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CONCEPT LATTICES AND CONCEPTUAL KNOWLEDGE SYSTEMS

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Abstract—"Concept Lattice" is the central notion of "Formal Concept Analysis," a new area of research which is based on a set-theoretical model for concepts and conceptual hierarchies. This model yields not only a new approach to data analysis but also methods for formal representation of conceptual knowledge. These methods are outlined on three levels. First, basics on concept lattices are explained starting from simple data contexts which consist of a binary relation between objects and attributes indicating which object has which attribute. On the second level, conceptual relationships are discussed for data matrices which assign attribute values to each of the given objects. Finally, a mathematical model for conceptual knowledge systems is described. This model allows us to study mathematically the representation, inference, acquisition, and communication of conceptual knowledge.

1. CONCEPT LATTICES AS KNOWLEDGE REPRESENTATION

Knowledge: What it is, how it is acquired, and how it can be represented, this has been discussed for more than two thousand years as central matter of epistemology and the discussion is far from being exhausted [1]. Thus, there cannot be any hope that mathematical models will capture the rich variety of ideas and understandings about knowledge and its representations. Although we restrict our considerations in this paper to conceptual knowledge, we are still confronted with a multitude of substantial views and theories about grasping knowledge by concepts and their relations. Already the different understandings of a concept, as a unit of thoughts, as the meaning of some word, as a cognitive structure, etc. [2], makes it clear that a formalization of conceptual knowledge has to concentrate on some specific type of abstraction which enables us to fulfill specified aims.

Formal concept analysis, which has been developed during the last ten years and shall be explained in this paper [3], is supposed to achieve aims as they are formulated in the German standards on concepts and conceptual systems (see [4,5]); these standards are seen as a general aid in sciences, economy and administration for a better understanding and use of "conceptual tools." The standards are based on the philosophical understanding of a concept as a unit of thoughts consisting of two parts: the extension and the intension (comprehension); the extension covers all objects (or entities) belonging to the concept while the intension comprises all attributes (or properties) valid for all those objects [6]. A set-theoretic model for these relationships is the root of formal concept analysis. This model yields not only a new approach to data analysis, but also methods for formal representation of conceptual knowledge.

Formal concept analysis starts with the primitive notion of a *(formal) context* which is defined as a triple (G, M, I) where G and M are sets while I is a binary relation between G and M, i.e., $I \subseteq G \times M$; the elements of G and M are called *objects* (in German: *Gegenstände*) and *attributes* (in German: *Merkmale*), respectively, and gIm, i.e., $(g, m) \in I$, is read: the object g has the attribute m. Frequently used are the following *derivation operators* represented by "prime":

$$X \mapsto X' = \{m \in M \mid gIm \text{ for all } g \in X\},$$
$$Y \mapsto Y' = \{g \in G \mid gIm \text{ for all } m \in Y\}.$$

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These operators form a so-called *Galois connection* between the power sets of G and M which can be expressed by the following conditions indicating a natural "duality" between objects and attributes [7, pp. 122–125]:

(1) $X_1 \subseteq X_2$ implies $X_2' \subseteq X_1'$ for $X_1, X_2 \subseteq G$;
(1′) $Y_1 \subseteq Y_2$ implies $Y_2' \subseteq Y_1'$ for $Y_1, Y_2 \subseteq M$;
(2) $X \subseteq X''$ and $X' = X'''$ for $X \subseteq G$;
(2′) $Y \subseteq Y''$ and $Y' = Y'''$ for $Y \subseteq M$;
(3) $(\bigcup_{t\in T} X_t)' = \bigcap_{t\in T} X_t'$ for $X_t \subseteq G (t \in T)$;
(3′) $(\bigcup_{t\in T} Y_t)' = \bigcap_{t\in T} Y_t'$ for $Y_t \subseteq M (t \in T)$;

In the frame of a formal context (G, M, I), the philosophical view of a concept as a unit of thoughts constituted by its extension and its intension can be formalized by the following definition: a pair (A, B) is said to be a *(formal) concept* of the context (G, M, I) if $A \subseteq G$, $B \subseteq M$, $A = B'$, and $B = A'$; A and B are called the *extent* and the *intent* of the concept (A, B), respectively. The set of all concepts of (G, M, I) is denoted by $\mathfrak{B}(G, M, I)$. The most important structure on $\mathfrak{B}(G, M, I)$ is given by the subconcept-superconcept-relation which is defined as follows: the concept (A_1, B_1) is a *subconcept* of the concept (A_2, B_2) if $A_1 \subseteq A_2$ which is equivalent to $B_2 \subseteq B_1$ by (1) and (1′) ((A_2, B_2) is then a *superconcept* of (A_1, B_1)). Since this definition yields an order relation, the subconcept-superconcept-relation is denoted by $\leq$; furthermore, let $\underline{\mathfrak{B}}(G, M, I) := (\mathfrak{B}(G, M, I), \leq)$. For the formulation of the basic theorem about the ordered set $\underline{\mathfrak{B}}(G, M, I)$, the following lattice-theoretical notions are needed: a subset of D of a complete lattice L is called *infimum-dense (supremum-dense)* if each element of L is the infimum (supremum) of some subset of D. An element a of a lattice L is said to be $\wedge$-irreducible ($\vee$-irreducible) if $a = b \wedge c (a = b \vee c)$ always implies $a = b$ or $a = c$; the set of all $\wedge$-irreducible ($\vee$-irreducible) elements of L is denoted by $J(L)$ $(M(L))$. For further lattice-theoretical notions see [7–10]. Now, we are ready to describe and characterize the hierarchy of all formal concepts of a formal context.

BASIC THEOREM FOR CONCEPT LATTICES [11]. *Let (G, M, I) be a context. Then $\underline{\mathfrak{B}}(G, M, I)$ is a complete lattice, called the concept lattice of (G, M, I), for which infimum and supremum can be described as follows:*

$$\bigwedge_{t\in T}(A_t, B_t) = \left(\bigcap_{t\in T} A_t, \left(\bigcup_{t\in T} B_t\right)''\right), \qquad \bigvee_{t\in T}(A_t, B_t) = \left(\left(\bigcup_{t\in T} A_t\right)'', \bigcap_{t\in T} B_t\right).$$

In general, a complete lattice L is isomorphic to $\underline{\mathfrak{B}}(G, M, I)$ if and only if there exist mappings $\tilde{\gamma} : G \to L$ and $\tilde{\mu} : M \to L$ such that $\tilde{\gamma}G$ is supremum-dense in L, $\tilde{\mu}M$ is infimum-dense in L, and gIm is equivalent to $\tilde{\gamma}g \leq \tilde{\mu}m$; in particular, $L \cong \underline{\mathfrak{B}}(L, L, \leq)$ and, if L has finite length, $L \cong \underline{\mathfrak{B}}(J(L), M(L), \leq)$.

A formal context can be considered as an elementary model for formal representation of knowledge yielding even a more structural representation of conceptual knowledge by its concept lattice; graphically, contexts are usually described by *cross-tables* while concept lattices are effectively visualized by *labelled line diagrams (Hasse diagrams)*. The power of this approach to formal representation of conceptual knowledge is explained in this article on three levels: concept lattices of formal contexts, concept lattices of many-valued contexts, and conceptual knowledge systems. Let us begin with an example from sociology given by the cross-table in Figure 1 (see [12], p. 148). This table can be understood as a description of a formal context: its objects are the eighteen ladies from Old City whose names are heading the rows and its attributes are the fourteen social events which are represented by the columns; furthermore, the crosses indicate when an object has an attribute, i.e., which lady has participated in which social event.

A labelled line diagram of the concept lattice of the context given by Figure 1 is shown in Figure 2. The little circles represent the 65 concepts of the context and the ascending paths

	Code Numbers of Social Events Reported in Old City Herald													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Evelyn	x	x	x	x	x	x		x	x					
Laura	x	x	x		x	x	x	x						
Theresa		x	x	x	x	x	x	x	x					
Brenda	x		x	x	x	x	x	x						
Charlotte			x	x	x		x							
Frances			x		x	x		x						
Eleanor					x	x	x	x						
Pearl						x		x	x					
Ruth					x		x	x	x					
Verne							x	x	x			x		
Myra								x	x	x		x		
Katherine								x	x	x		x	x	x
Sylvia							x	x	x	x		x	x	x
Nora						x	x		x	x	x	x	x	x
Helen							x	x		x	x	x		
Dorothy								x	x					
Olivia									x		x			
Flora									x		x			

Figure 1. Participation of social events by some ladies in Old City.

