\mathbf{X}}}(t_0) = -\int_{t_0}^{t} [(\alpha - \gamma)' + 2\beta]\, x_1^2. \quad \blacksquare \tag{6}$$

PROPOSITION 2. *If* $(\alpha - \gamma)' + 2\beta \geqslant k > 0$ $(\leqslant -k < 0)$ *and if* $\bar{\mathbf{X}} \in S$ *is such that* $l(\bar{\mathbf{X}}) > -\infty$ $(< +\infty)$, *then* $\bar{\mathbf{X}} \in S_2$.

Proof. If $(\alpha - \gamma)' + 2\beta \geqslant k > 0$, then from (6) we have

$$k \int_0^t x_1^2(s)\, ds \leqslant V_{\bar{\mathbf{X}}}(0) - V_{\bar{\mathbf{X}}}(t). \tag{7}$$

Since $l(\bar{\mathbf{X}}) > -\infty$ it follows that $x_1 \in L^2$. Similarly if $(\alpha - \gamma)' + 2\beta \leqslant -k < 0$ from (6) and $l(\bar{\mathbf{X}}) < +\infty$, it follows that $x_1 \in L^2$. For both cases from Lemma 2 it follows that $\bar{\mathbf{X}} \in S_2$. $\blacksquare$

PROPOSITION 3. *If* $\bar{\mathbf{X}} \in S_2$ *then* $\bar{\mathbf{X}} \in S_0$.

Proof. Assume $\bar{\mathbf{X}} = (x_1, x_2, x_3) \in S_2$, that is, $x_i(t) \in L^2$, $i = 1, 2, 3$. From (1) it follows that $x_i' \in L^2$, and then $x_i x_i' \in L^1$, $i = 1, 2, 3$. Next from $x_i^2 |_0^t = 2 \int_0^t x_i x_i'$ we obtain that $x_i(t)$ is bounded on $[0, \infty)$, $i = 1, 2, 3$. Again from (1) it follows that x_i' is bounded on $[0, \infty)$, $i = 1, 2, 3$. Finally from Lemma 3 if $x_i \in L^2$ and x_i' is bounded, then $\lim_{t\to\infty} x_i(t) = 0$, $i = 1, 2, 3$. Thus $\bar{\mathbf{X}} \in S_0$. $\blacksquare$

From Propositions 2 and 3 the following proposition follows immediately.

PROPOSITION 4. *If* $(\alpha - \gamma)' + 2\beta \geqslant k > 0$ $(\leqslant -k < 0)$ *then for* $\bar{\mathbf{X}} \in S$ *we have* $l(\bar{\mathbf{X}}) > -\infty$ $(l(\bar{\mathbf{X}}) < +\infty) \Leftrightarrow \bar{\mathbf{X}} \in S_2 \Leftrightarrow \bar{\mathbf{X}} \in S_0 \Leftrightarrow l(\bar{\mathbf{X}}) = 0$.

PROPOSITION 5. *If* $(\alpha - \gamma)' + 2\beta \geqslant k > 0$ $(\leqslant -k < 0)$ *then* $S_0 = S_+$ $(S_0 = S_-)$. *Then from Proposition* 4 *it follows that* $S_0 = S_2 = S_+$ $(S_0 = S_2 = S_-)$.

Proof. We assume $\bar{\mathbf{X}}$ nontrivial. Then if $(\alpha - \gamma)' + 2\beta \geqslant k > 0$ $(\leqslant -k < 0)$, $V_{\bar{\mathbf{X}}}(t)$ is strictly decreasing (strictly increasing). If now $\bar{\mathbf{X}} \in S_0$ then $l(\bar{\mathbf{X}}) = 0$, and we must have $V_{\bar{\mathbf{X}}}(t) > 0$ $(V_{\bar{\mathbf{X}}}(t) < 0)$ $\forall t \in [0, \infty)$. Then $\bar{\mathbf{X}} \in S_+$ (S_-). Assume next that $\bar{\mathbf{X}} \in S_+$ (S_-); then $V_{\bar{\mathbf{X}}}(t) \geqslant 0$ $(V_{\bar{\mathbf{X}}}(t) \leqslant 0)$. Since $l(\bar{\mathbf{X}})$ exists we must have $l(\bar{\mathbf{X}}) \geqslant 0$ $(l(\bar{\mathbf{X}}) \leqslant 0)$. From Proposition 4 we then have $l(\bar{\mathbf{X}}) = 0$. Thus S_+ $(S_-) \subset S_0$. This finishes the proof. $\blacksquare$

