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Relative morsification theory

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Dedicated to Dirk Siersma on the occasion of his 60th birthday

Abstract

In this paper we develop a Morsification Theory for holomorphic functions defining a singularity of finite codimension with respect to an ideal, which recovers most previously known morsification results for non-isolated singularities and generalise them to a much wider context. We also show that deforming functions of finite codimension with respect to an ideal within the same ideal respects the Milnor fibration. Furthermore we present some applications of the theory: we introduce new numerical invariants for non-isolated singularities, which explain various aspects of the deformation of functions within an ideal; we define generalisations of the bifurcation variety in the versal unfolding of isolated singularities; applications of the theory to the topological study of the Milnor fibration of non-isolated singularities are presented. Using intersection theory in a generalised jet-space we show how to interpret the newly defined invariants as certain intersection multiplicities; finally, we characterise which invariants can be interpreted as intersection multiplicities in the above mentioned generalised jet space.

MSC: 32S30; 32S55

Keywords: Non-isolated singularities; Morsifications; Milnor fibration

0. Introduction

Let $\mathcal{O}_{\mathbb{C}^n,O}$ be the ring of germs of holomorphic functions at the origin O of $\mathbb{C}^n$. Two germs f and g are R-equivalent (right-equivalent) if there exists a germ of biholomorphism $\phi:(\mathbb{C}^n,O)\to(\mathbb{C}^n,O)$ such that $f\circ\phi=g$. One of the main aims of singularity theory is the classification of

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germs of holomorphic functions up to R-equivalence; this includes giving normal forms for each of the equivalence classes and invariants to decide whether two functions belong to the same class, but also studying the adjacencies (hierarchy) between equivalence classes (a class C is adjacent to a class C' if every function of C' can be deformed into a function of C by arbitrarily small deformation). When Arnold worked out the beginning of the classification of isolated singularities he observed that infinite series of classes of singularities occur (like A_k or D_k), it appeared clear that the series where associated with non-isolated singularities, and that the hierarchy of the series reflects the hierarchy of non-isolated singularities; therefore the study of classification and hierarchy of non-isolated singularities is interesting not just by itself, but also in connection with the study of series of isolated singularities.

Many useful invariants for the classification and adjacency problems stem from the study of the Milnor fibration associated to a holomorphic function (vanishing (co)-homology, monodromy, intersection form, spectrum, homotopy type of the Milnor fibre, etc). This is quite well understood in case that the function has an isolated singularity, but when non-isolated singularities are present our knowledge is still very limited. See [23] for a recent survey of known results and open questions.

A fruitful way of studying isolated singularities is the so called morsification method: Denote by B_ϵ the closed ball of radius ϵ centered at the origin of $\mathbb{C}^n$; denote by D_η the closed disk of radius η centered at 0 in $\mathbb{C}$. Given a holomorphic germ $f \in \mathcal{O}_{\mathbb{C}^n,O}$ with an isolated singularity, let ϵ, η be a pair of radii for which the Milnor fibration of f is defined; then, any (small enough) generic perturbation g, has, as critical locus inside B_ϵ, as many Morse-type singularities as its Milnor number; moreover the Milnor fibration of f is preserved by the perturbation in the following sense: the restriction

$$g : B_\epsilon \cap g^{-1}(\partial D_\eta) \to \partial D_\eta$$

is C^∞-equivalent to the Milnor fibration of f. This gives a powerful method to study the Milnor fibration: for example it allows to compute the homotopy type of the Milnor fibre, and to relate the monodromy with the intersection form via Picard-Lefschetz theory.

The goal of this paper is to generalise this method to a wide class of non-isolated singularities. However, an arbitrary small perturbation of a function f with non-isolated singularities does not preserve the Milnor fibration in the sense explained above; therefore, we need to restrict the type of deformations that we will allow. The main idea is allow only deformations of f within a suitable ideal I of $\mathcal{O}_{\mathbb{C}^n,O}$ (which in some cases is the ideal of functions that are singular where f is singular, with the same generic transversal type). This point of view has been used in [10,16–21,26] to prove generalised morsification theorems and study the Milnor fibration of functions with smooth 1-dimensional critical locus and simple transversal type, or with an i.c.i.s. (isolated complete intersection singularity) of dimension at most 2 as critical locus and transversal type A_1. Recently the morsification theorem has been generalised in [3] for critical locus an i.c.i.s. of any dimension with transversal type A_1. In all these works the morsification theorem and the preservation of the Milnor fibration are proved exploiting special properties of the ideal considered (low dimensionality of its zero set or defining a complete intersection), which do not generalise easily. In all the cases the functions that can be morsified are precisely the functions of finite extended codimension with respect to the ideal I in the sense of [18] (see Section 1 for a definition). Using a new conceptual approach, in this paper we generalise the morsification and preservation of Milnor fibration for functions of finite extended codimension with respect to any ideal of $\mathcal{O}_{\mathbb{C}^n,O}$. Moreover we introduce and study new numerical invariants for non-isolated singularities. Below we summarise informally the main results of the paper.

In Section 2 we establish the preservation of the Milnor fibration: given *any* ideal I of $\mathcal{O}_{\mathbb{C}^n,O}$ and a function f of finite extended codimension with respect to I, we prove that any small enough perturbation of f within the ideal preserves the Milnor fibration (see Theorem 2.2).

The main and most difficult result of this paper is a relative morsification theorem (see Theorem 8.6) that roughly states the following: given any function f of finite extended codimension with respect to an ideal I, a generic small perturbation g of f within the ideal only has singularities that are simplest with respect to I in a certain natural sense. Moreover, in a small neighbourhood of the origin there are finitely many points in which g has positive extended codimension, with the property that the germ of g at any of them is unsplittable, in the sense that it cannot be split into several points of positive codimension by further perturbation. Furthermore, the points of positive extended codimension that are outside $V(I)$ are always Morse points of g. The proof uses the theory developed in Sections 1 and 3–8. We now summarise the content of these sections. Fix an ideal I.

Section 1 is preliminary. Some known definitions and results are recalled. Among them is the concept of extended codimension of a function of I with respect to I. This is an invariant that generalises the Milnor number to our setting. Indeed, the functions of finite extended codimension with respect to I will be those that can be studied via the relative morsification theorem (as isolated singularities are the functions whose properties can be studied via the usual morsification theory). In particular, by a theorem of Pellikaan (see Theorem 1.12), any function of finite extended codimension has an unfolding which is versal within I. On the other hand the extended codimension does not have all the good properties that the Milnor number has. The Milnor number is conservative in the following sense: let f be a function with finite Milnor number at O; there is a small neighbourhood U of the origin such that, given any deformation $f+tg$, if t is small enough, the sum of the Milnor numbers of $f+tg$ at the points of U equals the Milnor number of f at the origin. We show with an example that the extended codimension with respect to an ideal is not conservative in general.

Consider a coherent ideal sheaf $\tilde{I}$ defined in a neighbourhood U of the origin such that $\tilde{I}_O = I$, define

$$J^{\infty}(U,\tilde{I}) := \coprod_{x\in U} \tilde{I}_x. \tag{1}$$

In Section 3 we give a infinite-dimensional analytic structure to $J^{\infty}(U,\tilde{I})$, viewing it as a generalised ∞-jet space associated to $\tilde{I}$. For this we view $J^{\infty}(U,\tilde{I})$ as a the projective limit of

$$J^{m}(U,\tilde{I}) := \coprod_{x\in U} \tilde{I}_x/(\mathbf{m}^{m+1}\cap \tilde{I}_x),$$

when $m \geqslant 0$, and give a certain finite-dimensional generalised analytic structure to each $J^{m}(U,\tilde{I})$. In particular we endow $J^{\infty}(U,\tilde{I})$ with a topology and define such concepts as finite-determined set (the finite determinacy of a set implies that it can be understood as a subset of a finite-dimensional analytic space), analytic set, irreducibility, smoothness and codimension in $J^{\infty}(U,\tilde{I})$ of a closed analytic subset. We prove the existence of decompositions of analytic subsets in irreducible components.

In Section 4 we consider the filtration of $J^{\infty}(U,\tilde{I})$ by sets consisting of germs of given extended codimension. We prove that the levels of the filtration are finitely-determined closed analytic subsets of $J^{\infty}(U,\tilde{I})$. Although this is more or less clear from the intuitive point of view, the proof gets rather technical. We also give a lower bound for the codimension in $J^{\infty}(U,\tilde{I})$ of the set of germs of given finite extended codimension.

In Section 5 we stratify the space $J^{\infty}(U,\tilde{I})$ in locally closed analytic strata containing germs of the same topological type. This is done by combining the results of [4] with the theory developed in Section 3.

In Section 6 the concept of Whitney regularity is defined for stratifications of $J^{\infty}(U,\tilde{I})$. Given any locally finite partition of $J^{\infty}(U,\tilde{I})$ by locally closed analytic subsets, the existence of a canonical Whitney stratification refining it is proved.

In Section 7 we define analytic mappings from an analytic set to $J^{\infty}(U,\tilde{I})$. The concept of transversality of a mapping to a submanifold of $J^{\infty}(U,\tilde{I})$ is introduced. The parametric transversality theorem is extended to our setting. It is shown that versality implies transversality in the following sense: let $f\in I$ be a function of finite extended codimension and $F:(\mathbb{C}^n,O)\times(\mathbb{C}^r,O)\to\mathbb{C}$ be a versal unfolding of f within I, then mapping

$$\rho_F:(\mathbb{C}^n,O)\times(\mathbb{C}^r,O)\to J^{\infty}(U,\tilde{I})$$

assigning to (x,s) the germ of $F_{|\mathbb{C}^n\times\{s\}}$ at x is analytic and transversal to any submanifold of $J^{\infty}(U,\tilde{I})$ which is invariant by the natural action of holomorphic diffeomorphims preserving $\tilde{I}$.

Finally, in Section 8 the relative morsification theorem (see Theorem 8.6) is stated. A certain Whitney stratification of $J^{\infty}(U,\tilde{I})$ (which is built up from the filtration by ascending extended codimension and the stratification considered in Section 5 using the geometric tools developed in Sections 3 and 6) is needed. The proof is a transversality argument using the results developed in Section 7. As a by-product of our method we introduce generalisations of the bifurcation variety in the base of a versal unfolding.

The rest of the paper presents applications of our theory and several examples.

In Section 9 we present two immediate applications of the morsification theorem. In the first application we define new numerical invariants for functions of finite codimension with respect to an ideal, namely the *splitting function*, the *corrected extended codimension* and the *Morse number*. All of them are conservative in the sense explained above. The corrected extended codimension is close to the extended codimension, and could be thought as a conservative version of it. The splitting function is a finer invariant. A consequence of the relative morsification theorem is that the singularities outside $V(I)$ of a generic deformation within I of a function of finite extended codimension form a finite set of Morse points; the Morse number is the cardinality of this set. The second application, based also on the results of Section 2, shows how to study the topology of a function of finite extended codimension using a morsification. In particular, homology splitting and bouquet decomposition theorems are presented (see Theorem 9.3).

In Section 10 we study further numerical invariants. Given any $f\in I$ we have a jet-extension mapping $\rho_f:(\mathbb{C}^n,O)\to J^{\infty}(U,\tilde{I})$. We develop a bit of intersection theory on the generalised jet space $J^{\infty}(U,\tilde{I})$ so that the intersection number at the origin of the mapping ρ_f with any n-codimensional subvariety of $J^{\infty}(U,\tilde{I})$ is well defined and has some natural properties. This will enable us to interpret the numerical invariants introduced above as intersection numbers, and give a formula of them in terms of dimensions of certain complex vector spaces (this becomes more explicit in the case of the Morse number). Using intersection theory we are able to prove that any numerical invariant which is conservative and satisfies two other natural properties is actually a linear combination of intersection multiplicities with n-codimensional subvarieties of $J^{\infty}(U,\tilde{I})$ which are invariant by the natural action of holomorphic diffeomorphims preserving $\tilde{I}$ (see Theorem 10.13). A classification of such subvarieties yields in many examples in Section 11 an expression of any conservative invariant as a linear combination of well known invariants (for example in terms of the Morse number and number of D_{∞}-points in the case of isolated-line singularities; see [20]).

In Section 11 we analyse several applications and examples. In particular we point out previous morsification theorems that are now consequences of this theory. Among them is the morsification theorem for singularities whose critical locus is an i.c.i.s. and have generic transversal type A_1; we show how to extend this theorem to the case in which the critical locus is not necessarily a complete intersection (see Proposition 11.3).

1. Functions of finite codimension with respect to an ideal

Let $\mathbf{m}$ be the maximal ideal of $\mathcal{O}_{\mathbb{C}^n,O}$, let $x_1,\ldots,x_n$ be coordinate functions for $\mathbb{C}^n_O$. The module of germs of vector fields at the origin O of $\mathbb{C}^n$ will be denoted by Θ; then $\mathbf{m}\Theta$ are the vector fields vanishing at the origin. Denote by $\mathscr{D}$ the group of germs of holomorphic diffeomorphims of $\mathbb{C}^n$ fixing the origin, and by $\mathscr{D}_e$ the set of germs of holomorphic diffeomorphims at the origin that not necessarily fix it (we have no group structure because composition need not be defined). Let $I \subset \mathcal{O}_{\mathbb{C}^n,O}$ be an ideal; given U, a small enough neighbourhood of the origin, there is a coherent sheaf of ideals $\tilde{I}$ whose stalk at O is I. Following [18] we define $\mathscr{D}_I$ and $\mathscr{D}_{I,e}$ to be, respectively, the subgroup of $\mathscr{D}$ and the subset of $\mathscr{D}_e$ of elements preserving the ideal:

Definition 1.1. Define $\mathscr{D}_{I,e}$ as the set of all $\phi \in \mathscr{D}_e$ that have a representative $\phi : V \to W$, with V and W open subsets in U and $O \in V$, such that

$$\phi^*(\Gamma(W,\tilde{I})) = \Gamma(V,\tilde{I}).$$

Define $\mathscr{D}_I = \mathscr{D}_{I,e} \cap \mathscr{D}$.

Clearly, the action of $\mathscr{D}$ on $\mathcal{O}_{\mathbb{C}^n,O}$ by composition on the right restricts to an action $\sigma_I : I \times \mathscr{D}_I \to I$. Given $f \in I$ we denote by $\mathrm{Orb}(f)$ its orbit by σ_I.

Let ϕ_t be a 1-parameter family of holomorphic diffeomorphims of $\mathscr{D}_e$ smoothly depending on t, such that $\phi_0 = \mathrm{Id}_{\mathbb{C}^n}$; let $\phi_{1,t},\ldots,\phi_{n,t}$ be its components; consider $f \in \mathcal{O}_{\mathbb{C}^n,O}$. The chain rule gives:

$$\left.\frac{\mathrm{d} f\circ\phi_t}{\mathrm{d}t}\right|_{t=0} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \left.\frac{\mathrm{d}\phi_{i,t}}{\mathrm{d}t}\right|_{t=0} = X(f), \tag{2}$$

where X is the holomorphic vector field given by $X = \sum_{i=1}^{n} \mathrm{d}\phi_{i,t}/\mathrm{d}t_{|t=0}\partial/\partial x_i$. If $\phi_t \in \mathscr{D}$ for any t, then $X \in \mathbf{m}\cap\Theta$. If $\phi_t \in \mathscr{D}_{I,e}$ for any t, then $X(I) \subset I$. Define $\Theta_{I,e}$ by the formula $\Theta_{I,e} := \{X \in \Theta : X(I) \subset I\}$. If $\phi_t \in \mathscr{D}_I$ for any t then $X \in \mathbf{m} \cap \Theta_{I,e}$; define $\Theta_I := \mathbf{m} \cap \Theta_{I,e}$.

Conversely, integration associates to any $X \in \Theta$ a 1-parameter flow ϕ_t of holomorphic diffeomorphims of $\mathscr{D}_e$, with $\phi_0 = \mathrm{Id}_{\mathbb{C}^n}$, such that if $X \in \Theta_{I,e}$ then $\phi_t \in \mathscr{D}_{I,e}$, and if $X \in \Theta_I$ then $\phi_t \in \mathscr{D}_I$.

Given a family $\phi_t \subset \mathscr{D}_I$ as above, and $f \in I$, we may regard the family of functions $f \circ \phi_t$ as a smooth path in $\mathrm{Orb}(f)$. This motivates the following:

Definition 1.2. Consider $f \in I$, define the *tangent space* and the *extended tangent space* at f to its orbit respectively by

$$\tau_I(f) := \Theta_I(f) \quad \tau_{I,e} := \Theta_{I,e}(f).$$

Moreover we define the (possibly infinite) *I-codimension* and *extended I-codimension* respectively by

$$c_I(f) := \dim_{\mathbb{C}} \frac{I}{\tau_I(f)} \quad c_{I,e}(f) := \dim_{\mathbb{C}} \frac{I}{\tau_{I,e}(f)}.$$

Notice that both $\tau_I(f)$ and $\tau_{I,e}(f)$ are ideals of $\mathcal{O}_{\mathbb{C}^n,O}$. It is easy to see (cf. [18]) that the I-codimension is finite if and only if the extended I-codimension is so.

Our setting is more general that the one studied by Pellikaan in [17,18]. There the ideal I is asked to be the primitive of another ideal $I' \subset \mathcal{O}_{\mathbb{C}^n,O}$: consider an ideal $I' \subset \mathcal{O}_{\mathbb{C}^n,O}$, we define $\int I'$, the *primitive ideal of* I', as

$$\int I' := \{f \in \mathcal{O}_{\mathbb{C}^n,O} : (f) + J_f \subset I'\},$$

where $J_f := \{\partial f/\partial x_1, \ldots, \partial f/\partial x_n\}$ is the Jacobian ideal of f. Heuristically $\int I'$ is the ideal of functions that vanish and "are singular at" the analytic space defined by I'. For many applications it is sufficient to work with the class of primitive ideals: for example in the study of non-isolated singularities with generic transversal type A_1. However the ideals considered by de Jong in [10] for the cases of line singularities with transversal types A_3, E_7, and E_6 (this last case for ambient space of dimension at least 4) are not primitives of any other ideals.

Observe that, when $I = \int I'$, there are two possibly different definitions of (extended) I-codimension: we can consider diffeomorphims that preserve I' (as Pellikaan does) instead of diffeomorphisms that preserve $\int I'$; hence we use the modules $\Theta_{I',e}$ and $\Theta_{I'}$ instead of $\Theta_{\int I',e}$ and $\Theta_{\int I'}$. It is easy to see that $\Theta_{I',e} \subset \Theta_{\int I',e}$, but the equality is not known in general, to the author's knowledge. This give rise to two, a priori different, morsification theories for the ideal $\int I$. The constructions and results of this paper are valid for either of them. Anyhow, the equality $\Theta_{I',e} = \Theta_{\int I',e}$ holds when I' is a radical ideal (it is easy to prove that any $X \in \Theta_{\int I',e}$ must be tangent to the analytic space $V(I')$ defined by I' at each of its smooth points; this implies that $X \in \Theta_{I',e}$), and in all the examples that we have considered.

Notation 1.3. Let $\mathscr{F}$ be a coherent sheaf on U. Denote by $\mathscr{F}_x$ its stalk at x; given any section $\varphi \in \Gamma(\mathscr{F}, U)$ denote by φ_x its germ at x.

Let $\tilde{\Theta}$ be the free $\mathcal{O}_U$-module of vector fields over U, and $\Theta_{\tilde{I},e}$ the coherent $\mathcal{O}_U$-module of vector fields preserving $\tilde{I}$. The stalk of $\Theta_{\tilde{I},e}$ at any $x \in U$ is the $\mathcal{O}_{U,x}$-module $\Theta_{\tilde{I}_x,e}$. Therefore, for any f holomorphic in U, we have that $\Theta_{\tilde{I},e}(f)$ is a coherent sheaf of ideals whose stalk at any $x \in U$ is

$\tau_{I_x,e}(f_x)$. Define the coherent $\mathscr{O}_U$-module

$$\mathscr{F} := \frac{\tilde{I}}{\Theta_{\tilde{I},e}(f)}. \tag{3}$$

Standard properties of coherent sheaves imply that $c_{I,e}(f) < \infty$ if and only if $\mathscr{F}$ is concentrated at the origin, i.e. $\mathscr{F}_x = 0$ for $x \neq O$.

Notation 1.4. Let $\pi : \mathbb{C}^n \times \mathbb{C}^r \to \mathbb{C}^r$ be the projection to the second factor; take any $s \in \mathbb{C}^r$. If $\mathscr{G}$ is a coherent analytic sheaf on $\mathbb{C}^n \times \mathbb{C}^r$ we denote by $\mathscr{G}_{|s}$ the pullback of $\mathscr{G}$ to $\pi^{-1}(s)$.

If F is an analytic function on $\mathbb{C}^n \times \mathbb{C}^r$ we denote by $F_{|s}$ the restriction of F to $\pi^{-1}(s)$. Therefore, $F_{|s,x}$ denotes the germ at x of the restriction of F to $\pi^{-1}(s)$. If X is an analytic subset of $\mathbb{C}^n \times \mathbb{C}^r$ we denote by X_s the fibre of X over s.

Definition 1.5. For $f \in I$, an *r-parametric I-unfolding of f* is a holomorphic germ $F : (\mathbb{C}^n \times \mathbb{C}^r, (O,O)) \to \mathbb{C}$ such that $F_{|O,O} = f$ and $F_{|s,O}$ belongs to I for any $s \in \mathbb{C}^r$. We denote by $I(r)$ the module formed by all the r-parametric I-unfoldings. Define $\Theta_{I,e}(r) := \pi^*\Theta_{I,e} = \mathscr{O}_{\mathbb{C}^n \times \mathbb{C}^r}\Theta_{I,e}$.

Lemma 1.6. *The following equalities hold*: $I(r) = \pi^* I = I\mathscr{O}_{\mathbb{C}^n \times \mathbb{C}^r,(O,O)}$.

Proof. The only non-trivial statement is that $I(r) \subset \pi^* I$. Consider the coordinates $x_1, \ldots, x_n$ for $\mathbb{C}^n$ and fix coordinates $s_1, \ldots, s_r$ for $\mathbb{C}^r$. Let $f_1, \ldots, f_k$ be a set of generators for I. Take $F \in I(r)$; we have to find $G_1, \ldots, G_k$, convergent power series in $x_1, \ldots, x_r, s_1, \ldots, s_r$, such that

$$F = \sum_{i=1}^{k} G_i f_i. \tag{4}$$

By Artin's approximation theorem it is enough to find formal power series G_i satisfying the last equation.

Express each G_i as $G_i = \sum g^i_{i_1,\ldots,i_n} x_1^{i_1} \ldots x_n^{i_n}$ and F as $F = \sum A_{j_1,\ldots,j_n} x_1^{j_1} \ldots x_n^{j_n}$ where each $g^i_{j_1,\ldots,j_n}$ and each $A_{j_1,\ldots,j_n}$ is a power series in $s_1, \ldots, s_k$; express each f_i as $f_i = \sum a^i_{j_1,\ldots,j_n} x_1^{j_1} \cdots x_n^{j_n}$ where each $a^i_{j_1,\ldots,j_n}$ is a complex number. For any positive integer N the truncation of Equality (4) to its N-jet with respect of $x_1, \ldots, x_n$ may be seen as a linear system whose variables are $\{g^i_{j_1,\ldots,j_n} : j_1 + \cdots + j_n \leqslant N\}$, whose coefficients are $\{a^i_{j_1,\ldots,j_n} : j_1 + \cdots + j_n \leqslant N\}$ (complex numbers), and whose independent terms are $\{A_{j_1,\ldots,j_n} : j_1 + \cdots + j_n \leqslant N\}$ (holomorphic functions in $s_1, \ldots, s_r$). The fact that F is a I-unfolding implies that for any value of $s_1, \ldots, s_r$ close enough to the origin the system has a solution; using this and the fact that the rank of the fundamental matrix of the system does not depend on $s_1, \ldots, s_r$ (the $a^i_{j_1,\ldots,j_n}$'s are complex numbers), we deduce that there exists a solution of the system depending holomorphically on $s_1, \ldots s_r$ in a neighbourhood of the origin of $\mathbb{C}^r$. This provides a solution for the truncation of Equality (4) to its N-jet. Applying Krull's intersection theorem we deduce the existence of formal solutions. □

Remark 1.7. In the last lemma we have shown the following statement: consider $f_1, \ldots, f_k$, $F \in \mathscr{O}_{\mathbb{C}^n \times \mathbb{C}^r,(O,O)}$ such that $f_1, \ldots, f_k$ are independent of $s_1, \ldots, s_r$; if for any $s \in \mathbb{C}^r$ close enough to the origin $F_{|s} \in (f_{1|s}, \ldots, f_{k|s})$ then $F \in (f_1, \ldots, f_k)$. The independence of the f_i's on $s_1, \ldots, s_r$ is needed, as the following example shows: $f(x,s) = xs^2$, $F(x,s) = xs$.

Given U and V, neighbourhoods of the origin of $\mathbb{C}^n$ and $\mathbb{C}^r$ respectively, we define the coherent $\mathcal{O}_{U\times V}$-modules $\tilde{I}(r)$ and $\Theta_{\tilde{I},e}(r)$ as $\tilde{I}(r) := \tilde{I}\mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^r}$ and $\Theta_{\tilde{I},e}\mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^r}$.

Lemma 1.8. *Consider a representative $F: U\times V \to \mathbb{C}$ of a r-parametric I-unfolding of a germ $f \in \int I$. Define a coherent $\mathcal{O}_{U\times V}$-module by the formula*

$$\mathcal{G} := \frac{\tilde{I}(r)}{(\Theta_{\tilde{I},e}(r))(F)}. \tag{5}$$

Then,

(1) *We have $\mathcal{G}_{(x,s)} = 0$ if and only if $c_{\tilde{I}_x,e}(F_{|s,x}) = 0$.*
(2) *If $c_{I,e}(f)=0$ then U and V can be shrunk enough so that $c_{\tilde{I}_x,e}(F_{|s,x})=0$ for any $(x,s)\in U\times V$.*
(3) *Let $Z \subset U\times V$ be the support of $\mathcal{G}$; let $p: U\times V \to V$ be the projection to the second factor; define $\varphi := p_{|Z}$. If $c_{I,e}(f) < \infty$ then we can shrink U and V so that φ is finite and $p_*\mathcal{G}$ is a coherent $\mathcal{O}_V$ module.*

Proof. Denote by $\mathbf{m}_s$ the maximal ideal of $\mathcal{O}_{\mathbb{C}^r,s}$. A standard commutative algebra argument shows

$$\frac{\mathcal{G}_{(x,s)}}{\mathbf{m}_s\mathcal{G}_{(x,s)}} \cong \frac{\tilde{I}_x}{\tau_{\tilde{I}_x,e}(F_{|s,x})}. \tag{6}$$

Therefore $\mathcal{G}_{(x,s)} = 0$ implies $c_{\tilde{I}_x,e}(F_{|s,x}) = 0$; conversely, if $c_{\tilde{I}_x,e}(F_{|s,x}) = 0$ then $\mathcal{G}_{(x,s)} = \mathbf{m}_s\mathcal{G}_{(x,s)}$, and hence $\mathcal{G}_{(x,s)} = 0$ by Nakayama's Lemma. We have shown the first assertion.

Suppose that $c_{I,e}(f)=0$; then, as $f = F_{|O,O}$ we have that $\mathcal{G}_{(O,O)}=0$; as the support of a coherent $\mathcal{O}_{U\times V}$-module is closed, the second assertion follows.

Suppose $c_{I,e}(f) < \infty$. Shrinking U we can assume that $\mathcal{F}$ is concentrated at the origin of $\mathbb{C}^n$. Therefore, by the previous assertions, we have that $\mathcal{G}_{(x,O)}=0$ for any $x\in U\setminus\{O\}$. By the projection lemma of [8, p. 62], we can shrink U and V so that the third assertion follows. □

A corollary of Lemma 1.8 is the upper semicontinuity of extended codimension:

Corollary 1.9. *Let $f\in I$ with $c_{I,e}(f) < \infty$; consider $F: U\times V \to \mathbb{C}$ a representative of a I-unfolding of f such that the second statement of Lemma* 1.8 *holds. Let φ be the mapping introduced in the last lemma. Then for s close enough to the origin,*

$$\sum_{x\in\varphi^{-1}(s)} c_{\tilde{I}_x,e}(F_{s,x}) \leqslant c_{I,e}(f). \tag{7}$$

Proof. After Lemma 1.8 we are reduced to proving that, for s close enough to the origin, $\dim_{\mathbb{C}}(\varphi_*\mathcal{G})\otimes(\mathcal{O}_{V,O}/\mathbf{m}_O) \geqslant \dim_{\mathbb{C}}(\varphi_*\mathcal{G})\otimes(\mathcal{O}_{V,s}/\mathbf{m}_s)$, which is standard since $\varphi_*\mathcal{G}$ is coherent. □

When $I = \mathcal{O}_{\mathbb{C}^n,O}$ then the extended codimension is equal to the Milnor number, in this case the inequality (7) becomes an equality. Although this happens in many other cases such as isolated line singularities (see [20,19]) and most of the examples that we have computed, the following example shows that the equality does not hold in general.

Example 1.10. Let $I=(x^2,y)\subset \mathcal{O}_{\mathbb{C}^2,O}$ (ideal of curves tangent to the x-axis); then $\Theta_{I,e}$ is generated by $x\partial/\partial x, y\partial/\partial x, x^2\partial/\partial y, y\partial/\partial y$. Consider

$$f(x,y)=y^2+x^3;$$

then $\tau_{I,e}(f)=(x^3,x^2y,y^2)$ and $c_{I,e}(f)=3$. Any unfolding $F:\mathbb{C}^2\times\mathbb{C}\to\mathbb{C}$ of f such that for any s the curve $F_{|s}=0$ is tangent to the x-axis at the origin is a I-unfolding. Choose an I-unfolding such that $F_{|s}=0$ is smooth at the origin; then one checks easily that $c_{I,e}(F_{|s})=0$. Therefore the only terms that can contribute to the left hand side of Inequality (7) are the extended codimensions of $F_{|s}$ at points x outside the support of I. As $I_x=\mathcal{O}_{\mathbb{C}^2,x}$ at these points, the extended codimension coincides with the Milnor number. As $\mu(f)=2$, Inequality (7) is strict in this case.

Let $F\in I(r)$, $G\in I(q)$ be I-unfoldings of $f\in I$. A morphism $\xi:F\to G$ of I-unfoldings is a pair (Φ,λ) consisting of holomorphic germs $\lambda:(\mathbb{C}^r,O)\to(\mathbb{C}^q,O)$ and $\Phi:(\mathbb{C}^n\times\mathbb{C}^r,(O,O))\to\mathbb{C}^n$ with the following properties:

(1) $\Phi_{|\mathbb{C}^n\times\{s\}}\in\mathcal{D}_{I,e}$ for any s,
(2) $\Phi_{|\mathbb{C}^n\times\{O\}}=\mathrm{Id}_{\mathbb{C}^n}$, and
(3) $G(\Phi(x,s),\lambda(s))=F$.

Definition 1.11. Let $f\in I$ and F an I-unfolding of it. We say that F is versal if for any other I-unfolding G of f there exists a morphism of I-unfoldings from G to F.