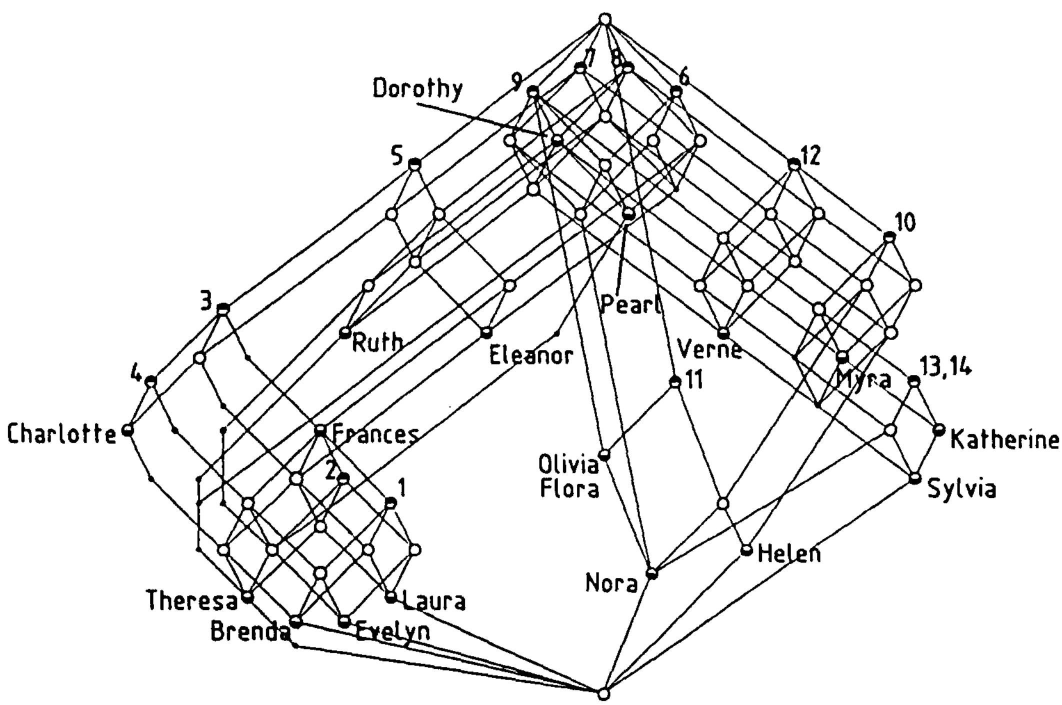

Figure 2. Concept lattice of the formal context in Figure 1.

of line segments represent the subconcept-superconcept-relation (such a path may change its direction at a meeting of lines only if there is a little circle or a dot). A name of an object g is attached to the little circle representing $\gamma g := (\{g\}'', \{g\}')$ which is the smallest concept having g in its extent; a name of an attribute m is attached to the little circle representing $\mu m := (\{m\}', \{m\}'')$ which is the largest concept having m in its intent. This labelling allows us to read from the diagram for each concept (A, B) its extent A and its intent B because $g \in A \Leftrightarrow \gamma g \leq (A, B)$ and $m \in B \Leftrightarrow (A, B) \leq \mu m$. In Figure 2, for instance, the unlabelled

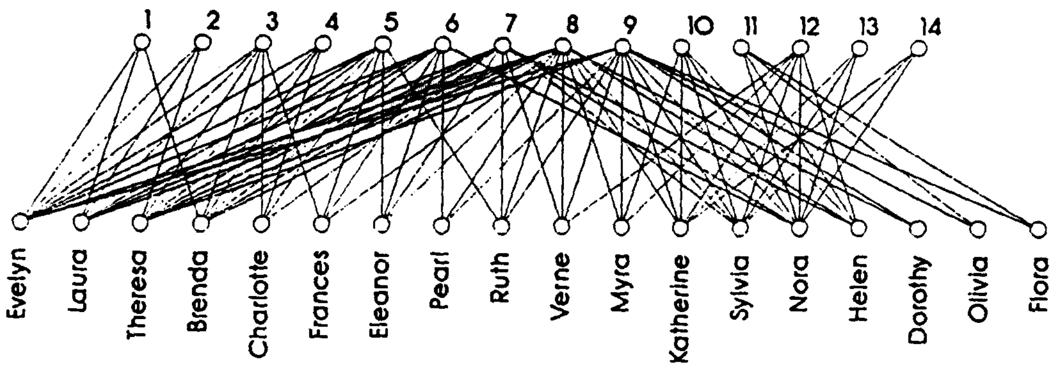

Figure 3. Representation of the table in Figure 1 as a bipartite graph.

circle most to the right represents the concept with the extent {Myra, Katherine, Sylvia, Helen} and the intent {8, 10, 12}. The labelling, in particular, preserves the underlying context because $gIm \Leftrightarrow \gamma g \leq \mu m$ (notice that γ and μ are the mappings $\tilde{\gamma}$ and $\tilde{\mu}$ of the Basic Theorem in the case $L = \underline{\mathfrak{B}}(G, M, I)$). Therefore, concept lattices constitute a structural analysis of data contexts without reducing the data. A labelled line diagram of a concept lattice still represents all knowledge coded in the underlying context and, furthermore, unfolds (and reveals to the eye) the inherent conceptual structure of the coded knowledge. Of course, a context can also be represented as bipartite graph as in Figure 3, but mostly such representation is less readable and less informative.

Before we discuss how to derive further information from a concept lattice, there should be an answer to the basic question: How can one determine the concept lattice of a given context (G, M, I)? The derivation operators yield for each $X \subseteq G$ the concept (X'', X') by (2) and for each $Y \subseteq M$ the concept (Y', Y'') by (2'). This construction method is often used to generate single concepts, but it is too costly for determining all concepts of a given context if one starts from arbitrary subsets of objects or attributes. It is better to use first the formulas $X' = \bigcap_{t\in T} \{g\}'$ or $Y' = \bigcap_{m\in Y} \{m\}'$, which are special cases of (3) and (3'), and then to form (X'', X') or (Y', Y''). Thus, one can start with the special intents $\{g\}'$ $(g \in G)$ or the special extents $\{m\}'$ $(m \in M)$ since, by the formulas, every intent is the intersection of some special extents $\{g\}'$ and every extent is the intersection of some special extents $\{m\}'$. Although this improved construction method works quite well for treating small contexts by hand, for computer programs another method has been proven more successful; its basic idea is to construct from a given extent the next extent with respect to a lexicographic order fixed for all subsets of objects (see [13]). A comparative study of different algorithms for determining concept lattices can be found in [14].

The drawing of labelled line diagrams has been the most successful graphical method for representing concept lattices. Up to now, it is still a kind of art which requires experience. Nevertheless, there is a major effort to develop computer programs which assist in drawing adequate line diagrams for concept lattices; basic ideas and existing programs are discussed in [15]. Since diagrams should not only represent the conceptual structure but also unfold views for interpretations, there is not a unique way to draw concept lattices. Different aims and meanings require different drawings, where often, for a single concept lattice, several line diagrams are desirable. As a general strategy, it has been proven successful to decompose a concept lattice in smaller and more easily understood parts which can be visualized by more or less standardized graphical patterns [15,16]. As is indicated by the diagram in Figure 2, the most elementary graphical patterns are the parallelogram [17] and the straight line formed by several line segments. These patterns are often combined to two- or higher-dimensional grids; in Figure 2, for instance, we have on the lower left the grid of a four-dimensional cube (hypercube) which also occurs (incomplete) at the top of the diagram. Recognition and arrangement of the patterns can be supported by a geometric representation of the concept lattice which is described in [18]. In general, the decomposition strategy is dependent on methods for the structural analysis of concept lattices which are developed in great variety [11,19–25]. An essential advantage of the representation by labelled line diagrams is that their correctness can be controlled without knowing how they are constructed: By the second part of the Basic Theorem, for a finite context (G, M, I), one has to check that the line diagram describes a finite lattice, that the circles of the $\vee$-irreducible and $\wedge$-irreducible elements are labelled by object names and attribute names, respectively, and that gIm is equivalent to $\gamma g \leq \mu m$ for all $g \in G$ and $m \in M$; this is necessary and sufficient to determine that the labelled line diagram represents $\mathfrak{B}(G, M, I)$.

Let us now come back to the discussion about further information represented in a concept lattice. First of all, a concept lattice can be viewed as a **hierarchical conceptual clustering** of the objects (via its extents). The concept lattice in Figure 2, for instance, shows that the conceptual hierarchy classifies the ladies in mainly two directions with some marginal deviations caused by some ladies who participated in only a few events. This results also from other studies of the same data set by methods of social network analysis: one outcome is shown in Figure 4 (see [26]) where the resulting graph has just three maximal cliques which are the extents $\{5\}'$, $\{11\}'$, and $\{12\}'$. (Dorothy and Pearl are missing in Figure 4). Obviously, the concept lattice is more informative than the graph (even with further labellings for the edges). Other clustering methods also have failed to differentiate knowledge as fully as concept lattices.

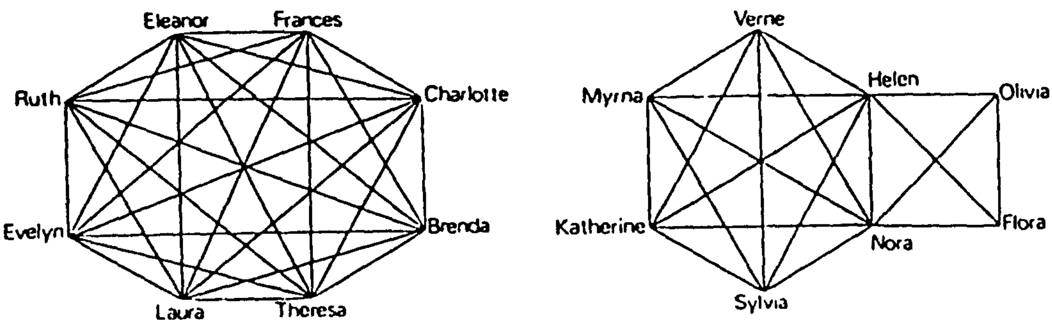

Figure 4. Some reduced representation of the data in Figure 1 by a graph.

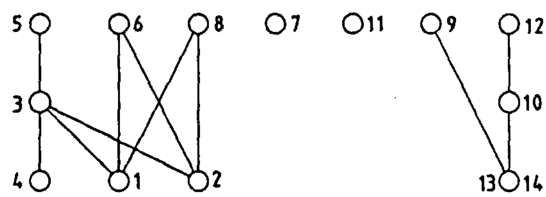

Figure 5. The ordered set of all attributes in Figure 1.

