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Extremals of the supereigenvector cone in max algebra: A combinatorial description

Sergeĭ Sergeev [1]

University of Birmingham, School of Mathematics, Edgbaston B15 2TT, UK

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ABSTRACT

We give a combinatorial description of extremal generators of the supereigenvector cone $\{x: A \otimes x \geq x\}$ in max algebra.

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1. Introduction

By max algebra we understand the semiring of nonnegative numbers $\mathbb{R}_+$ equipped with arithmetical operations of "tropical addition" $a \oplus b = \max(a, b)$ (instead of the usual one), and the ordinary multiplication. See Butkovič [4] for one of the recent textbooks, as well as Heidergott, Olsder and van der Woude [8] for another textbook explaining

E-mail address: sergiej@gmail.com.
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a typical application of max algebra to scheduling problems. These arithmetical operations are extended to matrices and vectors in the usual way: for two matrices A and B of appropriate sizes, we have $(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}$ and $(A \otimes B)_{ik} = \bigoplus_j a_{ij} b_{jk}$. We also consider the max-algebraic powers of matrices: $A^{\otimes t} = \underbrace{A \otimes \ldots \otimes A}_{t}$.

With each square matrix $A \in \mathbb{R}_+^{n\times n}$ we can associate a weighted directed digraph $\mathcal{G}(A) = (N, E)$ with set of nodes $N = \{1, \ldots, n\}$ and edges $E = \{(i, j) \mid a_{ij} \neq 0\}$. Each matrix entry a_{ij} is the weight of edge (i, j).

A sequence of edges $(i_1, i_2), \ldots, (i_{k-1}, i_k)$ of $\mathcal{G}(A)$ is called a *walk*. The *length* of this walk is $k-1$, and the *weight* of this walk is defined as $a_{i_1 i_2} \cdot \ldots \cdot a_{i_{k-1} i_k}$. Node i_1 is called the *beginning node*, and i_k is called the *final node* of that walk. If $i_1 = i_k$ then the walk is called a *cycle*.

It is easy to see that the i, j entry of the max-algebraic power $A^{\otimes t}$ is equal to the greatest weight of a walk of length t beginning at i and ending at j. Considering the formal series

$$A^* = I \oplus A \oplus A^{\otimes 2} \oplus \ldots \oplus A^{\otimes k} \oplus \ldots, \tag{1}$$

called the *Kleene star* of A we see that the i, j entry of A^* is equal to the greatest weight among all walks connecting i to j with no restriction on weight. This greatest weight is defined for all i, j if and only if $\mathcal{G}(A)$ does not have cycles with weight exceeding 1, otherwise (1) diverges, or more precisely, some entries of A^* diverge to $+\infty$.

In this paper we consider the set of supereigenvectors of a given square matrix $A \in \mathbb{R}_+^{n\times n}$. These are vectors x satisfying $A \otimes x \geq x$, so we are interested in the set

$$V^*(A) = \{x \colon A \otimes x \geq x\}. \tag{2}$$

Supereigenvectors are of interest for several reasons. Let us first mention a work of Butkovič, Schneider and Sergeev [6], where the supereigenvectors were shown to be instrumental in the analysis of the sequences $\{A^k \otimes x \colon k \geq 1\}$, and where a problem of describing the sets of supereigenvectors was posed. Wang and Wang [10] solved this problem by describing a generating set for (2). The goal of the present paper is to describe those generators which are extremals, or loosely speaking, “essential generators”.

The set of max-algebraic eigenvectors of A (here, associated with eigenvalue 1) and the set of subeigenvectors of A defined, respectively, as

$$V(A) = \{x \colon A \otimes x = x\}, \quad V_*(A) = \{x \colon A \otimes x \leq x\}, \tag{3}$$

have been well studied and thoroughly described in the literature. Let us also mention that $A \otimes x \geq x$ belongs to the class of two-sided systems $A \otimes x \leq B \otimes x$, whose polynomial solvability is still under question, while it is known that the problem is in the intersection of NP and co-NP classes, see for instance Bezem, Nieuwenhuis and

Rodriguez-Carbonell [3]. A number of algorithms solving this general problem and describing the full solution set have been designed: see, in particular, the double description method of Allamigeon, Gaubert and Goubault [1].

$V(A)$, $V^*(A)$ and $V_*(A)$ are examples of max cones. Recall that a subset of $\mathbb{R}_+^n$ is called a max cone if it is closed under addition $\oplus$ of its elements, and under the usual scalar multiplication. The description that we work with is in terms of max-algebraic generating sets and bases. Let us recall some definitions that are necessary here.

An element $u \in \mathbb{R}_+^n$ is called a *max combination* of elements $v^1, \ldots, v^m \in \mathbb{R}_+^n$ if there exist scalars $\lambda_1, \ldots, \lambda_m \in \mathbb{R}_+$ such that $u = \bigoplus_{i=1}^m \lambda_i v^i$. Further $S \subseteq \mathbb{R}_+^n$ is called a *generating set* for a max cone $\mathcal{K}$ if every element of $\mathcal{K}$ can be represented as a max combination of some elements of S. If S is a generating set of $\mathcal{K}$, we write $\mathcal{K} = \operatorname{span}_\oplus(S)$. Further, S is called a *basis* if none of the elements of S is a max combination of other elements of S.