The following theorem was proved by Pellikaan in [18] as an application of the general results of Damon in [2]; although Pellikaan proves it for primitive ideals, his arguments extend without change to our setting.

Theorem 1.12 (Unfolding theorem). *Let $F:\mathbb{C}^n\times\mathbb{C}^r\to\mathbb{C}$ be an I-unfolding of a function $f\in I$. Let $s_1,\ldots,s_r$ be the coordinates of the base space $\mathbb{C}^r$. The following statements are equivalent*

(1) $\tau_{I,e}(f)+\mathbb{C}(\partial F/\partial s_1)_{|s=0}+\cdots+\mathbb{C}(\partial F/\partial s_r)_{|s=0}=I$.
(2) *F is a versal I-unfolding of f.*

A corollary of this is that $f\in I$ has a versal unfolding if and only if $c_{I,e}(f)$ is finite. Versality is open in the following sense:

Proposition 1.13. *Let F be a versal I-unfolding of a function $f\in I$; let $F:U\times V\to\mathbb{C}$ be a representative of the germ F. Then U and V can be shrunk to smaller neighbourhoods of O so that F is a versal $\tilde{I}_x$-unfolding of $F_{|s,x}$ for any $(x,s)\in U\times V$.*

Proof. Let $\mathcal{F}$ and $\mathcal{G}$ be the sheaves associated to f and F by the formulae (3) and (5), respectively. As f has a versal unfolding then $c_{I,e}(f)<\infty$. Hence, by the second assertion of Lemma 1.8, we can shrink U and V so that φ is finite and $p_*\mathcal{G}=\varphi_*\mathcal{G}$ is a coherent $\mathcal{O}_V$-module.

Let $s_1,\dots,s_r$ the coordinates of $\mathbb{C}^r$ and (s) the ideal they generated. The functions $\partial F/\partial s_1,\dots,\partial F/\partial s_r$, defined on the whole $U\times V$, can be seen as sections of $p_*\tilde{I}(r)$ over V; denote by $\partial_i F$ the image of $\partial F/\partial s_i$ by the natural homomorphism $p_*\tilde{I}(r)\to p_*\mathscr{G}$. Define $\mathscr{M}\subset p_*\mathscr{G}$ to be the coherent $\mathcal{O}_V$-module generated by $\partial_1 F,\dots,\partial_r F$. We claim that $\mathscr{M}=p_*\mathscr{G}$ if we shrink V enough. To prove the claim we only need to show that $\mathscr{M}_O=(p_*\mathscr{G})_O$. By Nakayama, this reduces to proving the equality $p_*\mathscr{G}_O=\mathscr{M}_O+(s)p_*\mathscr{G}_O$. As $\varphi^{-1}(O)=\{(O,O)\}$, then $(p_*\mathscr{G})_O=\mathscr{G}_{(O,O)}$; therefore $(p_*\mathscr{G})_O/(s)(p_*\mathscr{G})_O$ is equal to $\mathscr{G}_{(O,O)}/(s)\mathscr{G}_{(O,O)}$, which, by formula (6), is isomorphic to $I/\tau_{I,e}(f)$. We have constructed an isomorphism $\psi:(p_*\mathscr{G})_O/(s)(p_*\mathscr{G})_O\to I/\tau_{I,e}(f)$. It is easy to see that the image of $\partial_i F$ by ψ is the class of $(\partial F/\partial s_i)_{|O}$ in $I/\tau_{I,e}(f)$. As F is a versal I-unfolding of f, by Theorem 1.12 we conclude that the restriction of ψ to $(\mathscr{M}_O+(s)(p_*\mathscr{G})_O)/(s)(p_*\mathscr{G})_O$ is surjective. This shows the claim.

Consider $(x,s)\in U\times V$, if $\mathscr{G}_{(x,s)}=0$ then, by Lemma 1.8, we have the equality $\tau_{\tilde{I}_x,e}(F_{|s,x})=\tilde{I}_x$; hence, by Theorem 1.12 F is a versal $\tilde{I}_x$-unfolding of $F_{|s,x}$. Suppose that $(x,s)\in\operatorname{Supp}(\mathscr{G})$. Let $\mathbf{m}_s$ be the maximal ideal of $\mathcal{O}_{\mathbb{C}^r,s}$. By the finiteness of φ we have an equality $(p_*\mathscr{G})_s=\oplus_{y\in\varphi^{-1}(s)}\mathscr{G}_{(y,s)}$. Using formula (6) we obtain an isomorphism

$$\psi_s:(p_*\mathscr{G})_s/\mathbf{m}_s(p_*\mathscr{G})_s=\bigoplus_{y\in\varphi^{-1}(s)}\mathscr{G}_{(y,s)}/\mathbf{m}_s\mathscr{G}_{(y,s)}\cong\bigoplus_{y\in\varphi^{-1}(s)}\tilde{I}_y/\tau_{\tilde{I}_y,e}(F_{|s,y}).$$

Using that $\mathscr{M}=p_*\mathscr{G}$, noting that the image of $\partial_i F$ by ψ_s has $(\partial F/\partial x_i)_{|s,y}$ as component in $\tilde{I}_y/\tau_{\tilde{I}_y,e}(F_{|s,y})$, and taking into account that $x\in\varphi^{-1}(s)$ we obtain that $\tilde{I}_x=\tau_{\tilde{I}_x,e}(F_{s,x})+\mathbb{C}(\partial F/\partial x_1)_{|s,x}+\cdots+\mathbb{C}(\partial F/\partial x_r)_{|s,x}$. Then, by Theorem 1.12 the mapping F is a versal unfolding of $F_{|s,x}$. $\square$

2. Topology of unfoldings of functions of finite I-codimension

Denote by $\dot{D}_\eta$ the punctured disk of radius η centered at the origin and by B_ϵ, $\bar{B}_\epsilon$ and S_ϵ the open ball, closed ball and sphere of radius ϵ centered at the origin of $\mathbb{C}^n$; let $f\in\mathcal{O}_{\mathbb{C}^n,O}$. Lê proved in [12] that if $\epsilon>0$ is small enough and $\epsilon\gg\eta>0$ then

$$f_{|\bar{B}_\epsilon\cap f^{-1}(\dot{D}_\eta)}:\bar{B}_\epsilon\cap f^{-1}(\dot{D}_\eta)\to\dot{D}_\eta \tag{8}$$

is a locally trivial fibration, and, moreover, if (ϵ',η') is another pair with $\epsilon'\leqslant\epsilon$ and such that $f_{|\bar{B}_{\epsilon'}\cap f^{-1}(\dot{D}_{\eta'})}$ is also a locally trivial fibration, then both fibrations are equivalent. Moreover, in view of Hironaka [9, Section 5], if we consider in $\mathbb{C}$ the stratification $\{\mathbb{C}\setminus\{0\},\{0\}\}$, then there exist an analytic Whitney stratification of a neighbourhood U of the origin of $\mathbb{C}^n$ containing $\bar{B}_\epsilon$ such that $U\cap f^{-1}(\mathbb{C}\setminus\{0\})$ is a stratum, the mapping $f:U\to\mathbb{C}$ satisfies the Thom A_f-condition respect to this stratification and, for each stratum $X\subset U$ and each point $x\in X\cap S_\epsilon$, we have $T_xX\pitchfork S_\epsilon$.

Definition 2.1. A pair (ϵ,η) with all the properties above is called a *good system of radii* for f.

The fibration

$$f_{|\bar{B}_\epsilon\cap f^{-1}(\partial D_\eta)}:\bar{B}_\epsilon\cap f^{-1}(\partial D_\eta)\to\partial D_\eta \tag{9}$$

is called the *Milnor fibration* of f.

The main result of this section is the preservation of transversality with the Milnor sphere for unfoldings of functions of finite codimension:

Theorem 2.2. *Let $f \in I$ such that $c_{I,e}(f) < \infty$; consider a 1-parametric I-unfolding F of f; let (ϵ, η) be a good system of radii for f. Given a value s of the parameter consider the restriction*

$$F_{|s} : X_{D_\eta,s} := F_{|s}^{-1}(D_\eta) \cap \bar{B}_\epsilon \to D_\eta. \tag{10}$$

Then δ can be chosen small enough so that, given $s, s' \in D_\delta$,

(1) *If $t \in \dot{D_\eta} \setminus \{0\}$ then $F_{|s}^{-1}(t) \pitchfork S_\epsilon$.*
(2) *The locally trivial fibrations that $F_{|s}$ and $F_{|s'}$ induce over ∂D_η are equivalent.*
(3) *$X_{D_\eta,s}$ and $X_{D_\eta,s'}$ (where $X_{D_\eta,s} := F_{|s}^{-1}(D_\eta) \cap \bar{B}_\epsilon$) are homeomorphic.*

This result is a generalisation of analogous statements in [20,21,10,26], and [3]; in those papers the result is proved in the case in which the dimension of $V(I)$ is at most 2, and/or I is of a rather particular type, in all the cases the idea is to use the special properties of I to control explicitly the geometry of F at S_ϵ. Next we show how to control the geometry of F in a neighbourhood of S_ϵ for any I, without making use of any geometric property of I.

Lemma 2.3. *Let $F : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$ be a 1-parametric I-unfolding, consider a smooth path $\gamma : (-v, v) \to \mathbb{C}$ such that $\gamma(0) = 0$. There exist a positive number $\mu < v$, a neighbourhood V_0 of S_ϵ in $\mathbb{C}^n$, a neighbourhood W of $S_\epsilon \times (-\mu, \mu)$ in $\mathbb{C}^n \times (-\mu, \mu)$, and a C^∞-diffeomorphism*

$$\Psi : V_0 \times (-\mu, \mu) \to W \tag{11}$$

of the form $\Psi(x, t) = (\psi_t(x), t)$ (i.e. a C^∞-family Ψ of diffeomorphisms ψ_t depending on t), such that $\psi_0 = \mathrm{Id}_{\mathbb{C}^n}$ and $F_{|\gamma(t)} \circ \psi_t = f_{|V_0}$ for any $t \in (-\mu, \mu)$.

Proof. As $c_{I,e}(f) < \infty$, the sheaf $\mathscr{F}$ defined in Eq. (3) is concentrated at the origin of $\mathbb{C}^n$. This implies that, if ϵ is small enough, for any $x \in S_\epsilon$, we have $\mathscr{F}_x = 0$ and hence $c_{\tilde{I}_x,e}(f_x) = 0$. As $f_x = F_{|0,x}$, we have $c_{\tilde{I}_x,e}(F_{|0,x}) = 0$, which by Lemma 1.8, implies $\mathscr{G}_{(x,0)} = 0$ for any $x \in S_\epsilon$. By coherence, there is an open neighbourhood U of $S_\epsilon \times \{0\}$ in $\mathbb{C}^n \times \mathbb{C}$ such that $\mathscr{G}_{|U} = 0$.

Let s be a coordinate for the base space of the unfolding F; as the ideal sheaf $\tilde{I}(1)$ is closed under differentiation respect to s we have $\partial F/\partial s \in \tilde{I}(1)$. As $\mathscr{G}_{|U} = 0$, for any $x \in U$ there exist an open neighbourhood U^x of x in U and a vector field $X^x \in \Gamma(U^x, \Theta_{\tilde{I},e}(1))$ such that $X^x(F_{|U^x}) = \partial F/\partial s_{|U^x}$. Choose a finite collection $U^1, \ldots, U^k$ of the above open subsets such that $\bigcup_{i=1}^k U^i \supset S_\epsilon \times \{0\}$. Let X^i be the vector field associated to U^i. Redefine U to be the union of the chosen subsets. Consider a C^∞ partition of unity $\{\rho^1, \ldots, \rho^k\}$ subordinated to the cover $\{U^1, \ldots, U^k\}$ and define $X := \sum_{i=1}^k \rho^i X^i$. The vector field X is of the form $X = \sum_{i=1}^n \alpha_i \partial/\partial x_i$ where α_i are C^∞ functions over U, and satisfies $X(F_{|U}) = \sum_{i=1}^k \rho^i X^i(F_{|U^i}) = \partial F/\partial s_{|U}$.

Choose $0 < \mu < v$ and a neighbourhood V_0 of S_ϵ in $\mathbb{C}^n$ such that $V_0 \times \gamma(-\mu, \mu)$ is contained in U. Consider the C^∞ vector field tangent to V_0 and smoothly depending on t defined by $Y(x, t) := -(\mathrm{d}\gamma/\mathrm{d}t) X(x, \gamma(t))$. Express Y as $Y = \sum_{i=1}^n \beta_i \partial/\partial x_i$, then $\beta_i(x, t) = -(\mathrm{d}\gamma/\mathrm{d}t)\alpha_i(x, \gamma(t))$. Integrating Y (and perhaps shrinking V_0 and μ) we obtain a C^∞-family of diffeomorphisms $\psi : V_0 \times (-\mu, \mu) \to U$ such that $\psi_0 = \mathrm{Id}_{\mathbb{C}^n}$; define $\Psi(x, t) := (\psi(x, t), t)$ and $W := \Psi(V_0 \times (-\mu, \mu))$. We need to check that

$F_{|\gamma(t)} \circ \psi_t = f_{|V_0}$ for any $t \in (-\mu, \mu)$. This is obvious for $t = 0$. Hence if $G(x,t) := F(\psi(x,t), \gamma(t))$, it suffices to show that $\partial G/\partial t = 0$. By the chain rule,

$$\frac{\partial G}{\partial t}(x,t) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(\psi(x,t),\gamma(t))\beta_i(\psi(x,t),t) + \frac{\partial F}{\partial s}(\psi(x,t),\gamma(t))\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t)$$

$$= \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(\psi(x,t),\gamma(t))\left(-\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t)\right)\alpha_i(\psi(x,t),\gamma(t)) + \frac{\partial F}{\partial s}(\psi(x,t),\gamma(t))\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t)$$

$$= \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t)\left[\left(-X(F) + \frac{\partial F}{\partial s}\right)(\psi(x,t),\gamma(t))\right] = 0. \qquad \square$$

Proof of Theorem 2.2. We first show than (2) and (3) follow from (1). The pair (ϵ, η) is chosen so that $\mathrm{Sing}(f) \cap \bar{B}_\epsilon \subset f^{-1}(0)$, therefore, δ can be chosen small enough so that F_s has no critical points at $F_{|s}^{-1}(\partial D_\eta) \cap \bar{B}_\epsilon$. Define $\bar{F}: \mathbb{C}^n \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ by $\bar{F}(x,s) := (F(x,s), s)$. Assertion (2) follows from (1) by applying Ehresmann's fibration theorem to $\bar{F}_{|\bar{F}^{-1}(\partial D_\eta \times D_\delta)}$. Define $X_{D_\eta} := \bar{F}^{-1}(D_\eta \times D_\delta) \cap (\bar{B}_\epsilon \times D_\delta)$ and consider $p: X_{D_\eta} \to D_\delta$, the restriction to X_{D_η} of the projection to the second factor; then $p^{-1}(s) = X_{D_\eta,s}$ for any $s \in D_\delta$. Using (1), it is easy to show that X_{D_η} is a manifold with corners and that p is a proper differentiable map, whose restriction to the boundary and corners is submersive; using a version of the Ehresmann fibration theorem for manifolds with corners (3) follows.

Define $Y \subset S_\epsilon \times D_\delta$ to be the set of pairs (x,s) such that $F_{|s}(x) \neq 0$ and either $F_{|s}$ is critical at x or $F_{|s}^{-1}(F_{|s}(x))$ is not transversal to S_ϵ at x. Identifying $\mathbb{C}^n \times \mathbb{C}$ with $\mathbb{R}^{2n+2}$ it is easy to see that the closure $\bar{Y}$ of Y is a real analytic subset. The following claim implies assertion (1) (perhaps shrinking η): the intersection $\bar{Y} \cap F^{-1}(0,0)$ is empty.

Let us prove the claim. Suppose the contrary: let $(x,0) \in \bar{Y} \cap F^{-1}(0,0)$; by the curve selection lemma (see [15, Section 3]) there is an analytic path $\alpha: (-v,v) \to \bar{Y}$ such that $\alpha(-v,v) \subset Y$ and $\alpha(0) = (x,0)$. Define $\gamma: (-v,v) \to \mathbb{C}$ to be the second component of the composition $\bar{F} \circ \alpha$, and choose $\mu < v$ so that the statement of Lemma 2.3 holds; let Ψ be the family of diffeomorphisms predicted by Lemma 2.3. Consider a sequence $\{t_n\} \subset (\mu, \mu)$ convergent to 0; define x_n to be the first component of $\alpha(t_n)$ in $\mathbb{C}^n \times \mathbb{C}$ (by the formula $(x_n, \gamma(t_n)) = \alpha(t_n)$), and $y_n := \psi_{t_n}^{-1}(x_n)$. As $\{x_n\}$ and $\{\psi_{t_n}\}$ converge to x and $\mathrm{Id}_{\mathbb{C}^n}$, respectively, we deduce that $\{y_n\}$ converges to x. If $F_{|\gamma(t_n)}$ is singular at x_n then $f = F_{|\gamma(t_n)} \circ \psi_{t_n}$ is singular at y_n; then $f(y_n) = F_{|\gamma(t_n)}(x_n) = 0$, which is not true; therefore $F_{|\gamma(t_n)}$ is not singular at x_n. Then $F_{|\gamma(t_n)}^{-1}(F_{|\gamma(t_n)}(x_n))$ is not transversal to S_ϵ at x_n, which means that $T_{x_n}F_{|\gamma(t_n)}^{-1}(F_{|\gamma(t_n)}(x_n)) \subset T_{x_n}S_\epsilon$; this implies that $T_{y_n}f^{-1}(f(y_n)) \subset \mathrm{d}\psi_{t_n}^{-1}(x_n)(T_{x_n}S_\epsilon)$. Taking a subsequence we can assume that the sequence $T_{y_n}f^{-1}(f(y_n))$ converges to a linear subspace $T \subset T_xS_\epsilon$. On the other hand we have fixed Whitney stratifications of $\mathbb{C}$ and of an open neighbourhood U of the origin of $\mathbb{C}^n$ (containing $U \backslash f^{-1}(0)$ as a stratum) such that f satisfies the Thom A_f-condition respect to them and that $X \pitchfork S_\epsilon$ for any stratum X of U. Let X be the stratum containing x, as $f(x) = 0$ and $U \backslash f^{-1}(0)$ is a stratum we have that $X \subset f^{-1}(0)$; hence $\ker(\mathrm{d}f_{|X}(x)) = T_xX$. By Thom A_f-condition $T_xX \subset \lim T_{y_n}f^{-1}(f(y_n)) = T$; as T is included in T_xS_ϵ, we contradict the transversality $X \pitchfork S_\epsilon$. This proves the claim. $\square$

3. Generalised jet-spaces

The morsification theorem for isolated singularities can be proved in the following way (which is not the easiest but has the virtue of generalising to our setting): the subset $Z_1 \subset J^2(\mathbb{C}^n, \mathbb{C})$ consisting of singular 2-jets (jets whose linear parts vanish) is a closed analytic subset of codimension n; the subset $Z_2 \subset Z_1$ of singular 2-jets germs with degenerate Hessian is closed analytic of codimension bigger than n. On the other hand any germ $f \in \mathcal{O}_{\mathbb{C}^n,O}$ can be approximated by germs g such that the 2-jet extension $j^2 g : \mathbb{C}^n \to J^2(\mathbb{C}^n, \mathbb{C})$ is transversal to Z_1 and Z_2 in a neighbourhood of the origin; therefore the singularities that an approximation g can have close to the origin should have non-degenerate Hessian (which means being of Morse type).

Consider an ideal $I \subset \mathcal{O}_{\mathbb{C}^n,O}$. When trying to generalise the classical morsification method to functions of finite codimension with respect to I we meet (among others) the following new difficulties: consider $f \in I$ such that $c_{I,e}(f) = \infty$; it is not clear a priori which singularities a generic deformation within the ideal may have. On the other hand, due to the presence of non-isolated singularities, if we work with ordinary m-jet spaces the spaces parametrising singularity types (analogous to Z_1 and Z_2) will be of arbitrarily large codimension as m increases, which would make the above method collapse. The idea to overcome these problems will be to use a presentation of the ideal I by generators and relations to define a sort of generalised m-jet spaces, in which the codimensions the varieties parametrising the relevant singularity types remain stable as m increases.

We need to introduce the concept of ∞-jet spaces and give it an infinite dimensional analytic structure.

Let $U \in \mathbb{C}^n$ be an open subset. For any $m < \infty$ the mth jet-space $J^m(U, \mathbb{C}^r)$ is a vector bundle over U with projection mapping $pr_m : J^m(U, \mathbb{C}^r) \to U$. There is a natural analytic vector bundle epimorphism $pr^m_l : J^m(U, \mathbb{C}^r) \to J^l(U, \mathbb{C}^r)$ for any $m \geqslant l$. The set $J^\infty(U, \mathbb{C}^r) := \coprod_{x \in U} \mathcal{O}^r_{U,x}$ is clearly the projective limit of the system formed by the $J^m(U, \mathbb{C}^r)$'s and the π^m_l's. For any m there is a projection mapping $\pi^\infty_m : J^\infty(U, \mathbb{C}^r) \to J^m(U, \mathbb{C}^r)$.

Fix a coordinate system $\{x_1, \ldots, x_n\}$ in $\mathbb{C}^n$; denote by $\mathbb{C}\{x\}$ the ring of convergent power series in $\{x_1, \ldots, x_n\}$. There is a bijection

$$\tau_\infty : U \times \mathbb{C}\{x\}^r \to J^\infty(U, \mathbb{C}^r)$$

which assigns to $(x, (f_1, \ldots, f_r))$ the unique r-tuple of germs $(g_1, \ldots, g_r) \in \mathcal{O}^r_{\mathbb{C}^n,x}$ such that the Taylor expansion of g_i at x is f_i. Passing to m-jets this defines an analytic vector bundle trivialisation

$$\tau_m : U \times (\mathbb{C}\{x\}/\mathbf{m}^{m+1})^r \to J^m(U, \mathbb{C}^r) \tag{12}$$

for any $m < \infty$.

A subset $C \subset J^\infty(U, \mathbb{C}^r)$ is said *k-determined* if it satisfies $(\pi^\infty_k)^{-1}(\pi^\infty_k(C)) = C$; the subset C is said to be a *k-determined closed analytic subset of* $J^\infty(U, \mathbb{C}^r)$ if it is k-determined and $\pi^\infty_k(Y)$ is a closed analytic subset of $J^k(U, \mathbb{C}^r)$. Any k-determined (closed analytic) subset is also m-determined (closed analytic) for any $m \geqslant k$. A k-determined *locally closed analytic* subset is the difference between two k-determined closed analytic subsets.

Consider in each $J^m(U, \mathbb{C}^r)$ the transcendental topology; we endow $J^\infty(U, \mathbb{C}^r)$ with the initial topology for the family of projections π^∞_m. A family $\{C_j\}_{j \in J}$ of subsets of $J^\infty(U, \mathbb{C}^r)$ is locally finite if for any $x \in J^\infty(U, \mathbb{C}^r)$ there exists a positive integer m and a neighbourhood U of $\pi^\infty_m(x)$ in $J^m(U, \mathbb{C}^r)$ such that $(\pi^\infty_m)^{-1}(U)$ only meets finitely many C_j's. Therefore if each C_j is finite-determined, choosing m high enough, the union $\bigcup_{j\in J} C_j$ is locally a m-determined subset. This motivates the following definition: a *closed analytic* subset of $J^\infty(U,\mathbb{C}^r)$ is any union of a locally finite collection of finite-determined closed analytic subsets of X^∞; a *locally closed analytic* subset of X^∞ is the difference between two closed analytic subsets.

Let $\tilde{I}$ be an ideal sheaf defined on a neighbourhood U of the origin, having I as stalk at the origin. The natural candidates to substitute for the ordinary jet-spaces considered in the isolated-singularity case are the following: define the sets

$$J^\infty(U,\tilde{I}) := \coprod_{x\in U} \tilde{I}_x \quad J^m(U,\tilde{I}) := \coprod_{x\in U} \tilde{I}_x/\mathbf{m}_x^{m+1}\cap \tilde{I}_x$$

for any positive integer m. For any $\infty \geqslant m \geqslant k \geqslant 0$ consider the projection mapping $\pi^m_k : J^m(U,\tilde{I}) \to J^k(U,\tilde{I})$. The set of spaces $\{J^m(U,\tilde{I})\}_{m<\infty}$ together with the mappings $\{pr^m_k\}_{k<m<\infty})$ form a projective system of sets whose limit is $J^\infty(U,\tilde{I})$. There are natural projection mappings $\pi^\infty_m : J^\infty(U,\tilde{I}) \to J^m(U,\tilde{I})$ satisfying $\pi^\infty_l = \pi^m_l \circ \pi^\infty_m$. The concept of k-determined subset of $J^\infty(U,\tilde{I})$ is defined analogously to the case of systems of analytic varieties. For any $m \leqslant \infty$ we consider the natural projection $pr_m : J^m(U,\tilde{I}) \to U$; its fibre over $x \in U$ is the vector space $\tilde{I}_x/\tilde{I}_x \cap \mathbf{m}_x^{m+1}$.

For any $x\in U$ we define the function $\mu_x : \mathbb{Z}_{>0} \to \mathbb{Z}$ by the formula

$$\mu_x(m) := \dim_{\mathbb{C}}(\tilde{I}_x/\tilde{I}_x \cap \mathbf{m}_x^{m+1}). \tag{13}$$

The function μ_x is Zariski-lower semicontinuous in U (see [4]). There exists a stratification of U (which is called the *Zariski–Samuel stratification* with respect to $\tilde{I}$) by Zariski locally closed analytic subsets, which is the minimal partition such that $\mu_x = \mu_y$ for any x, y in the same stratum. We will refer to the strata as the $\tilde{I}$-strata of U, and we will denote them by $\Sigma_0,\dots,\Sigma_s$, where Σ_s is the stratum containing the origin, the stratum Σ_0 is the complement of the zero set of $\tilde{I}$, and if $\Sigma_i \subset \bar{\Sigma}_j$ then $i \geqslant j$.

Notation 3.1. Consider an analytic function $f : V \to \mathbb{C}$, with V an open subset of $\mathbb{C}^n$. Given any $m \leqslant \infty$ there is an associated *jet extension* $j^m f : J^m(V,\mathbb{C})$ assigning to $x \in V$ the class of f_x in $\mathcal{O}_{\mathbb{C}^n,x}/\mathbf{m}_x^{m+1}$ (we adopt the convention $\mathbf{m}_x^\infty = (0)$).

We define a certain kind of analytic atlas on $J^m(U,\tilde{I})$ in the following way:

Consider an open subset V of U and a set $\mathscr{H} = \{h_1,\dots,h_s\} \subset \Gamma(U,\tilde{I})$ which generate $\tilde{I}_{|V}$. Define

$$\varphi_{\mathscr{H}} : \mathcal{O}^s_V \to \mathcal{O}_{|V} \tag{14}$$

by the formula $\varphi_{\mathscr{H}}(f_1,\dots,f_s) := \sum_{i=1}^s f_i h_i$. For any $m \leqslant \infty$, taking m-jets we obtain a mapping

$$j^m\varphi_{\mathscr{H}} : J^m(V,\mathbb{C}^s) \to J^m(V,\mathbb{C}). \tag{15}$$

where $j^m\varphi_{\mathscr{H}}(j^m f_1(x),\dots,j^m f_s(x)) := \sum_{i=1}^s j^m(f_i h_i)(x)$ for any $(f_1,\dots,f_s) \in (\mathcal{O}_{V,x})^s$ and $x \in V$. The mapping $j^m\varphi$ has $J^m(V,\tilde{I})$ as image, and it is a homomorphism of analytic vector bundles if $m < \infty$.

Definition 3.2. For any $m \leqslant \infty$, a *chart* of $J^m(U,\tilde{I})$ is a surjective mapping of the form

$$j^m\varphi_{\mathscr{H}} : J^m(V,\mathbb{C}^s) \to J^m(V,\tilde{I}). \tag{16}$$

for a certain open subset V (which will be called the *base subset* of the chart) and a set $\mathscr{H}$ of holomorphic functions on V generating $\mathscr{H}$ of $\tilde{I}_{|V}$. The *canonical analytic atlas* of $J^m(U,\tilde{I})$ is the collection of all the charts of $J^m(U,\tilde{I})$.

Given another open subset V' and a set of generators $\mathscr{H}' = \{h'_1,\dots,h'_{s'}\}$ of $\tilde{I}_{|V'}$, there exists an open covering $\{V''_l\}_{l\in L}$ of $V\cap V'$ such that, given any $l\in L$ for any $i\leqslant s'$, we have analytic functions $\theta_{i,1},\dots,\theta_{i,s}$ defined on V''_l such that $h'_i := \sum_{j=1}^{s}\theta_{i,j}h_j$. Define the $\mathcal{O}_{V''_l}$-module homomorphism $\theta:\mathcal{O}^{s'}_{V''_l}\to\mathcal{O}^{s}_{V''_l}$ by the formula $\theta(f_1,\dots,f_{s'})=(f_1,\dots,f_{s'})M_\theta$ where M_θ is the matrix whose (i,j)-entry is $\theta_{i,j}$. For any $m\leqslant\infty$ let

$$j^m\theta:J^m(V''_l,\mathbb{C}^{s'})\to J^m(V''_l,\mathbb{C}^{s}) \tag{17}$$

be the associated mapping of jet-spaces, which is analytic for $m<\infty$. We have clearly the compatibility relation

$$j^m\varphi_{\mathscr{H}'}=j^m\varphi_{\mathscr{H}}\circ j^m\theta. \tag{18}$$

The mapping (17) is called *a transition function* between the charts $j^m\varphi_{\mathscr{H}}$ and $j^m\varphi_{\mathscr{H}'}$. Notice that in contrast with the case of manifolds, transition functions between charts are not globally defined on the intersection of the two charts and need not be unique.

As we work locally around the origin of $\mathbb{C}^n$ we can take U small enough so that there is a set $\mathscr{G}=\{g_1,\dots,g_r\}$ of analytic functions defined on U which generate $\tilde{I}_x$ for any $x\in U$. We will denote the homomorphism $\varphi_{\mathscr{G}}$ simply by

$$\varphi:\mathcal{O}^r_U\to\mathcal{O}_U, \tag{19}$$

and, for any $m\leqslant\infty$, the mapping $j^m\varphi_{\mathscr{G}}$ by

$$J^m(U,\mathbb{C}^r)\xrightarrow{j^m\varphi}J^m(U,\mathbb{C}). \tag{20}$$

Notation 3.3. Let $X\subset U$ be any subset. For any $m\leqslant\infty$, we will denote by $J^m(X,\tilde{I})$ or $J^m(U,\tilde{I})_{|X}$ the inverse image of X under $pr_m:J^m(U,\tilde{I})\to U$. An analogous notation works for the jet space $J^m(U,\mathbb{C}^r)$.

Remark 3.4. For $m<\infty$ and any $\tilde{I}$-stratum, the restriction $j^m\varphi_{|J^m(\Sigma_i,\mathbb{C}^r)}$ is a constant rank homomorphism of trivial analytic vector bundles over Σ_i, whose image is $J^m(\Sigma_i,\tilde{I})$. This gives a natural structure of trivial analytic vector bundle to $J^m(\Sigma_i,\tilde{I})$, for $m<\infty$ (see [4] for an application).