Second, a concept lattice can be understood as a **representation of all implications** between the attributes (via its intents). An *implication* of a context (G, M, I) is a pair of subsets of M, denoted by $Y \rightarrow Z$, for which $Y' \subseteq Z'$, i.e., each object from G having all attributes of Y has also all attributes of Z. This corresponds to "inheritance" of attributes in conventional semantic networks. Since $Y' \subseteq Z' \Leftrightarrow \forall n \in Z: (Y', Y'') \leq \mu n$ in $\mathfrak{B}(G, M, I)$ and since $(Y', Y'') = \bigwedge_{m \in Y} \mu m$ by the Basic Theorem, the implication $Y \rightarrow Z$ can be read from a labelled line diagram of $\mathfrak{B}(G, M, I)$. In Figure 2, for instance, we see that $\{11, 12\} \rightarrow \{7, 10\}$ is an implication of the context in Figure 1, because $\mu 11 \wedge \mu 12 \leq \mu 7$ and $\mu 11 \wedge \mu 12 \leq \mu 10$ (any lady who attended parties 11 and 12 also attended 7 and 10). The implications with a one-element premise are of special interest because they yield a natural ordering on the set of all attributes which is defined by $m \leq n :\Leftrightarrow m \rightarrow n (\Leftrightarrow \mu m \leq \mu n)$. A line diagram of this ordering for our example is shown in Figure 5. A basis for all implications (i.e. a minimal generating set of all implications) of a finite context (G, M, I) is given by $\{P \rightarrow (P'' \backslash P) \mid P \text{ pseudo-intent}\}$ where a pseudo-intent is recursively defined as follows: a set P of attributes is a *pseudo-intent* in (G, M, I) if $P \neq P''$ and $Q'' \subseteq P$ for all pseudo-intents Q with $Q \subset P$ (see [13,27,28]). The described basis of all implications of the context in Figure 1 is listed in Figure 6.

Let us briefly summarize that formal contexts and their concept lattices are substantial tools for formal representation of conceptual knowledge. These tools activate the rich source of mathematical developments in order and lattice theory for knowledge representation. In particular,

```
 1:  M14  ==>  M9   M10  M12  M13
 2:  M13  ==>  M9   M10  M12  M14
 3:  M11  M12  ==>  M7   M10
 4:  M10  ==>  M12
 5:  M8   M11  ==>  M7   M10  M12
 6:  M7   M11  ==>  M10  M12
 7:  M7   M9   M10  M12  ==>  M13  M14
 8:  M7   M9   M10  M11  M12  M13  M14  ==>  M6
 9:  M6   M12  ==>  M7   M9   M10  M11  M13  M14
10:  M6   M11  ==>  M7   M9   M10  M11  M13  M14
11:  M6   M7   M8   ==>  M5
12:  M5   M12  ==>  M1   M2   M3   M4   M6   M7   M8
          M9   M10  M12  M13  M14
13:  M5   M11  ==>  M1   M2   M3   M4   M6   M7   M8
          M9   M10  M12  M13  M14
14:  M5   M9   ==>  M8
15:  M5   M6   ==>  M8
16:  M5   M6   M8   M9   ==>  M2   M3   M4
17:  M4   ==>  M3   M5
18:  M3   ==>  M5
19:  M3   M5   M8   ==>  M6
20:  M2   ==>  M3   M5   M6   M8
21:  M2   M3   M4   M5   M6   M8   ==>  M9
22:  M1   ==>  M3   M5   M6   M8
23:  M1   M2   M3   M4   M5   M6   M7   M8   M9   ==>
          M10  M11  M12  M13  M14
```

Figure 6. A basis of all implications of the formal context in Figure 1.

the representation by labelled line diagrams is a powerful instrument if it is combined with the structure theory of concept lattices. Then these diagrams can make transparent the different meanings of concept lattices as, for instance, the hierarchical classification of objects or the logic of attribute implications (further basic meanings of concept lattices are discussed in [29]).

2. CONCEPT LATTICES OF MANY-VALUED CONTEXTS

Often data are not given by cross-tables so that it is not obvious how to understand them as formal context. In formal concept analysis, a general approach has been developed to interpret data as formal contexts (see [11,30,31]). This approach is based on the set-theoretic model of many-valued context formalizing data structures which are represented in statistics by data matrices and in computer science by relational databases. A *many-valued context* is defined to be a quadruple (G, M, W, I) where G, M, and W are sets and I is a ternary relation between G, M, and W (i.e., $I \subseteq G \times M \times W$) such that $(g, m, w_1) \in I$ and $(g, m, w_2) \in I$ imply $w_1 = w_2$. The elements of G, M, and W are called *objects, (many-valued) attributes*, and *attribute values* (in German: Merkmalswerte), respectively, and $(g, m, w) \in I$ is read: the object g has the value w for the attribute m; (G, M, W, I) is called an *n-valued context* if $|W| = n$. A formal context may be understood in this terminology as a special case: a 1-valued context. An attribute m of a many-valued context (G, M, W, I) may be considered as a partial map from G into W, which suggests writing $m(g) = w$ instead of $(g, m, w) \in I$ and defining the domain of m by $\text{dom}(m) := \{g \in G \mid (g, m, w) \in I \text{ for some } w \in W\}$. The attribute m is said to be *complete* if $\text{dom}(m) = G$, and a many valued context is called *complete* if all its attributes are complete. Let us remark that the object-attribute-value triad is represented in frame-based semantic network as frames, slots and values, respectively.

In general, there is no automatic way to derive from a many-valued context a suitable formal context. Such a derivation is always an action of interpretation. In formal concept analysis, this interpretation is performed by a method called *conceptual scaling* (see [31]). The first step of conceptual scaling is to interpret for each attribute m its values as objects of some separate formal context $\mathbb{S}_m := (G_m, M_m, I_m)$, i.e., the attribute m of the many-valued context $\mathbb{K} := (G, M, W, I)$ is understood as a partial map from G into G_m. The contexts $\mathbb{S}_m$ and their concept lattices should have a clear structure and should reflect some meaning for interpretation: such contexts are therefore called *(conceptual) scales*. In a second step, the scales $\mathbb{S}_m (m \in M)$ are combined to a common scale $\mathbb{S}$ by some product operator; for the sake of simplicity, we restrict ourselves here to the operator $\underline{\times}$ called *semiproduct* which yields $\mathbb{S} := \underline{\times}_{m \in M} \mathbb{S}_m := (\times_{m \in M} G_m, \bigcup_{m \in M} (M_m \times \{m\}), \nabla)$ where ∇ is the relation with $(g_m)_{m \in M} \nabla (n,p) :\Leftrightarrow g_p I_p n$. Finally, we obtain the formal context $(G, \bigcup_{m \in M} (M_m \times \{m\}), J)$ with $gJ(n,p) :\Leftrightarrow (m(g))_{m \in M} \nabla (n,p)$ (or equivalently $p(g) I_p n$); the extents of this context are exactly the pre-image of the extents of $\mathbb{S}$ under the map $g \mapsto (m(g))_{m \in M}$. Let us recall that the resulting formal context is determined by the chosen scales $\mathbb{S}_m$ interpreting the attributes m of $\mathbb{K}$ and by the product operator $\underline{\times}$; therefore, $(G, \bigcup_{m \in M} (M_m \times \{m\}), J)$ is called the *derived context* of the *scaled context* $(\mathbb{K}, \underline{\times}_{m \in M} \mathbb{S}_m)$ and the concept lattice of the derived context is also taken as the concept lattice of the scaled context, i.e., $\underline{\mathfrak{B}}(\mathbb{K}, \underline{\times}_{m \in M} \mathbb{S}_m) := \underline{\mathfrak{B}}(G, \bigcup_{m \in M} (M_m \times \{m\}), J)$.

The notion of a scaled context opens a new level for formal representation of conceptual knowledge which will be demonstrated by a second example. Figure 7 shows a data table which lists the amounts of absorption for nine colour stimuli for eleven receptors in the goldfish retina (see [32]). This table can be understood as the description of a many-valued context: its objects are the eleven receptors; its attributes are the nine colours (wavelengths); and its attribute values are the numbers measuring the amount of absorption; furthermore, $m(g) = w$ means that the receptor g has w as amount of absorption for the colour m. Let us denote the many-valued context of Figure 7 by $\mathbb{K} := (G, M, W, I)$ with $G := \{r_1, r_2, \ldots, r_{11}\}$, $M := \{v_{430}, b_{458}, b_{485}, b_{498}, g_{530}, g_{540}, y_{585}, o_{610}, r_{660}\}$, and $W := \{0, 1, 2, \ldots, 199\}$.

Receptor	Violet 430	Blue 458	Blue 485	Blue-Green 498	Green 530	Blue 540	Yellow 585	Orange 610	Red 660
1	147	153	89	57	12	4	0	0	0
2	153	154	110	75	32	24	23	17	0
3	145	152	125	100	14	0	0	0	0
4	99	101	122	140	154	153	93	44	0
5	46	85	103	127	152	148	116	75	26
6	73	78	85	121	151	154	109	57	0
7	14	2	46	52	97	106	137	92	45
8	44	65	77	73	84	102	151	154	120
9	87	59	58	52	86	79	139	153	146
10	60	27	23	24	56	72	136	144	111
11	0	0	40	39	55	62	120	147	132

Figure 7. Colour absorption of eleven receptors in the goldfish retina.