An element u of a max cone $\mathcal{K} \subseteq \mathbb{R}_+^n$ is called an *extremal*, if whenever $u = v \oplus w$ and $v, w \in \mathcal{K}$, we have $u = v$ or $u = w$. An element $u \in \mathbb{R}_+^n$ is called *scaled* if $\max_{i=1}^n u_i = 1$. A basis of a max cone is called *scaled* if so is every element of that basis.

Proposition 1.1. *(See [5,7].) For any closed max cone $\mathcal{K} \subseteq \mathbb{R}_+^n$, let E be the set of scaled extremals. Then E is non-empty, $\mathcal{K} = \operatorname{span}_\oplus(E)$ and, furthermore, E is a unique scaled basis of $\mathcal{K}$.*

It is easy to see that the max cones $V^*(A)$, $V_*(A)$ and $V(A)$ are closed, so that Proposition 1.1 applies to them. In fact, all these cones have a finite number of scaled extremals, which constitute their essentially unique bases.

The rest of the paper is organized as follows. In Section 2 we describe a generating set of the supereigenvector cone. This description, obtained in Theorem 2.1, is equivalent to the description given by Wang and Wang [10] (see, in particular, [10] Theorem 3.2), but it is obtained using a more geometric "cellular decomposition" technique. In Section 3 we give criteria under which the generators described in Section 2 are extremals. These criteria are combinatorial in nature, and expressed in terms of certain cycles of the digraph associated with the matrix (namely, cycles whose weight is not less than 1). These criteria are the main result of the paper, formulated in Theorems 3.1, 3.2 and 3.3.

The combinatorial description of extremals given in this paper also leads to a test of extremality of a given generator. This test does not require any knowledge of other generators and runs in $O(m^2)$ time, where m is the number of nonzero components of a given generator.

2. Generating sets

Let $A \in \mathbb{R}_+^{n\times n}$, and let $[n] := \{1, \ldots, n\}$. A mapping τ of a subset of $[n]$ into itself will be called a *(partial) strategy* of $\mathcal{G}(A)$. Given a strategy τ we can define the matrix $A^\tau = (a_{i,j}^\tau)$ by

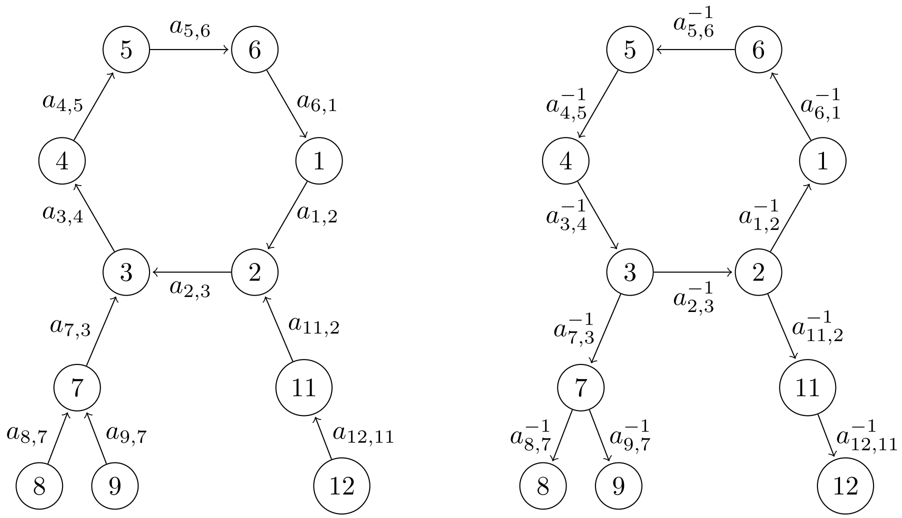

Fig. 1. Digraphs $\mathcal{G}(A^\tau)$ and $\mathcal{G}(A^{\tau-})$ for $A \in \mathbb{R}_+^{11}$ (an example).

$$a_{i,j}^\tau = \begin{cases} a_{i,j}, & \text{if } j = \tau(i), \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

By domain of τ, denoted by $\text{dom}(\tau)$, we mean the set of indices i for which $\tau(i)$ is defined, that is, the index subset which τ maps into itself.

If τ is a strategy then its inverse, denoted by $\tau-$, is, in general, a multivalued mapping of a subset of $\text{dom}(\tau)$ to the whole $\text{dom}(\tau)$. Define the matrix $A^{\tau-} = (a_{i,j}^{\tau-})$ by

$$a_{i,j}^{\tau-} = \begin{cases} a_{j,i}^{-1}, & \text{if } i = \tau(j), \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Consider the associated digraphs $\mathcal{G}(A^\tau)$ and $\mathcal{G}(A^{\tau-})$ (see Fig. 1). Let us list some properties of $\mathcal{G}(A^\tau)$.

Lemma 2.1.

(i) *For every pair of nodes of $[n]$, either there is a unique walk in $\mathcal{G}(A^\tau)$ connecting one of these nodes to the other, or there is no such walk.*

(ii) *$\mathcal{G}(A^\tau)$ contains at least one cycle.*

(iii) *For each node of* $\text{dom}(\tau)$, *there is a unique cycle of $\mathcal{G}(A^\tau)$ that can be accessed from this node via a walk in $\mathcal{G}(A^\tau)$, which is also unique.*

(iv) *For each node of* $\text{dom}(\tau)$, *there are no nodes that can be accessed from it by a walk of $\mathcal{G}(A^\tau)$ other than the nodes of the unique cycle and the unique access walk mentioned in (iii).*

A strategy τ is called *admissible* if there is no cycle in $\mathcal{G}(A^\tau)$ whose weight is smaller than 1. In this case, there is no cycle of $\mathcal{G}(A^{\tau-})$ whose weight is greater than 1, hence we have $\lambda(A^{\tau-}) \leq 1$.