PROPOSITION 6. *If* $(\alpha - \gamma)' + 2\beta \geqslant k > 0$ *and if* $\bar{\mathbf{X}} \in S - S_0$, *then there exists a* $t_0 \geqslant 0$ *such that* $x_1(t) \neq 0$ $\forall t \geqslant t_0$. *If* $(\alpha - \gamma)' + 2\beta \leqslant -k < 0$ *and if* $\bar{\mathbf{X}} \in S_0 - \mathbf{O}$, *then* $x_1(t) \neq 0$ $\forall t \in [0, \infty)$.

Proof. Let $(\alpha - \gamma)' + 2\beta \geqslant k > 0$ and $\bar{\mathbf{X}} \in S - S_0 = S - S_+$. Then there exists a $t_0 \geqslant 0$ such that $V_{\bar{\mathbf{X}}}(t_0) < 0$. Since $V_{\bar{\mathbf{X}}}(t)$ is strictly decreasing we have that $V_{\bar{\mathbf{X}}}(t) < 0\ \forall t \in [t_0, \infty)$. Then $x_1(t) \neq 0\ \forall t \in [t_0, \infty)$, since if $x_1(t^*) = 0$ for some $t^* \in (t_0, \infty)$ we would have $0 \leqslant x_2^2(t^*) = V_{\bar{\mathbf{X}}}(t^*) < 0$, which is a contradiction. Assume now that $(\alpha - \gamma)' + 2\beta \leqslant -k < 0$ and that $\bar{\mathbf{X}} \in S_0 - \mathbf{O} \subset S_-$. Thus $V_{\bar{\mathbf{X}}}(t) < 0\ \forall t \in [0, \infty)$. Then if there exists a t^* such that $x_1(t^*) = 0$, we have $V_{\bar{\mathbf{X}}}(t^*) = x_2^2(t^*) \geqslant 0$. Contradiction, and $x_1(t) \neq 0\ \forall t \in [0, \infty)$. ∎

Next we note that for a function ϕ: $[0, \infty) \to \mathbb{R}^3$ either $\lim_{t\to\infty} \|\phi(t)\| = +\infty$ or ϕ has a limit point, that is, there exists a sequence $\{tn\}$, $tn \to +\infty$ $n \to \infty$, such that $\lim_{t\to\infty} \bar{\mathbf{X}}(tn) = p \in \mathbb{R}^3$.

PROPOSITION 7. *If* $(\alpha - \gamma)' + 2\beta \geqslant k > 0$ $(\leqslant -k < 0)$, *then* (i) $\bar{\mathbf{X}} \in S_0 \Leftrightarrow \bar{\mathbf{X}}$ *is bounded* $\Leftrightarrow \bar{\mathbf{X}}$ *has a limit point;* (ii) $\bar{\mathbf{X}} \in S - S_0 \Leftrightarrow \lim_{t\to\infty} \|\bar{\mathbf{X}}(t)\| = +\infty$.

Proof. (i) We only prove that if $\bar{\mathbf{X}}$ has a limit point then $\bar{\mathbf{X}} \in S_0$. If $\bar{\mathbf{X}}$ has a limit point then $l(\bar{\mathbf{X}}) > -\infty$ $(< +\infty)$ and from Proposition 4, $\bar{\mathbf{X}} \in S_0$. (ii) It follows from (i). ∎

PROPOSITION 8. (i) *If* $(\alpha - \gamma)' + 2\beta \geqslant k > 0$ *then* $\dim S_0 = 2$; (ii) *If* $(\alpha - \gamma)' + 2\beta \leqslant -k < 0$ *then* $\dim S_0 = 1$.