To obtain a conceptual structure of the many-valued context $\mathbb{K}$, a conceptual scaling of $\mathbb{K}$ shall be performed. For a conceptual interpretation of the attributes and their values, a variety of conceptual scales may be considered (a list of standardized scales can be found in [31, p. 150]. The most simple scale would be the *nominal scale* $(W, W, =)$ which, at least, would conceptually separate different values; but it would not reflect the important order "smaller-higher" of the values. Thus, the *one-dimensional ordinal scale* $(W, W, \geq)$ is more appropriate to capture the ordinal nature of the attribute values. The non-empty extents of $(W, W, \geq)$ are the integer intervals $[n, 199]$ with $n \in W$ which are more interesting for interpreting the data than the extents $[0, n]$ $(n \in W)$ of the one-dimensional ordinal scale $(W, W, \leq)$ because one is interested in large amounts of absorption. If one wants both types of intervals for the interpretation, one has to choose the *one-dimensional interordinal scale* $(W, \{\leq, \geq\} \times W, \lozenge)$ with $w \lozenge (\leq, n) :\Leftrightarrow w \leq n$

	Violet 430										Blue 458										Blue 485										Blue-Green 498								
	≥14	≥44	≥46	≥60	≥73	≥87	≥90	≥145	≥147	≥163	≥2[illegible]	≥57	≥69	≥66	≥78	≥85	≥101	≥162	≥163	≥164	≥40	≥46	≥68	≥77	≥86	≥89	≥103	≥110	≥122	≥126	≥39	≥57	≥67	≥73	≥75	≥100	≥121	≥127	≥140
1	×	×	×	×	×	×	×	×	×		×	×	×	×	×	×	×	×	×		×	×	×	×	×	×					×	×	×						
2	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×			×	×	×	×	×				
3	×	×	×	×	×	×	×	×			×	×	×	×	×	×	×	×			×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×			
4	×	×	×	×	×	×	×				×	×	×	×	×	×	×				×	×	×	×	×	×	×	×	×		×	×	×	×	×	×	×	×	×
5	×	×	×								×	×	×	×	×	×					×	×	×	×	×	×	×				×	×	×	×	×	×	×	×	
6	×	×	×	×	×						×	×	×	×	×						×	×	×	×	×						×	×	×	×	×	×	×		
7	×										×										×	×									×	×							
8	×	×									×	×	×	×							×	×	×	×							×	×	×	×					
9	×	×	×	×	×	×					×	×	×								×	×	×								×	×							
10	×	×	×	×							×	×																											
11																					×										×								

	Green 530										Green 540										Yellow 585									Orange 610									Reed 660					
	≥19	≥32	≥55	≥66	≥84	≥86	≥97	≥151	≥152	≥164	≥4	≥24	≥62	≥72	≥79	≥102	≥106	≥148	≥163	≥164	≥23	≥93	≥109	≥116	≥120	≥136	≥137	≥139	≥151	≥17	≥44	≥67	≥76	≥92	≥144	≥147	≥163	≥164	≥26	≥45	≥111	≥120	≥132	≥146
1											×																																	
2	×	×									×	×									×									×														
3	×																																											
4	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×		×	×								×	×													
5	×	×	×	×	×	×	×	×	×		×	×	×	×	×	×	×	×			×	×	×	×						×	×	×	×						×					
6	×	×	×	×	×	×	×	×			×	×	×	×	×	×	×	×	×	×	×	×	×							×	×	×												
7	×	×	×	×	×	×	×				×	×	×	×	×	×	×				×	×	×	×	×	×	×			×	×	×	×	×					×	×				
8	×	×	×	×	×						×	×	×	×	×	×					×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×		
9	×	×	×	×	×	×					×	×	×	×	×						×	×	×	×	×	×	×	×		×	×	×	×	×	×	×	×		×	×	×	×	×	×
10	×	×	×	×							×	×	×	×							×	×	×	×	×	×				×	×	×	×	×	×				×	×	×			
11	×	×	×								×	×	×								×	×	×	×	×					×	×	×	×	×	×	×			×	×	×	×	×	

Figure 8. A derived context of the many-valued context in Figure 7.

and $w\Diamond(\geq, n) :\Leftrightarrow w \geq n$. Let us remark that even the algebraic structure of the real numbers can be captured by a suitable conceptual scale.

For this paper, we restrict ourselves to the ordinal scale $\mathbf{W} := (W, W, \geq)$. If we scale our many-valued context $\mathbf{K}$ by the semi-product of nine copies of $\mathbf{W}$, we obtain as the derived context the formal context shown in Figure 8. Its concept lattice in Figure 9, which consists of 137 concepts, represents a rich but clear conceptual structure of our many valued context. The structure is dominated by two main dimensions which are represented by the long chains on the left and right in Figure 9: the left chain leads from low to high absorption of blue and violet, while the right chain ranges from low absorption of green, yellow and orange to high absorption of red. Two

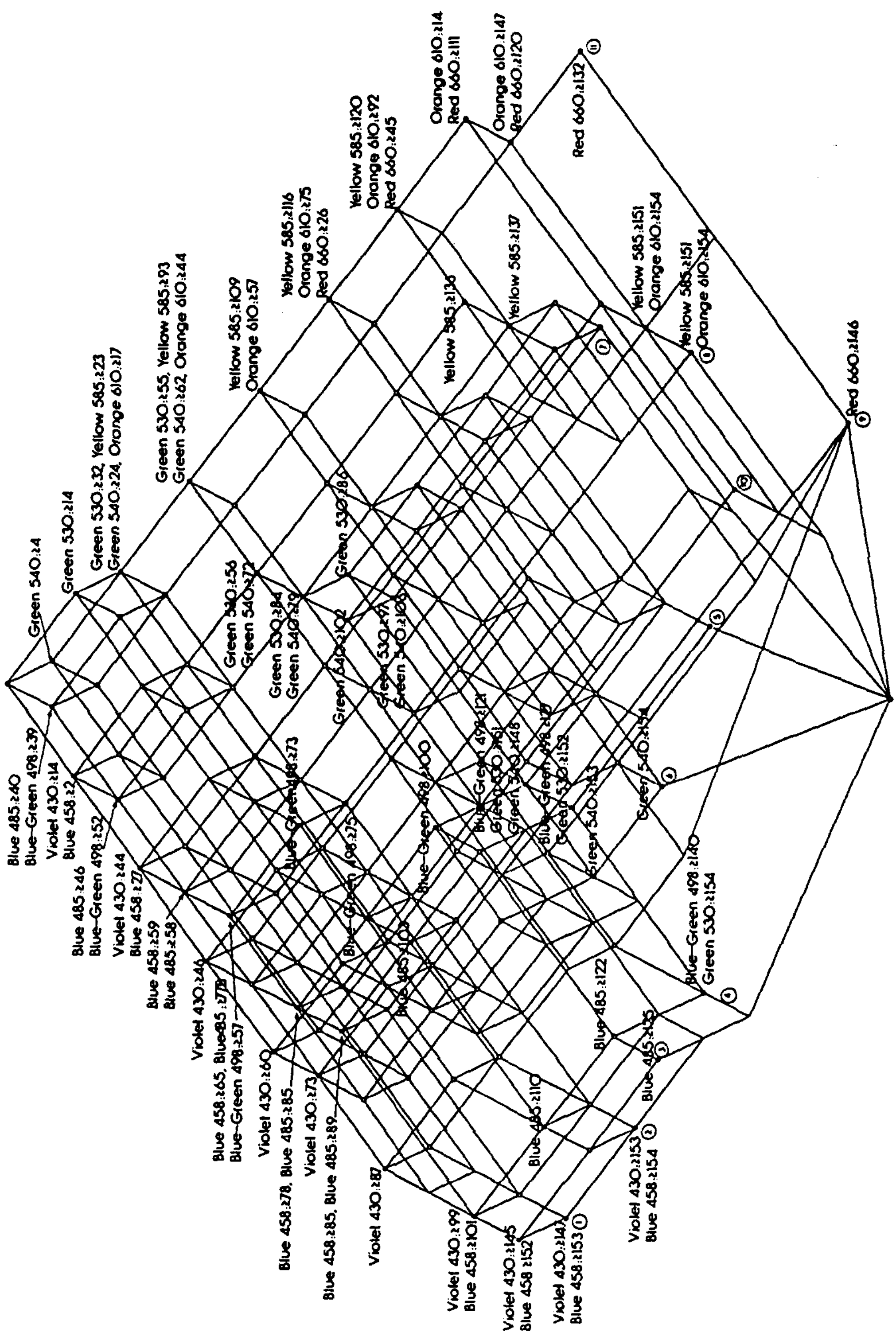

Figure 9. Concept lattice of the formal context in Figure 8.

minor dimensions of the 4-dimensional conceptual structure are caused by green and blue-green. The richness of the structure indicates that the eleven receptors receive a differentiated spectrum of the colours. The classification of the receptors described in [32] is formed by the extents $\{b_{458} :\geq 152\}' = \{r_1, r_2, r_3\}$, $\{g_{540} :\geq 148\}' = \{r_4, r_5, r_6\}$, and $\{r_{660} :\geq 111\}' = \{r_8, r_9, r_{10}, r_{11}\}$ (the receptor r_7 is excluded).

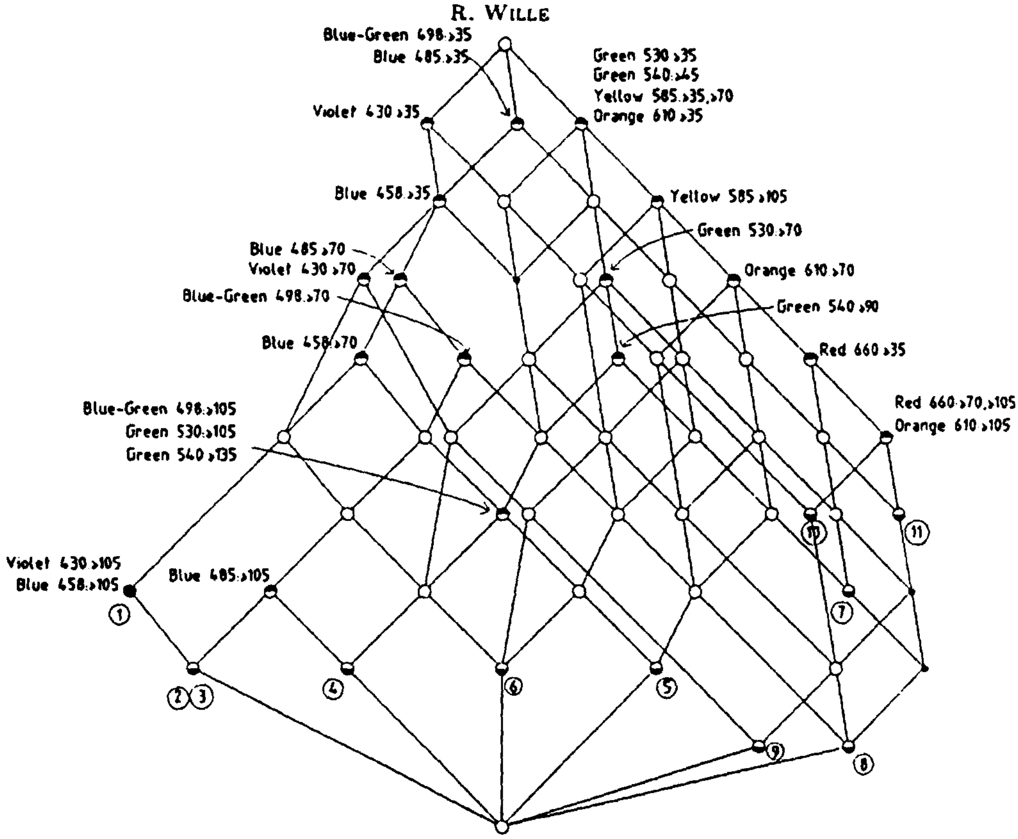

Figure 10. Concept lattice of a subcontext in Figure 8.