The set of all admissible strategies is denoted by $\mathcal{T}_{\text{adm}}(A)$. Let us argue that the set of all supereigenvectors can be represented as a union of the sets of subeigenvectors of $A^{\tau-}$ with τ ranging over all admissible strategies.

Proposition 2.1.

$$V^*(A) = \bigcup_{\tau \in \mathcal{T}_{\text{adm}}(A)} V^*(A^\tau) = \bigcup_{\tau \in \mathcal{T}_{\text{adm}}(A)} V_*(A^{\tau-}). \tag{6}$$

Proof. To prove that

$$V^*(A) = \bigcup_{\tau \in \mathcal{T}_{\text{adm}}(A)} V^*(A^\tau), \tag{7}$$

observe that every vector x satisfying $A \otimes x \geq x$ also satisfies $A^\tau \otimes x \geq x$ for some (partial) mapping τ which can be defined as follows:

$$\operatorname{dom}(\tau) = \{i\colon x_i \neq 0\}, \quad \tau(i)\colon\ a_{i,\tau(i)} x_{\tau(i)} = \max_j a_{i,j} x_j. \tag{8}$$

The choice of $\tau(i)$ among the indices attaining maximum is free, any such index can be taken for $\tau(i)$.

It can be verified that if $x_i > 0$ then $a_{i,\tau(i)} x_{\tau(i)} > 0$, hence $x_{\tau(i)} > 0$, thus τ maps $\operatorname{dom}(\tau)$ into itself, so it is a strategy. To check that it is admissible let $i, \tau(i), \ldots, \tau^\ell(i) = i$ constitute a cycle, so we have $a_{i,\tau(i)} x_{\tau(i)} \geq x_i, \ldots, a_{\tau^{\ell-1}(i),i} x_i \geq x_{\tau^{\ell-1}(i)}$. Multiplying up all these inequalities and canceling the product of x_i's we get that the cycle weight is not less than 1. This shows that τ is admissible. To complete the proof of (7) observe that $A \otimes x \geq A^\tau \otimes x \geq x$ for every mapping τ and every vector x satisfying $A^\tau \otimes x \geq x$.

It remains to check that $V^*(A^\tau) = V_*(A^{\tau-})$ for every partial mapping τ. We have

$$\begin{aligned} V^*(A^\tau) = \{x\colon a_{i,\tau(i)} x_{\tau(i)} \geq x_i\ \forall i \in \operatorname{dom}(\tau)\} = \\ \{x\colon (a_{i,\tau(i)})^{-1} x_i \leq x_{\tau(i)}\ \forall i \in \operatorname{dom}(\tau)\} = V_*(A^{\tau-}). \end{aligned} \tag{9}$$

Combined with (7), this implies (6). □

Thus the cones $V^*(A^\tau) = V_*(A^{\tau-})$, with τ ranging over all admissible strategies, can be considered as building blocks of $V^*(A)$. Hence the generating set of $V^*(A)$ can be formed as the union of all generating sets of $V^*(A^\tau) = V_*(A^{\tau-})$: these are the generating sets of subeigenvector cones. A generating set for a general subeigenvector cone $V_*(A)$ is easy to find.

Proposition 2.2. *(E.g. [2,4,9].) Let $A \in \mathbb{R}_+^{n\times n}$ be such that the weight of any cycles of $\mathcal{G}(A)$ does not exceed one. Then $V_*(A) = \operatorname{span}_\oplus(A^*)$.*

We now specialize this description to $V_*(A^{\tau-})$. For this purpose, let us denote by $k \longrightarrow_\tau i$ the situation when $k = i$ or k can be connected to i by a walk on $\mathcal{G}(A^\tau)$. In this case, the unique walk connecting k to i on $\mathcal{G}(A^\tau)$ will be denoted by P^τ_{ki}.

Proposition 2.3. *Let τ be an admissible strategy. Then $V^*(A^\tau) = V_*(A^{\tau-})$ is generated by the vectors $x^{(\tau,k)}$ for $k = 1,\dots,n$, whose coordinates are defined as follows:*

$$x_i^{(\tau,k)} = \begin{cases} 1, & \text{if } i = k, \\ (w(P^\tau_{ki}))^{-1}, & \text{if } i \neq k \text{ and } k \longrightarrow_\tau i, \\ 0, & \text{if } i \neq k \text{ and } k \not\longrightarrow_\tau i. \end{cases} \tag{10}$$

Proof. By Proposition 2.2, $V_*(A^{\tau-}) = \operatorname{span}_\oplus((A^{\tau-})^*)$, so it amounts to argue that the columns of $(A^{\tau-})^*$ are exactly $x^{(\tau,1)},\dots,x^{(\tau,n)}$. This claim follows by the optimal walk interpretation of the entries $\alpha_{i,k}$ of $(A^{\tau-})^*$: we obtain that $\alpha_{i,k} = x_i^{(\tau,k)}$ as defined in (10). Indeed, recalling that $\alpha_{kk} = 1$ for all k, i accesses k in $\mathcal{G}(A^{\tau-})$ if and only if k accesses i in $\mathcal{G}(A^\tau)$ and that the weight of the unique access walk from i to k in $\mathcal{G}(A^{\tau-})$ is the reciprocal of the weight of the unique access walk from k to i in $\mathcal{G}(A^\tau)$, we obtain the claim from the optimal walk interpretation of the entries of the Kleene star. □

Denote by $\mathcal{C}^{\geq 1}(A)$, respectively by $\mathcal{C}^{>1}(A)$, the set of cycles in $\mathcal{G}(A)$ whose weight is not less than 1, respectively greater than 1.