Proof. (i) Let $\bar{\mathbf{X}}_1, \bar{\mathbf{X}}_2, \bar{\mathbf{X}}_3$ be a basis for S such that $\bar{\mathbf{X}}_3(0) = (0, 0, 1)$. From $V_{\bar{\mathbf{X}}_3}(0) = 0$ it follows that $\bar{\mathbf{X}}_3 \notin S_+ = S_0$. Thus $S - S_0 \neq \phi$ and $\dim S_0 \leqslant 2$. Next we show the existence of two linearly independent elements in S_0. Let us denote by $H \subset \mathbb{R}^3$ the hyperplane $x_1 = 0$. We note that if $\bar{\mathbf{X}} \in S$ is such that $\bar{\mathbf{X}}(t_0) \in H$ then $V_{\bar{\mathbf{X}}}(t_0) = x_2^2(t_0) \geqslant 0$. Now for each integer $m \geqslant 1$ it is easy to see the existence of $\alpha_{m1}, \alpha_{m3}, \beta_{m2}, \beta_{m3} \in \mathbb{R}$ such that $\alpha_{m1}^2 + \alpha_{m3}^2 = \beta_{m2}^2 + \beta_{m3}^2 = 1$ and such that the elements $Z_{m1}, Z_{m2} \in S$ defined by $Z_{m1} = \alpha_{m1}\bar{\mathbf{X}}_1 + \alpha_{m3}\bar{\mathbf{X}}_3$, $Z_{m2} = \beta_{m2}\bar{\mathbf{X}}_2 + \beta_{m3}\bar{\mathbf{X}}_3$ satisfy $Z_{m1}(m)$, $Z_{m2}(m) \in H$. Then $V_{Z_{mi}}(m) \geqslant 0$, for $m = 1, 2,\ldots$, and $i = 1, 2$. Since $V_{\bar{\mathbf{X}}}$ is strictly decreasing for $\bar{\mathbf{X}}$ nontrivial, it follows that $V_{Z_{mi}}(t) > 0$ for $t \in [0, m)$, $m = 1, 2,\ldots$, $i = 1, 2$. From a classical argument we can assume that $Z_{mi}(t) \to Z_i(t)$ as $m \to \infty$, $i = 1, 2$, $t \geqslant 0$, where Z_1 and Z_2 are two nontrivial elements of S. Then $V_{Z_i}(t) \geqslant 0$, $t \geqslant 0$, $i = 1, 2$, and since V_{Z_1}, V_{Z_2} are strictly decreasing, we obtain $V_{Z_i}(t) > 0$ for $t \geqslant 0$, $i = 1, 2$. We claim that Z_1 and Z_2 are linearly independent. Assume they are not. Then $Z_i = \pm\bar{\mathbf{X}}_3$, $i = 1, 2$. This implies $V_{Z_i}(t) = V_{\pm\bar{\mathbf{X}}_3}(t) = V_{\bar{\mathbf{X}}_3}(t) < 0$, $\forall t \geqslant 0$. Contradiction. This finishes the proof of (i).

(ii) We first prove that $\dim S_0 \leqslant 1$. Let us define by $P = \{\bar{\mathbf{X}} \in S \mid x_1(0) = 0\}$; then P is a two-dimensional subspace of S, such that if $\bar{\mathbf{X}} = (x_1, x_2, x_3) \in P$, then $V_{\bar{\mathbf{X}}}(0) = x_2^2(0) \geqslant 0$. Since $S_0 = S_-$ we have that $P \cap S_0 = \{0\}$. Then $\dim S_0 \leqslant 1$. Next let $\bar{\mathbf{X}}_1, \bar{\mathbf{X}}_2, \bar{\mathbf{X}}_3$ be a basis for S such that $\bar{\mathbf{X}}_2, \bar{\mathbf{X}}_3$ are also a basis for P. For each integer $m \geqslant 1$ let $\alpha_{m1}, \alpha_{m2}, \alpha_{m3}$ be numbers such that $\sum_{i=1}^3 \alpha_{mi}^2 = 1$ and such that $Z_m = \sum_{i=1}^3 \alpha_{mi}\bar{\mathbf{X}}_i \in S$ satisfies $Z_m(m) \in \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = x_2 = 0\}$. Then $V_{Z_m}(m) = 0$ and hence

$V_{Z_m}(t) < 0$ for $t \in [0, m)$. Again from a classical argument we can assume that $Z_m(t) \to Z(t)$ as $m \to \infty$, $t \geqslant 0$, where $Z \in S$ and is nontrivial. Then $Z \in S_- = S_0$ and $\dim S_0 \geqslant 1$. We conclude $\dim S_0 = 1$. ∎

The next two theorems follow from the propositions we have proved.