The chosen conceptual scaling has the advantage that it unfolds a conceptual structure without losing the original data. Therefore, the concept lattice in Figure 9 can be viewed as a complete knowledge representation of the data. Of course, an appropriate reduction of this lattice may also be a useful knowledge representation. For instance, Figure 10 shows the (reduced) concept lattice, which is derived by suitable partitions of the attribute values into four segments; in particular, Green 540 is scaled by $(W, \{45, 90, 135\}, \geq)$ and all the other colours are scaled by $(W, \{35, 70, 105\}, \geq)$. This conceptual scaling yields as the derived context the subcontext in Figure 8, which consists of the columns headed by: Violet 430: ≥ 44, ≥ 73, ≥ 145; Blue 458: ≥ 59, ≥ 78, ≥ 152; Blue 485: ≥ 40, ≥ 77, ≥ 110; Blue-Green 498: ≥ 30, ≥ 73, ≥ 121; Green 530: ≥ 55, ≥ 84, ≥ 151; Green 540: ≥ 62, ≥ 102, ≥ 148; Yellow 585: ≥ 93, ≥ 93, ≥ 109; Orange 610: ≥ 44, ≥ 75, ≥ 144; Red 660: ≥ 45, ≥ 111, ≥ 111. Since the extents of such a subcontext (consisting of a collection of columns) are also extents of the underlying context, the concept lattice of Figure 10 has a natural $\wedge$-embedding into the concept lattice of Figure 9. From this it is clear that the lattice of Figure 10 shows a similar dimensional structure as the lattice of Figure 9. The appropriateness of the segmentation of the attribute values is confirmed by the fact that the natural classes of the receptors described in [32] are also extents of the derived subcontext. Instead of segmenting the attribute values into disjoint intervals, one can also use overlapping blocks of some tolerance relation [33].

Concept lattices can also be used to represent dependencies between the attributes of a many-valued context. A general method for such representation is to deduce, from the many-valued context, a suitable formal context whose attribute implications are exactly the specified dependencies between the attributes of the many-valued context (see [31,34,35]). Let us first consider functional dependency in a complete many-valued context $\mathbf{K} := (G, M, W, I)$. For $Y, Z \subseteq M$, Z is called *functionally dependent* on Y if, for all $g, h \in G, y(g) = y(h)$, for all $y \in Y$, implies $z(g) = z(h)$ for all $z \in Z$, i.e., there is a function $f : W^Y \to W^Z$ such that $f(y(g))_{y \in Y} = (z(g))_{z \in Z}$ for all $g \in G$ [36, p. 43]. To represent functional dependency, we deduce the formal context $\mathbf{K}_f := (\mathfrak{P}_2(G), M, I_f)$ where $\mathfrak{P}_2(G)$ is the set of all two-element subsets of G and $\{g, h\} I_f m :\Leftrightarrow m(g) = m(h)$. It is easy to prove that, for $Y, Z \subseteq M$, Z is functionally

	Violet 430	Blue 458	Blue 485	Blue-Green 498	Green 530	Green 540	Yellow 585	Orange 610	Red 660
{1,2}									×
{1,3}							×	×	×
{7,9}				×					

Figure 11. A reduced context for determining all functional dependencies between the attributes in Figure 7.

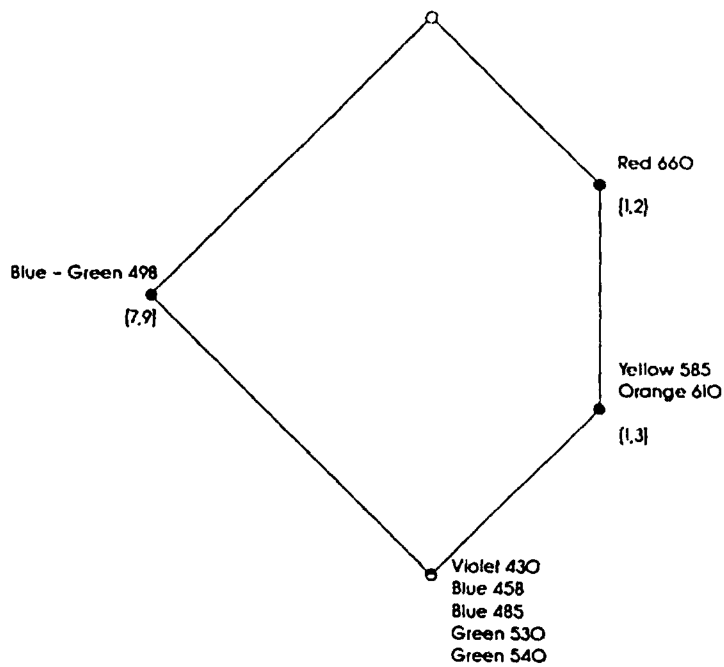

Figure 12. Concept lattice of the formal context in Figure 11.

dependent on Y in $\mathbf{K}$ if and only if $Y \to Z$ is an implication of $\mathbf{K}_f$. Thus, the canonical basis of all implications of $\mathbf{K}_f$, which is described in Section 1, can also be taken as a basis of all functional dependencies of $\mathbf{K}$. For the many-valued context $\mathbf{K}$ in Figure 7, the deduced context $\mathbf{K}_f$ has 55 objects which may be reduced to only 3 objects by the following argument: by the Basic Theorem for Concept Lattices, a finite formal context (G, M, I) has the same intents and so the same implications as a reduced context $(G_r, M, I \cap (G_r \times M))$ for which $G_r \subseteq G$, and $\{\gamma g \mid g \in G_r\}$ is the set of all $\vee$-irreducible elements of $\underline{\mathfrak{B}}(G, M, I)$. Such a reduced context for $\mathbf{K}_f$ is shown in Figure 11; its concept lattice in Figure 12 represents all implications of $\mathbf{K}_f$ and hence all functional dependencies on $\mathbf{K}$. Recall from Section 1 that, for instance, μ(Red 660) $\wedge\mu$(Blue-Green 498) $\leq \mu$(Yellow 585) in Figure 12 means: the attribute "Yellow 585" is functionally dependent on the pair of attributes "Red 660", "Blue-Green 498."

The small concept lattice in Figure 12 indicates that there is a large number of functional dependencies in the many-valued context of Figure 7. Many of these dependencies might be considered less meaningful since they are caused by only small differences between the attribute values. This suggests that we introduce the following notion of dependency in a complete many-valued context $\mathbf{K} := (G, M, W, I)$ with $W \subseteq \mathbf{R}$ (see [33]). For $Y, Z \subseteq M$ and $\delta \in \mathbf{R}$ with $\delta \geq 0$, Z is called δ-dependent on Y if, for all $g, h \in G, |y(g) - y(h)| \leq \delta$, for all $y \in Y$, implies $|z(g) - z(h)| \leq \delta$

	Violet 430	Blue 458	Blue 485	Blue–Green 498	Green 530	Green 540	Yellow 585	Orange 610	Red 660
{1,2}	×	×							×
{1,3}	×	×			×	×	×	×	×
{1,6}			×						×
{4,6}					×	×			×
{5,6}		×		×	×	×			
{5,8}	×								
{7,9}				×			×		
{7,11}		×	×						
{8,9}		×			×			×	
{8,10}								×	×
{9,10}						×	×	×	
{10,11}					×			×	

Figure 13. A reduced context for determining all δ-dependencies between the attributes in Figure 7 ($\delta := 10$).