If $\mathcal{G}(A^\tau)$, for a strategy τ, consists of one cycle and one non-empty walk connecting its origin to a node of that cycle, then τ is called a *germ*. The origin of that walk will be denoted by o_τ. If the weight of the cycle is no less than 1 then the germ is called *admissible*. The set of all admissible germs in $\mathcal{G}(A)$ will be denoted by $\mathcal{T}_{\text{ag}}(A)$.

Obviously, both $\mathcal{C}^{\geq 1}(A) \subseteq \mathcal{T}_{\text{adm}}(A)$ and $\mathcal{T}_{\text{ag}}(A) \subseteq \mathcal{T}_{\text{adm}}(A)$. The following theorem describes a generating set of $V^*(A)$ by means of nonnegative cycles and admissible germs.

Theorem 2.1. *We have $V^*(A) = \operatorname{span}_\oplus(S)$ where*

$$S = \{x^{(\tau,o_\tau)}: \tau \in \mathcal{T}_{\text{ag}}(A)\} \cup \{x^{(\tau,k)}: \tau \in \mathcal{C}^{\geq 1}(A),\ k \in \operatorname{dom}(\tau)\}. \tag{11}$$

Proof. Since every admissible germ and every nonnegative cycle is an admissible strategy, inclusion $V_*(A^{\tau-}) \subseteq V^*(A)$ and Proposition 2.3 imply that

$$\operatorname{span}_\oplus\{x^{(\tau,o_\tau)}: \tau \in \mathcal{T}_{\text{ag}}(A)\} \subseteq V^*(A)$$

and

$$\operatorname{span}_\oplus\{x^{(\tau,k)}: \tau \in \mathcal{C}^{\geq 1}(A),\ k \in \operatorname{dom}(\tau)\} \subseteq V^*(A).$$

To prove the opposite inclusion, observe that for each $x^{(\tau,k)}$ of (10) we can define a new strategy τ' by

$$\operatorname{dom}(\tau') = \{i: k \longrightarrow_\tau i\}, \quad \tau'(l) = \tau(l) \text{ if } l \in \operatorname{dom}(\tau'), \tag{12}$$

and then we have $x^{(\tau',k)} = x^{(\tau,k)}$. We now argue that τ' is a more simple strategy than τ.

By Lemma 2.1 part (iv), in $\mathcal{G}(A^\tau)$ node k only accesses one nonnegative cycle and the nodes on the unique walk leading to that cycle. It follows that either $\tau' \in \mathcal{C}^{\geq 1}(A)$ and then $k \in \operatorname{dom}(\tau')$, or $\tau' \in \mathcal{T}_{\text{adm}}(A)$ and $k = o_{\tau'}$. Hence for every generator of $V^*(A)$, expressed as $x^{(\tau,k)}$, there exists τ' which is either a nonnegative cycle with one of the nodes being k, or it is an admissible germ and $k = o_{\tau'}$, in any case such that $x^{(\tau',k)} = x^{(\tau,k)}$. This implies that

$$V^*(A) \subseteq \operatorname{span}_\oplus\{x^{(\tau,o_\tau)}: \tau \in \mathcal{T}_{\text{ag}}(A)\} \oplus \operatorname{span}_\oplus\{x^{(\tau,k)}: \tau \in \mathcal{C}^{\geq 1}(A),\ k \in \operatorname{dom}(\tau)\}.$$

The theorem is proved. □

Remark 2.1. Algorithm 3.2 in Wang and Wang [10] is implicitly based on a theoretical result equivalent to Theorem 2.1 above. To help the reader compare Theorem 2.1 with the results of [10], let us mention that [10] uses different terminology and notation. In particular, the set $V^*(A)$ is denoted by $\mathscr{X}$, and the notation $\mathscr{X}'_\sigma$ stands for the set $\{x^{(\sigma,k}: k \in \operatorname{dom}(\sigma)\}$, where $\sigma \in \mathcal{C}^{\geq 1}(A)$. A germ with a cycle σ appears as a union of cycle σ and a J_σ-path, in the terminology of [10].

3. Extremals

Let us introduce the following partial order relation.

$$y \leq_i x \text{ if } x_i \neq 0,\ y_i \neq 0 \text{ and } y_k y_i^{-1} \leq x_k x_i^{-1}\ \forall k. \tag{13}$$

In particular, this relation is transitive:

$$x \leq_i y \leq_i z \Rightarrow x \leq_i z. \tag{14}$$

The following fact is known, see [4, Proposition 3.3.6] or [5, Theorem 14], and also [9].