For our purposes it is convenient to consider subsets of $J^\infty(U,\tilde{I})$ parametrising germs with certain geometric properties, just as the subset $Z_1\subset J^2(U,\mathbb{C})$ considered at the beginning of the section parametrises Morse singularities. To have a geometric understanding of these subsets we will look at their inverse image by the charts of the canonical analytic atlas of $J^m(U,\tilde{I})$.

For any $m<\infty$ we give to $J^m(U,\tilde{I})$ the final topology for the set of all charts of the canonical analytic atlas. As the transition functions are continuous the topology on $J^m(U,\tilde{I})$ is just the final topology for the mapping $j^m\varphi:J^m(U,\mathbb{C}^r)\to J^m(U,\tilde{I})$. We give to $J^\infty(U,\tilde{I})$ the topology obtained viewing it as projective limits of the systems of topological spaces $\{J^m(U,\tilde{I})\}_{m\in\mathbb{N}}$. It is easy to check that the charts for $m=\infty$ are continuous.

Remark 3.5. As $j^m\varphi_{|J^m(\Sigma_i,\mathbb{C}^r)}$ is a submersion for any $m<\infty$ and any $\tilde{I}$-stratum, the restriction of the topology of $J^m(U,\tilde{I})$ to $J^m(\Sigma_i,\tilde{I})$ coincides with the restriction of the topology of $J^m(\Sigma_i,\mathbb{C})$ to $J^m(\Sigma_i,\tilde{I})$. Therefore the topology considered in [4] for $J^m(\Sigma_i,\tilde{I})$ is the restriction of the topology considered here for $J^m(U,\tilde{I})$.

Definition 3.6. Consider $0<k\leqslant m\leqslant\infty$. A k-determined subset $C\subset J^m(U,\tilde{I})$ is *closed analytic* if for any chart $\varphi_{\mathscr{H}}$ with base subset V, the set $(j^k\varphi_{\mathscr{H}})^{-1}(\pi^m_k(C))$ is an m-determined closed analytic subset of $J^k(V,\mathbb{C}^m)$. A *locally closed* k-determined analytic subset of $J^m(U,\tilde{I})$ is the difference between two k-determined closed analytic subsets. A *closed analytic* subset of $J^\infty(U,\tilde{I})$ is a locally finite union of finitely determined closed analytic subsets of $J^\infty(U,\tilde{I})$. A *locally closed analytic* subset of $J^\infty(U,\tilde{I})$ is the difference between two closed analytic subsets.

Consider a k-determined subset of $J^m(U,\tilde{I})$. Given any $k\leqslant k'\leqslant m$ it is easy to check that it is a k-determined (locally) closed analytic subset if and only if it is a k'-determined (locally) closed analytic subset. Let Z be a closed m-determined subset of $J^m(U,\tilde{I})$, and let $j^m\varphi_{\mathscr{H}}:J^m(V,\mathbb{C}^s)\to J^m(U,\tilde{I})$ and $j^m\varphi_{\mathscr{H}'}:J^m(V',\mathbb{C}^{s'})\to J^m(U,\tilde{I})$ be two charts. By the compatibility (18), if $(j^k\varphi_{\mathscr{H}})^{-1}(Z)$ is a closed analytic subset, then $(j^k\varphi_{\mathscr{H}'})^{-1}(Z_{|V''_l})$ is closed analytic in $J^m(V''_l,\mathbb{C}^{s'})$. Therefore to prove the analyticity of a m-determined subset it is enough to check the condition for a set of charts whose base subsets cover U. In particular it is enough to check that $(j^m\varphi)^{-1}(Z)$ is analytic.

Definition 3.7. A subset $Y\subset J^m(U,\mathbb{C}^r)$ satisfying $(j^m\varphi)^{-1}(j^m\varphi(Y))=Y$ is called $j^m\varphi$-*saturated.*

Closed analytic subsets of $J^m(U,\tilde{I})$ are in a bijective correspondence with $j^m\varphi$-saturated closed analytic subsets of $J^m(U,\mathbb{C}^r)$.

Definition 3.8. A closed analytic subset of $J^\infty(U,\tilde{I})$ is *irreducible* if it cannot be expressed as the union of two closed analytic subsets of $J^\infty(U,\tilde{I})$ not containing it.

It follows that any irreducible closed analytic subset is finite-determined.

We shrink U so that its closure is compact and contained in an open subset where $\tilde{I}$ is defined. Then the following uniform Artin–Rees theorem holds (see [1] for a proof): there exists $\lambda\in\mathbb{Z}_{\geqslant 0}$ such that

$$\tilde{I}_x\cap\mathbf{m}_x^{m+\lambda}\subset\mathbf{m}_x^m\tilde{I}_x \tag{21}$$

for any $x\in U$ and any m. The minimal λ so that the last inclusion holds is called the *uniform Artin–Rees constant*.

Lemma 3.9. *Let $C\subset J^\infty(U,\tilde{I})$ be a k-determined closed analytic subset. For any irreducible component K of $(j^k\varphi)^{-1}(\pi^\infty_k(C))$, the preimage $(\pi^{\lambda+k}_k)^{-1}(K)$ is $j^{k+\lambda}\varphi$-saturated.*

Proof. Set $m=k+\lambda$ and

$$A:=(\pi^m_k)^{-1}((j^k\varphi)^{-1}(\pi^\infty_k(C))).$$

Let $K' := (\pi_k^m)^{-1}(K)$. Clearly A is closed analytic and K' is one of its irreducible components. We have to show that

$$K'_x + \ker(j^m\varphi_x) \subset K'_x \tag{22}$$

for any $x \in U$ (where $K'_x = K' \cap pr_m^{-1}(x)$, and $j^m\varphi_x$ is the restriction of $j^m\varphi$ to $J^m(U,\mathbb{C}^r)_x$). An element of $\ker j^m\varphi_x$ is the class in $J^m(U,\mathbb{C}^r)_x$ of a r-tuple $(f_1,\dots,f_r) \in \mathcal{O}^r_{U,x}$ such that $\varphi_x(f_1,\dots,f_r)$ belongs to $\tilde{I}_x \cap \mathbf{m}_x^{m+1}$. It follows from uniform Artin-Rees that $\varphi_x(f_1,\dots,f_r)$ belongs to $\mathbf{m}_x^{k+1}\tilde{I}_x$. Therefore there exists $(h_1,\dots,h_r) \in \mathbf{m}_x^{k+1}\mathcal{O}^r_{U_x}$ so that $\varphi_x(h_1,\dots,h_r) = \varphi_x(f_1,\dots,f_r)$. We deduce that $(f_1,\dots,f_r)$ belongs to $\mathbf{m}_x^{k+1}\mathcal{O}^r_{U,x} + \ker(\varphi_x)$. Hence

$$\ker(j^m\varphi_x) \subset \mathbf{m}_x^{k+1}\mathcal{O}^r_{U,x}/\mathbf{m}_x^{m+1}\mathcal{O}^r_{U,x} + \pi_m^\infty(\ker(\varphi_x)). \tag{23}$$

As $\varphi^{-1}(C)$ is k-determined we have

$$A_x + \mathbf{m}_x^{k+1}\mathcal{O}^r_{U,x}/\mathbf{m}_x^{m+1}\mathcal{O}^r_{U,x} \subset A_x \tag{24}$$

for any $x \in U$. The set $E := \coprod_{x\in U} \mathbf{m}_x^{k+1}\mathcal{O}^r_{U,x}/\mathbf{m}_x^{m+1}\mathcal{O}^r_{U,x}$ is a trivial vector sub-bundle of $J^m(U,\mathbb{C}^r)$. Let r_1 be the rank of E. Consider an analytic trivialisation $\tau_1 : G_1 \times U \to E$, where $G_1 \cong \mathbb{C}^{r_1}$. View G_1 as an additive analytic group. We have an analytic action

$$G_1 \times J^m(U,\mathbb{C}^r) \xrightarrow{\sigma_1} J^m(U,\mathbb{C}^r)$$

defined by $\sigma_1(g,f) := \tau_1(g) + f$. By (24) the subset A is left invariant by the action.

Since $\tilde{I}$ is coherent we can take U small enough so that we have an exact sequence as follows

$$\mathcal{O}^s_U \xrightarrow{\psi} \mathcal{O}^r_U \xrightarrow{\varphi} \tilde{I} \to 0. \tag{25}$$

Taking m-jets we obtain a homomorphism $j^m\psi : J^m(U,\mathbb{C}^s) \to J^m(U,\mathbb{C}^r)$ of trivial analytic vector bundles. The exactness of (25) implies

$$\pi_m^\infty(\ker\varphi_x) = \pi_m^\infty(\operatorname{im}\psi_x) = \operatorname{im} j^m\psi_x \tag{26}$$

for any $x \in U$. This, together with the $j^m\varphi$-saturation of A implies

$$A_x + \operatorname{im}(j^m\psi_x) \subset A_x. \tag{27}$$

Let r_2 be the rank of the vector bundle $J^m(U,\mathbb{C}^s)$. Consider an analytic trivialisation $\tau_2 : G_2 \times U \to J^m(U,\mathbb{C}^s)$, where $G_1 \cong \mathbb{C}^{r_2}$. View G_2 as an additive analytic group. We have an analytic action

$$G_2 \times J^m(U,\mathbb{C}^r) \xrightarrow{\sigma_2} J^m(U,\mathbb{C}^r)$$

defined by $\sigma_2(g,f) := j^m\psi \circ \tau_2(g) + f$. By (27) the subset A is left invariant by the action.

Define the additive analytic group $G := G_1 \oplus G_2$ and the action

$$\sigma : G \times J^m(U,\mathbb{C}^r) \to J^m(U,\mathbb{C}^r) \tag{28}$$

by $\sigma := \sigma_1 \oplus \sigma_2$. The subset A is invariant by the action σ. As G is irreducible we conclude that each of the irreducible components of A are also invariant by the action. This, together with (23) and (26) imply (22). □

Proposition 3.10. *Let $C \subset J^\infty(U,\tilde{I})$ be a k-determined closed analytic subset. The following are equivalent*:

(1) *The set C is irreducible.*
(2) *The set C is finite determined and $(j^m\varphi)^{-1}(\pi_m^\infty(C))$ is irreducible for a certain $m \geqslant k$.*
(3) *The set C is finite determined and $(j^m\varphi)^{-1}(\pi_m^\infty(C))$ is irreducible for any $m \geqslant k$.*

Moreover, for any closed analytic subset C of $J^\infty(U,\tilde{I})$ there exists a unique irredundant locally finite decomposition of C in irreducible closed analytic subsets. If C is k-determined, then each of its irreducible components is $k+\lambda$-determined.

Proof. For any $m \geqslant k$ the morphism $\pi_k^m : J^m(U,\mathbb{C}^r) \to J^k(U,\mathbb{C}^r)$ is a vector bundle epimorphism. As $(j^m\varphi)^{-1}(\pi_m^\infty(C)) = (\pi_k^m)^{-1}((j^k\varphi)^{-1}(\pi_k^\infty(C)))$, the irreducibility of $(j^m\varphi)^{-1}(\pi_m^\infty(C))$ and $(j^k\varphi)^{-1}(\pi_k^\infty(C))$ are equivalent. This shows (2) $\Leftrightarrow$ (3).

Let us prove (3) $\Rightarrow$ (1). Let C satisfy the property (3). Suppose $C = C_1 \cup C_2$, where C_1 and C_2 are closed analytic. Let C be k-determined. Using the ideas of Lemma 3.9 and the fact that C_1 and C_2 are locally finite-determined closed analytic sets, it is easy to show that C_1 and C_2 are $(k+\lambda)$-determined. From here proving (1) is straightforward.

Suppose C is k-determined. Let $\{K_j\}_{j\in J}$ be the decomposition in irreducible components of $(j^k\varphi)^{-1}(\pi_k^\infty(C))$. For any $j \in J$ the set $K_j' := (\pi_k^{k+\lambda})^{-1}(K_j)$ is a irreducible component of $(j^{k+\lambda}\varphi)^{-1}(\pi_{k+\lambda}^\infty(C))$. By Lemma 3.9 the set K_j' is $j^{k+\lambda}\varphi$-saturated. Therefore each $C_j := (\pi_{k+\lambda}^\infty)^{-1} j^{k+\lambda} \varphi(K')$ is a $(k+\lambda)$-determined closed subset of $J^\infty(U,\tilde{I})$ and $\bigcup_{j\in J} C_j = C$. As we have already shown (2) $\Rightarrow$ (1) we know that each C_j is irreducible. This shows the existence of a unique irredundant decomposition in irreducible components for finite-determined subsets, and also proves (1) $\Rightarrow$ (2).

The existence and unicity of irredundant decompositions in irreducible components for non-necessarily finite-determined closed analytic subsets is easily deduced from the same property in the special case of finite-determined subsets. $\square$

Let C be a k-determined closed analytic subset of $J^\infty(U,\tilde{I})$. The irreducible components of C are not k-determined in general. This can be seen already in the simplest examples:

Example 3.11. Consider $I \subset \mathbb{C}\{z\}$ generated by z^2. Let $\varphi : \mathcal{O}_\mathbb{C} \to \mathcal{O}_\mathbb{C}$ be defined by $\varphi(f) := z^2 f$. The set $\coprod_{x\in\mathbb{C}} \mathbf{m}_x$ is a 1-determined closed subset of $J^\infty(\mathbb{C},\tilde{I})$. It has two irreducible components $C_1 = J(\mathbb{C},\tilde{I})_0$ and $C_2 = \mathbf{m}_0^3 \cup \coprod_{x\neq 0} \mathbf{m}_x$. Clearly C_2 is 2-determined, but not 1-determined.

Definition 3.12. The *codimension* $\operatorname{codim}(C)$ of a k-determined irreducible locally closed subset of $J^\infty(U,\tilde{I})$ is the codimension of $(j^k\varphi)^{-1}(pr_k^\infty(C))$ in $J^k(U,\mathbb{C}^r)$.

In the situation of the last definition, if $m \geqslant k$, then clearly $\operatorname{codim}(C)$ equals the codimension of $(j^m\varphi)^{-1}(pr_m^\infty(C))$ in $J^m(U,\mathbb{C}^r)$. It is also clear that if $C \subset C'$ are closed analytic subsets of $J^\infty(U,\tilde{I})$, and C' is irreducible, then either $C = C'$ or $\operatorname{codim}(C_i) > \operatorname{codim}(C')$ for any irreducible component C_i of C.

We now show that the codimension of C does not depend on the chosen chart:

Lemma 3.13. *Let C be a k-determined irreducible locally closed subset of $J^\infty(U,\tilde{I})$. Given any open subset $V \subset U$ and any system of s generators $\mathscr{H}$ of $\tilde{I}_{|V}$, the codimension of $(j^k\varphi_{\mathscr{H}})^{-1}(pr_k^\infty(C))$ in $J^k(V,\mathbb{C}^r)$ equals* $\operatorname{codim}(C)$.

Proof. Consider another set of generators $\mathscr{H}' = \{h'_1,\dots,h'_{s'}\}$ of $\tilde{I}_{|V'}$. If $\mathscr{H}'$ contains $\mathscr{H}$, in order to provide transition functions between the charts associated to $\mathscr{H}$ and $\mathscr{H}'$, the functions $\theta_{i,j}$ can be chosen to be $\theta_{i,j}=0$ if $j\neq i$ and $\theta_{i,i}=1$ for $i\leqslant s$. Note that the $\theta_{i,j}$'s are defined in the whole $V\cap V'$ (the set where both $\mathscr{H}$ and $\mathscr{H}'$ are defined). In this case the transition function $j^k\theta$ is a submersion for any $k<\infty$. Therefore, as $(j^k\varphi_{\mathscr{H}'})^{-1}(\pi_k^\infty(C)) = (j^k\theta)^{-1}((j^k\varphi_{\mathscr{H}})^{-1}(\pi_k^\infty(C)))$, the codimension of $(j^k\varphi_{\mathscr{H}'})^{-1}(\pi_k^\infty(C))$ in $J^k(V,\mathbb{C}^{s'})$ equals the codimension of $(j^k\varphi_{\mathscr{H}})^{-1}(\pi_k^\infty(C))$ in $J^k(V,\mathbb{C}^s)$.

Given any two sets of generators of $\tilde{I}_{|V'}$ their union gives a set of generators containing both of them. Therefore the codimension does not depend on the set of generators giving rise to the chart. □

The following natural fact is easily deduced as a consequence of the existence of decomposition in irreducible components:

Lemma 3.14. *The topological closure of a locally closed analytic subset of $J^\infty(U,\tilde{I})$ is closed analytic.*

Proof. Let C be a locally closed analytic subset of $J^\infty(U,\tilde{I})$. We have $C=A\backslash B$, with A and B closed analytic subsets of $J^\infty(U,\tilde{I})$. Let $A=\bigcup_{j\in J}A_j$ be a decomposition of A in irreducible components. Let $J'\subset J$ be the subset consisting of the indices j such that A_j is not included in B. By the local finiteness of the decomposition in irreducible components we have $\bar{C}=\bigcup_{j\in J'}\overline{A_j\backslash B}$. Therefore we are reduced to the case in which A is irreducible. We can cover $J^\infty(U,\tilde{I})$ by open subsets $\{U_l\}_{l\in L}$ such that A, U_l and $B\cap U_l$ are m_l-determined for a certain integer m_l. This means that for any $l\in L$ there exists an open subset $V_l\subset U$, a $j^{m_l}\varphi$-saturated open subset $W_l\subset J^{m_l}(V_l,\mathbb{C}^r)$ and $j^{m_l}\varphi$-saturated closed analytic subsets A_l and B_l of $J^{m_l}(V_l,\mathbb{C}^r)$ such that

$$(\pi_{m_l}^\infty)^{-1}(j^{m_l}\varphi(W_l)) = U_l,$$

$$(\pi_{m_l}^\infty)^{-1}(j^{m_l}\varphi(A_l)) = A\cap U_l,$$

$$(\pi_{m_l}^\infty)^{-1}(j^{m_l}\varphi(B_l)) = B\cap U_l.$$

As A is irreducible, any irreducible component of $B\cap A$ has strictly bigger codimension. Therefore $\overline{A_l\backslash B_l}=A_l$, and consequently (by the definition of the topology in $J^\infty(U,\tilde{I})$) the set $A\cap U_l$ equals $\bar{C}\cap U_l$. It follows that $A=\bar{C}$. □

After this lemma it makes sense to define:

Definition 3.15. A locally closed analytic subset is *irreducible* if its closure is irreducible.

Lemma 3.16. *Let C be a k-determined irreducible locally closed analytic subset of $J^{\infty}(U,\tilde{I})$. Given any open subset $V \subset U$ and any system of generators $\mathscr{H}$ of $\tilde{I}_{|V}$ we consider $A := (j^{k+\lambda}\varphi_{\mathscr{H}})^{-1}(\pi^{\infty}_{k+\lambda}(C))$. Then the set $\mathrm{Sing}(A)$ is $j^{k+\lambda}\varphi_{\mathscr{H}}$-saturated.*

Proof. In the proof of Lemma 3.9 we have shown that A is invariant by the group action $\sigma: G \times J^{k+\lambda}(V,\mathbb{C}^s) \to J^{k+\lambda}(V,\mathbb{C}^s)$ (where s is the number of generators of $\mathscr{H}$). Therefore for any $g \in G$ we have $\sigma(g,\mathrm{Sing}(A)) = \mathrm{Sing}(A)$. This, together with (23) and (26) implies

$$\mathrm{Sing}(A) + \ker(j^{k+\lambda}\varphi_{\mathscr{H},x}) \subset \mathrm{Sing}(A) \tag{29}$$

for any $x \in V$. □

Definition 3.17. Let C be locally closed analytic in $J^{\infty}(U,\tilde{I})$ and $f \in C$. We say that C *is irreducible at* f if there is only one irreducible component of C containing f. We say that C is *smooth* at f if

(1) It is irreducible at f.
(2) Let C' be the unique irreducible component of C containing f; let C' be k-determined. For any open subset $V \subset U$ containing $pr_{\infty}(f)$, any system of generators $\mathscr{H}$ of $\tilde{I}_{|V}$ and any $m \geqslant k+\lambda$ the locally closed analytic $(j^m\varphi_{\mathscr{H}})^{-1}(\pi^{\infty}_m(C'))$ is smooth at any h such that $j^m(h) = \pi^{\infty}_m(f)$.

Using an argument similar to the proof of Lemma 3.13 it is easy to check that it is enough to check smoothness at a single chart.

Remark 3.18. Taking into account Lemma 3.16 it is easy to check that it is enough to check the second condition for smoothness for a particular $m \geqslant k+\lambda$ and a particular h such that $j^m(h)=\pi^{\infty}_m(f)$. Moreover, for any locally closed analytic subset C the set $\mathrm{Sing}(C)$ of singular points is closed analytic in C, and it is $(k+\lambda)$-determined if C is k-determined.

4. The filtration by extended codimension

The natural generalisation of the subvariety of $Z_1 \subset J^2(U,\mathbb{C})$ parametrising Morse singularities is the set of germs in $J^m(U,\tilde{I})$ parametrising singularities of extended codimension equal to 1, this motivates the following:

Definition 4.1. Suppose that m is either a non-negative integer or ∞; define

$$C_m := \{f \in J^{\infty}(U,\tilde{I}):\ c_{\tilde{I}_{\pi^{\infty}(f)},e}(f) \geqslant m\},$$

$$\dot{C}_m := \{f \in J^{\infty}(U,\tilde{I}):\ c_{\tilde{I}_{\pi^{\infty}(f)},e}(f) = m\},$$

$$K_m := (j^{\infty}\varphi)^{-1}(C_m) \quad \dot{K}_m := (j^{\infty}\varphi)^{-1}(\dot{C}_m).$$

Clearly $C_{\infty} = \dot{C}_{\infty} = \bigcap_{m\in\mathbb{N}} C_m$ and $K_{\infty} = \dot{K}_{\infty} = \bigcap_{m\in\mathbb{N}} K_m$.

Lemma 4.2. *For any $x \in U$ and any $f \in \tilde{I}_x$ such that $c_{\tilde{I}_x,e}(f) = m$ we have $\mathbf{m}_x^m\tilde{I}_x \subset \Theta_{\tilde{I}_x,e}(f)$.*

Proof. The ith graded piece of $\tilde{I}_x/\Theta_{\tilde{I}_x,e}(f)$ by the $\mathbf{m}_x$-adic filtration is the module

$$M_i := \frac{\mathbf{m}_x^i\tilde{I}_x + \Theta_{\tilde{I}_x,e}(f)}{\mathbf{m}_x^{i+1}\tilde{I}_x + \Theta_{\tilde{I}_x,e}(f)}.$$

As $m = c_{\tilde{I}_x,e}(f) = \dim_{\mathbb{C}}(\tilde{I}_x/\Theta_{\tilde{I}_x,e}(f)) = \sum_{i=0}^{\infty}\dim_{\mathbb{C}}(M_i)$ we deduce that there exists $l \leqslant m$ such that $M_l = 0$, which is the same that $\mathbf{m}_x^l\tilde{I}_x \subset \Theta_{\tilde{I}_x,e}(f) + \mathbf{m}_x^{l+1}\tilde{I}_x$. Applying Nakayama's Lemma to the module $\mathbf{m}_x^l\tilde{I}_x + \Theta_{\tilde{I}_x,e}(f)/\Theta_{\tilde{I}_x,e}(f)$ we conclude $\mathbf{m}_x^l\tilde{I}_x \subset \Theta_{\tilde{I}_x,e}(f)$. □

Lemma 4.3. *For any $x \in U$ the subset*

$$A_m := \{f \in \tilde{I}_x : \mathbf{m}_x^m\tilde{I}_x \subset \Theta_{\tilde{I}_x,e}(f)\}$$

satisfies $A_m + \mathbf{m}_x^{m+2}\tilde{I}_x = A_m$.

Proof. Consider $f \in A_m$ and $g \in \mathbf{m}_x^{m+2}\tilde{I}_x$. Clearly

$$\Theta_{\tilde{I}_x,e}(f) \equiv \Theta_{\tilde{I}_x,e}(f+g) \pmod{\mathbf{m}_x^{m+1}\tilde{I}_x}.$$

Then, as $\mathbf{m}_x^m\tilde{I}_x \subset \Theta_{\tilde{I}_x,e}(f)$, we have that $\mathbf{m}_x^m\tilde{I}_x \subset \Theta_{\tilde{I}_x,e}(f+g) + \mathbf{m}_x^{m+1}\tilde{I}_x$; by Nakayama's Lemma we have $\mathbf{m}_x^m\tilde{I}_x \subset \Theta_{\tilde{I},e}(f+g)$, and hence $f+g \in A_m$. □

Lemma 4.4. *The subsets C_m and K_m are $(\lambda+m)$-determined; the subsets $\dot{C}_m$ and $\dot{K}_m$ are $(\lambda+m+1)$-determined, where λ is the uniform Artin-Rees constant.*

Proof. Obviously it is enough to prove the lemma for C_m and $\dot{C}_m$. To prove that $\dot{C}_m$ is $(\lambda+m+1)$-determined we have to check the following claim: consider $f \in \dot{C}_m$, let $pr_\infty(f) = x$, consider $g \in \mathbf{m}_x^{\lambda+m+2} \cap \tilde{I}_x$, then $f + g \in \dot{C}_m$. By the uniform Artin-Rees theorem $g \in \mathbf{m}_x^{m+2}\tilde{I}_x$; Lemma 4.2 implies $\Theta_{\tilde{I}_x,e}(f) \supset \mathbf{m}_x^m\tilde{I}_x$; then, by Lemma 4.3, we have $\Theta_{\tilde{I}_x,e}(f+g) \supset \mathbf{m}_x^m\tilde{I}_x$, and hence

$$\frac{\tilde{I}_x}{\Theta_{\tilde{I}_x,e}(f+g)} = \frac{\tilde{I}_x/\mathbf{m}_x^m\tilde{I}_x}{\Theta_{\tilde{I}_x,e}(f+g)/\mathbf{m}_x^m\tilde{I}_x}.$$

On the other hand, as $g \in \mathbf{m}_x^{m+2}\tilde{I}_x$ we have $X(g) \in \mathbf{m}_x^{m+1}\tilde{I}_x$ for any $X \in \Theta_{\tilde{I}_x,e}$. Therefore $\Theta_{\tilde{I}_x,e}(f+g)/\mathbf{m}_x^m\tilde{I}_x = \Theta_{\tilde{I}_x,e}(f)/\mathbf{m}_x^m\tilde{I}_x$, and hence $c_{\tilde{I}_x,e}(f+g) = c_{\tilde{I}_x,e}(f) = m$; this shows the claim. As $C_m = I \setminus \bigcup_{i<m}\dot{C}_i$, and each $\dot{C}_i$ is $(\lambda+i+1)$-determined, we conclude that C_m is $(\lambda+m)$-determined. □

Proposition 4.5. (1) *The subsets C_m and K_m are $(\lambda+m)$-determined closed analytic subsets of $J^\infty(U,\tilde{I})$ and $J^\infty(U,\mathbb{C}^r)$, respectively.*

(2) *The subset $\dot{C}_m$ and $\dot{K}_m$ are $(\lambda+m+1)$-determined open subsets in the analytic Zariski topology of C_m and K_m, respectively.*

Proof. It is enough to prove the statements for K_m and $\dot{K}_m$. The determinacy statements are proved in Lemma 4.4; for the rest we work by induction on m. We take as initial step $m=-1$, in this case everything is trivial. Assume that the proposition is true for any $k < m$. Then K_{m-1} is closed analytic in $J^\infty(U,\mathbb{C}^r)$ and $\dot{K}_{m-1} \subset K_{m-1}$ is an open inclusion in the analytic Zariski topology of K_{m-1}; as $K_m := K_{m-1}\setminus\dot{K}_{m-1}$ the subset K_m is closed analytic in K_{m-1}, and hence in $J^\infty(U,\mathbb{C}^r)$. This shows the first assertion.

Let $\{\xi_1,\ldots,\xi_k\}$ be a system of generators of $\Theta_{I,e}$ as a $\mathcal{O}_{\mathbb{C}^n,O}$-module; we can assume (perhaps shrinking U) that each ξ_i is defined on U and that the germs $\{\xi_{1,x},\ldots,\xi_{k,x}\}$ generate $\Theta_{\tilde{I}_x,e}$ for any $x\in U$. Let $h_1,\ldots,h_l\in\mathcal{O}_{\mathbb{C}^n,O}$ be the set of monomials in $x_1,\ldots,x_n$ of degree lower or equal than $\lambda+m+2$; then $J^{\lambda+m+1}(U,\mathbb{C})_x$ is generated by $\{j^{\lambda+m+1}h_1(x),\ldots,j^{\lambda+m+1}h_s(x)\}$ for any $x\in U$. Therefore, defining $\{\theta_1,\ldots,\theta_d\}:=\{h_i\xi_j:1\leqslant i\leqslant l,\ 1\leqslant j\leqslant k\}$, the set $\{\theta_{1,x},\ldots,\theta_{s,x}\}$ generates $\Theta_{\tilde{I}_x,e}/\mathbf{m}_x^{\lambda+m+2}\Theta_{\tilde{I}_x,e}$ as a complex vector space for any $x\in U$; hence it also generates $\Theta_{\tilde{I}_x,e}/\mathbf{m}_x^{\lambda+m+2}\cap\Theta_{\tilde{I}_x,e}$.

Recall that we have fixed a set of functions $g_1,\ldots,g_r$ generating $\tilde{I}$ in U. For any $k\leqslant d$ define an analytic vector bundle homomorphism

$$\alpha_k:J^{\lambda+m+2}(U,\mathbb{C}^r)\to J^{\lambda+m+1}(U,\mathbb{C})$$

by the formula

$$\alpha_k(j^{\lambda+m+2}f_1(x),\ldots,j^{\lambda+m+2}f_r(x))$$
$$:=\sum_{i=1}^{r}j^{\lambda+m+1}f_i(x)j^{\lambda+m+1}\theta_k(g_i)(x)+j^{\lambda+m+1}\theta_k(f_i)(x)j^{\lambda+m+1}g_i(x),$$

where $x\in U$, $f_1,\ldots,f_r\in\mathcal{O}_{\mathbb{C}^n,x}$. The mapping α_k is defined so that

$$j^{\lambda+m+1}\theta_k(\varphi(f_1,\ldots,f_r))(x)=\alpha_k(j^{\lambda+m+2}f_1(x),\ldots,j^{\lambda+m+2}f_r(x)). \tag{30}$$

Given any $f\in J^{\lambda+m+2}(U,\mathbb{C}^r)$ such that $pr_{\lambda+m+2}(f)=x$, denote by $S(f)$ the subspace of $J^{\lambda+m+1}(U,\mathbb{C})_x$ spanned by $\{\alpha_k(f):1\leqslant k\leqslant d\}$. For any $f\in J^{\lambda+m+2}(U,\mathbb{C}^r)$ define $\mathrm{rk}(f):=\dim_{\mathbb{C}}(S(f))$. We can consider $J^{\lambda+m+1}(U,\tilde{I})_x$ as a subspace of $J^{\lambda+m+1}(U,\mathbb{C})_x$ via the canonical isomorphism $\tilde{I}_x/\mathbf{m}_x^{\lambda+m+2}\cap\tilde{I}_x\cong\tilde{I}_x+\mathbf{m}_x^{\lambda+m+2}/\mathbf{m}_x^{\lambda+m+2}$. Then clearly $S(f)\subset J^{\lambda+m+1}(U,\tilde{I})_x$. Taking into account (30) we deduce easily

$$J^{\lambda+m+1}(U,\tilde{I})_x/S(f)=\tilde{I}_x/(\Theta_{\tilde{I}_x,e}(\varphi_x(h))+\mathbf{m}_x^{\lambda+m+2}\cap\tilde{I}_x) \tag{31}$$

for any $h\in\mathcal{O}^r_{\mathbb{C}^n,x}$ such that $\pi^\infty_{\lambda+m+2}(h)=f$.