THEOREM 1. *If* $(\alpha - \gamma)' + 2\beta \geqslant k > 0$ *in Equation* (1), *then*

(i) $S_2 = S_+ = S_0$.

(ii) *If* $\bar{\mathbf{X}} = (x_1, x_2, x_3) \in S - S_0$, *then there exists a* $t_0 \geqslant 0$ *such that* $x_1(t) \neq 0$, $\forall t \geqslant t_0$.

(iii) $\bar{\mathbf{X}} \in S_0$ *if and only if* $x_1(t)$ *is bounded*, $t \in [0, \infty)$.

(iv) *If* $\bar{\mathbf{X}} \notin S_0$ *then* $\lim_{t\to\infty}(x_1^2 + x_2^2 + x_3^2) = +\infty$.

(v) $\dim S_0 = 2$.

THEOREM 2. *If* $(\alpha - \gamma)' + 2\beta \leqslant -k < 0$ *in Equation* (1), *then*

(i) $S_0 = S_2 = S_-$.

(ii) *If* $\bar{\mathbf{X}} = (x_1, x_2, x_3) \in S_0$, $\bar{\mathbf{X}}$ *nontrivial, then* $x_1(t) \neq 0$ $\forall t \in [0, \infty)$.

(iii) $\bar{\mathbf{X}} \in S_0$ *if and only if* $x_1(t)$ *is bounded*, $t \in [0, \infty)$.

(iv) *If* $\bar{\mathbf{X}} \notin S_0$ *then* $\lim_{t\to\infty}(x_1^2 + x_2^2 + x_3^2) = +\infty$.

(v) $\dim S_0 = 1$.

3. APPLICATION TO EQUATION (5)

In this section we will apply Theorems 1 and 2 to Equation (5). For this equation we assume that q is continuous and that p is C^1 and bounded on $[0, \infty)$. We denote by T the solution space of (5) and by

$$T_0 = \{y \in T \mid \lim_{t\to\infty} y^{(i)}(t) = 0, i = 0, 1, 2\}.$$

From Theorems 1 and 2 the following theorem can be deduced for Equation (5),

THEOREM 3. *Assume there exist bounded functions* α, β, γ: $[0, \infty) \to \mathbb{R}$, β *continuous*, α *and* γ *of class* C^1 *such that* $\alpha + \gamma = p$, $\alpha' + \beta = q$. *Then if* $2q - p' \geqslant k > 0$ *we have*

(i) $\dim T_0 = 2$,

(ii) *if* $y \notin T_0$ *then there exists a* $t_0 \geqslant 0$ *such that* $y(t) \neq 0$ $\forall t \geqslant t_0$.

If $p' - 2q \geqslant k > 0$ *we have*

(iii) $\dim T_0 = 1$,

(iv) *if* $y \in T_0$, *y nontrivial, then* $y(t) \neq 0$ $\forall t \geqslant 0$.

If now $2q - p' \geqslant k > 0$ *or* $p' - 2q \geqslant k > 0$ *we have*

(v) $T_0 = \{y \in T \mid y^{(i)} \in L^2, i = 0, 1, 2\}$,

(vi) $y \in T_0$ *if and only if* $y(t)$ *is bounded*, $t \in [0, \infty)$,

(vii) *if* $y \notin T_0$ *then* $\lim_{t\to\infty}[y^2 + y'^2 + (y'' - \alpha y)^2] = +\infty$.

Proof. From the hypothesis of the theorem it follows that if $2q - p' \geqslant k > 0$ ($p' - 2q \geqslant k > 0$), then $(\alpha - \gamma)' + 2\beta \geqslant k > 0$ ($\leqslant -k < 0$). Then the proof follows from the fact that the spaces T and S are isomorphic under the transformation $y \to (y, y', y'' - \alpha y)$ and from Theorems 1 and 2. ∎

Note 1. We point out that the hypothesis on the functions α, β, γ of Theorem 3 are satisfied, for instance, for each of the following (nonidentical) cases:

(1) q bounded ($\alpha = 0$, $\gamma = p$, $\beta = q$),

(2) $q - p'$ bounded ($\alpha = p$, $\gamma = 0$, $\beta = q - p'$),

(3) $2q - p'$ bounded ($\alpha = \gamma = p/2$, $2\beta = 2q - p'$).