Proposition 3.1. *Let $S \subseteq \mathbb{R}^n_+$ and $\mathcal{K} = \operatorname{span}_\oplus(S)$. Then x is not an extremal of $\mathcal{K}$ if and only if for each $i \in \operatorname{supp}(x)$ there exists $y^i \in S$ such that $y^i \leq_i x$ and $y^i \neq x$.*

We consider the case when $\mathcal{K} = V^*(A)$. A generating set of this max cone is given in Theorem 2.1. Our purpose is to identify extremals, which yield an essentially unique basis of $V^*(A)$, by means of the criterion described in Proposition 3.1.

We first show that for each τ and k, there is a relation between $x^{(\tau,k)}$ and $x^{(\tau,\tau(k))}$, with respect to every preorder relation except for $\leq_k$.

Lemma 3.1. *Let* $\tau \in \mathcal{T}_{\mathrm{ag}}(A)$ *and* $k = o_\tau$ *or* $\tau \in \mathcal{C}^{\geq 1}(A)$ *and* $k \in \mathrm{dom}(\tau)$*.*

(i) $x^{(\tau,\tau(k))} \leq_j x^{(\tau,k)}$ *for all* $j \in \mathrm{dom}(\tau)$ *and* $j \neq k$*.*
(ii) $x^{(\tau,\tau(k))} \neq x^{(\tau,k)}$ *if and only if* $\tau \in \mathcal{T}_{\mathrm{ag}}(A)$ *or* $\tau \in \mathcal{C}^{>1}(A)$*.*

Proof. Let $\tau \in \mathcal{T}_{\mathrm{ag}}(A)$ and $k = o_\tau$. Then $\mathrm{supp}(x^{(\tau,\tau(k))}) = \mathrm{dom}(\tau)\backslash\{k\}$, and in particular, $x^{(\tau,\tau(k))} \neq x^{(\tau,k)}$. As we also have $(x_j^{(\tau,\tau(k))})^{-1} x_i^{(\tau,\tau(k))} = (x_j^{(\tau,k)})^{-1} x_i^{(\tau,k)} = w(P_{ij}^\tau)$ for all $i, j \neq k$, claim (i) follows, in the case when $\tau \in \mathcal{T}_{\mathrm{ag}}(A)$.

Let $\tau \in \mathcal{C}^{\geq 1}(A)$. Then $\mathrm{supp}(x^{(\tau,\tau(k))}) = \mathrm{supp}(x^{(\tau,k)}) = \mathrm{dom}(\tau)$, and

$$\left(x_j^{(\tau,\tau(k))}\right)^{-1} x_i^{(\tau,\tau(k))} = \left(x_j^{(\tau,k)}\right)^{-1} x_i^{(\tau,k)} = w(P_{ij}^\tau) \text{ for } i, j \neq k.$$

However, we also have

$$\left(x_i^{(\tau,\tau(k))}\right)^{-1} \cdot x_k^{(\tau,\tau(k))} \leq \left(x_i^{(\tau,k)}\right)^{-1} \cdot x_k^{(\tau,k)} \text{ for } i \neq k, \tag{15}$$

since

$$\begin{aligned} &x_k^{(\tau,k)} \left(x_i^{(\tau,k)}\right)^{-1} = w(P_{ki}^\tau), \quad x_k^{(\tau,\tau(k))} \left(x_i^{(\tau,\tau(k))}\right)^{-1} = (w(P_{ik}^\tau))^{-1}, \\ &w(P_{ki}^\tau) w(P_{ik}^\tau) = w(\tau) \geq 1. \end{aligned} \tag{16}$$

Furthermore, we have $x^{(\tau,\tau(k))} \neq x^{(\tau,k)}$ if and only if the inequality in (15) is strict, which happens if and only if $w(\tau) > 1$. Hence both claims. □

If $\tau \in \mathcal{T}_{\mathrm{ag}}(A)$ and $k = o_\tau$ or $\tau \in \mathcal{C}^{\geq 1}(A)$ and $k \in \mathrm{dom}(\tau)$, there is a unique walk issuing from k and containing all nodes of $\mathrm{dom}(\tau)$. Denote the final node of that walk by $\mathrm{endn}(\tau, k)$.

Corollary 3.1. *Let* $\tau \in \mathcal{T}_{\mathrm{ag}}(A)$ *and* $k = o_\tau$ *or* $\tau \in \mathcal{C}^{\geq 1}(A)$ *and* $k \in \mathrm{dom}(\tau)$*. Let* $i \neq k$ *be any index in* $\mathrm{dom}(\tau)$*. Then* $x^{(\tau,i)} \leq_i x^{(\tau,k)}$*.*

Proof. Without loss of generality we will assume that the nodes of τ, where τ is a cycle or a germ, are numbered in such a way that $k = 1$ and $\tau(i) = i + 1$ for all $i \in \mathrm{dom}(\tau)$ except for the node $\mathrm{endn}(\tau, k)$ which has the greatest number m. Note that if τ is a cycle then $\tau(m) = 1$.