Let Σ_i be an $\tilde{I}$-stratum; consider the positive integer $N_i:=\mu_x(\lambda+m+1)$, where x is any point in Σ_i and μ_x is the function defined in Formula 13. Define the closed analytic subset $T_i\subset J^{\lambda+m+2}(U,\mathbb{C}^r)$ as follows:

$$T_i:=\{f\in J^{\lambda+m+2}(\bar{\Sigma}_i,\mathbb{C}^r):\ \mathrm{rk}(f)<N_i-m\}.$$

Suppose that $h\in\dot{K}_m$, consider the $\tilde{I}$-stratum Σ_i such that $x:=pr_\infty(h)\in\Sigma_i$. Lemma 4.2 implies that $\Theta_{\tilde{I}_x,e}(\varphi(h)_x)$ contains $\mathbf{m}_x^m\tilde{I}_x$, which, by uniform Artin-Rees, contains $\mathbf{m}_x^{\lambda+m+2}\cap\tilde{I}_x$. This, together with equality (31), implies that

$$N_i-\mathrm{rk}(j^{\lambda+m+2}h(x))=c_{\tilde{I}_x,e}(\varphi(h)_x)=m. \tag{32}$$

Therefore

$$\dot{K}_{m|\Sigma_i}\subset K_m\backslash(\pi^\infty_{\lambda+m+2})^{-1}(T_i). \tag{33}$$

The union

$$M := \coprod_{x \in U} (\mathbf{m}_x^m / \mathbf{m}_x^{\lambda+m+2})^r$$

is clearly an analytic sub-bundle of $J^{\lambda+m+1}(U, \mathbb{C}^r)$. Consider the vector bundle homomorphism $j^{\lambda+m+1}\varphi$: $J^{\lambda+m+1}(U, \mathbb{C}^r) \to J^{\lambda+m+1}(U, \mathbb{C})$. The image of $j^{\lambda+m+1}\varphi_{|M}$ is the union

$$L := \coprod_{x \in U} L_x,$$

where $L_x := \mathbf{m}_x^m \tilde{I}_x + \mathbf{m}_x^{\lambda+m+2}/\mathbf{m}_x^{\lambda+m+2}$. We consider the stratification of U in locally closed analytic subsets $\Sigma_{i,j}$ defined to be the minimal common refinement of the Hilbert-Samuel stratification with respect to $\tilde{I}$, and the minimal stratification such that the rank of $j^{\lambda+m+1}\varphi_{|M}$ restricted to each stratum is constant. Given any $\Sigma_{i,j}$, the restriction $L_{|\Sigma_{i,j}}$ is an analytic vector bundle over it.

Consider any stratum $\Sigma_{i,j}$. Let Y be an irreducible component of the locally closed $(\lambda + m + 2)$-determined subset $K_{m|\Sigma_{i,j}} \backslash (\pi^\infty_{\lambda+m+2})^{-1}(T_i)$.

Claim 1. *If Y and $\dot{K}_m$ have non-empty intersection then $Y \subset \dot{K}_m$.*

Let us prove the claim: as $\mathrm{rk}(f) \geqslant N_i - m$ for any $f \in J^{\lambda+m+2}(\Sigma_i, \mathbb{C}^r) \backslash T_i$, the $\lambda+m+2$ determined set

$$Y' := \{f \in Y : \mathrm{rk}(\pi^\infty_{\lambda+m+2}(f)) = N_i - m\}$$

is closed analytic in Y (it is the locus where $\mathrm{rk}(pr^\infty_{\lambda+m+2}(f))$ is minimal in Y). By (32) we have $\dot{K}_m \cap Y \subset Y'$.

By Lemma 4.2, if $f \in \dot{K}_m$ and $x = pr_{\lambda+m+2}(f)$ then $S(\pi^\infty_{\lambda+m+2}(f))$ contains the linear subspace L_x. Conversely, suppose that $f \in Y'$ satisfies

$$S(\pi^\infty_{\lambda+m+2}(f)) \supset L_x.$$

Using equality (31), we deduce

$$\Theta_{\tilde{I}_x,e}(\varphi(f)_x) + \mathbf{m}_x^{\lambda+m+2} \cap \tilde{I}_x \supset \mathbf{m}_x^m \tilde{I}_x.$$

By uniform Artin-Rees $\mathbf{m}_x^{\lambda+m+2} \cap \tilde{I}_x \subset \mathbf{m}_x^{m+2}\tilde{I}_x$, and therefore

$$\mathbf{m}_x^m \tilde{I}_x \subset \Theta_{\tilde{I}_x,e}(\varphi(f)_x) + \mathbf{m}_x^{m+2}\tilde{I}_x.$$

Using Nakayama's lemma we conclude that $\Theta_{\tilde{I}_x,e}(\varphi(f)_x)$ contains $\mathbf{m}_x^m \tilde{I}_x$, which in turn contains $\mathbf{m}_x^{\lambda+m+2} \cap \tilde{I}_x$. Therefore

$$c_{\tilde{I}_x,e}(\varphi(f)_x) = \dim_{\mathbb{C}} \left(\frac{\tilde{I}_x}{\Theta_{\tilde{I}_x,e}(\varphi(f)_x) + \mathbf{m}_x^{\lambda+m+2} \cap \tilde{I}_x} \right),$$

which, by the equality (31) is equal to $N_i - \mathrm{rk}(f) = N_i - (N_i - m) = m$; consequently $f \in \dot{K}_m$. We have shown

$$\dot{K}_m \cap Y = \{f \in Y' \colon S(\pi^\infty_{\lambda+m+2}(f)) \supset L_x\}.$$

We consider the restriction $pr_{\lambda+m+2}:\pi^{\infty}_{\lambda+m+2}(Y'_{|\Sigma_{i,j}}) \to \Sigma_{i,j}$, the pullback vector bundle $pr^*_{\lambda+m+2}(J^{\lambda+m+1}(U,\mathbb{C})_{|\Sigma_{i,j}})$, and the sub-bundle $pr^*_{\lambda+m+2}(L_{|\Sigma_{i,j}})$. Taking into account that $S(f)$ has constant dimension when f ranges in Y' it is easy to check that the union

$$S := \coprod_{f\in\pi^{\infty}_{\lambda+m+2}(Y'_{|\Sigma_{i,j}})} S(f)$$

is an analytic vector sub-bundle of $pr^*_{\lambda+m+2}(J^{\lambda+m+1}(U,\mathbb{C})_{|\Sigma_{i,j}})$. Let Y'' be the closed analytic subset of Y' formed by points f such that the fibre $S_{\pi^{\infty}_{\lambda+m+2}(f)}$ contains the fibre $(pr^*_{\lambda+m+2}L_{|\Sigma_{i,j}})_{\pi^{\infty}_{\lambda+m+2}(f)}$. As $\dot{K}_m \cap Y$ is the subset of Y' formed by germs such that $S(\pi^{\infty}_{\lambda+m+2}(f))$ contains L_x, we deduce that $\dot{K}_{m|\Sigma_{i,j}} = Y''$. We have shown that $\dot{K}_m \cap Y$ is a closed analytic subset of Y', and hence of Y.

Since Y is included in K_m, if we have $f\in Y\backslash\dot{K}_m$ and $x := pr_{\infty}(f)$ then $c_{\tilde{I}_x,e}(\varphi(f)_x) > m$. If $Y \not\subset \dot{K}_m$ then $\dot{K}_m \cap Y$ is a $(\lambda+m+2)$-determined proper analytic subset of the irreducible analytic subset Y. Then it is possible to find an analytic path $\gamma: D \to \pi^{\infty}_{\lambda+m+2}(Y)$, from a small disk D to Y such that $\gamma^{-1}(\pi^{\infty}_{\lambda+m+2}(\dot{K}_m)) = \{0\}$.

Consider the analytic trivialisation

$$\tau_{\lambda+m+2}: U\times(\mathbb{C}\{x\}/\mathbf{m}^{\lambda+m+3})^r \to J^{\lambda+m+2}(U,\mathbb{C}^r)$$

defined in (12). There exist an analytic path $\xi: D\to U$ and polynomials $h_{1,t},\ldots,h_{r,t}$ of degree $\lambda+m+2$ in $x_1,\ldots,x_n$, with coefficients depending analytically on t such that $\gamma(t)=\tau_{\lambda+m+2}(\xi(t),(h_{1,t},\ldots,h_{r,t}))$ for any $t\in D$. For any t,i let $\tilde{h}_{i,r}$ be the unique function in $\mathcal{O}_{\mathbb{C}^n,O}$ whose Taylor expansion at $\xi(t)$ equals $h_{i,t}$; clearly $\tilde{h}_{i,t}$ is a polynomial whose the coefficients depend analytically on t. The I-unfolding

$$F:\mathbb{C}^n\times D\to\mathbb{C}$$

defined by $F(x,t) := \varphi(\tilde{h}_{1,t},\ldots,\tilde{h}_{r,t})(x)$ satisfies

$$c_{\tilde{I}_{\xi(O)},e}(F_{|0,\xi(O)}) = m \quad\text{and}\quad c_{\tilde{I}_{\xi(t)},e}(F_{|0,\xi(t)}) > m,$$

which contradicts the upper-semicontinuity of the extended codimension (see Corollary 1.9). This establishes Claim 1.

Decompose each $K_m\cap(\pi^{\infty}_{\lambda+m+2})^{-1}(T_i)$ (for $1\leqslant i\leqslant s$) in irreducible components; we say that an irreducible component Z is *relevant* if $pr_{\lambda+m+2}(Z)\cap\Sigma_i\neq\emptyset$. For any stratum $\Sigma_{i,j}$, decompose $K_{m|\bar{\Sigma}_{i,j}}$ in irreducible components, and say that an irreducible component is *excessive* if it is contained in K_{m+1}. Define the closed analytic subset of $T\subset K_m$ to be the union of all the excessive components of each of the $K_{m|\bar{\Sigma}i,j}$'s and all the relevant components of each of the $K_m\cap(\pi^{\infty}_{\lambda+m+2})^{-1}(T_i)$'s.

Claim 2. $T = K_{m+1}$.

If the claim is true the induction step is complete, and hence the proposition is proved.

To prove the claim consider $f\in K_m\backslash T$, and let $\Sigma_{i,j}$ be the stratum such that $x = pr_{\infty}(f)\in\Sigma_{i,j}$. As f does not belong to T, it cannot be in $(\pi^{\infty}_{\lambda+m+2})^{-1}(T_i)$. Let Y be the irreducible component of $K_{m|\bar{\Sigma}_{i,j}}$ to which f belongs. As $f\notin T$, the component Y cannot be excessive, and therefore $Y\cap\dot{K}_m$ is non-empty. If $Y_{|\Sigma_{i,j}}\cap\dot{K}_m\neq\emptyset$ then, by Claim 1 we have $Y_{|\Sigma_{i,j}}\subset\dot{K}_m$; therefore $f\in\dot{K}_m$. If $Y_{|\Sigma_{i,j}}\cap\dot{K}_m=\emptyset$ then $Y_{|\Sigma_{i,j}}\subset K_{m+1}$. Hence $Y\cap\dot{K}_m$ is included in the proper analytic subset $Y_{|\partial\Sigma_{i,j}}$, where $\partial\Sigma_{i,j}$ is the difference $\bar{\Sigma}_{i,j}\setminus\Sigma_{i,j}$. As $Y_{|\Sigma_{i,j}} \subset K_{m+1}$ is Zariski open in the finitely-determined irreducible closed analytic subset Y, it is possible to find an analytic path $\gamma: D \to \pi^{\infty}_{\lambda+m+2}(Y)$, from a disk to Y such that $\gamma^{-1}(\dot{K}_m) = \{0\}$. As we have seen previously this leads to a contradiction; we conclude $Y_{|\Sigma_{i,j}} \cap \dot{K}_m \neq \emptyset$, and hence $f \in \dot{K}_m$. This shows $K_{m+1} \subset T$.

Suppose $f \in T$. If f is in a excessive component of $K_{m|\bar{\Sigma}i,j}$ for some i,j then $f \in K_{m+1}$. Otherwise $\pi^{\infty}_{\lambda+m+2}(f)$ is in a relevant component Y of $K_m \cap (\pi^{\infty}_{\lambda+m+2})^{-1}(T_i)$ for some i. By (33) we have $K_m \cap (\pi^{\infty}_{\lambda+m+2})^{-1}(T_{i|\Sigma_i}) \subset K_{m+1}$; therefore $Y \cap \dot{K}_m$ is contained in the proper analytic subset $Y_{|\partial\Sigma_i}$. As Y is irreducible we conclude as before that $Y \subset K_{m+1}$. This completes the proof of the claim. □

The next proposition gives a codimension bound for the irreducible components of K_1 or C_1 that will be interesting for us (those corresponding to functions of finite extended codimension).

Proposition 4.6. *If Y is an irreducible component of K_1 (resp. C_1) of codimension strictly smaller than n, then Y is included in K_∞ (resp. C_∞).*

Proof. It is enough to work with components of K_1. Let $Y \subset K_1$ be such a component. Suppose that $Y \not\subset K_\infty$; then there is $h \in Y$ such that $c_{\tilde{I}_x,e}(j^\infty\varphi(h)(x))$ is finite, where $x = pr_\infty(h)$. The element h is the germ at x of an analytic function from a neighbourhood W of x to $\mathbb{C}^r$, with components $h_1,\ldots,h_r$; the function $g := \varphi(h)$ is a section of $\tilde{I}$ over W, whose germ g_x at x equals $j^\infty\varphi(h)$. Consider the analytic mapping $j^\infty h: W \to J^\infty(W,\mathbb{C}^r)$; the locus $(j^\infty h)^{-1}(Y)$ consists of points in which g has positive extended codimension. As $c_{\tilde{I}_x,e}(g_x)$ is finite, the associated sheaf $\mathscr{F}$, defined in formula (3), is skyscraper at x, and hence $\dim_x((j^\infty h)^{-1}(Y)) = 0$. On the other hand, the following easy statement implies that the last dimension is positive, giving a contradiction.

(†) *Let $\rho: (X,x) \to (Z,z)$ be an analytic morphism of germs of complex spaces, with (Z,z) smooth. Consider an analytic subgerm (Y,z) of (Z,z) such that* $\operatorname{codim}_z(Y,Z) < \dim_x(X)$, *then* $\dim_x(\rho^{-1}(Y)) > 0$.

If $\dim_x(\rho^{-1}(z)) > 0$ we are done. Assume $\dim_x(\rho^{-1}(z)) = 0$; by the Proposition on p. 63 of [8], shrinking X and Z we can assume that ρ is finite, and therefore $(\rho(X),z)$ is a closed analytic subspace of dimension equal to $\dim_x(X)$. As (Z,z) is smooth, $\dim_z(Y \cap \rho(X)) \geqslant \dim_z(\rho(X)) - \operatorname{codim}_z(Y,Z) > 0$. Then $\dim_x(\rho^{-1}(Y)) > 0$. □

5. The topological partition

Functions of the same extended codimension do not need to have the same topological type, even if they lie in a connected family. For example, when I defines a smooth subvariety of dimension d, all the singularity types $D(d,k)$ defined in [18] have zero extended codimension with respect to I^2 and pairwise different topological types. We need to subdivide the subvarieties $\dot{K}_m$ so that the functions on the (relevant) resulting pieces have constant topological type. As we will show there is a canonical way to do it.

Any representative $\phi: V \to U$ of a germ $\phi \in \mathscr{D}_{\tilde{I}_x,e}$ induces by push-forward a bijection

$$j^\infty\phi_*: J^\infty(V,\tilde{I}) \to J^\infty(\phi(V),\tilde{I}), \tag{34}$$

defined by $j^\infty\phi_*(f_y) := (f_y \circ \phi^{-1})_{\phi(y)}$ for any $y \in V$ and $f_y \in \tilde{I}_y$. As the $\mathbf{m}_y$-adic filtration is transformed by push-forward into the $\mathbf{m}_{\phi(y)}$-adic filtration, for any $m < \infty$, the mapping $j^\infty\phi_*$ descends to a bijection

$$j^m\phi_* : J^m(V,\tilde{I}) \to J^m(\phi(V),\tilde{I}). \tag{35}$$

The functions $g_1 \circ \phi^{-1},\dots,g_r \circ \phi^{-1}$ are sections of $\tilde{I}$ defined over $\phi(V)$ (where $g_1,\dots,g_r$ is the fixed set of generators for $\tilde{I}$ at U). As $\tilde{I}$ is generated over U by $g_1,\dots,g_r$, if we shrink V enough we can assume that for any $i \leqslant r$ we have an expression of the form

$$g_i \circ \phi^{-1} = \sum_{j=1}^{r} h_{i,j} g_j,$$

where each $h_{i,j}$ is an analytic function defined on $\phi(V)$. Consequently, if $f \in \Gamma(V,\tilde{I})$ is of the form $f = \sum_{i=1}^{r} f_i g_i$, then

$$f \circ \phi^{-1} = \sum_{i=1}^{r} (f_i \circ \phi^{-1}) g_i \circ \phi^{-1} = \sum_{i=1}^{r} \left(\sum_{j=1}^{r} f_j \circ \phi^{-1} h_{j,i} \right) g_i. \tag{36}$$

For any $m \leqslant \infty$ the mapping

$$j^m\tilde{\phi}_* : J^k(V,\tilde{I}) \to J^m(\phi(V),\tilde{I}), \tag{37}$$

defined by the formula

$$j^m\tilde{\phi}_*(j^m f_1(x),\dots,j^m f_r(x)) := \left(\dots, j^m \left(\sum_{j=1}^{r} f_j \circ \phi^{-1} h_{j,i} \right) (\phi(x)), \dots \right)$$

defines a mapping which is analytic for $m < \infty$, and, by Eq. (36), satisfies $j^m\varphi \circ j^m\tilde{\phi}_* = j^m\phi_* j^m\varphi$. Therefore we say that $j^m\tilde{\phi}_*$ is an *analytic local lifting* of $j^m\phi_*$.

Definition 5.1. Let $T \subset J^\infty(U,\tilde{I})$ be a (locally) closed analytic subset. We say that T is $\mathscr{D}_{\tilde{I}_e}$*-invariant* if for any analytic diffeomorphism $\phi : V \to U$ preserving $\tilde{I}$ and any $x \in V$, we have $\phi_{t*}(T_x) = T_{\phi_t(x)}$.

Clearly the set C_m is $\mathscr{D}_{\tilde{I},e}$-invariant for any $m \leqslant \infty$.

Now we recall briefly the results of [4]:

Definition 5.2. Two germs $f : (\mathbb{C}^n,x) \to \mathbb{C}$ and $g : (\mathbb{C}^n,y) \to \mathbb{C}$ are called *topologically equivalent* if there exit germs of homeomorphisms $\phi : (\mathbb{C}^n,x) \to (\mathbb{C}^n,y)$ and $\alpha : (\mathbb{C},f(x)) \to (\mathbb{C},g(x))$ such that $\alpha \circ f = g \circ \phi$.

A closed subset $A \subset J^\infty(U,\tilde{I})$ is called *residual* if for any positive integer c there is a closed analytic subset T' containing A, and such that all its irreducible components are of codimension at least c.

Let T be an irreducible locally closed subset of $J^\infty(U,\tilde{I})$, by [4, Proposition 10], there exists a unique closed subset $\Gamma \subset T$ with the following properties:

(i) We have a decomposition $\Gamma = \Gamma^{(a)} \cup \Gamma^{(r)}$ where $\Gamma^{(a)}$ is a closed analytic subset of T, and $\Gamma^{(r)}$ is a residual closed subset.
(ii) Any $f, g \in T$ in the same path-connected component of $T \setminus \Gamma$ are topologically equivalent.
(iii) The subset Γ is minimal among the subsets of T satisfying Properties (i) and (ii).

The subset Γ is called the *topological discriminant* of T. The decomposition $\Gamma = \Gamma^{(a)} \cup \Gamma^{(r)}$ is unique provided that Γ is chosen to be minimal; we say that $\Gamma^{(a)}$ and $\Gamma^{(r)}$ are respectively the *analytic and residual parts* of Γ.

In [4] it was proved that if V is an open subset of U then $\Gamma_{|V}$ is the topological discriminant of $T_{|V}$.

Lemma 5.3. *If T is $\mathscr{D}_{\tilde{I},e}$-invariant then Γ is also $\mathscr{D}_{\tilde{I},e}$-invariant.*

Proof. Let $\phi : V \to \phi(V)$ be an analytic diffeomorphism preserving $\tilde{I}$. It is enough to show

$$\phi_*(\Gamma_{|V}) \supset \Gamma_{|\phi(V)} \tag{38}$$

(for the opposite inclusion we consider the analogous statement for ϕ_*^{-1} and apply ϕ_*). The set $\Gamma' := \phi_*(\Gamma_{|V}) \cap \Gamma_{|\phi(V)}$ is a closed subset which is the union of the analytic subset $\Gamma^{(a)}_{\phi(V)} \cap \phi_*(\Gamma^{(a)}_{|V})$ and the residual subset

$$[\Gamma^{(r)}_{|\phi(V)} \cap \phi_*(\Gamma_{|V})] \cup [\Gamma_{|\phi(V)} \cap \phi_*(\Gamma^{(r)}_{|V})].$$

If any two germs that can be connected by a continuous path in $T_{|\phi(V)} \setminus \Gamma'$ have the same topological type then Γ' must contain the topological discriminant $\Gamma_{|\phi(V)}$ of $T_{|\phi(V)}$, and therefore inclusion (38) holds.

Let $\gamma : [0,1] \to T_{|\phi(V)} \setminus \Gamma'$ be a continuous path. The interval $[0,1]$ is the union of the open subsets $\gamma^{-1}(T_{|\phi(V)} \setminus \Gamma_{|\phi(V)})$ and $\gamma^{-1}(T_{|\phi(V)} \setminus \phi_*(\Gamma_{|V})$. As over each of these subsets the topological type clearly remains constant, then it also does along γ. □

Shrink U so that its closure is contained in an open subset where $\Theta_{\tilde{I},e}$ is generated by global sections, then the Main Theorem of [4], applied to $T = J^\infty(U,\tilde{I})$, gives:

Theorem 5.4. *There exist a unique filtration (which we call the filtration by successive discriminants)*

$$J^\infty(U,\tilde{I}) = A_0 \supset A_1 \supset \cdots \supset A_i \supset \cdots$$

by closed analytic subsets, and a residual subset $\Gamma^{(r)}$ (called the cumulative residual topological discriminant *of $J^\infty(U,\tilde{I})$), with the following properties*:

(1) *We have $\bigcap_{i \geqslant 0} A_i \subset \Gamma^{(r)}$.*
(2) *For any $i \geqslant 0$ the set $A_{i+1} \cup (\Gamma^{(r)} \cap A_i)$ is the topological discriminant of A_i.*
(3) *Any irreducible component of A_i has codimension at least i.*

(4) *If T is $\mathscr{D}_{\tilde{I},e}$-invariant then A_i is $\mathscr{D}_{\tilde{I},e}$-invariant for any $i \geqslant 0$. The set $\Gamma^{(r)}$ is contained in a residual subset which is an intersection of $\mathscr{D}_{\tilde{I},e}$-invariant closed analytic subsets of C.*

We need to make a remark concerning the level of generality of this paper in comparison to [4]. Instead of working with analytic subsets of $J^\infty(U,\tilde{I})$, in [4] our attention was restricted to closed analytic subsets of $J^\infty(\Sigma_i,\tilde{I})$, where Σ_i is a certain $\tilde{I}$-stratum of U. Once the existence of decompositions in irreducible components has been proved in our setting, all the arguments of [4] can be translated with minimal changes to prove the statements given here. Another difference is in Property 4: here we state the $\mathscr{D}_{\tilde{I},e}$-invariance of certain subsets, and in [4] we state flow-invariance, which is a weaker property. The stronger property holds because here we work over $\mathbb{C}$, and in [4] we considered also the real case; actually Lemma 5.3 provides the required additional arguments to [4].

By Theorem 5.4(4) there exists a $\mathscr{D}_{\tilde{I},e}$-invariant closed analytic subset Δ_1 containing $\Gamma^{(r)}$, whose irreducible components have codimension at least $n+1$. We can suppose $\Gamma^{(r)} = \bigcap_{k\in\mathbb{N}} \Gamma_k$, being each Γ_k closed analytic with the properties $\Delta_1 \supset \Gamma_k$ and $\Gamma_k \supset \Gamma_{k+1}$. For any $i \leqslant n$ let $\{A_{i,j}\}_{j\in J_i}$ be the set of irreducible components of $A_i \backslash A_{i+1}$ of codimension smaller or equal than n. The subset $A_{i,j}\backslash\Gamma^{(r)}$ is path-connected for any i,j, as it is the union $\bigcup_{k\in\mathbb{N}} A_{i,j}\backslash\Gamma_k$, where the $A_{i,j}\backslash\Gamma_k$ are an increasing sequence of irreducible (and hence connected) locally closed subsets. We conclude that all the germs in $A_{i,j}\backslash\Gamma^{(r)}$ have the same topological type.

We say that two components $A_{i,j}$ and $A_{i,j'}$ are equivalent if their respective generic germs have the same topological type. Non-equivalent $A_{i,j}$ and $A_{i,j'}$ are disjoint. If this were not the case let Y be an irreducible component of their intersection. As $\Gamma^{(r)}$ is contained in closed subsets of arbitrarily large codimension we deduce that Y is not contained in $\Gamma^{(r)}$. Therefore the topological type of a generic germ in Y coincides with the generic topological type in $A_{i,j}$ and $A_{i,j'}$. This contradicts the non-equivalence of $A_{i,j}$ and $A_{i,j'}$. Let L_i be the set of equivalence classes and for any $l \in L_i$ define $B_{i,l}$ as the union of all the subsets of the class l. Consider the decomposition

$$\bigcup_{j\in J_i} A_{i,j} = \coprod_{l\in L_i} B_{i,l}.$$

Clearly the sets $B_{i,l}$ are closed analytic in $A_i\backslash A_{i+1}$ and the elements of $B_{i,l}\backslash\Gamma^{(r)}$ and $B_{i,l}\backslash\Delta_1$ are germs which have pairwise the same topological type.

Define $\Delta_{2,i}$ to be the union of all the irreducible components of A_i of codimension at least $n+1$. Define $\Delta_2 := \bigcup_{j\in\mathbb{N}} \Delta_{2,j}$. As Δ_2 contains A_{n+1} the union is easily seen to be locally finite, and hence Δ_2 is an analytic closed subset.

We have a locally finite partition

$$J^\infty(U,\tilde{I}) := \left[\coprod_{i\leqslant n, l\in L_i} B_{i,l}\backslash(\Gamma^{(r)}\cup\Delta_2)\right]\coprod(\Gamma^{(r)}\cup\Delta_2) \tag{39}$$

in disjoint subsets such that for any $i \leqslant n$ and $l \in L_i$ the set $B_{i,l}$ is a $\mathscr{D}_{\tilde{I},e}$-invariant locally closed analytic subset in $J^\infty(U,\tilde{I})$ such that any two germs in $B_{i,l}\backslash\Gamma^{(r)}$ have the same topological type. All the subsets of the partition are canonically defined.

Definition 5.5. The partition introduced above is called the *topological partition of* $J^\infty(U,\tilde{I})$ up to codimension n, and is canonically defined.

6. Whitney stratifications in $J^\infty(U,\tilde{I})$

Let C be a locally closed analytic subset of $J^\infty(U,\tilde{I})$. A *stratification of* C is a partition of C in a locally finite family $\{X_j\}_{j\in J}$ of disjoint smooth irreducible locally closed analytic subsets of $J^\infty(U,\tilde{I})$. Given two smooth irreducible locally closed analytic subsets X and Y of $J^\infty(U,\tilde{I})$, we say that X is *Whitney-regular* over Y if for any k such that both X and Y are k-determined, and for any open subset $V\subset U$ and any system of generators $\mathscr{H}$ of $\tilde{I}_{|V}$, the subset $(j^k\varphi_{\mathscr{H}})^{-1}(X)$ is Whitney-regular over $(j^k\varphi_{\mathscr{H}})^{-1}(Y)$ (this makes sense as $(j^k\varphi_{\mathscr{H}})^{-1}(X)$ and $(j^k\varphi_{\mathscr{H}})^{-1}(Y)$ are submanifolds of $J^k(U,\mathbb{C}^r)$).

It is easy to show that given two smooth locally closed analytic subsets $X,Y\subset J^k(U,\mathbb{C}^r)$, a positive integer $m\geqslant k$, and a point $y\in(\pi^m_k)^{-1}(Y)$ then $(\pi^m_k)^{-1}(X)$ is Whitney-regular over $(\pi^m_k)^{-1}(Y)$ at a point y if and only if X is Whitney-regular over Y at $\pi^m_k(y)$. Therefore it is enough to check the Whitney regularity condition for a particular k. An argument similar to the proof of Lemma 3.13 shows that to prove Whitney regularity at a point it is enough to check it at a single chart containing the point.

A *Whitney* stratification of a locally closed analytic subset of $J^\infty(U,\tilde{I})$ is a stratification of it such that any stratum is Whitney regular over any other stratum.

We will make use of the following fact (see [24]): let X and Y be two irreducible locally closed analytic subsets of a complex manifold, such that $\dim(X)>\dim(Y)$. Denote by X_{sm} and Y_{sm} the set of smooth points of X and Y. There exists a unique minimal proper closed analytic subset $W(X,Y)$ of Y containing $\mathrm{Sing}(Y)$ such that X_{sm} is Whitney regular over $Y_{sm}\backslash W(X,Y)$. Moreover the set of points $y\in Y_{sm}$ such that X is not Whitney regular over Y at y is dense in $W(X,Y)\cap Y_{sm}$.