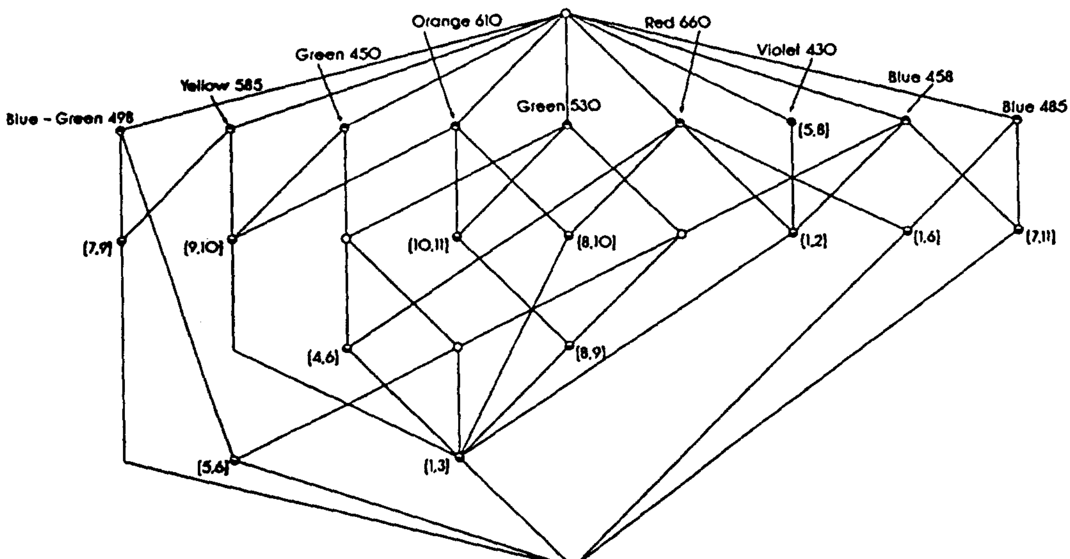

Figure 14. Concept lattice of the formal context in Figure 13.

for all $z \in Z$. To represent δ-dependency, we deduce the formal context $\mathbf{K}_\delta := (\mathfrak{P}_2(G), M, I_\delta)$ where $\{g,h\} I_\delta m :\Leftrightarrow |m(g) - m(h)| \leq \delta$. Again, it is easy to prove that, for $Y, Z \subseteq M$, Z is δ-dependent on Y in $\mathbf{K}$ if and only if $Y \to Z$ is an implication of $\mathbf{K}_\delta$. For the many-valued context $\mathbf{K}$ in Figure 7 and for $\delta := 10$, a reduced context of $\mathbf{K}_\delta$ is shown in Figure 13; its concept lattice in Figure 14 represents all implications of $\mathbf{K}_\delta$ and hence all δ-dependencies of $\mathbf{K}$.

To increase the meaningfulness, functional dependency may also be modified in another direction: We require not only the existence of the function f, but also that f preserves a specified structure on the attribute values. This will be outlined by one of the most important examples of a dependency notion for a complete many-valued context $\mathbf{K} := (G, M, W, I)$ in which the value set W carries an order relation $\leq$ (see [34]). For $Y, Z \subseteq M$, Z is called *ordinally dependent* on Y

if, for all $g, h \in G, y(g) \leq y(h)$, for all $y \in Y$ implies $z(g) \leq z(h)$ for all $z \in Z$; i.e., there is an order-preserving function $f : W^Y \rightarrow W^Z$ such that $f((y(g))_{y \in Y}) = (z(g))_{z \in Z}$. To represent ordinal dependency, we deduce the formal context $\mathbf{K}_0 := (G^2, M, I_0)$ where $(g, h) I_0 m :\Leftrightarrow m(g) \leq m(h)$. It easily follows that, for Y, $Z \subseteq M$, Z is ordinally dependent on Y in $\mathbf{K}$ if and only if $Y \rightarrow Z$ is an implication of $\mathbf{K}_0$. For the many-valued context $\mathbf{K}$ in Figure 7, a reduced context of $\mathbf{K}_0$ is shown in Figure 15.

	Violet 430	Blue 458	Blue 485	Blue-Green 498	Green 530	Green 540	Yellow 585	Orange 610	Red 660
(3,1)	×	×				×	×	×	×
(1,4)			×	×	×	×	×	×	×
(1,6)				×	×	×	×	×	×
(6,1)	×	×	×						×
(3,2)	×	×			×	×	×	×	×
(6,4)	×	×	×	×	×				×
(5,6)	×					×			
(6,5)		×	×	×	×		×	×	×
(5,9)	×						×	×	×
(7,8)	×	×	×	×			×	×	×
(11,7)	×	×	×	×	×	×	×		
(8,9)	×				×		×	×	
(9,8)		×	×	×		×			×
(10,8)		×	×	×	×	×	×	×	×
(11,8)	×	×	×	×	×	×		×	×

Figure 15. A reduced context for determining all ordinal dependencies between the attributes in Figure 7.

From Figure 15 we derive the canonical basis of all implications of $\mathbf{K}_0$ and hence of all ordinal dependencies of $\mathbf{K}$ (this basis is listed in Figure 16). Since the context of Figure 15 has 134 concepts, it is better to represent the ordinal dependencies of $\mathbf{K}$ by the small concept lattice (see Figure 17) of the context which is complementary to the one in Figure 15. In general, for a formal context (H, N, J), the *complementary context* is defined by $(H, N, (H \times N) \backslash J)$. For Y where $Z \subseteq N$, $Y \rightarrow Z$ is an implication of (H, N, J) if and only if $(g, m) \in (H \times Z) \backslash J$ always implies $(g, n) \in (H \times Z) \backslash J$ for some $n \in Y$. This yields the consequence that, for instance, the first implication in the list of Figure 16 can be read from the diagram in Figure 17 as follows: The extent of μ(Orange 610) is contained in the union of the extents of μ(Yellow 585) and of μ(Red 660); hence {Yellow 585, Red 660} $\rightarrow$ {Orange 610} is an implication of $\mathbf{K}_0$ and therefore an ordinal dependency on $\mathbf{K}$.

Let us stop here our discussion of the second level of representing conceptual knowledge and emphasize once more that such a representation is always based on a conceptually scaled many-valued context. This basic idea has been recently elaborated to the notion of a "conceptual file" to support interactive procedures with the represented knowledge (see [37]). Let us also remark that the conceptual scaling of many-valued contexts is tightly connected with the theory of conceptual measurement (see [30,31]) which also offers tools for the representation of conceptual knowledge. Of course, the tools of knowledge representation already described in Section 1

1:	Y585	R660	==>	O610			
2:	G540	O610	==>	R660			
3:	G530	G540	R660	==>	O610		
4:	BG498	O610	==>	R660			
5:	BG498	G540	O610	R660	==>	G530	
6:	BG498	G540	Y585	==>	G530		
7:	B485	O610	==>	BG498	R660		
8:	B485	Y585	==>	BG498			
9:	B485	G540	==>	BG498			
10:	B485	G530	==>	BG498			
11:	B458	O610	==>	R660			
12:	B458	BG498	==>	B485			
13:	V430	G540	R660	==>	B458	O610	
14:	V430	G540	Y585	==>	B458		
15:	V430	G530	R660	==>	B458		
16:	V430	G530	G540	==>	B458		
17:	V430	BG498	==>	B458	B485		
18:	V430	B485	==>	B458			
19:	V430	B458	G530	O610	R660	==>	G540
20:	V430	B458	G530	Y585	==>	G540	
21:	V430	B458	B485	BG498	G540	==>	G530

Figure 16. Basis of all ordinal dependencies between the attributes in Figure 7.

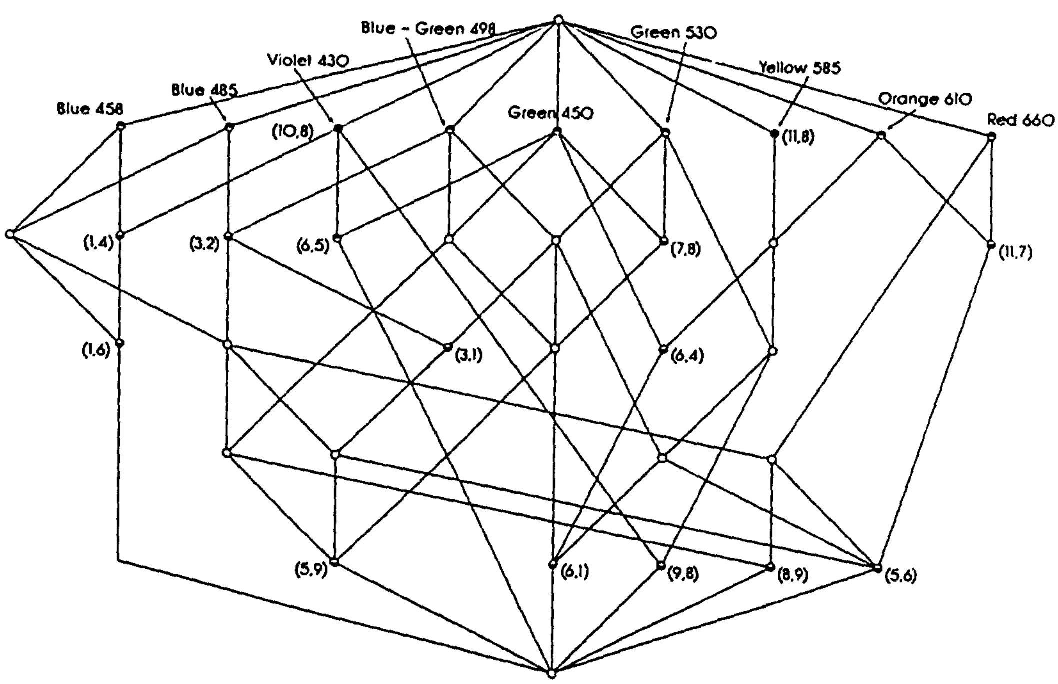

Figure 17. Concept lattice of the formal context in Figure 15.

may be applied on our second level of knowledge representation too, since conceptual scaling of many-valued contexts yields formal contexts and concept lattices. As a new content on our second representation level, we have the dependencies between many-valued attributes (for the representation of partial dependencies and implications see [38].