Repeatedly applying Lemma 3.1 part (i), we have

$$x^{(\tau,i)} \leq_i x^{(\tau,i-1)} \leq_i x^{(\tau,i-2)} \leq_i \ldots \leq_i x^{(\tau,1)}. \qquad \square \tag{17}$$

We now formulate and prove the main results of the paper, which constitute a combinatorial characterization of the supereigenvector cone $V^*(A)$. Let us distinguish between germs whose unique cycle has weight strictly greater than 1, whose set we denote by

$\mathcal{T}_{\text{ag}}^{>1}(A)$, and the set of germs whose unique cycle has weight 1, whose set we denote by $\mathcal{T}_{\text{ag}}^{=1}(A)$.

Theorem 3.1. *Let $\tau \in \mathcal{T}_{\text{ag}}^{>1}(A)$ and $k = o_\tau$ or $\tau \in \mathcal{C}^{>1}(A)$ and $k \in \text{dom}(\tau)$. Then $x^{(\tau,k)}$ is not an extremal if and only if one of the following conditions holds:*

(i) *there exist i, l and j such that $i \neq l \neq j$, $i \longrightarrow_\tau l \longrightarrow_\tau j$ and $a_{i,j} \geq w(P^\tau_{ij})$;*
(ii) *there exist i and j such that $i \neq j$, $i \longrightarrow_\tau j$, $j \neq \text{endn}(\tau, k)$ and $a_{j,i} \geq (w(P^\tau_{ij}))^{-1}$.*

In the case of $\mathcal{T}_{\text{ag}}^{=1}(A)$, we have to replace condition (i) by a more elaborate one. For a germ τ consisting of a cycle c and a walk connecting the origin o_τ to c, denote by $o_{c\tau}$ the (unique) node which this cycle and this walk have in common.

Theorem 3.2. *Let $\tau \in \mathcal{T}_{\text{ag}}^{=1}(A)$ and $k = o_\tau$. Then $x^{(\tau,k)}$ is not an extremal if and only if one of the following conditions holds:*

(ĭ) *there exist i, l and j such that $i \neq l \neq j$, $i \longrightarrow_\tau l \longrightarrow_\tau j$, and either $\tau(i) \neq o_{c\tau}$ and $a_{i,j} \geq w(P^\tau_{ij})$ or $\tau(i) = o_{c\tau}$ and $a_{i,j} > w(P^\tau_{ij})$;*
(ii) *there exist i and j such that $i \neq j$, $i \longrightarrow_\tau j$, $j \neq \text{endn}(\tau, k)$ and $a_{j,i} \geq (w(P^\tau_{ij}))^{-1}$.*

Proof of Theorem 3.1 and Theorem 3.2. Without loss of generality we will assume that the nodes of τ, where τ is a cycle or a germ, are numbered in such a way that $k = 1$ and $\tau(i) = i + 1$ for all $i \in \text{dom}(\tau)$ except for the node $\text{endn}(\tau, k)$ which has the greatest number m. Note that if τ is a cycle then $\tau(m) = 1$, and otherwise $\tau(m) = o_{c\tau}$. With such numbering, conditions (i), (ii) and (ĭ) take the following form:

(i') there exist i, j such that $i + 1 < j$ and $a_{i,j} \geq w(P^\tau_{ij})$;
(ii') there exist i, j such that $i < j$, $j \neq m$ and $a_{j,i} \geq (w(P^\tau_{ij}))^{-1}$;
(ĭ') there exist i, j such that $i < j$, and either $o_{c\tau} \neq i+1$ and $a_{i,j} \geq w(P^\tau_{ij})$ or $o_{c\tau} = i+1$ and $a_{i,j} > w(P^\tau_{ij})$.

The "only if" part: Suppose that $x^{(\tau,1)}$ is not an extremal. As $V^*(A) = \text{span}_\oplus(S)$ where S is defined in (11), by Proposition 3.1 and Theorem 2.1, there exist τ' and s such that $x^{(\tau',s)} \leq_1 x^{(\tau,1)}$ and $x^{(\tau',s)} \neq x^{(\tau,1)}$. As $x^{(\tau',s)} \leq_1 x^{(\tau,1)}$, it follows that $\text{dom}(\tau') \subseteq \text{dom}(\tau)$ and that $1 \in \text{dom}(\tau')$ so that $s \longrightarrow_{\tau'} 1$. Also since $x^{(\tau',1)} \leq_1 x^{(\tau',s)}$ by Corollary 3.1, and since $\leq_1$ is transitive, we can assume $s = 1$.

Now suppose there exist i, j such that $i + 1 < j$ and $j = \tau'(i)$. Consider the least such i and j. Condition $x^{(\tau',1)} \leq_1 x^{(\tau,1)}$ means that $x_l^{(\tau',1)} \leq x_l^{(\tau,1)}$ for all $l \in \text{dom}(\tau)$. In terms of walks, this means that $w(P^{\tau'}_{1l})^{-1} \leq w(P^\tau_{1l})^{-1}$, or equivalently, $w(P^{\tau'}_{1l}) \geq w(P^\tau_{1l})$ for all $l \in \text{dom}(\tau)$. In particular, this implies $a_{i,j} \geq w(P^\tau_{ij})$, thus we have (i').

Suppose that there are no such i, j. Then it can be verified that we have $\tau'(s) = s+1$ for all $s \in \text{dom}(\tau')$ except for one node j for which $i = \tau'(j) < j$. However, if $j = m$ then

$x^{(\tau',1)} = x^{(\tau,1)}$, a contradiction. Hence $j < m$, and the edge (j,i) belongs to the unique cycle of τ'. The other edges of that cycle form the walk P_{ij}^{τ} and the cycle is in $\mathcal{C}^{\geq 1}(A)$, hence we have (ii').