Lemma 6.1. *Let X and Y be two k-determined irreducible locally closed analytic subsets of $J^\infty(U,\tilde{I})$ such that* $\mathrm{codim}(X,J^\infty(U,\tilde{I}))<\mathrm{codim}(Y,J^\infty(U,\tilde{I}))$. *There exists a unique minimal $(k+\lambda)$-determined proper closed analytic subset $W(X,Y)$ of Y such that X_{sm} is Whitney regular over $Y_{sm}\backslash W(X,Y)$. Moreover the set of points $y\in Y_{sm}$ such that X is not Whitney regular over Y at y is dense in $W(X,Y)\cap Y_{sm}$. In addition, if X and Y are both $\mathscr{D}_{\tilde{I},e}$-invariant, then $W(X,Y)$, $\mathrm{Sing}(X)$ and $\mathrm{Sing}(Y)$ are $\mathscr{D}_{\tilde{I},e}$-invariant.*

Proof. Let k be such that both X and Y are k-determined. Define

$$X':=(j^k\varphi)^{-1}(\pi^\infty_k(X))\quad Y':=(j^k\varphi)^{-1}(\pi^\infty_k(Y)).$$

The sets X' and Y' are locally closed analytic subsets of $J^k(U,\mathbb{C}^r)$ such that $\dim(X')>\dim(Y')$; consider the set $W(X',Y')$. If we prove that for a certain $m\geqslant k$ the set $W:=(\pi^m_k)^{-1}(W(X',Y'))$ is $j^m\varphi$-saturated, then the set $W(X,Y):=(\pi^\infty_m)^{-1}(j^m\varphi(W))$ clearly satisfies all the desired properties, with the exception of $\mathscr{D}_{\tilde{I},e}$-invariance, which will be proved later.

We will prove the $j^m\varphi$-saturation for $m=k+\lambda$, where λ is the uniform Artin-Rees constant. We have to check

$$W_x+\ker(j^m\varphi_x)\subset W_x \tag{40}$$

for any $x\in U$. Because of inclusion (23) it is enough to show that W is invariant by the group action (28). The sets W and $W((\pi^m_k)^{-1}(X'),(\pi^m_k)^{-1}(Y'))$ are clearly equal; therefore, the subset Z

of $(\pi_k^m)^{-1}(Y')$ formed by the singular locus of $(\pi_k^m)^{-1}(Y')$ and the points where $(\pi_k^m)^{-1}(X')$ is not Whitney-regular over $(\pi_k^m)^{-1}(Y')$ is dense in W. As $(\pi_k^m)^{-1}(X')$ and $(\pi_k^m)^{-1}(Y')$ are invariant by the action (28) and Whitney regularity is preserved by diffeomorphisms, then Z is also invariant by the action. This implies in turn the invariance of W.

It remains to be prove the $\mathscr{D}_{\tilde{I},e}$-invariance of $W(X,Y)$.

Consider an open subset V of U and a set $\mathscr{H}=\{h_1,\dots,h_s\}$ which generate $\tilde{I}_{|V}$. As Whitney regularity can be checked with respect to any chart, we have

$$W(X,Y)\cap J^{\infty}(V,\tilde{I})=(\pi_m^{\infty})^{-1}(j^m\varphi_{\mathscr{H}}(W')),$$

where $W'=(\pi_k^{k+\lambda})^{-1}(W(X'_{\mathscr{H}},Y'_{\mathscr{H}}))$, with $X'_{\mathscr{H}}:=(j^k\varphi_{\mathscr{H}})^{-1}(\pi_k^{\infty}(X))$ and $Y'_{\mathscr{H}}:=(j^k\varphi_{\mathscr{H}})^{-1}\times(\pi_k^{\infty}(Y))$.

Suppose that X and Y are $\mathscr{D}_{\tilde{I},e}$-invariant. Consider an analytic diffeomorphism $\phi:V\to U$ preserving $\tilde{I}$. We consider the associated push-forward mapping

$$j^{\infty}\phi_*\colon J^{\infty}(V,\tilde{I})\to J^{\infty}(\phi(V),\tilde{I}). \tag{41}$$

As $\tilde{I}$ is generated over U by $g_1,\dots,g_r$, the functions $g_1\circ\phi^{-1},\dots,g_r\circ\phi^{-1}$ generate $\tilde{I}$ over $\phi(V)$. Define $\alpha:\mathscr{O}^r_{\phi(V)}\to\tilde{I}_{|\phi(V)}$ by the formula $\alpha(f_1,\dots,f_r):=\sum_{i=1}^r f_i(g_i\circ\phi^{-1})$. For any $k\leqslant\infty$ we let

$$j^k\alpha\colon J^k(\phi(V),\mathbb{C}^r)\to J^k(\phi(V),\tilde{I})$$

be the associated mapping of k-jets. For any $k\leqslant\infty$ the mapping

$$j^k\tilde{\phi}_*:J^k(V,\mathbb{C}^r)\to J^k(\phi(V),\mathbb{C}^r), \tag{42}$$

defined by the formula

$$j^k\tilde{\phi}_*(j^kf_1(x),\dots,j^kf_r(x)):=(j^k(f_1\circ\phi^{-1})(\phi(x)),\dots,j^k(f_r\circ\phi^{-1})(\phi(x)))$$

satisfies $j^k\alpha\circ j^k\tilde{\phi}_*=j^k\phi_*\circ j^k\varphi$, and, if $k<\infty$, defines an analytic isomorphism. Letting $k=m$ and taking into account the fact that the definition of the sets $\mathrm{Sing}(X_{|\phi(V)})$, $\mathrm{Sing}(Y_{|\phi(V)})$ and $W(X_{|\phi(V)},Y_{|\phi(V)})$ does not depend on the chosen set of generators of $\tilde{I}_{\phi(V)}$, we obtain easily that invariance of $\mathrm{Sing}(X),\mathrm{Sing}(Y)$ and $W(X,Y)$ by ϕ_*. $\square$

Theorem 6.2. *Let X be a closed analytic subset of $J^{\infty}(U,\tilde{I})$, consider a locally finite partition*

$$X:=\coprod_{j\in J}X_j \tag{43}$$

by closed analytic subsets of $J^{\infty}(U,\tilde{I})$. There is a canonical Whitney stratification of X such that each stratum is a locally closed analytic subset contained in one of the sets X_i. Moreover, if each of the subsets X_j is $\mathscr{D}_{\tilde{I},e}$-invariant, then the strata are $\mathscr{D}_{\tilde{I},e}$-invariant.

Proof. The construction is inductive: suppose that for a certain N we have constructed a locally finite partition in disjoint locally closed analytic subsets

$$X:=\left(\coprod_{i\in I_N}Z_i\right)\coprod\left(\coprod_{i\in I'_N}Y_i\right) \tag{44}$$

refining the original partition (43) and with the following properties: for any $i \in I_N$ the set Z_i is smooth and irreducible of codimension at most N, for any $i_1, i_2 \in I_N$ the stratum Z_{i_1} is Whitney regular over Z_{i_2}, for any $i \in I_{N'}$ the set Y_i is irreducible, and the union $\bigcup_{i\in I'_N} Y_i$ is closed analytic with all the irreducible components of codimension at least $N+1$. Moreover if each of the subsets X_j is $\mathscr{D}_{\tilde{I},e}$-invariant, then the Z_i's and Y_i's are also $\mathscr{D}_{\tilde{I},e}$-invariant.

Let $L \subset I'_N$ be the set of indices parametrising components of codimension precisely $N+1$. For any $i \in L$ we define

$$A_i := \mathrm{Sing}(\bar{Y}_i) \bigcup (\bar{Y}_i \backslash Y_i) \bigcup \left[\bigcup_{j \neq i} \bar{Y}_j \cap \bar{Y}_i \right] \bigcup \left[\bigcup_{j \in I_N} W(Z_j, \bar{Y}_i) \right].$$

To check that it is a closed analytic subset of $\bar{Y}_i$ we only have to show that the two possibly infinite unions involved in its definition are locally finite; this follows easily from the local finiteness of the family (44). We define $I_{N+1} := I_N \cup L$ and $Z_i := Y_i \backslash A_i$ for any $i \in L$. Let $\{Y_i\}_{i \in I'_{N+1}}$ be the set of disjoint subsets consisting of the union of $\{Y_i\}_{i\in I'_N \backslash L}$ and the sets of irreducible components of $A_i \cap Y_i$ for any $i \in L$. The partition

$$X := \left(\coprod_{i \in I_{N+1}} Z_i \right) \coprod \left(\coprod_{i \in I'_{N+1}} Y_i \right) \tag{45}$$

has the properties of the partition (44) substituting N by $N+1$. Repeating this process infinitely we obtain the desired Whitney stratification. □

7. Transversality in generalised jet-spaces

Definition 7.1. Let W be an open subset of $\mathbb{C}^m$. A mapping $\alpha : W \to J^\infty(U, \mathbb{C}^r)$ is *analytic* if the composition $\pi_k^\infty \circ \alpha$ is analytic for any positive integer k. A mapping $\alpha : W \to J^\infty(U, \tilde{I})$ is *analytic* if there exists a neighbourhood V around each point $x \in W$, and an analytic mapping $\tilde{\alpha} : V \to J^\infty(U, \mathbb{C}^r)$ such that $\varphi \circ \tilde{\alpha} = \alpha_{|V}$. We say that $\tilde{\alpha}$ is a *local lifting* of α at x.

Definition 7.2. Let $C \subset J^\infty(U, \tilde{I})$ be a closed analytic subset endowed with a stratification $\mathscr{X} = \{X_j\}_{j\in J}$ by smooth irreducible locally closed subsets. Consider an analytic mapping $\alpha : W \to J^\infty(U, \tilde{I})$. We say that α is transversal to the stratification at a point $x \in W$, and we denote it by $\alpha \pitchfork_x \mathscr{X}$ if either $\alpha(x) \notin C$, or, when $\alpha(x) \in X_j$ for a certain $j \in J$, there exists a local lifting $\tilde{\alpha} : V \to J^\infty(U, \mathbb{C}^r)$ around x, and a positive integer m such that X_j is m-determined and

$$\mathrm{d}(\pi_m^\infty \circ \tilde{\alpha})_x(T_x V) + T_{\pi_m^\infty \circ \tilde{\alpha}(x)} \pi_m^\infty(\varphi^{-1}(X_i)) = T_{\pi_m^\infty \circ \tilde{\alpha}(x)} J^m(U, \mathbb{C}^r). \tag{46}$$

It is easy to check that Condition (46) holds for a certain $m \geqslant k$ if and only if it holds for any $m \geqslant k$. As usual, transversality to a Whitney stratification is an open condition:

Lemma 7.3. *Suppose that the stratification $\mathscr{X}$ considered in the last definition is a Whitney stratification. If $\alpha \pitchfork_x \mathscr{X}$ then there is a neighbourhood V of x in W such that $\alpha \pitchfork_y \mathscr{X}$ for any $y \in V$.*

Proof. As $\mathscr{X}=\{X_j\}_{j\in J}$ is locally finite there is a neighbourhood Ω of $\alpha(x)$ in $J^\infty(U,\tilde{I})$ meeting only finitely many strata $\{X_j\}_{j\in J'}$. Let k be a positive integer such that all these strata are k-determined. Consider a local lifting $\tilde{\alpha}: V_1 \to J^\infty(U,\mathbb{C}^r)$ with $V_1 \subset \alpha^{-1}(\Omega)$ and such that condition (46) is satisfied for $m=k$. This means that the mapping $\pi_m^\infty \circ \tilde{\alpha}: V_1 \to J^k(U,\mathbb{C}^r)$ is transversal to the complex manifold $\pi_k^\infty(\varphi^{-1}(X_i))$ at x. Since $\mathscr{X}$ is a Whitney stratification the partition $\mathscr{X}' := \{\pi_k^\infty(\varphi^{-1}(X_j))\}_{j\in J'}$ forms a Whitney stratification in the usual sense, and therefore there exists an open neighbourhood V of x in V_1 such that $\pi_m^\infty \circ \tilde{\alpha}$ is transversal to any stratum of $\mathscr{X}'$. Clearly $\alpha \pitchfork_y \mathscr{X}$ for any $y\in V$. $\square$

Theorem 7.4 (Generalised parametric transversality). *Let M and S be complex manifolds. Let $\varphi: M\times S \to J^\infty(U,\tilde{I})$ be a analytic mapping. Consider a closed analytic subset C of $J^\infty(U,\tilde{I})$ endowed with a Whitney stratification $\mathscr{X}$. Suppose that φ is transversal to $\mathscr{X}$. Denote by $\Delta\subset S$ the set of points where $\varphi_{|s} := \varphi_{|M\times\{s\}}$ is not transversal to $\mathscr{X}$. Then $S\backslash\Delta$ is dense in S. Moreover, if there exists a compact subset $K\subset M$ such that φ_s is transversal to $\mathscr{X}$ at any point of $(M\backslash K)\times S$, then Δ is a proper closed analytic subset of S.*

Proof. Using local liftings it is easy to show that as φ is analytic, the subset $C' := \varphi^{-1}(C)$ is closed analytic in $M\times S$, and since φ is transversal to $\mathscr{X}$, the stratification $\mathscr{Y} := \{Y_j\}_{j\in J}$ (where $Y_j = \varphi^{-1}(X_j)$) is a Whitney stratification of $\varphi^{-1}(C)$.

Let $\pi: M\times S\to S$ be the projection to the second factor. Take $(x,s)\in M\times S$; if $(x,s)\in C'$ let Y_j be the stratum of $\mathscr{Y}$ to which it belongs. A straightforward argument shows that the mapping $\varphi_{|s}$ is transversal to $\mathscr{X}$ at (x,s) if and only either $(x,s)\notin C'$ or $\pi_{|Y_j}$ is a submersion in (x,s). For any $j\in J$ we let Z_j be the set of points where $\pi_{|Y_j}$ fails to be a submersion. By Sard's Theorem $\pi(Z_j)$ is a set of measure 0 in S. The set Δ is equal to the union $\bigcup_{j\in J}\pi(Z_j)$, and has measure 0 as J is denumerable. This shows that $S\backslash\Delta$ is dense.

Suppose that there exists a compact K with the property stated in the theorem. We claim that the union $Z=\bigcup_{j\in J} Z_j$ is a closed analytic subset of $M\times S$. Then, as $Z\subset K\times S$, the restriction $\pi_{|Z}$ is proper and $\Delta=\pi(Z)$ is a closed analytic subset. It only remains to prove the claim. Using the fact that $\mathscr{Y}$ is a *Whitney* stratification it is easy to show that Z is closed. Therefore $Z=\bigcup_{j\in J}\bar{Z}_j$. Due to the local finiteness of $\{Y_j\}_{j\in J}$, if $\bar{Z}_j$ is a closed analytic subset for any $j\in J$ the claim is true. For any $j\in J$ we consider the closure of the stratum $\bar{Y}_j$. Consider $(x,s)\in\bar{Y}_j$; choose local coordinates $(y_1,\ldots,y_k)$ of S around s, let $(\pi_1,\ldots,\pi_k)$ be the components of π with respect to it; let $f_1,\ldots,f_N$ be a set of analytic equations in a neighbourhood V of (x,s) in $M\times S$ defining $\bar{Y}_j\cap V$. If the codimension of $\bar{Y}_i$ is c then the rank of the set $\{\mathrm{d}f_1,\ldots,\mathrm{d}f_N,\mathrm{d}\pi_1,\ldots,\mathrm{d}\pi_k\}$ of 1-forms defined over V is at most $c+k$. Define

$$Z_j' := \{z\in\bar{Y}_j\cap V:\ \operatorname{rank}\{\mathrm{d}f_1(z),\ldots,\mathrm{d}f_N(z),\mathrm{d}\pi_1(z),\ldots,\mathrm{d}\pi_k(z)\} < c+k\}.$$

Clearly Z_j' is a closed analytic subset of V such that $Z_j'\cap Y_j = Z_j\cap V$. Therefore $\bar{Z}_j\cap V$ is analytic in V. $\square$

Next we show how the versality of an unfolding implies the transversality of its associated jet extension with respect to any $\mathscr{D}_{\tilde{I},e}$-invariant subset.

Let $F: (\mathbb{C}^n\times\mathbb{C}^s,(O,O)) \to \mathbb{C}$ be an s-parametric I-unfolding. By Lemma 1.6 there are open neighbourhoods V and W of the origin in $\mathbb{C}^n$ and $\mathbb{C}^s$ respectively, such that $V\subset U$ and we can write $F=\sum_{i=1}^r g_iF_i$, with $\{g_1,\ldots,g_r\}$ our fixed set of generators of $\tilde{I}$, and each F_i an analytic function on $V \times W$. Then the jet extension

$$\rho_F : V \times W \to J^{\infty}(U, \tilde{I}) \tag{47}$$

defined by $\rho_F(x,s) = j^{\infty}F_{|s}(x)$ is an analytic mapping, as it admits the analytic lifting

$$\tilde{\rho}_F : V \times W \to J^{\infty}(U, \mathbb{C}^r) \tag{48}$$

defined by $\tilde{\rho}_F(x,s) = (j^{\infty}F_{1|s}(x), \ldots, j^{\infty}F_{r|s}(x))$.

Proposition 7.5. *Consider $F : (\mathbb{C}^n \times \mathbb{C}^s, (O,O)) \to \mathbb{C}$ as above. If F is versal at $(x,s) \in V \times W$, then ρ_F is transversal at (x,s) to any $\mathscr{D}_{\tilde{I},e}$-invariant smooth locally closed subset of $J^{\infty}(U, \tilde{I})$.*

Proof. Let C be a k-determined $\mathscr{D}_{\tilde{I},e}$-invariant locally smooth closed subset. Suppose that F is versal at (x,s). If $\rho_F(x,s) \notin C$ there is nothing to prove. Suppose $\rho_F(x,s) \in C$. The set $A := (j^k\varphi)^{-1}(\pi_k^{\infty}(C))$ is a smooth analytic subset of $J^k(U, \mathbb{C}^r)$.

To shorten formulas we denote the function $F_{|s}$ by f, the point $\pi_k^{\infty}(\rho_F(x,s)) \in J^k(U, \tilde{I})$ by p, and the point $\pi_k^{\infty}(\tilde{\rho}_F(x,s)) \in J^k(U, \mathbb{C}^r)$ by q. Notice that $\rho_F(x,s) = f_x = j^{\infty}f(x)$, and $p = j^k f(x) = \pi_k^{\infty}(f_x)$.

We have to show that

$$\mathrm{d}(\pi_k^{\infty} \circ \tilde{\rho}_F)_{(x,s)}(T_{(x,s)}V \times W) + T_qA = T_qJ^k(U, \mathbb{C}^r). \tag{49}$$

As $J^k(U, \mathbb{C}^r)$ is a trivial vector bundle, its tangent space at q splits as the direct sum

$$T_qJ^k(U, \mathbb{C}^r) = T_xU \oplus E, \tag{50}$$

where E denotes $(\mathcal{O}_{U,x}/\mathbf{m}_x^{k+1})^r$, the fibre of the vector bundle $J^k(U, \mathbb{C}^r)$ over x. Then the differential $\mathrm{d}(pr_k)_q$ is the projection homomorphism to the first summand. Denote by β the projection to the second summand. We have an epimorphism $j^k\varphi_x : E \to \tilde{I}_x/(\tilde{I}_x \cap \mathbf{m}_x^{k+1})$. As A is $j^k\varphi$-saturated, we have

$$\{0\} \oplus \ker(j^k\varphi_x) \subset T_qA. \tag{51}$$

Consider any germ of vector field $X \in \Theta_{\tilde{I}_x,e}$. Let $\phi : V_x \times (-\delta, \delta) \to U$ be the flow obtained by integrating a representative of X in a neighbourhood V_x of x in U. As f_x belongs to the $\mathscr{D}_{\tilde{I},e}$-invariant subset C, $\phi_{t*}f_x$ belongs to C for any $t \in (-\delta, \delta)$. Therefore the mapping

$$\gamma : (-\delta, \delta) \to J^k(U, \mathbb{C})$$

defined by $\gamma(t) := \pi_k^{\infty}(\phi_{t*}f_x) = j^k(\phi_{t*}f)(\phi_t(x))$ is a differentiable curve whose image lies in $\pi_k^{\infty}(C) \subset J^k(U, \tilde{I})$ and such that $\gamma(0) = p$. Let Σ_i be the $\tilde{I}$-stratum to which x belongs. Clearly if $\tau := pr_k \circ \gamma$ then $\tau(t) = \phi_t(x)$. As any diffeomorphism preserving $\tilde{I}$ leaves invariant the $\tilde{I}$-strata, the integral curve τ maps $(-\delta, \delta)$ into Σ_i, and that image of γ is contained in $J^k(\Sigma_i, \tilde{I})$. By Remark 3.4, the mapping $j^k\varphi$ restricts to an epimorphism of analytic vector bundles

$$j^k\varphi_{|\Sigma_i} : J^k(\Sigma_i, \mathbb{C}^r) \to J^k(\Sigma_i, \tilde{I}).$$

As $j^k\varphi_{|\Sigma_i}(q) = p$ there exists a differentiable lifting

$$\tilde{\gamma} : (-\delta, \delta) \to J^k(\Sigma_i, \mathbb{C}^r) \tag{52}$$

such that $\gamma = j^k\varphi \circ \tilde{\gamma}$ and $\tilde{\gamma}(0) = q$. As any lifting of γ has its image contained in A, the tangent vector $\tilde{\gamma}'(0)$ belongs to T_qA. Clearly $d(pr_k)_q(\tilde{\gamma}'(0)) = \tau'(0) = X(x)$.

Consider the pullback vector bundle $\tau^*J^k(\Sigma_i, \tilde{I}) \to (-\delta, \delta)$. Its fibre over 0 is $\tilde{I}_x/\tilde{I}_x \cap \mathbf{m}_x^{k+1}$. Denote by $\eta : \tau^*J^k(\Sigma_i, \tilde{I}) \to J^k(\Sigma_i, \tilde{I})$ the mapping defined by $\eta(h, t) := h$ for any $t \in (-\delta, \delta)$ and $h \in (\tau^*J^k(\Sigma_i, \tilde{I}))_t = J^k(\Sigma_i, \tilde{I})_{\tau(t)}$. We consider the trivialisation

$$\psi : (-\delta, \delta) \times (\tilde{I}_x/\tilde{I}_x \cap \mathbf{m}_x^{k+1}) \to \tau^*J^k(\Sigma_i, \tilde{I}) \tag{53}$$

defined by $\psi(t, \pi_k^\infty(h_x)) := (t, \pi_k^\infty(\phi_{t*}h_x))$ for any $h_x \in \tilde{I}_x$ and any $t \in (-\delta, \delta)$. Define the curve

$$\alpha_1 : (-\delta, \delta) \to (-\delta, \delta) \times (\tilde{I}_x/\tilde{I}_x \cap \mathbf{m}_x^{k+1})$$

by the formula $\alpha_1(t) := (t, \pi_k^\infty(f_x))$. Observe that $\gamma = \eta \circ \psi \circ \alpha_1$.

On the other hand we consider the curve

$$\sigma : (-\delta, \delta) \to V \times W$$

defined by $\sigma(t) := (\tau(t), s)$. Then $\pi_k^\infty \circ \rho_F \circ \sigma(t) = \pi_k^\infty(f_{\tau(t)}) = j^k f(\tau(t))$. Defining

$$\alpha_2 : (-\delta, \delta) \to (-\delta, \delta) \times (\tilde{I}_x/\tilde{I}_x \cap \mathbf{m}_x^{k+1})$$

as $\alpha_2(t) := (t, \pi_k^\infty(\phi_{-t*}f_{\tau(t)}))$ we have $\pi_k^\infty \circ \rho_F \circ \sigma = \eta \circ \psi \circ \alpha_2$.

Consider the following direct sum decomposition:

$$T_{(0,p)}[(-\delta, \delta) \times \tilde{I}_x/\tilde{I}_x \cap \mathbf{m}_x^{k+1}] = \mathbb{R} \oplus \tilde{I}_x/\tilde{I}_x \cap \mathbf{m}_x^{k+1}. \tag{54}$$

With respect to it we have $\alpha_1'(0) = (1, 0)$ and $\alpha_2'(0) = (1, \pi_k^\infty(X(f_x)))$. Then, decomposing the tangent space of the vector bundle $J^k(U, \mathbb{C})$ at $\pi_k^\infty(f_x)$ as

$$T_pJ^k(U, \mathbb{C}) = T_xU \times \mathcal{O}_{U,x}/\mathbf{m}^{k+1},$$

we have

$$(\pi_k^\infty \circ \rho_F \circ \sigma)'(0) - \gamma'(0) = d(\eta \circ \psi)_{(0,p)}(\alpha_2'(0) - \alpha_1'(0)) = (0, \pi_k^\infty(X(f_x))).$$

As $\pi_k^\infty \circ \rho_F = j^k\varphi \circ \pi_k^\infty \circ \tilde{\rho}_F$ and $\gamma = j^k\varphi \circ \tilde{\gamma}$ we conclude that

$$(0, \pi_k^\infty(X(f_x))) \in d(j^k\varphi)_q(d(\pi_k^\infty \circ \tilde{\rho}_F)_{(x,s)}(T_{(x,s)}V \times W) + T_qA).$$

Taking into account inclusion (51) we obtain

$$\{0\} \oplus (j^k\varphi_x)^{-1}(\pi_k^\infty(\Theta_{\tilde{I}_x,e}(f_x)) \subset d(\pi_k^\infty \circ \tilde{\rho}_F)_{(x,s)}(T_{(x,s)}V \times W) + T_qA. \tag{55}$$

Let $\partial/\partial w_1, \ldots, \partial/\partial w_d$ be a basis of the tangent space T_sW. The versality of F at (x, s) means

$$\mathbb{C}(\partial F/\partial w_1)_{|s,x} + \cdots + \mathbb{C}(\partial F/\partial w_d)_{|s,x} + \Theta_{\tilde{I}_x,e}(f_x) = \tilde{I}_x. \tag{56}$$

As $d(j^k\varphi \circ \pi_k^\infty \tilde{\rho}_F)_{(x,s)}(0, \partial/\partial w_i) = (0, \pi_F^\infty((\partial F/\partial w_i)_{|s,x})$, equality (56) together with inclusion (55) imply

$$\{0\} \oplus E \subset d(\pi_k^\infty \circ \tilde{\rho}_F)_{(x,s)}(T_{(x,s)}V \times W) + T_qA. \tag{57}$$

After this it is sufficient to prove that $\mathrm{d}(pr_k)_q$ maps the space

$$\mathrm{d}(\pi_k^{\infty} \circ \tilde{\rho}_F)_{(x,s)}(T_{(x,s)}V \times W)$$

surjectively over T_xU, but this is trivial because $pr_k \circ \pi_k^{\infty} \circ \tilde{\rho}_F$ is the projection of $V \times W$ to its first factor. □

8. The relative morsification theorem

Let Δ_1 be a $\mathscr{D}_{\tilde{I},e}$-invariant closed analytic subset of $\Gamma(U,\tilde{I})$ with all its irreducible components of codimension at least $n+1$, and containing $\Gamma^{(r)}$ (where $\Gamma^{(r)}$ was introduced in Theorem 5.4). Recall the set Δ_2, appearing in the topological partition (39). Define $\Delta := \Delta_1 \cup \Delta_2$. The partition $\mathscr{P}_0$ defined by

$$J^{\infty}(U,\tilde{I}) := \left[\coprod_{i \leqslant n, l \in L_i} B_{i,l} \backslash \Delta \right] \coprod \Delta, \tag{58}$$

is a partition of $J^{\infty}(U,\tilde{I})$ by locally closed analytic $\mathscr{D}_{\tilde{I},e}$-invariant subsets, which is closely related with the topological partition: they only differ in subsets of codimension strictly bigger than n. It has the advantage that all the terms involved in its definition are locally closed analytic subsets. On the other hand Δ is not canonically defined, but its irreducible components have codimension at least $n+1$. As we will see, subsets of codimension bigger than $n+1$ are too small to affect the singularity types appearing in a generic deformation of any function of I of finite extended codimension.

By Proposition 4.5 the level sets C_m of the filtration by extended codimension are finite-determined closed analytic subsets if $m < \infty$. Let $\{C_{1,j}\}_{j\in J}$ be the set of irreducible components of C_1. Given $C_{1,j}$ we let $m(j)$ be the extended codimension of a generic element in it, that is, the minimal m such that the intersection $C_{i,j} \cap \dot{C}_m$ is not empty. Define $\partial C_{1,j} := C_{1,j} \cap C_{m(j)+1}$. Observe that if $m(j) = \infty$ then $\partial C_{1,j} = C_{1,j}$. The local finiteness of $\{\partial C_{1,j}\}_{j\in J}$ follows from the locally finiteness of $\{C_{1,j}\}_{j\in J}$. Therefore $\partial C_1 := \bigcup_{j\in J} \partial C_{1,j}$ is a closed analytic subset. We have a canonically defined partition by $\mathscr{D}_{\tilde{I},e}$-invariant locally closed subsets

$$J^{\infty}(U,\tilde{I}) = \dot{C}_0 \coprod (C_1 \backslash \partial C_1) \coprod \partial C_1 \tag{59}$$

such that the irreducible components of $C_1 \backslash \partial C_1$ are at least n-codimensional and the irreducible components of ∂C_1 are either of codimension strictly bigger than n or are contained in C_{∞}. Indeed, the irreducible components of $C_1 \backslash \partial C_1$ are of the form $C_{1,j} \backslash \partial C_1$ for $j \in J$. If $\mathrm{codim}(C_{1,j}) < n$, by Proposition 4.6 we have $C_{1,j} \subset C_{\infty}$, and hence $m(j) = \infty$; in this case $\partial C_{1,j} = C_{1,j}$ and therefore $C_{1,j} \backslash \partial C_1$ is empty. Suppose that we have a component C' of ∂C_1 not contained in C_{∞}; then C' is a component of $\partial C_{1,j}$ for a certain j such that $C_{1,j}$ is not contained in C_{∞}; in this case $\mathrm{codim}(C_{1,j}) \geqslant n$ and C' is a proper closed analytic subset of the irreducible $C_{1,j}$; we deduce $\mathrm{codim}(C') \geqslant n+1$.

The partition

$$J^{\infty}(U,\tilde{I}) = \coprod_{k\in K} Z_k, \tag{60}$$

whose strata are the subsets of the form $X \cap Y$, where X and Y are, respectively, strata of the partitions (58) and (59), is a locally finite partition by locally closed analytic $\mathscr{D}_{\tilde{I},e}$-invariant subsets refining the partitions (58) and (59). It is canonically defined up to codimension n: the only non-canonical element involved in its definition is the subset Δ_1, whose irreducible components are at least $n+1$-codimensional. Therefore, given a different choice Δ_1' for this set, and a stratum Z_i' of codimension smaller or equal than n of the resulting partition, it is easy to see that there exists a stratum Z_k in the original partition with the same closure $\bar{Z}$ than Z_i' such that

$$\operatorname{codim}(\bar{Z}\setminus(Z_k \cap Z_i') \geqslant n+1.$$

If Z_k is a stratum such that $\operatorname{codim}(Z_k) \leqslant n$ then any two germs in Z_k are topologically equivalent and have the same extended codimension.

Definition 8.1. The canonical Whitney stratification $\mathscr{X} := \{X_j\}_{j\in J}$ associated (by Theorem 6.2) to the partition (60) is called the *n-canonical Whitney stratification* of $J^\infty(U,\tilde{I})$.

Remark 8.2. The *n-canonical Whitney stratification* of $J^\infty(U,\tilde{I})$ is canonically defined up to codimension n, in the sense explained above. If X_j is a stratum such that $\operatorname{codim}(X_j) \leqslant n$ then any two germs in X_j are topologically equivalent and have the same extended codimension.

Definition 8.3. A stratum of the Topological Partition or of the n-canonical Whitney stratification of $J^\infty(U,\tilde{I})$ is called *thick* if it has a irreducible component of codimension at most n.