3. CONCEPTUAL KNOWLEDGE SYSTEMS

A systematic treatment of knowledge would not be satisfying if it is only concerned with knowledge representation; it should cover inference and acquisition of knowledge and should

		Lattice Concepts				Lattice Properties											
		Boolean Lattice	CDS-Lattice	Geometric Lattice	Metric Lattice	atomistic	Brouwerian	complemented	distributive	dually semimodular	graded	modular	relatively complemented	sectionally complemented	semimodular	Stonean	uniquely complemented
		$c \wedge d$	$c \wedge e$	$a \wedge j$	g		d		$d \wedge g$	$e \wedge f$		$e \wedge j$	$h \wedge i$	$i \wedge a \wedge c$	$j \wedge f$	$k \wedge d$	$c \wedge d$
		I	II	III	IV	a	b	c	d	e	f	g	h	i	j	k	l
Lattice Concepts	I	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×
	II	·	×	·	·	×	·	×	·	×	×	·	·	×	·	·	·
	III	·	·	×	·	×	·	×	·	·	×	·	×	×	×	·	·
	IV	·	·	·	×	·	·	·	·	×	×	×	·	·	×	·	·
Lattice Examples	1	·	·	·	×	·	×	·	×	×	×	×	·	·	×	×	·
	2	·	×	×	×	×	·	×	·	×	×	×	×	×	×	·	·
	3	·	·	·	·	×	·	×	·	·	·	·	×	×	·	·	·
	4	·	·	×	·	×	·	×	·	·	×	·	×	×	×	·	·
	5	·	·	·	·	×	·	·	·	×	×	·	·	·	·	·	·
	6	·	·	·	×	·	×	·	×	×	×	×	·	·	×	·	·
	7	·	·	·	·	×	·	×	·	·	×	·	·	·	·	·	·
	8	·	·	·	·	·	·	×	·	·	×	·	·	·	×	·	·
	9	·	×	·	·	×	·	×	·	×	×	·	·	×	·	·	·
	10	·	×	·	·	×	·	×	·	×	×	·	×	×	·	·	·

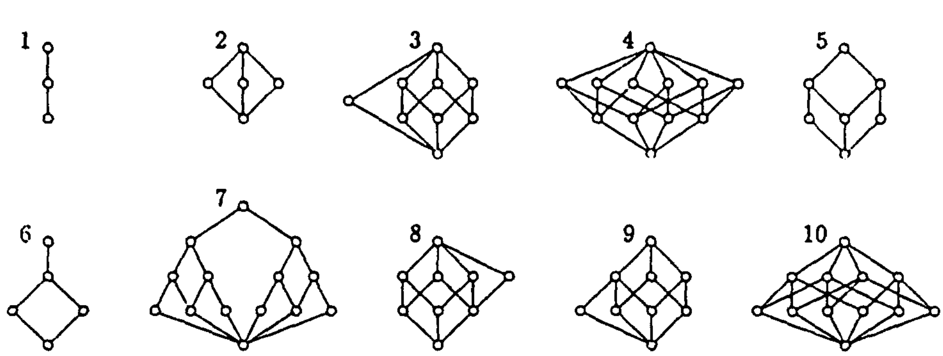

Figure 18. A conceptual knowledge system of finite lattices.

also establish tools for a communication about knowledge. To model conceptual knowledge systems which fulfill these tasks, we need a specification of conceptual knowledge. In [39], such a specification is founded on three basis notions, namely objects, attributes, and concepts, which are linked by four basic relations: an object has an attribute, an object belongs to a concept, an attribute abstracts from a concept, and a concept is a subconcept of another concept. These structural elements are well mathematized in formal concept analysis: In the frame of a formal context(G, M, I) (and its concept lattice), we have objects $g \in G$, attributes $m \in M$, and concepts $(A, B) \in \mathfrak{B}(G, M, I)$ for which $gIm(\Leftrightarrow \gamma g \leq \mu m)$ means "the object g has the attribute m", $g \in A(\Leftrightarrow \gamma g \leq (A, B))$ means "the object g belongs to the concept (A, B)", $m \in B(\Leftrightarrow (A, B) \leq \mu m)$ means "the attribute m abstracts from the concept (A, B)", and $(A_1, B_1) \leq (A_2, B_2)$ means "the concept (A_1, B_1) is a subconcept of the concept (A_2, B_2)". This underlines that formal contexts and their concept lattices are the appropriate mathematical structures for a formal representation of the basic elements of conceptual knowledge.

Knowledge inference and acquisition are performed to extend given knowledge. Therefore, inference and acquisition can only be mathematically treated within a system which models the represented knowledge as part of some knowledge universe. The notion of a conceptual knowledge system introduced in [39] relates to some "conceptual universe" which comprises the basic elements of all the conceptual knowledge within a field of interest. Formally, a *conceptual universe* is defined as a formal context $\mathsf{U} := (G_\mathsf{U}, M_\mathsf{U}, I_\mathsf{U})$. Before we define what is a conceptual knowledge system related to U, we want to discuss ideas for such systems by an example.

The table in Figure 18 represents a conceptual knowledge system within the conceptual universe U of all finite lattices and their properties. More precisely, G_U consists of all finite lattices (up to isomorphism), M_U comprises all documented properties of finite lattices, and $gI_\mathsf{U}m$ is valid in U if and only if the finite lattice g has the property m. The conceptual knowledge system in Figure 18 is composed by a set B of four concepts of U (i.e., $B \subseteq \mathfrak{B}(\mathsf{U})$), a set G of ten objects of U (i.e., $G \subseteq G_\mathsf{U}$), and a set M of twelve attributes of U (i.e., $M \subseteq M_\mathsf{U}$). The crosses in the table of Figure 18 represent four relations, namely $I_1 \subseteq G \times M$, $I_2 \subseteq G \times B$, $I_3 \subseteq B \times M$, and $I_4 \subseteq B \times B$, which are "part" of the four basic relations of the conceptual knowledge coded in U. The dots in Figure 18 describe the negations (complements) J_k of the relations I_k in $\mathsf{U}(k = 1, 2, 3, 4)$. Each cell of the table contains either a cross or a dot which means that the conceptual knowledge system represents the full restrictions of the four basic relations of U to B, G, and M.

Often there are also empty cells [39] so that the corresponding relationships of the conceptual universe are not fully represented by the present system. There are two possibilities for such an empty cell: either one can infer or one cannot infer how to fill it from the already coded knowledge in the system. For instance, the cross in Figure 18 indicating that lattice 1 is modular can be inferred by the crosses indicating that lattice 1 is a metric lattice and that a metric lattice is modular. On the other hand, it could not be inferred that lattice 3 is relatively complemented if there would not be a cross representing exactly this relationship; hence, such a cross results from knowledge acquisition exceeding the given system (without the considered cross).

Let us briefly sketch how the example in Figure 18 was elaborated by a method of knowledge acquisition called *attribute exploration* (see [40]). First we took the listed lattice concepts and properties from the index in [7] and analysed some finite lattices with respect to these concepts and properties. The resulting information was stored in a computer with an implemented program for attribute exploration (see [41,42]). Based on the present information, the program asks whether certain implications between the properties (and concepts) are valid in the conceptual universe. A typical question was: "Is a relatively complemented finite lattice always atomistic, complemented, graded and sectionally complemented?" We answered "No" and justified this by lattice 3 which is relatively complemented, but not graded. After finishing the interactive procedure of questions and answers, we had ended with the conceptual knowledge system in Figure 18. The confirmed implications are coded by certain crosses and the lattice terms above the columns which may be considered as further descriptors of the concepts and properties; for instance, the term $c \wedge d$ above the first column indicates that the complemented distributive lattices are the Boolean lattices. Since the coded implications form a basis of all implications of U between the listed concepts and properties, it can be concluded that the concept lattice of the resulting context described by

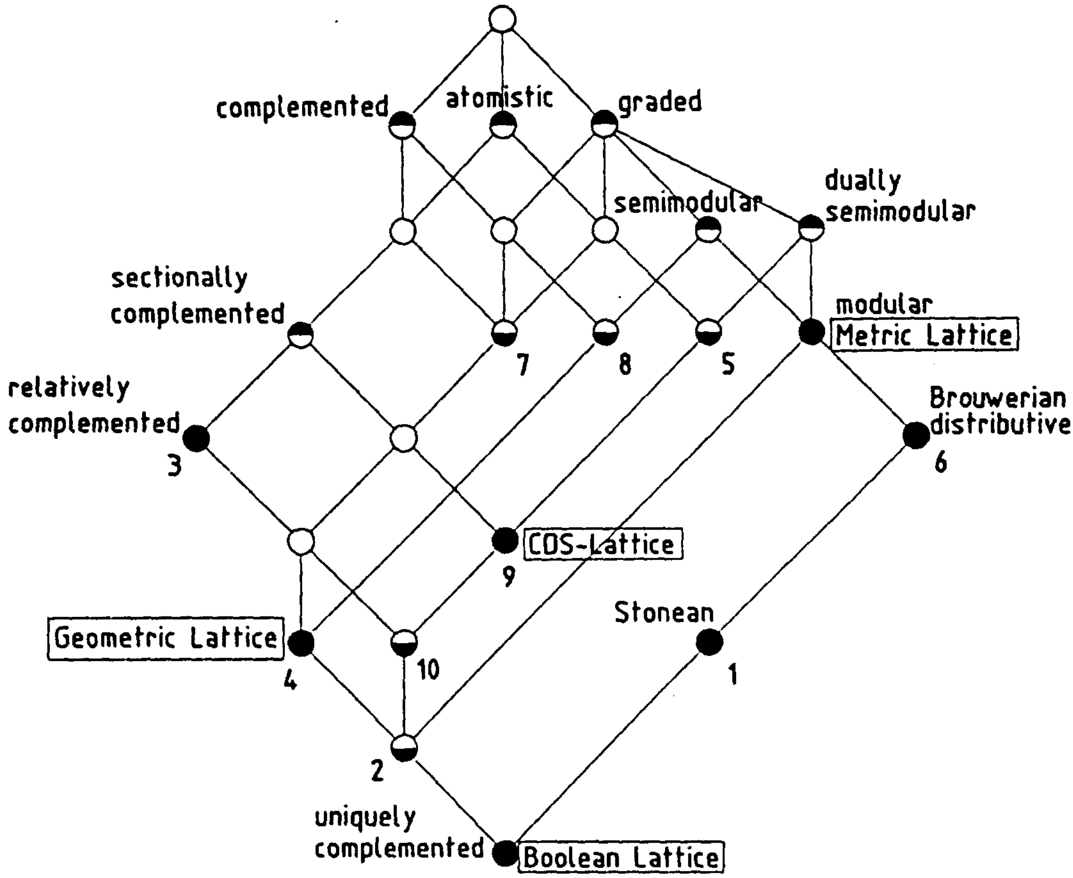

Figure 19. Concept lattice of the formal context in Figure 18.