4. Application to the Oscillatory Behavior of Equation (5)

In this section we study the oscillatory behavior of Equation (5), assuming that p, q: $[0, \infty) \to \mathbb{R}$ satisfy, p is C^1 and bounded, and q is continuous. Together with (5) we consider its formal adjoint,

$$z''' = p^*z' + q^*z, \tag{8}$$

where $p^* = p$ and $q^* = p' - q$. We also assume the hypothesis of Theorem 3, that is, there exist bounded functions α, β, γ: $[0, \infty) \to \mathbb{R}$, β continuous, α, γ of class C^1, such that $\alpha + \gamma = p$, $\alpha' + \beta = q$ is satisfied by Equation (5). We note that this is equivalent to saying the hypothesis is satisfied by Equation (8). In fact we just take $\alpha^* = \gamma$, $\gamma^* = \alpha$, and $\beta^* = -\beta$.

We define by T^* the solution space of (8) and by T_0^* the subspace, $T_0^* = \{z \in T^* \mid \lim_{t\to\infty} z^{(i)}(t) = 0, i = 0, 1, 2\}$. Thus T^* and T_0^* play for (8) the same role as T and T_0 do for (5).

We recall here that $f: [a, \infty) \to \mathbb{R}$, f nontrivial, is said to be oscillatory if $\sup\{t \geqslant a \mid f(t) = 0\} = +\infty$. We denote by θ (θ^*) the subset of T (T^*) formed by all the oscillatory solutions of (5) ((8)).

For $y \in T$ and $w \in T^*$ we define

$$[y, w] = wy'' - w'y' + (w'' - pw)\,y. \tag{9}$$

It is clear that $[y, w]$ is $\mathbb{R}$-bilinear and that $[y, w]$ is constant.

For each $w \in T^*$ we consider the linear operator $L_w\colon T \to \mathbb{R}$ defined by $L_w(y) = [y, w]$, clearly $L_w \neq \mathbf{O}$ if w is nontrivial.

For $w \in T^*$, nontrivial, we define

$$H(w) = \operatorname{Ker} L_w = \{y \in T \mid [y, w] = 0\}. \tag{10}$$

Then $H(w)$ is a two-dimensional subspace of T. Also if w_1 and w_2 are two nontrivial elements of T^*, then $H(w_1) = H(w_2)$ if and only if $w_2 = \lambda w_1$, where $\lambda \in \mathbb{R}$. If now $w \in T^*$ is such that $w(t) \neq 0$, $\forall t \geqslant t_0$, then clearly $H(w)$ coincides with the solution space of the following second-order O.D.E.,

$$y'' - \frac{w'}{w}\,y' + \frac{(w'' - pw)}{w}\,y = 0, \qquad \forall t \geqslant t_0. \tag{11}$$

Next we will consider separately the two cases: (1) $2q - p' \geqslant k > 0$ and (2) $p' - 2q \geqslant k > 0$.

(1) $2q - p' \geqslant k > 0$

In this case Equation (5) satisfies (i), (ii), (v), (vi), and (vii) of Theorem 3, and Equation (8) satisfies (iii), (iv), (v), (vi) and (vii) of Theorem 3.

PROPOSITION 9. *If $\theta \neq \phi$ then $\theta + \{0\} = T_0$.*

Proof. From Theorem 3(ii) we have $\theta \subset T_0$. To show that $T_0 - \{0\} \subset \theta$ let u_0, v_0 be a basis for T_0. Then $w_0 = u_0 v_0' - u_0' v_0 \in T^*$ and satisfies $\lim_{t\to\infty} w_0^{(i)}(t) = 0$, $i = 0, 1, 2$. Thus w_0 is a basis for T_0^* and $w_0(t) \neq 0\ \forall t \geqslant 0$. Then since u_0 and v_0 satisfy $[y, w_0] = 0$, we have that T_0 coincides with the solution space of (11), for $w = w_0$ and $t \geqslant 0$. Finally since $\theta \neq \phi$, $\theta \subset T_0$, we conclude that all the elements of $T_0 - \{0\}$ are oscillatory, that is, $T_0 - \{0\} \subset \theta$. ∎

Note 2. We note that from the proof of Proposition 9 it follows that T_0 coincides with the solution space of an O.D.E. of second order.