Consider $f \in I$ defined on a neighbourhood V of the origin O in U such that $c_{I,e}(f) < \infty$. Let $\rho_f : V \to J^\infty(U,\tilde{I})$ be its associated jet-extension. By the local finiteness of $\mathscr{X}$ we can shrink V such that $\rho_f(V)$ only meets finitely many strata. Therefore, there exists a radius ϵ such that $\rho_f(B_{2\epsilon})$ only meets the strata of $\mathscr{X}$ whose closures contain $\rho_f(O)$. Let $X_1,\ldots,X_s$ be such strata. If ϵ is small enough we know that the only point where f has positive extended codimension in $B_{2\epsilon}$ is the origin. Therefore, f is its own versal unfolding for any $x \neq O$, and, by Proposition 7.5, the jet extension ρ_f is transversal to each stratum X_i at x. Consequently $\rho_f^{-1}(X_i)$ is smooth outside the origin and $\{\rho^{-1}_{f|B_{2\epsilon}}(X_i)\}_{i\leqslant s}$ defines a Whitney stratification in $B_{2\epsilon}\setminus\{O\}$. Hence, if ϵ is small enough we can assume that S_ϵ is transversal to any stratum $\rho^{-1}_{f|B_\epsilon}(X_i)$. Any such radius ϵ satisfying the above properties is called a *good radius* for f. From this moment we say that a *good system of radii* for f is a pair (ϵ,η) satisfying the conditions imposed in Definition 2.1, and such that ϵ is a good radius for f.

Definition 8.4. A stratum of the topological partition or of the n-canonical Whitney stratification of $J^\infty(U,\tilde{I})$ is called *unavoidable by f* if it is thick and $\rho_f(O)$ belongs to the closure of one of its components of codimension at most n.

Definition 8.5. Let $f \in \tilde{I}_x$ such that $c_{\tilde{I}_x,e} < \infty$. We say that f_x is *codimensionally irreducible* if there not exist an unfolding $F:(\mathbb{C}^n,0)\times(\mathbb{C},0)$ and a sequence $\{(x_n,s_n)\}_{n\in\mathbb{N}}$ converging to $(x,0)$ such that $c_{\tilde{I}_{x_n},e}(F_{|s_n,x_n}) < c_{\tilde{I}_x,e}$.

Let $(A_1,\ldots,A_s)$ and $(A_1',\ldots,A_s')$ be two tuples of topological spaces such that $A_i \subset A_1$ and $A_j' \subset A_1'$ for any $i,j \leqslant s$. We say that the two tuples are *topologically equivalent* if there exists a homeomorphism $h : A_1 \to A_1'$ such that $h(A_i) = A_i'$ for any i.

Theorem 8.6 (The relative morsification theorem). *Let $f \in I$ be such that $c_{I,e}(f) < \infty$, and let $F : U \times V \to \mathbb{C}$ be a representative of a versal unfolding of f. Let $\{X_1, \ldots, X_s\}$ and $\{B_1, \ldots, B_k\}$ be respectively the strata of the n-canonical Whitney stratification and of the topological partition that are unavoidable by f. Let ϵ be a good radius for f. The subset V can be chosen small enough so that there exists a closed analytic subset $\Delta \subset V$ (called the* discriminant *of F), and a positive number η such that*

$$\rho_{F_{|s}} : B_{\epsilon+\eta} \times \{s\} \to J^{\infty}(U, \tilde{I})$$

is transversal to the canonical Whitney stratification $\mathscr{X}$ for any $s \in V \setminus \Delta$. As a consequence

(1) *If $s \in V \setminus \Delta$ then the image of $\rho_{F_{|s}}$ meets only strata of $\mathscr{X}$ or of the Topological Partition that are unavoidable by f.*
(2) *The topological types of the tuples*

$$(\bar{B}_{\epsilon}, \rho_{F_s}^{-1}(X_1), \ldots, \rho_{F_s}^{-1}(X_s)), \tag{61}$$

$$(\bar{B}_{\epsilon}, \rho_{F_s}^{-1}(B_1), \ldots, \rho_{F_s}^{-1}(B_k)) \tag{62}$$

do not depend on $s \in V \setminus \Delta$. Moreover the topological types of these tuples do not depend on the chosen versal unfolding.
(3) *The germ of $F_{|s,x}$ is codimensionally irreducible for any $x \in B_{\epsilon}$ and any $s \in V \setminus \Delta$.*
(4) *The number of points of $\bar{B}_{\epsilon}$ where $F_{|s}$ has a fixed extended codimension depends neither on $s \in V \setminus \Delta$ nor on the choice of the versal unfolding. The points of $B_{\epsilon} \setminus V(I)$ where $F_{|s}$ has positive extended codimension are A_1-singularities. Moreover $F_{|s}$ has extended codimension* 0 *along ∂B_{ϵ}.*
(5) *The analytic type of the germ (V, Δ, O) (where O is the origin of V) does not depend on the choice of the unfolding as long it is universal (that is, versal with* $\dim(V) = c_{I,e}(f)$*).*

Proof. As F is versal, by Proposition 7.5, the mapping ρ_F is transversal to $\mathscr{X}$. Using the same argument as in Lemma 2.3, choosing V small enough we can assume that $c_{\tilde{I}_x,e}(F_{|s,x}) = 0$ for any $x \in B_{2\epsilon} \setminus B_{\epsilon/2} \times V$; therefore at such points the germ $F_{|s,x}$ is its own versal unfolding. Hence, by Proposition 7.5, the mapping $\rho_{F_{|s}}$ is transversal to $\mathscr{X}$ for any $x \in B_{2\epsilon} \setminus B_{\epsilon/2}$. Choose $\eta < \epsilon$. Applying Theorem 7.4 we conclude the existence of a closed analytic subset Δ such that $\rho_{F_s} \pitchfork \mathscr{X}$ for any $s \in V \setminus \Delta$. Now we derive the stated consequences.

As the n-canonical Whitney stratification is a subdivision of the topological partition the assertions concerning the former imply the analogous assertions concerning the later.

Let $s \in V \setminus \Delta$; as $\dim(B_{\epsilon}) = n$, by transversality, the image of $\rho_{F_{|s}}$ only can meet strata of $\mathscr{X}$ of codimension at most n, hence thick. By local finiteness of $\mathscr{X}$ and the fact that the radius ϵ is a *good radius* for f, we can choose V small enough so that the closure of any stratum of $\mathscr{X}$ which is met by $\rho_F(B_{\epsilon+\eta} \times V)$ necessarily contains the origin. Therefore, the image of $\rho_{F_{|s}}$ only can meet strata that are unavoidable by f. This proves assertion (1).

Let $X_1, \ldots, X_s, X_{s+1}, \ldots, X_l$ be the strata of $\mathscr{X}$ whose closures meet $\rho_f(O)$. It follows from the transversality $F \pitchfork \mathscr{X}$ that

$$\mathscr{Y} = \{Y_i := \rho_F^1(X_i)\}_{i \leqslant l}$$

is a Whitney stratification of $B_{\epsilon+\eta} \times V$. Let $\pi : B_{\epsilon+\eta} \times V$ be the projection to the second factor. Define $Y_{i|V\setminus\Delta} := Y_i \cap \pi^{-1}(V\setminus\Delta)$. For any $s \in V\setminus\Delta$, the transversality $\rho_{F_{|s}} \pitchfork \mathscr{X}$ implies that either $Y_{i|V\setminus\Delta}$ is empty or it projects submersively to $V\setminus\Delta$. On the other hand, as ϵ is a good radius, choosing V small enough we have that $S_\epsilon \pitchfork Y_{i|s}$ for any $i \leqslant l$ and any $s \in V$. This implies that the Whitney stratifications $\mathscr{Y}$ and $\{B_\epsilon, S_\epsilon\}$ meet transversely, and that, if $\mathscr{Z}$ is the Whitney stratification formed by pairwise intersections of their strata, the restriction of the projection

$$\pi : \bar{B}_\epsilon \times (V\setminus\Delta) \to V\setminus\Delta$$

to any stratum of $\mathscr{Z}$ is submersive. As π is proper, $\mathscr{Z}$ is topologically trivial over $V\setminus\Delta$. This proves the independence of the topological type of the tuple (61) of $s \in V\setminus\Delta$.

Let $F : U \times V \to \mathbb{C}$ and $F' : U' \times V' \to \mathbb{C}$ be two versal unfoldings. The mapping $F'' : U \times (V \times V') \to \mathbb{C}$ defined by $F_{|(v,v')} := F_v + F_{v'} - f$ is an I-unfolding of F such that $F''_{|V\times\{O'\}} = F'$ and $F''_{|\{O\}\times V'} = F'$ (where O and O' are the respective origins of V and V'). Clearly F'' is versal, and moreover its discriminant contains neither $V \times \{O'\}$ nor $\{O\} \times V'$. The independence on $s \in V \times V'\setminus\Delta$ of the topological type of the tuple (61) for the unfolding F'' implies the independence of the choice of versal unfolding. This finishes the proof of Assertion (2).

Given $s \in V$, the set of points of $B_{\epsilon+\eta}$ where $F_{|s}$ has extended codimension is $\rho_{F_{|s}}^{-1}(C_1)$. For any $i > 0$ we let Z_i be the union of the strata among $X_1, \ldots, X_s$ which contain germs of extended codimension precisely i; define $Z := \bigcup_{i>0} Z_i$. Assertion (1) implies that for any $s \in V\setminus\Delta$ we have $\rho_{F_{|s}}^{-1}(C_1) = \rho_{F_{|s}}^{-1}(Z)$ and that, for any $i > 0$, the set of points where $F_{|s}$ has extended codimension i equals $\rho_{F_{|s}}^{-1}(Z_i)$. The topological type of the tuple (61) determines the topological types of the subsets $\rho_{F_{|s}}^{-1}(Z)$ and $\rho_{F_{|s}}^{-1}(Z_i)$ for any $i > 0$. Consequently, by Assertion (2), these topological types are independent of $s \in V\setminus\Delta$.

The independence of the topological type of $\rho_{F_{|s}}^{-1}(Z)$ implies Assertion (3). As the topological type of $\rho_{F_{|s}}^{-1}(Z_i)$ determines the number of points where $F_{|s}$ has extended codimension i, Assertion (2) also implies the first part of Assertion (4). The fact that the points of $B_\epsilon\setminus V(I)$ where $F_{|s}$ has positive extended codimension are of type A_1 follows from Assertion (3): if x is such a point then $\tilde{I}_x = \mathcal{O}_{\mathbb{C}^n,x}$ and, by the morsification theory for isolated singularities, the only codimensionally irreducible singularities in this case are Morse points.

The fifth assertion is easy, taking into account that the strata of $\mathscr{X}$ are $\mathscr{D}_{\tilde{I},e}$-invariant. □

The fact that the topological type of the tuple (62) does not depend on $s \in V\setminus\Delta$ can be rephrased by saying that if g and h are any two generic close approximations of f in I, the partition of B_ϵ by topological type of g is homeomorphic to the analogous partition for h.

Addendum 8.7. *The projections to the second factor*

$$(\bar{B}_\epsilon \times V\setminus\Delta, \rho_{F|\bar{B}_\epsilon\setminus\Delta}^{-1}(X_1), , \ldots, \rho_{F|\bar{B}_\epsilon\setminus\Delta}^{-1}(X_s)) \to V\setminus\Delta,$$

$$(\bar{B}_\epsilon \times V\setminus\Delta, \rho_{F|\bar{B}_\epsilon\setminus\Delta}^{-1}(B_1), , \ldots, \rho_{F|\bar{B}_\epsilon\setminus\Delta}^{-1}(B_k)) \to V\setminus\Delta$$

are locally trivial fibrations of tuples of topological spaces with fibres the tuples (61) *and* (62), *respectively. The topological types of these fibrations are* $\mathscr{D}_I$*-invariant.*

Addendum 8.8. *With the notations of the last theorem, let r be the number of A_1-points in $B_\epsilon \setminus V(I)$ of a generic deformation $F_{|s}$ of f within I. There exists a closed analytic proper subset $\Delta' \subset V$ such that its complement $V\setminus\Delta'$ is the set of $s \in V$ such that the images by $F_{|s}$ of the critical points of $F_{|s}$ in $B_\epsilon\setminus V(I)$ are r different points which are also different from 0 if $V(I) \neq \emptyset$.*

Proof. First we show that the set of values of the parameter s with the property of the statement is dense in V. Consider $s \in V\setminus\Delta$. Then the only singularities of $F_{|s}$ in $B_\epsilon\setminus V(I)$ are r Morse points, which we denote it by $p_1, \dots, p_r$. Suppose that, for any $i \leqslant r$, we manage to construct an I-unfolding $G^i := F_{|s} + th$ such that for any t small enough the critical points of $G^i_{|t}$ in $B_\epsilon\setminus V(I)$ are $p_1, \dots, p_r$ and

$$G^i_{|t}(p_i) = F_{|s}(p_i) + t,$$

$$G^i_{|t}(p_j) = F_{|s}(p_j)$$

for $i \neq j$. Then, using the versality of F, we deduce easily that for s' close enough to s the critical points of $F_{|s'}$ in $B_\epsilon\setminus V(I)$ are $p_1, \dots, p_r$, their images are pairwise different and different from 0.

Let us construct the I-unfolding G^1. As $p_1, \dots, p_r$ are different and do not belong to $V(I)$ there exists a function $g \in I$ vanishing at $p_2, \dots, p_r$ and such that $g(p_1) = 1$. Then the function $h := (2 - g^2)g^2$ belongs to I, vanishes and is singular at $p_2, \dots, p_r$, takes the value 1 and is singular at p_1. Define $G^1 := F_{|s} + th$.

Consider the analytic mapping $\sigma : V\setminus\Delta \to \mathbb{C}^r$ whose ith component σ_i assigns to s the ith symmetric function of the images of the critical points of $F_{|s}$ under $B_\epsilon\setminus V(I) \to \mathbb{C}\setminus V(f)$. Let $\beta : \mathbb{C}^r \to \mathbb{C}$ be the analytic mapping associating to $a_1, \dots, a_r$ the discriminant of the polynomial $T^r + \sum_{i=1}^r a_i T^{r-i}$. Define $\varphi : V\setminus\Delta \to \mathbb{C}$ to be the composition $\varphi := \beta \circ \sigma$. The function F is defined on $\bar{B}_\epsilon \times V$, therefore, if V is small enough $F(B_\epsilon \times V)$ is a bounded set in $\mathbb{C}$. Hence σ, and consequently φ are bounded functions. By Riemann's extension theorem we can suppose that the analytic function φ is defined on the whole of V. It is easy to check that the set defined by $\Delta' := \varphi^{-1}(0)$ if $V(I) = \emptyset$ and $\Delta' := \varphi^{-1}(0) \cup \sigma_r^{-1}(0)$ if $V(I) \neq \emptyset$ has the desired properties. □

Definition 8.9. The minimal closed analytic subset $\Delta \subset V$ with the properties of Theorem 8.6 is called the *discriminant* of F in V. The subset Δ' introduced in Addendum 8.8 is called the *bifurcation variety* of F in V. The union $\Delta \cup \Delta'$ is called the *big discriminant* of F in V.

The bifurcation variety defined here generalises the bifurcation variety for isolated singularities and the one studied in [27].

Definition 8.10. Let $f \in I$ be a function of finite extended codimension and ϵ a good radius for it. An I-unfolding F: $(\mathbb{C}^n \times \mathbb{C}, (O, 0)) \to \mathbb{C}$ is called a *morsification* if for any small enough $s \neq 0$ the jet-extension $\rho_{F_{|s}}$: $(\mathbb{C}^n, O) \to J^\infty(U, \tilde{I})$ is transversal to the n-canonical Whitney stratification in a neighbourhood of the origin containing $\bar{B}_\epsilon$, and the images of the isolated critical points of $F_{|s}$ are pairwise different and, if $V(I) \neq \emptyset$, also different from 0.

A consequence of our result is that any generic 1-parameter I-unfolding is a morsification.

9. Applications

In this section we explain two applications of our relative morsification theorem.

9.1. Numerical invariants

In Example 1.10 we proved that in general the extended codimension does not behave in an conservative way with respect to deformation, unlike the Milnor number in the case of isolated singularities. We now introduce some numerical $\mathscr{D}_I$-invariants for functions of I which are conservative and related to the extended codimension.

Definition 9.1. Let $f \in I$ be a function such that $c_{I,e}(f) < \infty$. Let ϵ be small enough so that the origin is the only point of B_ϵ where f has positive extended codimension. Define the *splitting function*

$$\sigma_I[f] : \mathbb{N} \to \mathbb{Z}_{\geqslant 0} \tag{63}$$

of f with respect to I by letting $\sigma_f(k)$ be the number of points of B_ϵ where $F_{|s}$ has extended codimension k, where F is a versal I-unfolding and s is a parameter not contained in the discriminant. Define the *corrected extended codimension* $\tilde{c}_{I,e}(f)$ of f with respect to I by the formula

$$\tilde{c}_{I,e}(f) := \sum_{k \in \mathbb{N}} k\sigma_I[f](k). \tag{64}$$

Define the *Morse number* $\mathscr{M}_I(f)$ of f with respect to I as the number of points of $B_\epsilon \backslash V(I)$ where $F_{|s}$ has an A_1 singularity.

Clearly $\tilde{c}_{I,e}(f) \leqslant c_{I,e}(f)$. Example 1.10 shows that the inequality may be strict. By formula (7) we have $\sigma_I[f](k) \leqslant c_{I,e}(f)/k$. The Morse number, the extended codimension and the corrected extended codimension coincides with the Milnor number when $I = \mathcal{O}_{\mathbb{C}^n,O}$.

Using versal unfoldings it is easy to check the conservativity of the invariants defined above: consider $f \in I$, a function such that $c_{I,e}(f) < \infty$; let ϵ be small enough so that the origin is the only point of B_ϵ where f has positive extended codimension. Given any I-unfolding $F : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}$ and any small enough $s \in \mathbb{C}^k$ of the parameter we have

$$\sum_{x \in B_\epsilon} \alpha_{\tilde{I}_x}(f) = \alpha_I(f_O), \tag{65}$$

where $\alpha_{\tilde{I}_x}(f_x)$ stands for $\sigma_{\tilde{I}_x}[f_x]$, $\tilde{c}_{\tilde{I}_x}(f_x)$ or $\mathscr{M}_{\tilde{I}_x}(f_x)$.

Remark 9.2. Obviously, more refined numerical invariants can be defined taking into account the distribution of the points of a fixed extended codimension of a generic deformation $F_{|s}$ in the partitions given by the tuples (61) and (62), or even considering numerical topological invariants of the tuples.

9.2. Consequences on the topology of the Milnor fibre

In this section we deduce topological properties of Milnor fibres with the help of morsifications.

Let $V=V(I)$ be the analytic germ defined by I. Suppose $V\neq\emptyset$. Due to the conical structure of analytic germs there exist $\epsilon_0>0$ such that $V\cap B_\epsilon$ is contractible for any $\epsilon<\epsilon_0$.

Recall that we have fixed generators $g_1,\dots,g_r$ of $\tilde{I}$ in the open subset U. Consider the real analytic function $\kappa:\bar{B}_\epsilon\to\mathbb{R}$ by the formula $\kappa(x):=\sum_{i=1}^r|g_i(x)|^2$. We claim that 0 is an isolated critical value of κ: otherwise, by the Curve Selection Lemma, there exists a germ of analytic path $\gamma:(\mathbb{R},0)\to\bar{B}_\epsilon$ such that $\gamma^{-1}(V)=\{0\}$ and $(\kappa\circ\gamma)'(t)=0$ for any t. This is a contradiction, and hence our claim is true. An analogous reasoning yields that 0 is an isolated critical value of the restriction $\kappa_{|S_\epsilon}$. As a consequence we can choose a positive ξ_0 such that 0 is the only critical value of κ in $[0,\xi_0]$ and $\kappa^{-1}(\xi)\pitchfork S_\epsilon$ for any $\xi\in(0,\xi_0]$. Then, for any $\xi\in(0,\xi_0]$, the set $N_{\epsilon,\xi}:=\kappa^{-1}[0,\xi]$ is a compact neighbourhood of $V\cap\bar{B}_\epsilon$ in $\bar{B}_\epsilon$, whose boundary is a manifold with corners. Moreover, by Ehresmann fibration theorem the mapping

$$\kappa_{|N_{\epsilon,\xi}\setminus\kappa^{-1}(0)}:N_{\epsilon,\xi}\setminus\kappa^{-1}(0)\to(0,\xi] \tag{66}$$

is a locally trivial fibration.

Consider $f\in I$ with $c_{I,e}(f)<\infty$. Let F be a I-morsification of f, and $h:=F_{|s}$ for a certain s to be chosen later. We can assume that ϵ is small enough so that there is $\eta>0$ such that (ϵ,η) is a good system of radii for f. Take a small enough positive number δ so that the conclusions of Theorem 2.2 are fulfilled; choose $s\in D_\delta$. Then, for any $t\in D_\eta$ we have $h^{-1}(t)\pitchfork S_\epsilon$ (the transversality is meant in a stratified sense when $t=0$).

Define

$$Y_{\epsilon,\xi,\eta}:=N_{\epsilon,\xi}\cap h^{-1}(D_\eta)\quad \dot{Y}_{\epsilon,\xi,\eta}:=Y_{\epsilon,\xi,\eta}\setminus h^{-1}(0).$$

We claim that there is a positive $\xi\leqslant\xi_0$ small enough so that there exists η such that

$$h_{|\dot{Y}_{\epsilon,\xi,\eta}}:\dot{Y}_{\epsilon,\xi,\eta}\to D_\eta\setminus\{0\} \tag{67}$$

is a locally trivial fibration, and, moreover, if (ξ',η'), with $\xi'\leqslant\xi$, is a pair of radii defining an analogous fibration, then this fibration is equivalent to the one associated with (ξ,η).

As the critical points of h in $\bar{B}_\epsilon$ are either points contained in V or isolated points of positive extended codimension, it is clear that choosing ξ small enough we can assume that the only critical points that h has in $N_{\epsilon,\xi}$ are included in V. Therefore all the critical points have image 0.

The boundary of $N_{\epsilon,\xi}$ is a manifold with corners admitting the following decomposition in smooth strata

$$\partial N_{\epsilon,\xi}=(\kappa^{-1}([0,\xi))\cap S_\epsilon)\coprod(\kappa^{-1}(\xi)\cap S_\epsilon)\coprod(\kappa^{-1}(\xi)\cap B_\epsilon).$$

As $h^{-1}(t)\pitchfork S_\epsilon$ for any $t\in D_\epsilon\setminus\{0\}$ and $\kappa^{-1}([0,\xi))\cap S_\epsilon$ is open in S_ϵ we deduce

$$h^{-1}(t)\pitchfork\kappa^{-1}([0,\xi))$$

for any $t\in D_\eta\setminus\{0\}$.

On the other hand $h^{-1}(0)$ is smooth outside V. We show that there exists ξ such that

$$h^{-1}(0) \pitchfork (\kappa^{-1}(\xi') \cap B_\epsilon) \quad h^{-1}(0) \pitchfork (\kappa^{-1}(\xi') \cap S_\epsilon) \tag{68}$$

for any $\xi' < \xi$. Suppose that for any $\xi > 0$ the set of points where $h^{-1}(0)$ is not transversal to $(\kappa^{-1}(\xi) \cap B_\epsilon)$ is not void. Then by the curve selection lemma there is a germ of analytic path $\gamma : (\mathbb{R}, 0) \to h^{-1}(0)$ such that $\gamma^{-1}(V) = \{0\}$ and $(\kappa \circ \gamma)'(t) = 0$ for any $t \in (-\delta, \delta)$. Then the function $\kappa \circ \gamma$ has zero derivative, but is not constant. This gives a contradiction, which shows $h^{-1}(0) \pitchfork (\kappa^{-1}(\xi') \cap B_\epsilon)$ for ξ' small. The transversality $h^{-1}(0) \pitchfork (\kappa^{-1}(\xi') \cap S_\epsilon)$ when ξ' is small enough is proven analogously. Using the compactness of $(\kappa^{-1}(\xi) \cap S_\epsilon) \coprod (\kappa^{-1}(\xi) \cap B_\epsilon)$ it is easy to show that if η is small enough then

$$h^{-1}(t) \pitchfork (\kappa^{-1}(\xi) \cap B_\epsilon) \quad h^{-1}(t) \pitchfork (\kappa^{-1}(\xi) \cap S_\epsilon)$$

for any $t \in D_\eta$.

Summarising, we have found ξ and η such that $h^{-1}(t)$ is transversal to the manifold with corners $\partial N_{\epsilon,\xi}$ for any $t \in \dot{D}_\eta$, and such that the only critical points of h at $N_{\epsilon,\xi}$ are at $h^{-1}(0)$. By Ehresmann's fibration theorem, the mapping (67) is a locally trivial fibration. Using the fact that (68) holds for any $\xi' < \xi$ it is easy to check that the fibration analogous to (67) associated with any other suitable pair (ξ', η') satisfying $\xi' \leqslant \xi$ is equivalent to the one associated with (ξ, η). This shows the claim.

Using both the transversality conditions that we have checked up to now and the fibrations (66) and (67), it is easy to show that the set $\mathrm{Int}(Y_{\epsilon,\xi,\eta})$ of interior points of $Y_{\epsilon,\xi,\eta}$ is a dilation neighbourhood of $V \cap B_\epsilon$ in the sense of [14]. On the other hand, considering a Whitney stratification on $V \cap B_\epsilon$ and taking a controlled tube system for it (in the sense of [6]) we obtain a tubular neighbourhood T of $V \cap B_\epsilon$ which admits a deformation retract to it. By the uniqueness theorem of dilation neighbourhoods of [14] we deduce that T is diffeomorphic to $\mathrm{Int}(Y_{\epsilon,\xi,\eta})$. Therefore the contractibility of $V \cap B_\epsilon$ implies that $\mathrm{Int}(Y_{\epsilon,\xi,\eta})$, and hence $Y_{\epsilon,\xi,\eta}$, is contractible.

The following theorem shows how to use our theory to extract topological information about the Milnor fibre of f.

Let F denote the Milnor fibre of f. Denote by F_0 the fibre of the fibration (67).

Theorem 9.3. *Let F denote the Milnor fibre of f. Denote by F_0 the fibre of the fibration* (67). *Recall that $\mathscr{M}(f)$ denotes the Morse number of F. Then*

$$\mathrm{H}_k(F, \mathbb{Z}) = \mathrm{H}_k(F_0, \mathbb{Z}) \oplus [\mathrm{H}_k(S^{n-1}, \mathbb{Z})]^{\mathscr{M}(f)} \tag{69}$$

for any k.

Moreover, if either $n = 2$ or both F_0 and F have trivial fundamental group then F has the homotopy type of the bouquet

$$F_0 \bigvee \left[\bigvee_{\mathscr{M}(f)} S^{n-1} \right], \tag{70}$$

that is, the bouquet of F_0 with $\mathscr{M}(f)$ spheres of dimension $n - 1$.

Proof. By Theorem 2.2 the Milnor fibre F is equal to the generic fibre of the mapping (10). After this essentially the same arguments as in [22] (for which the contractibility of $Y_{\epsilon,\xi,\eta}$ is needed) allow us to conclude. $\square$

Addendum 9.4. *If the critical locus of f is at least of codimension* 3 *then both F_0 and F have trivial fundamental group, and hence, the homotopy decomposition of last Theorem holds. This happens, in particular, when* $\operatorname{codim}(V(I)) \geqslant 3$.

Proof. Suppose that the codimension of the critical locus of f is at least 3. Let $S \subset J^{\infty}(U,\tilde{I})$ be the closed analytic subset formed by the singular germs. Then any irreducible component of S which is met by the image of the jet extension $\rho_f : B_\epsilon \to J^{\infty}(U,\tilde{I})$ has at least codimension 3. The transversality properties of a morsification $F_{|s}$ imply $\operatorname{codim}(\rho_{F_{|s}}^{-1}(S)) \geqslant 3$, that is, the codimension of the critical locus of $F_{|s}$ is at least 3. Then the results of [11] imply the simple connectivity of F_0 and F. $\square$

Theorem 9.3 reduces the study of many properties the Milnor fibre of f to the study of the Milnor fibre of F_0. In the study of F_0 the fibration (67) our relative morsification theorem becomes very important: it tells us that the only singularities that $F_{|s}$ can have belong to strata of the topological partition which are unavoidable by f (this imposes conditions for example on the codimension of such sets of singularities). This has been already used successfully for certain classes of ideals (see [20,21,10,26,16]). The transversality of the jet extension $\rho_{F_{|s}}$ to the relevant strata of $J^{\infty}(U,\tilde{I})$ relates the relative positions of the different singularity types appearing in $\rho_{F_{|s}}$ to the relative positions of the relevant strata in $J^{\infty}(U,\tilde{I})$.

Similar bouquet decomposition for theorems has been already obtained by Siersma (see [22]) and Tibar (see [25]) for the case of functions with isolated singularities defined on singular spaces.

10. Numerical invariants and intersection multiplicities

In this section we give a characterisation of the invariants introduced in Definition 8.9 in terms of intersection multiplicities in the generalised jet space. We will use certain intersection theoretic constructions of [5]. It can be checked easily that remain valid in the analytic setting, at least in the very restricted degree of generality that we need them.

Let $V \subset J^{\infty}(U,\tilde{I})$ be an irreducible closed analytic subset of codimension n. Given any $f \in \tilde{I}_x$, its jet extension $\rho_f : (\mathbb{C}^n,x) \to J^{\infty}(U,\tilde{I})$ is a germ of analytic mapping. Suppose that x is an isolated point of $\rho_f^{-1}(V)$; as we work with germs at x we can actually assume $\rho_f^{-1}(V) = \{x\}$. Express $f = \sum_{i=1}^{r} f_i g_i$ where $g_1,\dots,g_r$ is our fixed set of generators and $f_1,\dots,f_r$ are holomorphic in x. We have the associated local lifting $\tilde{\rho}_f : (\mathbb{C}^n,x) \to J^{\infty}(U,\mathbb{C}^r)$ given by $\tilde{\rho}_f(y) = (j^{\infty}f_1(y),\dots,j^{\infty}f_r(y))$. Choose an integer k such that V is k-determined; then $V' := j^k\varphi^{-1}(\pi_k^{\infty}(V))$ is an irreducible variety in $J^k(U,\mathbb{C}^r)$. Define $\tilde{\rho}_f^k := \pi_k^{\infty} \circ \tilde{\rho}_f$; let $x' := \tilde{\rho}_f^k(x)$. Let $Z_{\dim(V')}(V')$ and $Z_0(\{x'\})$ be respectively the groups of analytic cycles of dimension $\dim(V')$ and 0 of V' and $\{x'\}$. These groups are obviously isomorphic to $\mathbb{Z}$ with respective generators $[V']$ and $\{x'\}$. By Definition 8.1.2. of [5] there is a refined Gysin homomorphism

$$(\tilde{\rho}_f^k)^! : Z_{\dim(V')}(V') \to Z_0(\{x'\}.$$

Definition 10.1. We define the *intersection multiplicity* of ρ_f and V at x to be the integer $i_x(\rho_f, V)$ characterised by

$$(\tilde{\rho}_f^k)^!([V']) = i_x(\rho_f, V)[x'].$$

We have to prove that this integer is independent of k, and of the functions $f_1, \dots, f_r$ giving rise to the local lifting. Moreover we want to give a formula to compute $i_x(\rho_f, V)$. For this we need to recall from [5] how the intersection product $(\tilde{\rho}_f^k)^!([V'])$ is defined.