It remains to prove that if $\tau \in \mathcal{T}_{\rm ag}^{=1}(A)$ and not (i') or (ii'), then we have (ĭ'). So suppose that condition (ii') does not hold, $\tau \in \mathcal{T}_{\rm ag}^{=1}(A)$ and there exist only i and j with $i < j$, $o_{c\tau} = i+1$ and, by contradiction, that $a_{i,j} = w(P_{ij}^{\tau})$ for all such i and j. Then we have $x^{(\tau',1)} \leq_1 x^{(\tau,1)}$ only for $\tau' = \tau$ (trivially), or for τ' such that $\mathrm{dom}(\tau') = \mathrm{dom}(\tau)$, $\tau'(i) = j$ for some selection of j and $\tau'(k) = \tau(k)$ for all $k \in \mathrm{dom}(\tau)\backslash\{i\}$. However, it can be checked that $x^{(\tau',1)} = x^{(\tau,1)}$ for all such τ' since $\mathrm{supp}(x^{(\tau',1)}) = \mathrm{supp}(x^{(\tau,1)})$ and the weight of the unique cycle of τ is 1. This implies that there are no vectors preceding $x^{(\tau,1)}$ with respect to $\leq_1$ and different from $x^{(\tau,1)}$, a contradiction. Hence we have (ĭ').

The "if" part: By Proposition 3.1 and Lemma 3.1, it is enough to show that there exists $\tau' \in \mathcal{T}_{\rm ag}(A) \cup \mathcal{C}^{\geq 1}(A)$ such that $x^{(\tau',1)} \leq_1 x^{(\tau,1)}$ and $x^{(\tau',1)} \neq x^{\tau,1}$.

Suppose that (i') or (ĭ') holds, and take any such i and j. Denote by c the (unique) cycle of τ. Define τ' by

$$\mathrm{dom}(\tau') = \begin{cases} \{1,\ldots,i\} \cup \{j,\ldots,m\}, & \text{if } o_{c\tau} \leq i \text{ or } o_{c\tau} \geq j, \\ \{1,\ldots,i\} \cup \{o_{c\tau},\ldots,m\}, & \text{if } i < o_{c\tau} < j. \end{cases}$$

$$\tau'(l) = \begin{cases} \tau(l), & \text{if } l \in \mathrm{dom}(\tau'),\ l \neq i, \\ j, & \text{if } l = i. \end{cases} \tag{18}$$

The definition of τ' and the inequality $a_{i,j} \geq w(P_{ij}^{\tau})$ immediately imply $w(P_{1l}^{\tau'}) \geq w(P_{1l}^{\tau})$ for all $l \in \{1,\ldots,i\} \cup \{j,\ldots,m\}$. For the case when $l \in \{o_{c\tau},\ldots,j\}$ (if $i < o_{c\tau} < j$), observe that $w(P_{1l}^{\tau'}) \geq w(P_{1l}^{\tau}) \cdot w(c) \geq w(P_{1l}^{\tau})$. Thus $w(P_{1l}^{\tau'}) \geq w(P_{1l}^{\tau})$ holds for all $l \in \mathrm{dom}(\tau')$, implying the inequalities $x_l^{(\tau',1)}(x_1^{(\tau',1)})^{-1} \leq x_l^{(\tau,1)}(x_1^{(\tau,1)})^{-1}$ for all $l \in \mathrm{dom}(\tau')$. Hence $x^{(\tau',1)} \leq_1 x^{(\tau,1)}$. It remains to show that $x^{(\tau',1)} \neq x^{(\tau,1)}$.

Observe that $\mathrm{dom}(\tau')$ is a proper subset of $\mathrm{dom}(\tau)$ unless when $o_{c\tau} = i+1$ (that is, the cycle begins at the next node after i). If $\mathrm{dom}(\tau')$ is a proper subset of $\mathrm{dom}(\tau)$ then clearly $x^{(\tau',1)} \neq x^{(\tau,1)}$. If $o_{c\tau} = i+1$, we verify that for all $l \in \{o_{c\tau},\ldots,j\}$, we have that either $w(P_{1l}^{\tau'} > w(P_{1l}^{\tau}) \cdot w(c) \geq w(P_{1l}^{\tau})$ (if $a_{i,j} > w(P_{ij}^{\tau})$) or $w(P_{1l}^{\tau'}) \geq w(P_{1l}^{\tau}) \cdot w(c) > w(P_{1l}^{\tau})$ (if $w(c) > 1$), and then also $x^{(\tau',1)} \neq x^{\tau,1}$.

If (i') or (ĭ') does not hold but (ii') does, then define τ' by

$$\mathrm{dom}(\tau') = \{1,\ldots,j\}, \quad \tau'(l) = \begin{cases} i, & \text{if } l = j, \\ \tau(l) = l+1, & \text{if } l < j. \end{cases} \tag{19}$$

Then the condition $a_{j,i} \geq w(P_{ij}^{\tau})^{-1}$ implies that P_{ij}^{τ} and (j,i) constitute a nonnegative cycle, hence $\tau' \in \mathcal{T}_{\rm ag}(A) \cup \mathcal{C}^{\geq 1}(A)$ and $\mathrm{dom}(\tau')$ is a proper subset of $\mathrm{dom}(\tau)$. Thus we have $x^{(\tau',1)} \leq_1 x^{(\tau,1)}$. □

It remains to consider the case when τ is a cycle with weight 1. The set of such cycles is denoted by $\mathcal{C}^{=1}(A)$. In this case all vectors $x^{(\tau,i)}$ are proportional to each other, for all $i \in \text{dom}(\tau)$. Therefore we will denote $x^\tau = x^{(\tau,i)}$, where i is an arbitrary index of $\text{dom}(\tau)$.