PROPOSITION 10. *If $\theta = T_0 - \{0\}$ then $\theta^* = T^* - T_0^*$.*

Proof. Let $w \in T^* - T_0^*$. Then $H(w) \cap T_0$ is a line of T (otherwise $w \in T_0^*$). Assume next that w is not oscillatory, that is, $w(t) \neq 0\ \forall t \geqslant t_0$. Then the elements of $H(w)$ will satisfy (11) for $t \geqslant t_0$. Let $u \in H(w) \cap T_0$, u nontrivial. Then $u \in \theta$ and hence all the elements of the solution space of (11) are oscillatory.

It follows that $H(w) \subset \theta + \{0\} = T_0$. But then $H(w) \cap T_0$ cannot be a line of T. Contradiction. Thus $T^* - T_0^* \subset \theta^*$. On the other hand, from Theorem 3(iv) we have $\theta^* \subset T^* - T_0^*$. We conclude that $\theta^* = T^* - T_0^*$. ∎

PROPOSITION 11. *If $\theta^* \neq \phi$ then $\theta \neq \phi$.*

Proof. Let $w \in \theta^*$; then $H(w) \cap T_0$ is a line of T. We will show that if $u \in H(w) \cap (T_0 - \{0\})$, then u is oscillatory. Let us define by

$$H^*(u) = \{z \in T^* \mid [u, z] = 0\}, \tag{12}$$

then $H^*(u)$ is a two-dimensional subspace of T^* and $w \in H^*(u)$.

Next let $w_0 \in T_0^* - \{0\}$, then $\lim_{t\to\infty} w_0^{(i)}(t) = 0$, $i = 0, 1, 2$ and $w_0(t) \neq 0$ $\forall t \geqslant 0$. Also since $u \in T_0$ we have that $\lim_{t\to\infty} u^{(i)}(t) = 0$, $i = 0, 1, 2$. From here and since $[u, w_0] =$ constant, we have $[u, w_0] = 0$. Then $w_0 \in H^*(u)$. If we now assume that u is not oscillatory, say $u(t) \neq 0$ for $t \geqslant t_0$, $t_0 \geqslant 0$, then the elements of $H^*(u)$ would be solutions of the second-order O.D.E.,

$$\frac{1}{u}[u, z] = z'' - \frac{u'}{u} z' + \frac{u'' - pu}{u} z = 0,. \tag{13}$$

for $t \geqslant t_0$. Since $w(t)$ is an oscillatory solution of (13), for $t \geqslant t_0$ then all the solutions of (13) should be oscillatory. But this is a contradiction, since $w_0(t)$ is also a solution of (13), for $t \geqslant t_0$ and is not oscillatory. Then u is oscillatory. This finishes the proof. ∎

From Propositions 9, 10, 11 we have

THEOREM 4. *Let us consider Equation* (5) *where p, q: $[0, \infty) \to \mathbb{R}$ are such that p is C^1 and bounded and q is continuous. Assume there exist bounded functions α, β, γ: $[0, \infty) \to \mathbb{R}$, β continuous, α, γ of class C^1 such that $\alpha + \gamma = p$, $\alpha' + \beta = q$; then if $2q - p' \geqslant k > 0$, the following propositions are equivalent:* (a) $\theta \neq \phi$; (b) $\theta + \{0\} = T_0$; (c) $\theta^* = T^* - T_0^*$; (d) $\theta^* \neq \phi$.

Proof. (a) ⇒ (b) by Proposition 9; (b) ⇒ (c) by Proposition 10; (c) ⇒ (d) trivially since $T_0^* \neq T^*$; and (d) ⇒ (a) by Proposition 11. ∎

(2) $p' - 2q \geqslant k > 0$

In this case we have the following:

THEOREM 5. *If in Theorem* 4 *we have $p' - 2q \geqslant k > 0$ instead of $2q - p' \geqslant k > 0$, then the following propositions are equivalent:* (a) $\theta \neq \phi$; (b) $\theta = T - T_0$; (c) $\theta^* + \{0\} = T_0^*$; (d) $\theta^* \neq \phi$.

Proof. We have that $p' - 2q \geqslant k > 0$ implies $2q^* - p^{*\prime} \geqslant k > 0$. Thus Equation (8) satisfies the conditions of Theorem 4 and the proof follows from this fact. ▮

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