Consider the subvariety $V'' := \mathbb{C}^n \times V'$ of the product $\mathbb{C}^n \times J^k(U, \mathbb{C}^r)$; denote by σ_k the projection to the first factor. Let $\gamma_f^k : \mathbb{C}^n \to \mathbb{C}^n \times J^k(U, \mathbb{C}^r)$ be defined by $\gamma_f(y) := (y, \tilde{\rho}_f^k(y))$; its image Γ_f^k is the graph of $\tilde{\rho}_f^k$; let $x'' := \gamma_f^k(x)$. Observe that $J^k(U, \mathbb{C}^r)$ is isomorphic to $U \times \mathbb{C}^N$ for a certain N; let pr_1 and pr_2 be the projections to the first and second factor. Recall that we have fixed coordinates $(x_1, \dots, x_n)$ for $\mathbb{C}^n$. Let $z_1, \dots, z_N$ be a coordinate system for $\mathbb{C}^N$; consider the coordinate system $(y_1, \dots, y_{n+N})$ of $J^k(U, \mathbb{C}^r)$ defined by $y_i := x_i \circ pr_1$ for $1 \leqslant i \leqslant n$, and $y_{n+i} := z_i \circ pr_2$ for $1 \leqslant i \leqslant N$; then $\{x_1, \dots, x_n, y_1, \dots, y_{n+N}\}$ is a coordinate system for $\mathbb{C}^n \times J^k(U, \mathbb{C}^r)$. Define $h_i := y_{n+i} \circ \gamma_f^k \circ \sigma_k$ for $i \leqslant N$. Then the subvariety $\Gamma_f^k \subset \mathbb{C}^n \times J^k(U, \mathbb{C}^r)$ is defined by the regular sequence

$$(y_1 - x_1, \dots, y_n - x_n, y_{n+1} - h_1, \dots, y_{n+N} - h_N).$$

Composing with the natural ring epimorphism $\mathcal{O}_{\mathbb{C}^n \times J^k(U, \mathbb{C}^n), x''} \to \mathcal{O}_{V'', x''}$ we obtain a sequence $\mathbf{s} = (s_1, \dots, s_{n+N})$ of elements of $\mathcal{O}_{V'', x''}$. Let $\mathrm{K}_\bullet(\mathbf{s})$ be the Koszul complex associated to $\mathbf{s}$; denote by $\mathrm{H}_i(\mathrm{K}_\bullet(\mathbf{s}))$ its ith homology module. Then unwinding the definition of $(\tilde{\rho}_f^k)^!$ and applying Example 7.1.2 of [5] we obtain:

$$i_x(\rho_f, V) := \sum_{i=1}^{n+N} (-1)^i \dim_{\mathbb{C}}(\mathrm{H}_i(\mathrm{K}_\bullet(\mathbf{s}))). \tag{71}$$

Remark 10.2. If $\mathbf{s}$ is a regular sequence then $\mathrm{H}_i(\mathrm{K}_\bullet(\mathbf{s})) = 0$ for $i > 0$. Then

$$i_x(\rho_f, V) = \dim_{\mathbb{C}}(\mathrm{H}_0(\mathrm{K}_\bullet(\mathbf{s}))) = \dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n, c} / \tilde{\rho}_f^{k*} J),$$

where J is the ideal sheaf of V'. This happens when V' is a Cohen–Macaulay variety.

Suppose that $l > k$. The projection $\pi_k^l : J^l(U, \mathbb{C}^r) \to J^k(U, \mathbb{C}^r)$ is a trivial fibration with fibre $\mathbb{C}^L$, for a certain L; hence we have a product decomposition

$$J^l(U, \mathbb{C}^r) = J^k(U, \mathbb{C}^r) \times \mathbb{C}^L, \tag{72}$$

let q_1 and q_2 be the projections to the first and second factors. Let $w_1, \dots, w_L$ be a coordinate system for $\mathbb{C}^L$. Define a coordinate system $Y_1, \dots, Y_{n+N+L}$ of $J^l(U, \mathbb{C}^r)$ by $Y_i := q_1^* y_i$ for $1 \leqslant i \leqslant n + N$ and $y_{n+N+i} := q_2^* w_i$ for $1 \leqslant i \leqslant L$. The subvariety $W' := (j^l \varphi)^{-1}(\pi_l^\infty(V))$ is clearly equal to $(\pi_k^l)^{-1}(V')$, which by the product decomposition (72) is equal to $V' \times \mathbb{C}^L$. Therefore the subvariety $W'' := \mathbb{C}^n \times W'$ of the product $\mathbb{C}^n \times J^l(U, \mathbb{C}^r)$ is equal to the product $\mathbb{C}^n \times W' \times \mathbb{C}^L$. Observe that $Y_i \circ \gamma_f^l = y_i \circ \gamma_f^k$ for $1 \leqslant i \leqslant n+N$. Define $H_i := Y_{n+i} \circ \gamma_f^l \circ \sigma_l$ (where σ_l is the projection of $\mathbb{C}^n \times J^l(U, \mathbb{C}^r)$ to the first factor) for $1 \leqslant i \leqslant N + L$. Consider the projection $\beta := (\mathrm{Id}_{\mathbb{C}^n}, \pi_k^l) : \mathbb{C}^n \times J^l(U, \mathbb{C}^r) \to \mathbb{C}^n \times J^k(U, \mathbb{C}^r)$; observe that $H_i = h_i \circ \beta$ for any $i \leqslant N$. Define $x''' := \gamma_f^l(x)$; observe that $\beta(x''') = x''$.

Let $\mathbf{s}' = (s'_1, \ldots, s'_{n+N+L})$ be the sequence of elements of $\mathcal{O}_{W'',x'''}$ obtained by projecting the regular sequence

$$(Y_1 - x_1, \ldots, Y_n - x_n, Y_{n+1} - H_1, \ldots, Y_{n+N+L} - H_{N+L}).$$

Using the product structure in W'' it is easy to check that $(s'_{n+N+1}, \ldots, s'_{n+N+L})$ is a regular sequence. Moreover if Z is the analytic subspace of W'' defined by the ideal generated by $(s'_{n+N+1}, \ldots, s'_{n+N+L})$, and s''_i denotes the class of s'_i in $\mathcal{O}_{Z,x'''}$, then the restriction $\sigma_{|Z} : Z \to V''$ is an isomorphism satisfying $\sigma^*_{|Z} s_i = s''_i$.

Let $(s_1, \ldots, s_r)$ be elements of a ring A. Let $\bar{s}_2, \ldots, \bar{s}_r$ be the classes of $s_2, \ldots, s_r$ in $A/(s_1)$. We have the decomposition $\mathrm{K}_\bullet(s_1, \ldots, s_r) = \mathrm{K}_\bullet(s_1) \otimes \mathrm{K}_\bullet(s_2, \ldots, s_r)$, in which

$$\mathrm{K}_p(s_1, \ldots, s_r) = [\mathrm{K}_0(s_1) \otimes \mathrm{K}_p(s_2, \ldots, s_r)] \oplus [\mathrm{K}_1(s_0) \otimes \mathrm{K}_{p-1}(s_2, \ldots, s_r)]. \tag{73}$$

Denote by $\beta_p : \mathrm{K}_p(s_2, \ldots, s_r) \to \mathrm{K}_p(\bar{s}_2, \ldots, \bar{s}_r)$ be the natural epimorphism. Consider the morphism of complexes $\alpha_\bullet : \mathrm{K}_\bullet(s_1, \ldots, s_r) \to \mathrm{K}_\bullet(\bar{s}_2, \ldots, \bar{s}_r)$ defined by $\alpha_p = \beta_p \oplus 0$ in terms of the decomposition (73).

Lemma 10.3. *If s_1 is not a zero divisor of A then $\alpha_\bullet$ is a quasi-isomorphism.*

Proof. The homomorphism α is clearly surjective in each level. It is easy to check that the complex formed by the kernels is acyclic. □

Using the last lemma repeatedly we obtain

$$\sum_{i=1}^{n+N+L} (-1)^i \dim_{\mathbb{C}}(\mathrm{H}_i(\mathrm{K}_\bullet(\mathbf{s}'))) = \sum_{i=1}^{n+N} (-1)^i \dim_{\mathbb{C}}(\mathrm{H}_i(\mathrm{K}_\bullet(s''_1, \ldots, s''_{n+N}))); \tag{74}$$

due to the fact that $\sigma_{|Z}$ is an isomorphism satisfying $\sigma^*_{|Z} s_i = s''_i$ the last quantity equals

$$\sum_{i=1}^{n+N} (-1)^i \dim_{\mathbb{C}}(\mathrm{H}_i(\mathrm{K}_\bullet(\mathbf{s}))).$$

This proves independence of k in Definition 10.1.

Let X be a smooth analytic variety and Y an irreducible closed analytic subset of codimension n in X. Consider an analytic mapping $G : B_\epsilon \times D_\delta \to X$. Suppose $G^{-1}_{|0}(Y) = \{O\}$. Choosing ϵ small enough and $0 < \delta \ll \epsilon$, the restriction $\pi : Z := G^{-1}(Y) \to D_\delta$ of the projection to the second factor is a finite map. Consequently Z is a 1-dimensional closed analytic subspace of $B_\epsilon \times D_\delta$. For any $t \in D_\delta$ the set $G^{-1}_{|t}(Y)$ is a finite number of points $\{p_1, \ldots, p_d\}$; therefore any cycle in $Z_0(G^{-1}_{|t}(V))$ has a unique expression of the form $\sum_{i=1}^{s} n_i p_i$, where n_i are integers. Define the *degree* of a cycle as $\deg(\sum_{i=1}^{d} n_i p_i) := \sum_{i=1}^{d} n_i$. Proposition 10.2 of [5] tells us

Lemma 10.4. *In the preceding situation* $\deg(G^!_{|t}([V])$ *does not depend on* t.

Now we prove the independence of Definition 10.1 on the functions $f_1, \ldots, f_r$. Suppose that we have another expression $f = \sum_{i=1}^{r} g_i h_i$. Choose ϵ so that f, the f_i's and the h_i's are defined in B_ϵ. Defining $F_i : B_\epsilon \times \mathbb{C} \to \mathbb{C}$ as $F_i := (1-t) f_i + t h_i$ we have $f = \sum_{i=1}^{r} g_i F_{i|t}$ for each t. Define an analytic mapping $\phi_{|t}: B_\epsilon \times \mathbb{C} \to J^\infty(U, \mathbb{C}^r)$ by the formula $\phi(x,t) := j^\infty F_{1|t}(x), \dots, j^\infty F_{r|t}(x)$. Choose ϵ small enough so that $B_\epsilon \cap \rho_f^{-1}(V) = \{O\}$. As $\phi_{|t}: B_\epsilon \to J^\infty(U, \mathbb{C}^r)$ lifts ρ_f we have $\phi_{|t}^{-1}(V) = \{O\}$ for any t. Therefore $i_x(\rho_f, V)$ computed in terms of $G_{1|t}, \dots, G_{r|t}$ is equal to $\deg(\pi_k^\infty \circ \phi_{|t})^!([\pi_k^\infty(V)]))$. Applying Lemma 10.4 we see that this number is independent on t. Hence Definition 10.1 is independent on the choice of the functions $f_1, \dots, f_r$.

Remark 10.5. If V is $\mathscr{D}_{\tilde{I},e}$-invariant the independence on choices of Definition 10.1 also could be proved using the versal unfolding to show that for a generic deformation $F_{|t}$ of f, the number of points of $\rho_{F_{|t}}^{-1}(V)$ is equal to $i_x(\rho_f, V)$. In this way also follows the independence of $i_O(\rho_f, V)$ on the choice of the system of generators of I.

Another easy consequence of Lemma 10.4 is the following "Conservation of Number Formula":

Proposition 10.6. *Let $V \subset J^\infty(U, \tilde{I})$ be an irreducible closed analytic subset of codimension n. Consider any $f \in I$ for such that O is an isolated point of $\rho_f^{-1}(V)$. Let $F: \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$ be any I-unfolding of f. For any positive and small enough ϵ there exists a positive δ such that for any $t \in \mathbb{C}^k$ with $\|t\| < \delta$ we have*

$$i_O(\rho_f, V) = \sum_{x \in B_\epsilon} i_x(\rho_{F_{|t}}, V). \tag{75}$$

Proof. Let $\tilde{\rho}_F: \mathbb{C}^n \times \mathbb{C} \to J^\infty(U, \mathbb{C}^r)$ be an analytic local lifting of ρ_F. Observe that

$$\deg(\pi_k^\infty \circ \tilde{\rho}_{F|t})^!([\pi_k^\infty(V)])) = \sum_{x \in B_\epsilon} i_x(\rho_{F_{|t}}, V)$$

and apply Lemma 10.4. □

This provides an algebraic formula for the splitting function and the Morse number:

Corollary 10.7. *Consider $f \in I$ with $c_{I,e}(f) < \infty$. Then*

$$\sigma_I[f](n) = i_O(\rho_f, Z_n),$$

where Z_n is the union of the irreducible components of C_1 such that the extended codimension of a generic member of them is precisely n.

If a germ $f_x \in J^\infty(U, \tilde{I}_x)$ has a Morse point at x then

(1) the projection $pr(x)$ does not belong to $V(I)$,
(2) the function f_x has a critical point in x,
(3) the function f_x has positive codimension with respect to $\tilde{I}_x$.

The germs having Morse points are a dense open subset among the germs having these three properties. Clearly conditions (1) and (2) hold simultaneously if and only if conditions (1) and (3) hold at the same time. In other words, if Z is the set of germs f_x having x as a critical point then

$$Z \setminus pr^{-1}(V(I)) = C_1 \setminus pr^{-1}(V(I)). \tag{76}$$

We define the finitely-determined closed analytic subset $M \subset J^{\infty}(U,\tilde{I})$ to be the union of the irreducible components of C_1 not contained in $pr_{\infty}^{-1}(V(I))$. We have

Corollary 10.8. *Consider $f \in I$ with $c_{I,e}(f) < \infty$. Then*

$$\mathscr{M}(f) = i_O(\rho_f, M).$$

Suppose that M is k-determined. A key step to effectively compute the Morse number is to have an explicit description of the ideal of $M' := (j^k\varphi)^{-1}(\pi_k^{\infty}(M))$ as a subset of $J^k(U,\mathbb{C}^r)$. Using (76) we can obtain such a description:

Observe that Z is 1-determined. We show that it is a closed analytic subset and give generators for the ideal defining $(j^1\varphi)^{-1}(\pi_1^{\infty}(Z))$.

Recall that $\mathbb{C}\{x\}$ denotes the space of convergent power series in n variables $x_1,\dots,x_n$ and $\mathbf{m}$ denotes its maximal ideal. If $\alpha = (\alpha_1,\dots,\alpha_r)$ we set $x^{\alpha} := x_1^{\alpha_1}\dots x_r^{\alpha_r}$. Given any element $h = (h_1,\dots,h_r) \in \mathbb{C}\{x\}^r$, for any $\alpha \in \mathbb{Z}^r_{\geqslant 0}$ we denote by $a^i_{\alpha}(g)$ the coefficient of x^{α} in the power series expansion of h_i. The set $U \times (\mathbb{C}\{x\}/\mathbf{m}^{k+1})^r$ is an affine space with coordinates $x_1,\dots,x_n$ and a^i_{α} (for $1 \leqslant i \leqslant n$ and $\alpha = (\alpha_1,\dots,\alpha_r)$ such that $|\alpha| := \sum_{i=1}^r \alpha_i \leqslant k$). We consider the trivialisation $\tau_k : U \times (\mathbb{C}\{x\}/\mathbf{m}^{k+1})^r \to J^k(U,\mathbb{C}^k)$ (see Formula (12)). Define $\beta_0,\dots,\beta_n \in \mathbb{Z}^r_{\geqslant 0}$ by: $\beta_0 = (0,\dots,0)$ and β_i is the n-tuple whose only non-zero component is the ith one and has value 1. An easy computation shows that $(j^k\varphi \circ \tau_k)^{-1}(\pi_k^{\infty}(Z))$ is the subset of $U \times (\mathbb{C}\{x\}/\mathbf{m}^{k+1})^r$ given by the set of common zeros of the functions $Q_1,\dots,Q_n$, where

$$Q_i(\dots,z_i,\dots,a^i_{\alpha},\dots) := \sum_{j=1}^{n}\left[a^j_{\beta_0}\frac{\partial f_j}{\partial x_i}(z_1,\dots,z_n) + a^j_{\beta_i} f_j(z_1,\dots,z_n)\right].$$

Define $J_1 := (Q_1,\dots,Q_r)$ and let J_2 be the pullback of I by the projection of $U \times (\mathbb{C}\{x\}/\mathbf{m}^{k+1})^r$ to its first factor, that is, the ideal generated by $\{g_i(z_1,\dots,z_n) : 1 \leqslant i \leqslant r\}$. Then the ideal of functions vanishing at M' is

$$J_{M'} = (\sqrt{J_1} : J_2). \tag{77}$$

This shows in particular that $(j^{\infty}\varphi)^{-1}(M)$ is 1-determined: we could have worked with $k = 1$.

Remark 10.9. In [3] a slightly different morsification result for hypersurface singularities with critical locus an i.c.i.s. and transversal type A_1 was proved. There the critical locus of a generic deformation of a function of finite codimension is the Milnor fibre of the i.c.i.s together with a finite number of A_1 points. The techniques of this section can easily be adapted to compute the number of A_1 points in such a generic deformation.

10.1. Conservative numerical invariants

The following proposition tells us how to associate a numerical invariant to any closed analytic subset V of pure codimension n of $J^{\infty}(U,\tilde{I})$:

Proposition 10.10. *Suppose that V is $\mathscr{D}_{\tilde{I},e}$-invariant. Then the intersection multiplicity $i_x(\rho_f, V)$ is a $\mathscr{D}_{\tilde{I},e}$-invariant defined in $J^{\infty}(U,\tilde{I})$.*

Proof. For simplicity we work at the origin. Let $f \in I$ and $\phi \in \mathscr{D}_I$. We have to prove the equality $i_O(\rho_f, V) = i_O(\rho_{\phi_* f}, V)$. Consider neighbourhoods U_1 and U_2 of the origin in $\mathbb{C}^n$ such that ϕ is defined in U_1 and $\phi(U_1) = U_2$. The mapping ϕ induces a bijection

$$\phi_* : J^k(U_1, \tilde{I}) \to J^k(U_2, \tilde{I})$$

and an analytic isomorphism

$$\phi_* : J^k(U_1, \mathbb{C}^r) \to J^k(U_2, \mathbb{C}^r)$$

for any positive k and r. A set of generators $\{f_{1,1}, \dots, f_{1,r}\}$ of $\tilde{I}_{|U_1}$ induces a set of generators $\{f_{2,1}, \dots, f_{2,r}\}$ of $\tilde{I}_{|U_2}$ defining $f_{2,i} := \phi_* f_{1,i}$. Consider the associated epimorphisms

$$\varphi_i : \mathscr{O}^r_{U_1} \to \tilde{I}_{U_1}$$

for $i = 1, 2$, defined by $\varphi_i(h_1, \dots, h_r) := \sum_{j=1}^r h_j f_{i,j}$. They induce respectively mappings

$$j^k\varphi_i : J^k(U_i, \mathbb{C}^r) \to J^k(U_i, \tilde{I})$$

satisfying $j^k\varphi_2 \circ j^k\phi_* = j^k\phi_* \circ j^k\varphi_1$.

Choose $h_1, \dots, h_r$ satisfying $f = \sum_{i=1}^r h_i f_{1,i}$. This induces an analytic lifting $\tilde{\rho}^k_f$ of ρ^k_f, defined by the formula $\tilde{\rho}^k_f(x) := (j^k h_1(x), \dots, j^k h_2(x))$. Noticing that $\phi_* f = \sum_{i=1}^r \phi_* h_i f_{2,i}$ we deduce that $j^k\phi_* \circ \tilde{\rho}^k_f$ is an analytic lifting of $\rho^k_{\phi^* f}$. As $j^k\phi_* : J^k(U_1, \mathbb{C}^r) \to J^k(U_2, \mathbb{C}^r)$ is an analytic isomorphism which satisfies

$$j^k\phi_*[(j^k\varphi_1)^{-1}(\pi^\infty_k(V))] = (j^k\varphi_2)^{-1}(\pi^\infty_k(V))$$

(because of the $\mathscr{D}_I$-invariance of V), the equality $i_O(\rho_f, V) = i_O(\rho_{\phi_* f}, V)$ holds. $\square$

Definition 10.11. A numerical $\mathscr{D}_{\tilde{I},e}$-invariant $\Xi : J^\infty(U, \tilde{I}) \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ is called a *conservative invariant* if, for any $f_x \in \tilde{I}_x$ with $c_{\tilde{I}_x,e}(f) < \infty$ and any I-unfolding $F : (\mathbb{C}^n, x) \times (\mathbb{C}^k, O) \to \mathbb{C}$ of f, the following properties are satisfied:

(1) Finiteness: $\Xi(f_x) < \infty$.
(2) Analyticity: for any integer N the set

$$\{(x,t) \in \mathbb{C}^n \times \mathbb{C}^k : \Xi(F_{|t,x}) > 0\}$$

is closed analytic.
(3) Discreteness and conservation of number: there exists a sufficiently small neighbourhood U_x of x such that $\Xi(f_y) = 0$ for any $y \in U_x \setminus \{x\}$ and, for any positive and small enough δ, we have

$$\Xi(f_x) = \sum_{y \in U_y} \Xi(F_{|t,y})$$

for any $t \in B_\delta$.

Remark 10.12. Any numerical $\mathscr{D}_{\tilde{I},e}$-invariant $\Xi : J^\infty(U, \tilde{I}) \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ constructed by intersection multiplicity with $\mathscr{D}_{\tilde{I},e}$-invariant closed analytic subsets of pure codimension n of $J^\infty(U, \tilde{I})$ is a conservative invariant.

Observe that the property of conservation of number implies shows upper semicontinuity of Ξ, that is $\Xi(F_{|t,x}) \leqslant \Xi(f_x)$ for any t close enough to x.

The following theorem tells that any conservative invariant is a (locally) finite sum of invariants constructed by intersection multiplicity:

Theorem 10.13. *Let $\Xi : J^\infty(U,\tilde{I}) \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ be a conservative numerical $\mathscr{D}_{\tilde{I},e}$-invariant. Then there exist a family of pairs (V_i, n_i), where V_i is a n-codimensional $\mathscr{D}_{\tilde{I},e}$-invariant closed analytic subset of $J^\infty(U,\tilde{I})$, and n_i is a positive integer, such that for any $f_x \in J^\infty(U,\tilde{I})$ satisfying $c_{\tilde{I}_x,e}(f_x) < \infty$ the family $\{V_i\}_{i\in\mathbb{N}}$ is locally finite at f_x and $\Xi(f_x) = \sum_{i\in\mathbb{N}} n_i i_x(\rho_f, V_i)$ (the sum is finite by the locally finiteness of the family at f_x).*

Proof. Recall that C_n denotes the subset of $J^\infty(U,\tilde{I})$ formed by germs of extended codimension at least n. Define the sets

$$Z := \{f_x \in J^\infty(U,\tilde{I}) : \Xi(f_x) > 0\} \quad Z_k := Z \backslash C_k,$$

for any $k > 0$.

We claim that Z_k is $(k+\lambda)$-determined, where λ is the uniform Artin-Rees constant. Consider $f_x \in Z\backslash C_k$ and $g_x \in \mathbf{m}_x^{k+\lambda+1} \cap \tilde{I}_x$. We have to show that $f_x + g_x$ belongs to Z_k. By Lemma 4.4 the subset C_k is $(k+\lambda)$-determined, and hence $f_x + g_x \notin C_k$. As $c_{\tilde{I}_x,e}(f_x) < k$, by Lemma 4.2 we have $\mathbf{m}_x^{k-1}\tilde{I}_x \subset \Theta_{\tilde{I}_x,e}(f_x)$. As $\mathbf{m}_x\Theta_{\tilde{I}_x,e} \subset \Theta_{\tilde{I}_x}$ we deduce $\mathbf{m}_x^{k+1}\tilde{I}_x \subset \mathbf{m}_x\Theta_{\tilde{I}_x}(f_x)$. By uniform Artin-Rees $\mathbf{m}_x^{k+\lambda+1} \cap \tilde{I}_x \subset \mathbf{m}_x\Theta_{\tilde{I}_x}(f_x)$. Then the finite I-determinacy theorem (see Theorem 6.5 of [18]) tells that f_x is a $(k+\lambda)$-determined in I; in other words, given any $g_x \in \mathbf{m}_x^{k+\lambda+1} \cap \tilde{I}_x$ then $f_x + g_x$ is in the orbit $\mathscr{D}_{\tilde{I}_x}(f)$. Consequently, as Z is clearly $\mathscr{D}_{\tilde{I},e}$-invariant and f_x belongs to Z, also $f_x + g_x$ belongs to Z. This proves the claim.

Our next claim is that the topological closure $\bar{Z}_k$ of Z_k in $J^\infty(U,\tilde{I})$ is a $(k+2\lambda)$-determined closed analytic subset of $J^\infty(U,\tilde{I})$, with no irreducible components contained in C_k. Moreover, the set Z_k is closed in $J^\infty(U,\tilde{I})\backslash C_k$. Let $\{f_1,\dots,f_N\}$ be monomials in $x_1,\dots,x_n$ forming a basis of $\mathcal{O}_{\mathbb{C}^n,O}/\mathbf{m}_O^{\lambda+k+1}$. Consider $S := (\mathcal{O}_{\mathbb{C}^n,O}/\mathbf{m}_O^{\lambda+k+1})^r$ viewed as an affine space. Consider the system of coordinates $\{s_j^i\}$ for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant N$ characterised by the property that the point with coordinates $(a_1^1,\dots,a_N^r)$ represents the r-tuple $(\sum_{j=1}^N a_j^1 f_j,\dots,\sum_{j=1}^N a_j^r f_j)$. Define the I-unfolding $F : U \times S \to \mathbb{C}$ by the formula

$$F(x,s_1^1,\dots,s_N^r) := \sum_{i=1}^r \sum_{j=1}^N s_j^i f_j(x) g_j(x), \tag{78}$$

where $g_1,\dots,g_r$ are the fixed generators for $\tilde{I}$ at U. According to the second property of Ξ the subset $A \subset U \times S$ formed by the pairs (x,s) such that $\Xi(F_{|s,x}) > 0$ is closed analytic. The mapping $\alpha : U \times S \to J^{k+\lambda}(U,\mathbb{C}^r)$ defined by

$$\alpha(x,s_1^1,\dots,s_N^r) = \left(\sum_{j=1}^N s_j^1 j^{k+\lambda} f_j(x),\dots,\sum_{j=1}^N s_j^r j^{k+\lambda} f_j(x)\right)$$

is an analytic vector bundle isomorphism. We have defined F and α so that the compatibility relation $\pi^{\infty}_{k+\lambda} \circ \rho_F = j^{k+\lambda}\varphi \circ \alpha$ holds. This implies

$$\alpha(A)\backslash (j^{k+\lambda}\varphi)^{-1}(\pi^{\infty}_{k+\lambda}(C_k)) = (j^{k+\lambda}\varphi)^{-1}(\pi^{\infty}_{k+\lambda}(Z_k)).$$

The set $B := \alpha(A) \cup (j^{k+\lambda}\varphi)^{-1}(\pi^{\infty}_{k+\lambda}(C_k))$ is a closed analytic subset of $J^{k+\lambda}(U,\mathbb{C}^r)$ which is $j^{k+\lambda}\varphi$-saturated, as it is the union

$$[\alpha(A)\backslash (j^{k+\lambda}\varphi)^{-1}(\pi^{\infty}_{k+\lambda}(C_k))] \cup (j^{k+\lambda}\varphi)^{-1}(\pi^{\infty}_{k+\lambda}(C_k)),$$

of $j^{k+\lambda}\varphi$-saturated closed analytic subsets. Let B' be the union of the irreducible components of B not contained in $(j^{k+\lambda}\varphi)^{-1}(\pi^{\infty}_{k+\lambda}(C_k))$. Due to Lemma 3.9 we know that $B'' := (\pi^{k+2\lambda}_{k+\lambda})^{-1}(B')$ is $j^{k+2\lambda}\varphi$-saturated, and therefore $Z'_k := \pi^{\infty}_{k+2\lambda+2}(j^{k+2\lambda}\varphi(B''))$ is a $(k+2\lambda)$-determined closed analytic subset of $J^{\infty}(U,\tilde{I})$ with no irreducible components contained in C_k. By construction, it is clear that $Z'_k\backslash C_k = Z_k$. By Lemma 3.14 and the fact that no irreducible component of Z'_k is contained in C_k we conclude that the topological closure of $Z_k = Z'_k\backslash C_k$ equals Z'_k. Obviously, the set $\bar{Z}_k\backslash Z_k = Z'_k\backslash(Z'_k\backslash C_k)$ is contained in C_k. Our claim is proved.

For any positive integers k and m we have

$$\bar{Z}_k\backslash C_m \subset Z_m. \tag{79}$$

Indeed, consider $f_x \in J^{\infty}(U,\tilde{I})$ such that $c_{\tilde{I},e}(f) < m$. Suppose that $\Xi(f_x)=0$. Then, as Z_m is closed in $J^{\infty}(U,\tilde{I})\backslash C_m$, there is an open neighbourhood of f_x in $J^{\infty}(U,\tilde{I})$ where Ξ vanishes. From this, the inclusion (79) in the case $k<m$ follows easily. If $k \geqslant m$ then

$$\bar{Z}_k\backslash C_m \subset \bar{Z}_k\backslash C_k = Z_k,$$

but as $Z_k\backslash C_m$ is clearly equal to Z_m, inclusion (79) follows.

Set $Y := \bigcup_{k>0} \bar{Z}_k$; it follows from (79) that

$$Y\backslash C_m = Z_m = \bar{Z}_m\backslash C_m \tag{80}$$

for any $m>0$. Therefore Y is a finitely determined closed analytic subset locally at each point of $J^{\infty}(U,\tilde{I})\backslash C_{\infty}$. This is expressed in other words as follows: let $\{V_i\}_{i\in\mathbb{N}}$ be the set whose elements are the irreducible components of any of the $\bar{Z}_k$'s. Then the family $\{V_i\}_{i\in\mathbb{N}}$ is locally finite in $J^{\infty}(U,\tilde{I})\backslash C_{\infty}$. Moreover, each V_i is $\mathscr{D}_{\tilde{I},e}$-invariant, since it is an irreducible component of some $\bar{Z}_k$, which is $\mathscr{D}_{\tilde{I},e}$-invariant as it contains the $\mathscr{D}_{\tilde{I},e}$-invariant dense Zariski open subset $Z\backslash C_k$.