Theorem 3.3. *Let $\tau \in \mathcal{C}^{=1}(A)$ and $i \in \text{dom}(\tau)$. Then x^τ is not an extremal if and only if there exist two edges (k_1, l_1) and (k_2, l_2) such that $k_1, l_1, k_2, l_2 \in \text{dom}(\tau)$, $l_1 \notin \tau(k_1)$, $l_2 \notin \tau(k_2)$, $k_1 \neq k_2$, $a_{k_1,l_1} \cdot w(P^\tau_{l_1 k_1}) \geq 1$ and $a_{k_2,l_2} \cdot w(P^\tau_{l_2 k_2}) \geq 1$.*

Proof. Let x^τ be not an extremal, then for each $i \in \text{dom}(\tau)$ there exist τ_i and i' such that $x^{(\tau_i,i')} \leq_i x^\tau$, and hence (k_i, l_i) with $k_i, l_i \in \text{dom}(\tau)$ and $a_{k_i l_i} \cdot w(P^\tau_{l_i k_i}) \geq 1$. Indeed, if there is no such edge then the domain of any cycle or germ other than τ includes a node not in $\text{dom}(\tau)$, while all generators derived from τ are proportional to x^τ. Furthermore, some k_i's should be different, at least for two values of i. Indeed, if all k_i are equal to the same index denoted by k, then we have $x^{(\tau_i,\tau(k))} = x^\tau$ for all i, while $\tau(k)$ does not belong to the support of any other vector derived from the germ τ_i, for any i.

For the converse implication, let (k_1, l_1) and (k_2, l_2) be the two edges satisfying given conditions, and let τ_1 and τ_2 be defined by

$$\tau_1(i) = \begin{cases} \tau(i), & \text{if } i \in \text{dom}(\tau)\backslash\{k_1\}, \\ l_1, & \text{if } i = k_1. \end{cases}, \quad \tau_2(i) = \begin{cases} \tau(i), & \text{if } i \in \text{dom}(\tau)\backslash\{k_2\}, \\ l_2, & \text{if } i = k_2. \end{cases} \tag{20}$$

Since $k_1 \neq k_2$, for each $i \in \text{dom}(\tau)$, either $i \neq \tau(k_1)$ or $i \neq \tau(k_2)$, and we define $\tau' := \tau_1$ or $\tau' := \tau_2$ respectively. Then we have $x^{(\tau',i)} \leq_i x^\tau$ and $x^{(\tau',i)} \neq x^\tau$. As such a vector can be found for any i, x^τ is not extremal. □

Let us now formulate a corollary concerning the computational complexity of verifying whether a given generator is an extremal or not.

Corollary 3.2. *Let a generator of the form $x^{(\tau,k)}$, with a known $\tau \in \mathcal{T}_{\text{adm}}$, be given. Then one can verify whether $x^{(\tau,k)}$ is an extremal or not in no more than $O(m^2)$ operations (maximization, multiplication and inverse), where m is the number of nonzero components of $x^{(\tau,k)}$.*

Proof. In order to prepare for checking whether the conditions of Theorems 3.1, 3.2 or 3.3 hold, let us compute all path weights $w(P^\tau_{ij})$ where $i \longrightarrow_\tau j$. All these weights and their inverses can be computed in $O(m^2)$ time and stored in $m \times m$ matrices. To describe the corresponding procedure, assume that $k = 1$ and that we have $i \longrightarrow_\tau j$ iff $i \leq j$. On the lth step of the procedure, we have $w(P^\tau_{1l})$ ready, and then we compute $w(P^\tau_{2l}) = w(P^\tau_{1l}) \cdot a_{12}^{-1}$, $w(P^\tau_{3l}) = w(P^\tau_{2l}) \cdot a_{23}^{-1}$ and so on until $w(P^\tau_{l-1,l}) = a_{l-1,l}$. After that we compute $w(P^\tau_{1,l+1}) = w(P^\tau_{1l}) \cdot a_{l,l+1}$ and proceed with the next $l + 1$th step.

Checking the extremality of z requires verifying the conditions of Theorems 3.1, 3.2 or 3.3, for each pair i, j with $i \longrightarrow_\tau j$. The choice of theorem depends on the type of the germ τ, and the verification takes $O(m^2)$ operations. □

Remark 3.1. Checking the extremality of $z = x^{(\tau,k)}$ relies on the knowledge of τ. If τ is not known, then reconstructing it from a given generator z poses a problem. To avoid the complications, we assume that τ is produced by an algorithm which provides the generator. To this end, let us remark that Algorithm 3.2 in [10] constructs a family of generators from a cycle and walk leading to that cycle (or J_σ-path in the terminology of [10]), which means that it can efficiently produce a strategy that we need. The extremality checking procedure described in the proof of Corollary 3.2, can be added to that Algorithm.

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