Choose a component V_i. Consider $f_x \in V_i$ with $c_{\tilde{I}_x,e}(f_x) < \infty$, which, by the local finiteness of the family $\{V_i\}_{i\in\mathbb{N}}$ at $J^{\infty}(U,\tilde{I})\backslash C_{\infty}$ can be chosen so that there is a neighbourhood W of f_x in $J^{\infty}(U,\tilde{I})$ satisfying $W \cap Z = V_i$. If $\operatorname{codim}(V_i) < n$, an easy argument using Statement (†) of the proof of Lemma 4.6 shows that $\dim_x(\rho_f^{-1}(V_i)) > 0$. As $\Xi(f_y) > 0$ at any $y \in \rho_f^{-1}(V_i)$ this contradicts the third property of Ξ. Hence $\operatorname{codim}(V_i) \geqslant n$. Consider the canonical Whitney stratification $\mathscr{X}$ of V_i (see Theorem 6.2); the strata are $\mathscr{D}_{\tilde{I},e}$-invariant. Consider a versal I-unfolding $F : U_x \times S \to \mathbb{C}$ of f_x at x (where U_x is a neighbourhood of x), let s_0 be the point in S such that $F_{|s_0,x} = f_x$. By Proposition 7.5, the mapping ρ_F is transversal to $\mathscr{X}$. By Theorem 7.4 the set of $s \in S$ such that $\rho_{F|s}$ is transversal to $\mathscr{X}$ is dense in S. Hence, if s is generic and $\operatorname{codim}(V_i) > n$, the set $\rho_{F|s}(U_x)$ cannot meet V_i. Choosing S and U_x small enough we can ensure that the image of ρ_F lies in W. Consequently $(\rho_{F|s})^{-1}(Z) = (\rho_{F|s})^{-1}(V_i) = \emptyset$. This means that $\Xi(F_{|s,y}) = 0$ for any $y \in U_x$, which, if s is close enough to s_0, contradicts the third property of Ξ, since $\Xi(f_x) > 0$. We conclude that $\operatorname{codim}(V_i) = n$.

Suppose that V_i is a component of $\bar{Z}_k$. As $\bar{Z}_k$ is $k+2\lambda$-saturated, then the component V_i is $k+3\lambda$-saturated. Let B be the union of all the irreducible components of $\bar{Z}_k$ different from V_i; then $B \cup C_k$ contains all the components V_j different from V_i. The set $\Delta_i := (B \cup C_k \cup \operatorname{Sing}(V_i)) \cap V_i$ is a proper $(k+4\lambda)$-determined closed analytic subset of V_i. As V_i is smooth outside Δ_i we can define $D_i \subset V_i$ to be the subset of F_x such that ρ_f is not transversal to V_i at x. The $k+3\lambda$-determinacy of V_i implies easily that D_i is $(k+4\lambda)$-determined: if f_x and g_x have the same $k+4\lambda$-jet, by uniform Artin-Rees, they differ by an element of $\mathbf{m}_x^{k+3\lambda+1}\tilde{I}_x$, and we can find liftings $\tilde{\rho}_f$ and $\tilde{\rho}_g$ of ρ_f and ρ_g which are equal up to $k+3\lambda$-jets.

The set $X := (j^{k+4\lambda+1}\varphi)^{-1}(\pi^{\infty}_{k+4\lambda+1}(V_i \setminus \Delta_i))$ is a smooth irreducible locally closed analytic subset of $J^{k+4\lambda+1}(U,\mathbb{C}^r)$. Let $\mathscr{T} \to X$ be its tangent bundle. Denote by $\imath : X \hookrightarrow J^{k+4\lambda+1}(U,\mathbb{C}^r)$ the inclusion mapping as $\mathscr{T}$ is a sub-bundle of the restriction to X of the tangent bundle of $J^{k+4\lambda+1}(U,\mathbb{C}^r)$, the mapping $\pi^{k+4\lambda+1}_{k+2\lambda}$ induces a natural homomorphism

$$q : \mathscr{T} \to \mathscr{T}',$$

where $\mathscr{T}'$ is the pullback by $\pi^{k+4\lambda+1}_{k+2\lambda} \circ \imath$ of the tangent bundle $\mathscr{T}''$ of $J^{k+2\lambda}(U,\mathbb{C}^r)$. Observe that, as $pr_{k+2\lambda} : J^{k+2\lambda}(U,\mathbb{C}^r) \to U$ defines a vector bundle, for any $p \in J^{k+2\lambda}(U,\mathbb{C}^r)$ the fibre $\mathscr{T}''_p$ decomposes naturally as

$$T_{pr_{k+2\lambda}(p)}U \oplus J^{k+2\lambda}(U,\mathbb{C}^r)_p.$$

Any vector of $\mathscr{T}''_p$ is decomposed accordingly in two components; the first is called the base component, and the second is called the fibre component. Consider the trivial vector bundle of rank n over X and choose $e_1,\ldots,e_n$ to be global trivialising sections. We define an analytic homomorphism

$$\sigma : \mathbb{C}^n \times X \to \mathscr{T}'$$

by letting $\sigma(e_i, j^{k+4\lambda+1}f_1(y),\ldots,j^{k+4\lambda+1}f_r(y))$ be the unique vector of

$$\mathscr{T}'_{(j^{k+4\lambda+1}f_1(y),\ldots,j^{k+4\lambda+1}f_r(y))} = \mathscr{T}''_{(j^{k+2\lambda}f_1(y),\ldots,j^{k+2\lambda}f_r(y))}$$

with fibre component $j^{k+2\lambda}(\partial f_1/\partial x_i)(y),\ldots,j^{k+2\lambda}(\partial f_r/\partial x_i)(y))$ and base component $\partial/\partial x_i$. The set

$$D' := \{p \in X : q(\mathscr{T}_p) + \sigma(\mathbb{C}^n) \neq \mathscr{T}'_p\}$$

is easily shown to be closed analytic in X. As D' has been defined so that $(j^{k+4\lambda+1}\varphi)^{-1}(\pi^{\infty}_{k+4\lambda+1}(D_i)) = D'$ we conclude that D_i is a proper closed analytic and $(k+3\lambda)$-determined in $V_i \setminus \Delta_i$.

We claim that Ξ is constant on $V_i \setminus (\Delta_i \cup D_i)$. The set $\bar{X}$ is an irreducible analytic subset of $J^{k+4\lambda+1}(U,\mathbb{C}^r)$, as V_i is irreducible. Then, by well known properties of complex analytic sets (see [13, Ch. IV, Section 2]), the set $X \setminus D'$ is path-connected. Therefore $V_i \setminus (\Delta_i \cup D_i)$ is path-connected. Consequently we only have to show that Ξ is locally constant on $V_i \setminus (\Delta_i \cup D_i)$. Consider $f_x \in V_i \setminus (\Delta_i \cup D_i)$. Then $f_x \in J^{\infty}(U,\tilde{I}) \setminus C_k$. We have seen that any $g_y \in J^{\infty}(U,\tilde{I}) \setminus C_k$ is $(k+\lambda)$-determined in $\tilde{I}_y$. Therefore, if we consider the I-unfolding $G : U_x \times S \to \mathbb{C}$ defined by $G(y,s) := f_x(y) + F(x,s)$ (where U_x is a neighbourhood of x in U and F is the I-unfolding defined by formula (78)), then any germ g_y belonging to $J^{\infty}(U_x,\tilde{I}) \setminus C_k$ is $\mathscr{D}_{\tilde{I},e}$-equivalent to a germ of the form $G_{|s,y}$ for a certain $s \in S$. Consequently, as $f_x = G_{|0,x}$, to prove our claim it is enough to show that $\Xi(G_{|s,y})$ is constant on a neighbourhood of $(x,0)$ in $U_x \times S$. As f_x does not belong to B, taking U_x and a neighbourhood S' of 0 in S small enough we can assume

$$\rho_G^{-1}(Z) \cap (U_x \times S') = \rho_G^{-1}(V_i) \cap (U_x \times S'),$$

and $\rho_f^{-1}(Z) = \{x\}$. On the other hand, as f_x does not belong to D_i, the mapping $\rho_{G_{|0}} = \rho_f$ is transversal to V_i at x; therefore, by the open nature of transversality, if we choose U_x and S' small enough, the mapping $\rho_{F_{|s}}$ is transversal to V_i for any $s \in S'$. Consequently, for any $s \in S'$ the set $\rho_{G_{|s}}^{-1}(V_i)$ consists of a unique point $y(s)$, which is the only point $y \in U_x$ where $\Xi_{G|s,y} > 0$. As Ξ has the property (3) of Definition 10.11 we conclude $\Xi(G_{|s,y(s)}) = \Xi(f_x)$. This shows the claim.

Let n_i the value assumed by Ξ at any point of $V_i \backslash (\Delta_i \cup D_i)$. It only remains to show that

$$\Xi(f_x) = \sum_{i \in \mathbb{N}} n_i i_x(\rho_f, V_i) \tag{81}$$

for any f_x of finite extended codimension. Given such f_x defined on U_x, consider the associated mapping $\rho_f : U_x \to J^\infty(U, \tilde{I})$. The local finiteness of the family $\{V_i\}_{i \in \mathbb{N}}$ implies that U_x can be taken small enough so that $\rho_f^{-1}(Z) = \{x\}$ and the image of ρ_f meets only finitely many components $V_{i_1}, \dots, V_{i_m}$. We consider a versal I-unfolding $F : U_x \times S \to \mathbb{C}$ of f_x. A transversality argument (similar to earlier ones) using Proposition 7.5 and Theorem 7.4 shows that the subset of $s \in S$ such that $\rho_{F_{|s}}$ only meets Z in $V_{i_j} \backslash (\Delta_{i_j} \cup D_{i_j})$ (for $1 \leqslant j \leqslant m$), and it does it transversally, is dense in S. Using this, property of (3) of Definition 10.11, and Proposition 10.6, the proof of Eq. (81) is straightforward. □

11. Examples

In this section we illustrate the morsification theory by spelling it out for certain classes of ideals.

11.1. Classical case: $I = \mathcal{O}_{\mathbb{C}^n, O}$

This is the case of isolated singularities. Here our theory recovers the classical morsification theorem. Any conservative invariant is a multiple of the Milnor number, as the set of singular germs M (which is the closure of the set of Morse germs) is the only n-codimensional closed analytic subset of $J^\infty(U, \mathcal{O}_{\mathbb{C}^n})$.

11.2. The analytic subspace $V(I)$ is smooth

Let k be the codimension of $V(I)$. We can give coordinate functions

$$y_1, \dots, y_k, x_1, \dots, x_p$$

(with $k+p=n$) of $\mathbb{C}^n$ such that $I=(y_1, \dots, y_k)$. We set $V := V(I)$. Let $\tilde{I}$ be the ideal sheaf generated by $y_1, \dots, y_k$. We study the $\mathscr{D}_{\tilde{I},e}$-invariant analytic subspaces of $J^\infty(\mathbb{C}^n, \tilde{I})$ which are of codimension at most n; let Z be such a subspace. Consider the projection mapping $pr_\infty : J^\infty(\mathbb{C}^n, \tilde{I}) \to \mathbb{C}^n$.

If $pr_\infty(Z)$ is not contained in V then $Z_{|\mathbb{C}^n \backslash V}$ is a non-empty $\mathscr{D}_{\tilde{I},e}$-invariant subset of $J^\infty(\mathbb{C}^n \backslash V, \tilde{I})$. As $\tilde{I}_{|\mathbb{C}^n \backslash V} = \mathcal{O}_{\mathbb{C}^n \backslash V}$ then, by the case $I = \mathcal{O}_{\mathbb{C}^n, O}$, the subspace $Z_{|\mathbb{C}^n \backslash V}$ is either the total space $J^\infty(\mathbb{C}^n \backslash V, \tilde{I})$

or the set of singular germs. Therefore Z is equal either to $J^{\infty}(\mathbb{C}^n,\tilde{I})$ or to the n-codimensional subvariety M which is the closure of the set of Morse points.

Suppose that $pr(Z) \subset V$. Let A be the set of integer multi-indexes $\alpha \in \mathbb{Z}^k_{\geqslant 0}$ such that $|\alpha| \geqslant 2$. Any function $f \in I$ can be written as

$$f(x_1,\ldots,x_p,y_1,\ldots,y_k) = \sum_{i=1}^{k} f_i(x_1,\ldots,x_p)y_i + \sum_{\alpha\in A} a_\alpha(x_1,\ldots,x_n)y^\alpha, \tag{82}$$

with the f_i's and a_α's convergent power series in $x_1,\ldots,x_p$. It is easy to check that if f is non-singular (that is $f_i(0,\ldots,0) \neq 0$ for a certain $i \leqslant k$) then f is $\mathscr{D}_I$-equivalent to $f = y_1$.

Let Z_1 be the subset of $J^{\infty}(\mathbb{C}^n,\tilde{I})_{|V}$ formed by germs f_x having x as a critical point. This condition, in terms of the expression (82), means precisely that $f_i(0,\ldots,0) = 0$ for any $i \leqslant k$. Therefore Z_1 is a k-codimensional 1-determined closed analytic subset of $J^{\infty}(\mathbb{C}^n,\tilde{I})_{|V}$. As the codimension of $J^{\infty}(\mathbb{C}^n,\tilde{I})_{|V}$ in $J^{\infty}(\mathbb{C}^n,\tilde{I})$ is k, the set Z_1 is $2k$-codimensional in $J^{\infty}(\mathbb{C}^n,\tilde{I})$. Consequently if $p < k$ then the only $\mathscr{D}_{\tilde{I},e}$-invariant proper analytic subsets of $J^{\infty}(\mathbb{C}^n,\tilde{I})$ which are of codimension at most n are M and $J^{\infty}(\mathbb{C}^n,\tilde{I})_{|V}$.

Assume $p \geqslant k$. Let $f \in Z_1$, express it as in (82). If the differentials

$$\{df_i(0,\ldots,0) : 1 \leqslant i \leqslant k\}$$

are linearly independent then f is easily seen to be $\mathscr{D}_I$-equivalent to

$$\sum_{i=1}^{k} x_i y_i. \tag{83}$$

We say that a function of I is of type $\tilde{D}(n-2k,0)$ if it is $\mathscr{D}_I$-equivalent to (83). Functions of type $\tilde{D}(n-2k,0)$ have a $2k$-codimensional critical locus and transversal type A_1, they are $\mathscr{D}$-equivalent to a function of type $D(n-2k,0)$ (see Definition 11.2).

The subset $Z_2 \subset Z_1$ consisting of germs for which $\{df_i(0,\ldots,0)\}_{1\leqslant i\leqslant k}$ are not linearly independent is a 2-determined analytic subset of codimension $p-k+1$ in Z_1, and hence of codimension $2k+p-k+1 = n+1$ in $J^{\infty}(\mathbb{C}^n,\tilde{I})$. Therefore if $p \geqslant k$, the only $\mathscr{D}_{\tilde{I},e}$-invariant analytic subsets of $J^{\infty}(\mathbb{C}^n,\tilde{I})$ which are of codimension at most n are M, $J^{\infty}(\mathbb{C}^n,\tilde{I})_{|V}$ and Z_1, and their codimensions are n, k and $2k$.

Our morsification theory tells us that any function f with $c_{I,e} < \infty$ can be approximated (preserving the geometry at the boundary of a Milnor ball) by a function whose singularities are finitely many Morse points and a smooth $2k$-codimensional set of points of type $\tilde{D}(n-2,0)$. Moreover any conservative invariant Ξ is of the form $\Xi = n_1\mathscr{M}$ if $n \neq 2k$ and $\Xi = n_1\mathscr{M} + n_2\Delta$ if $n = 2k$, where $\mathscr{M}$ is the Morse number, Δ is the number of $\tilde{D}(n-2,0)$ points appearing in a generic deformation of f.

Remark 11.1. Let I be arbitrary and consider $f \in I$ with $c_{I,e}(f) < 0$. The above reasoning shows that the only singularities that a generic I-deformation g close enough to f can have at a smooth point of $V(I)$ are of type $\tilde{D}(n-2k,0)$, where $k = \operatorname{codim}_x(V(I),\mathbb{C}^n)$. Moreover if such a singularity appear then there is a $2k$-codimensional locally closed subset of $\mathbb{C}^n$ where g has this singularity type.

11.3. Transversal type A_1

In [3,19,16,20,21,26], in order to study functions which are singular with transversal type A_1 and a certain fixed singular locus, the following point of view was taken. Consider the ideal J defining the singular locus (in all these works the singular locus is asked to be either an i.c.i.s. or a low dimensional isolated singularity at the origin) with its reduced structure and study the singularities appearing in generic deformations of functions which are finite codimensional with respect to the primitive ideal $\int J$.

Suppose that J is a radical ideal. Consider the variety $V = V(J)$ and the stratification $\mathbb{C}^n = W \coprod V_0 \coprod V_1$, where $U = \mathbb{C}^n \backslash V$, $V_0 = V \backslash \mathrm{Sing}(V)$ and $V_1 = \mathrm{Sing}(V)$. Let $\tilde{J}$ be the ideal sheaf of functions vanishing at V. Denote the primitive ideal $\int J$ by I and let $\tilde{I}$ be the ideal sheaf associated to it.

We study the $\mathscr{D}_{\tilde{I},e}$-invariant subspaces of $J^\infty(U,\tilde{I})$ of codimension at most n and whose image by the projection $pr: J^\infty(U,\tilde{I}) \to U$ is not contained in V_1. Let Z_1 be the closed 1-determined subset formed by germs $f_x \in J^\infty(U,\tilde{I})$ having a critical point at x. Clearly $Z_1 = M \cup \pi^{-1}(V)$, where M is the closure of the set of germs $f_x \in J^\infty(U,\tilde{I})$ having an isolated singularity at x; the set of germs f_x which have an A_1-critical point at x are a dense open subset in M.

Given any $f_x \in pr^{-1}(V)$ we define $rk(f_x)$ to be the rank of its Hessian matrix at x. The set $K_r := \{f_x \in pr^{-1}(V) : rk(f_x) \leqslant r\}$ is a 2-determined $\mathscr{D}_{\tilde{I},e}$-invariant closed subset for any integer r. Let $C_{1|V}$ be the closed analytic subset consisting of germs of $pr^{-1}(V)$ of extended codimension at least 1.

Consider any $x \in V_0$; if $d = \dim_x(V)$ there is a coordinate system

$$(x_1,\dots,x_d,y_1,\dots,y_m)$$

of $\mathbb{C}^n$ at x such that $\tilde{J}_x = (y_1,\dots,y_m)$. In Pellikaan [18] it is proved that $\tilde{I}_x = \int \tilde{J}_x = \tilde{J}_x^2$. Therefore, we can express any $g \in \tilde{I}_x$ as $g = \sum_{i,j \leqslant m} h_{i,j} y_i y_j$, where $h_{i,j} \in \mathcal{O}_{\mathbb{C}^n,x}$ and $h_{i,j} = h_{j,i}$; moreover the $h_{i,j}$ are unique modulo the ideal $\tilde{J}_x$, hence the matrix $(h'_{i,j})_{i,j \leqslant m}$ (where $h'_{i,j} := h_{i,j}(O) + \sum_{k=1}^{d} (\partial h_{i,j}/\partial x_k)(O) x_k$) is well defined.

Definition 11.2 (Pellikaan [18]). Let $f \in \mathcal{O}_{\mathbb{C}^n,x}$. We say that f is of type $D(d,k)$ if there is a coordinate system $\{x_1,\dots,x_d,y_1,\dots,y_m\}$ of $\mathbb{C}^n$ at x such that

$$f(x,y) = \sum_{i,j \leqslant k} l_{i,j} y_i y_j + \sum_{i=k+1}^{m} y_i^2,$$

where $\{l_{i,j} : i,j \leqslant k\}$ is a collection of linearly independent linear forms in $x_1,\dots,x_d$.

Let $f_x \in \tilde{I}_x$. In [18] it is proved that $c_{\tilde{I},e}(f_x) = 0$ if and only if f is of type $D(d,k)$ for a certain k. Consequently, if $f_x \in K_r$ then $c_{\tilde{I},e}(f_x) = 0$ if and only if f is of type $D(d,d-r)$.

In [3] it was shown that a generic deformation within $\tilde{I}_x$ of any $f_x \in \tilde{I}_x$ with $c_{\tilde{I}_x,e} < \infty$ only has A_1 and $D(d,k)$ points as critical points (for $k \leqslant r$); moreover the locus where the deformation has $D(d,k)$ points is a smooth subvariety of codimension $k(k+1)/2$ in V_x; in [3] the singular locus V_x of f_x is smoothed while deforming f_x, but in our case it is already smooth (since $x \in V_0$).

Let C' be an irreducible component of $C_{1|V}$ such that its image under pr is not contained in V_1. We claim that its codimension is strictly bigger than n: suppose that the codimension of C' is smaller or equal than n, consider $f_x \in C'$; if the codimension c of C' equals n, by the Conservation of Number Formula for intersection multiplicities, any generic deformation of f_x within $\tilde{I}_x$ must contain points of positive extended codimension within V_0. This contradicts the fact that in a generic deformation only points of type $D(d,k)$ arise in V_0. If the codimension of C' is strictly smaller than n we consider a subvariety C'' of C' of codimension n containing f_x and repeat the argument.

Define C to be the union of the components of $C_{1|V}$ with image under pr not contained in V_1. Let C' be the union of the other components.

Let V' be a connected component of V_0 of dimension d. Consider $x \in V'$ and $f_x \in \tilde{I}_x$ with $rk(f_x)= r$. Either f_x is of type $D(d,d-r)$ or f_x belongs to C. Hence any irreducible component Y of $K_{r|V}$ not contained in C can be expressed as $Y_0 \coprod (Y \cap C)$, where Y_0 is the open subset containing points of type $D(d,d-r)$. As the locus of $D(d,d-r)$-points of a generic function g_x within $\tilde{I}_x$ has codimension $(d-r)(d-r+1)/2$ in V_0 then the codimension of Y in $J^\infty(U,\tilde{I})$ is $n-d+(d-r)(d-r+1)/2$.

We have the following splittings in locally closed $\mathscr{D}_{\tilde{I},e}$-invariant subsets of $J^\infty(U,\tilde{I})$:

$$J^\infty(U,\tilde{I}) = Z_1 \coprod (J^\infty(U,\tilde{I}) \backslash Z_1),$$

$$Z_1 = M \backslash pr^{-1}(V) \coprod pr^{-1}(V),$$

$$pr^{-1}(V) = pr^{-1}(V_1) \coprod (C \backslash pr^{-1}(V_1)) \coprod \left(\coprod_{r \in \mathbb{Z}_{\geqslant 0}} [K_r \backslash (C \cup K_{r-1} \cup pr^{-1}(V_1))] \right).$$

Applying the relative morsification theorem to them we obtain

Proposition 11.3. *Let $I \subset \mathscr{O}_{\mathbb{C}^n,O}$ be a radical ideal and V be the subvariety defined by it. Let $V_1,\ldots,V_r$ be the connected components of $V \backslash \mathrm{Sing}(V)$; let d_i be the dimension of V_i. If $f \in \int I$ is such that $c_{\int I,e} < \infty$, then given a small neighbourhood U of the origin, any generic deformation of f sufficiently close to f*

- *has only A_1 singularities in $U \backslash V$.*
- *only has singularities of type $D(d_i,k)$ at V_i, for $k \leqslant d_i$. Moreover the locus of points of type $D(d_i,k)$ is a smooth subvariety of codimension $k(k+1)/2$ in V_i.*

Suppose that in addition I has an isolated singularity at the origin. Then the topological type at the origin of any generic deformation of f is the generic topological type of a function in $\int I$ (such a generic topological type exists by the results of [4]). Moreover any conservative invariant is the sum of an integer multiple of the Morse number, and integer multiples of the numbers of $D(d_i,k)$ points in V_i in a generic deformation, for $i \leqslant r$ and k such that $k(k+1)/2 = d_i$.

11.4. Line singularities with simple transversal type

In [10] line singularities with transversal types A_1, A_2, A_3, D_4, E_6, E_7 and E_8 were studied from a topological point of view using a morsification result. In each case an ideal $I(S)$ was constructed in [10] such that any singularity with transversal type S has a right representative in $I(S)$, and if two functions $f,g \in I(S)$ are R-equivalent, then they are $\mathscr{D}_{I(S)}$-equivalent. For any S the orbits of $\mathscr{D}_{I(S)}$ of codimension 1 in $J^{\infty}(V(I),I(\tilde{S}))$ are determined, and the singularity types determined by them are called F_iS, for $1 \leqslant i \leqslant 3$. It is proved that any function $f \in I(S)$ can be deformed within $I(S)$ to any function having only A_1-points outside the singular line L, a finite number $h_i(f)$ of points of type F_iS in L, for $1 \leqslant i \leqslant 3$, and the generic singularity in $I(S)$ along the rest of the points of L.

Our theory recovers the morsification result, interprets $h_i(f)$ as intersection multiplicities, and shows that any conservative invariant Ξ is of the form

$$\Xi(f) = \sum_{i=1}^{3} n_i h_i(f) + n_4 \mathscr{M}(f).$$

11.5. Other examples

Example 11.4. Take $I = \mathbf{m}^k$; then $\Theta_{I,e} = \mathbf{m}\Theta$. Give weight 1 to the variables x_i and -1 to the derivations $\partial/\partial x_i$'s. Then the 0-weight graded piece of $\Theta_{I,e}$ is generated by $x_i\partial/\partial x_j$, with $1 \leqslant i,j \leqslant n$. Therefore for any function f, the module $I/\tau_{I,e}(f) + \mathbf{m}I$ has complex dimension at least $N_k - 2n$, where N_k is the dimension of the space of homogeneous polynomials in n variables of degree k. This provides examples of ideals I for which the support of $\sigma_I[f]$ contains arbitrarily high integers for any $f \in I$.

Example 11.5. Choose a function $g \in \mathbb{C}\{x_1,\ldots,x_{n-1}\}$ with an isolated singularity at the origin. View g as an element of $\mathscr{O}_{\mathbb{C}^n,O}$; its zero-set is singular along the line L defined by $x_1 = \cdots = x_{n-1} = 0$. Define $I = (g^2)$. Any $X \in \Theta_{I,e}$ admits a unique decomposition as $X = X_1 + X_2$, where

$$X_1 = a_n\partial/\partial x_n \qquad X_2 = \sum_{i=1}^{n-1} a_i\partial/\partial x_i,$$

for $a_1,\ldots,a_n \in \mathscr{O}_{\mathbb{C}^n,O}$. There is a unique expression $X_2 = \sum_{k=0}^{\infty} x_n^k X_{2,k}$, where each $X_{2,k}$ belongs to $\mathbb{C}\{x_1,\ldots,x_{n-1}\}(\partial/\partial x_1,\ldots,\partial/\partial x_{n-1})$. Then, each $X_{2,k}$ belongs to $\Theta_{I',e}$ where I' is the ideal generated by g in $\mathbb{C}\{x_1,\ldots,x_{n-1}\}$. Define $Y_2 := X_{2,0}$ and $Y_3 = \sum_{k=1}^{\infty} x_n^{k-1} \times X_{2,k}$. We have the decomposition

$$X = X_1 + Y_2 + x_n Y_3. \tag{84}$$

Consider $f \in I$ of the form $f = pg^2$, where p is a polynomial in x_n of degree $k \geqslant 1$. The decomposition (84) implies

$$\left(\frac{\mathrm{d}p}{\mathrm{d}x_n} g^2\right) + \tau_{I',e}(g^2)\mathscr{O}_{\mathbb{C}^n,O} \subset \tau_{I,e}(f). \tag{85}$$

As $g^2 J(g) \subset \tau_{I',e}(g^2)$ (where $J(g)$ is the Jacobian ideal of g) and $(x_n^{k-1}g^2) \subset (p'(x_n)g^2)$ we have

$$c_{I,e}(f) \leqslant (k-1)\mu(g), \tag{86}$$

where $\mu(g)$ is the Milnor number of g.

If $\mathrm{d}p/\mathrm{d}x_n$ vanishes at 0, the decomposition (84) implies

$$\tau_{I,e}(f) \subset (x_n g^2) + \tau_{I',e}(g^2)\mathscr{O}_{\mathbb{C}^n,O}. \tag{87}$$

In this case we have

$$c_{I,e}(f) \geqslant \dim_{\mathbb{C}}\left(\frac{(g^2)}{\tau_{I',e}(g^2)}\right) = \dim_{\mathbb{C}}\left(\frac{(g)}{\tau_{I',e}(g)}\right). \tag{88}$$

We choose $n=3$, $g(x_1,x_2) := (x_2^2+x_1^3)^2+x_2^5$ and $p(x_3) := 1+x_3^k$ with $k>2$. We weight the variables by $wt(x_1)=2$, $wt(x_2)=3$, $wt(x_3)=0$. A computation shows that for any $X\in\Theta_{I',e}$ we have $X\in(x_2\partial/\partial x_1)+(x_1,x_2)^2(\partial/\partial x_1,\partial/\partial x_2)$ and $wt(X(g))\geqslant 15$. We can write $X(g)=hg$ for some $h\in\mathbb{C}\{x_1,x_2\}$, and $wt(h)\geqslant 3$. Summarising

$$\Theta_{I',e}\subset(x_2\partial/\partial x_1)+(x_1,x_2)^2(\partial/\partial x_1,\partial/\partial x_2) \quad \tau_{I',e}(g)\subset(x_1^2g,x_2g). \tag{89}$$

Consider $h_1,\ldots,h_m$ in $\mathbb{C}\{x_1,x_2\}$ forming a basis of the complex vector space $\mathbb{C}\{x_1,x_2\}/J(g)$. By (85), the I-unfolding

$$F=f(x_1,x_2,x_3)+\sum_{i=1}^{r}\sum_{j=0}^{k-1}t_{i,j}h_i(x_1,x_2)x_3^jg^2(x_1,x_2)$$

depending on parameters $t_{i,j}$ is versal. We can choose $h_1=1$ and $h_i\in(x_1,x_2)$ if $i>1$. Then for any value t in the parameter space, $F_{|t}$ is of the form

$$F_{|t}=[1+h_t(x_3)+q(x_1,x_2,x_3)]g^2,$$

with $q(x_1,x_2,x_3)\in(x_1,x_2)$ and where $h_t(x_n)$ is a polynomial in x_3 of degree k. Choose t generic; we study the points x in which $F_{|t,x}$ has positive extended codimension. Let L be the line defined by $x_1=x_2=0$. The set $V(g)$ is smooth outside L, by Proposition 11.3 the only points in which $F_{|t,x}$ can have extended codimension outside L are A_1-points. Using $\partial/\partial x_3\in\Theta_{I,e}$, we deduce that when $p'(x_n)$ is not zero, $(0,0,x_n)$ is a point of extended codimension 0. We can assume that the derivative $p'(x_n)$ has $k-1$ simple roots. Let a be a root; consider $x=(0,0,a)$. A computation taking into account inclusions (89) yields

$$\tau_{\tilde{I}_x,e}(F_{|t,x})\subset(x_1^2g^2,x_2g^2,(x_3+\mu x_1)g^2),$$

where μ is the coefficient of x_1x_3 in q. Therefore $\tau_{\tilde{I}_x,e}(F_{|t,x})\geqslant 2$.

This example shows how a point of positive extended codimension splits in a generic deformation in several points of positive extended codimension, such k-1 of them have extended codimension at least 2.

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