Figure 18 (see Figure 19) has a natural ∧-embedding into the concept lattice of the universe U between the listed concepts and properties (for an extended analysis of 50 properties of finite lattices see [43]).

In our example, components of the conceptual knowledge system are a set B of concepts, a set G of objects, a set M of attributes, a relation $I = I_1 \cup I_2 \cup I_3 \cup I_4 \subseteq (B\dot{\cup}G) \times (B\dot{\cup}M)$ represented by crosses, a relation $J \subseteq (B\dot{\cup}G) \times (B\dot{\cup}M)$ disjoint from I represented by dots ($\dot{\cup}$ denotes the disjoint union of sets), and lattice terms as descriptors of concepts and attributes. All these components obtain their semantical meaning by the underlying conceptual universe U. For the general definition of a conceptual knowledge system, this setting will only be slightly generalized. First we allow two sets of concepts, which might coincide in special cases. Secondly, the descriptor terms are taken from a richer algebraic structure than a lattice to capture the Boolean logic both of the objects and of the attributes in the conceptual universe.

The richer algebraic structure is given by the semi-concepts of the conceptual universe. In general, a *semi-concept* of a formal context (G, M, I) is defined as a pair (A, B) with $A \subseteq G$ and $B \subseteq M$ such that $A = B'$ *or* $B = A'$. Obviously, the concepts of (G, M, I) are special semi-concepts but not every semi-concept has to be a concept. The order-relation between concepts is extended to semi-concepts by the definition $(A_1, B_1) \sqsubseteq (A_2, B_2) :\Leftrightarrow A_1 \subseteq A_2$ and $B_2 \subseteq B_1$. The operations ∧ and ∨ have also natural extensions to semi-concepts, but they do not yield a lattice structure anymore. In [39], the following algebraic operations ⊓, ⊔, ⌝, and ⌟ are introduced for semi-concepts:

$$(A_1, B_1) \sqcap (A_2, B_2) := (A_1 \cap A_2, (A_1 \cap A_2)'),$$
$$(A_1, B_1) \sqcup (A_2, B_2) := ((B_1 \cap B_2)', B_1 \cap B_2),$$
$$\neg(A, B) := (G \backslash A, (G \backslash A)'), \text{ and}$$
$$\lrcorner(A, B) := ((M \backslash B)', M \backslash B);$$

the constants $(\emptyset, M)$ and $(G, \emptyset)$ are considered as nullary operations $\bot$ and $\top$, respectively. The set of all semi-concepts of (G, M, I) together with the operations $\sqcap$, $\sqcup$, $\urcorner$, $\lrcorner$, $\bot$, and $\top$ is called the *algebra of semi-concepts* of (G, M, I) and denoted by $\underline{\mathfrak{H}}(G, M, I)$. The semi-concepts of the type (A, A') form a Boolean algebra with respect to $\sqcap$, $\urcorner$, and $\bot$ which is obviously isomorphic to the Boolean algebra of all subsets of G, while the semi-concepts of the type (B', B) form a Boolean algebra with respect to $\sqcup$, $\lrcorner$, and $\top$ dually isomorphic to the Boolean algebra of all subsets of M. These two Boolean algebras of semi-concepts have $\underline{\mathfrak{H}}(G, M, I)$ as union and $\underline{\mathfrak{B}}(G, M, I)$ as intersection. For $X \subseteq \underline{\mathfrak{H}}(G, M, I)$, $T(X)$ denotes the set of all algebraic terms constructed from X by the operational symbols $\sqcap$, $\sqcup$, $\urcorner$, $\lrcorner$, $\bot$, and $\top$; for $t \in T(X)$, $t_{(G,M,I)}$ is then the semi-concept which we obtain by evaluating t in $\underline{\mathfrak{H}}(G, M, I)$ [44, p. 162ff].

As already pointed out, for the general definition of a conceptual knowledge system we presuppose a conceptual universe $\mathsf{U} := (G_\mathsf{U}, M_\mathsf{U}, I_\mathsf{U})$; furthermore, let us recall that, for $g \in G_\mathsf{U}$ and $m \in M_\mathsf{U}$, $\gamma_\mathsf{U} g := (\{g\}'', \{g\}')$ and $\mu_\mathsf{U} m := (\{m\}', \{m\}'')$. Now, a *conceptual knowledge system* with respect to U is defined to be a 7-tuple $(B_1, B_2, G, M, I, J, \tau)$ with the following properties (see [39]):

(i) $B_1, B_2 \subseteq \mathfrak{B}(\mathsf{U}), G \subseteq G_\mathsf{U}$, $M \subseteq M_\mathsf{U}$, and $I, J \subseteq (B_1 \dot\cup G) \times (B_2 \dot\cup M)$,

(ii)

$$xIy \text{ implies } \begin{cases} x \leq y, & \text{if } x \in B_1 \text{ and } y \in B_2, \\ \gamma_\mathsf{U} x \leq y, & \text{if } x \in G \text{ and } y \in B_2, \\ x \leq \mu_\mathsf{U} y, & \text{if } x \in B_1 \text{ and } y \in M, \\ x I_\mathsf{U} y, & \text{if } x \in G \text{ and } y \in M, \end{cases}$$

(iii)

$$xJy \text{ implies } \begin{cases} x \not\leq y, & \text{if } x \in B_1 \text{ and } y \in B_2, \\ \gamma_\mathsf{U} x \not\leq y, & \text{if } x \in G \text{ and } y \in B_2, \\ x \not\leq \mu_\mathsf{U} y, & \text{if } x \in B_1 \text{ and } y \in M, \\ (x,y) \notin I_\mathsf{U}, & \text{if } x \in G \text{ and} y \in M, \end{cases}$$

(iv) for $X := B_1 \cup B_2 \cup \gamma_\mathsf{U} G \cup \mu_\mathsf{U} M$, τ is a map from $B_1 \dot\cup B_2 \dot\cup G \dot\cup M$ into the power set of $T(X)$ such that, for all $t \in \tau(x)$,

$$t_\mathsf{U} = \begin{cases} x, & \text{if } x \in B_1 \text{ or } x \in B_2, \\ \gamma_\mathsf{U} x, & \text{if } x \in G, \\ \mu_\mathsf{U} x, & \text{if } x \in M. \end{cases}$$

The example of Figure 18 might have already elucidated that the model of a conceptual knowledge system allows a comprehensive representation of conceptual knowledge. The advantage of our formal model is that it opens the application of a rich variety of mathematical tools. Here we only mention that questions about inference can be transferred to algebraic word problems which can be successfully treated by elaborating methods developed for solving word problems in lattices (see [45,46]). Solving algebraic word problems is also essential for methods of knowledge acquisition like the mentioned attribute exploration and the so-called concept exploration [13,40,42]. Mathematically designed tools for knowledge communication applicable to conceptual knowledge systems are already developed for different aims in formal concept analysis [15,37,47,48].

The model of a conceptual knowledge system extends the knowledge representation discussed in Section 1. But it may also be based on the more general approach described in Section 2. This will again be outlined by an example. The table in Figure 20 represents a conceptual knowledge system which comprises all paintings of Rembrandt in the Rijksmuseum Amsterdam documented in [49]; the concepts are taken from [50]. As conceptual universe we assume the formal context $\mathsf{U} := (G_\mathsf{U}, M_\mathsf{U}, I_\mathsf{U})$, where G_U is the set of all paintings of Rembrandt and M_U is the set of all attributes which might be assigned to a painting of Rembrandt. Since the representation in Figure 20 uses many-valued attributes, we have to understand U as the derived context of a scaled many-valued context. In particular, let us assume a nominal scale for the attribute "Material," a one-dimensional interordinal scale for the attributes "Height" and "Width," and

		Portrait	Self-Portrait	Family-Portrait	Male Portrait	Female Portrait	Group Portrait	Genre	Landscape	Animal Study	Old Testament	New Testament	Material	Height	Width	Date	No. of Persons
Portrait		x	·	·	·	·	·	·	·	·	·	·					≥1
Self-Portrait		x	x	x	x	·	·	·	·	·	·	·					1
Family Portrait		x	·	x	·	·	·	·	·	·	·	·					≥1
Male Portrait		x	·	·	x	·	·	·	·	·	·	·					1
Female Portrait		x	·	·	·	x	·	·	·	·	·	·					1
Group Portrait		x	·	·	·	·	x	·	·	·	·	·					≥2
Genre		·	·	·	·	·	·	x	·	·	·	·					
Landscape		·	·	·	·	·	·	·	x	·	·	·					
Animal Study		·	·	·	·	·	·	·	·	x	·	·					
Old Testament		·	·	·	·	·	·	·	·	·	x	·					
New Testament		·	·	·	·	·	·	·	·	·	·	x					
The Night Watch	(2016)	x	·	·	·	·	x	·	·	·	·	·	Canvas	3590	4380	1642	22
De Staalmeesters	(2017)	x	·	·	x	·	x	·	·	·	·	·	Canvas	1910	2790	1662	6
The Anatomical Lesson	(2018)	x	·	·	·	·	x	·	·	·	·	·	Canvas	1000	1340	1665	4
Sketch of "The Anatom. L."	(2018 A1)	x	·	·	·	·	x	·	·	·	·	·	Paper	109	132		9
The Bridal Couple	(2019)	x	·	·	·	·	x	·	·	·	·	·	Canvas	1215	1665	>1665	2
The Stone Bridge	(2020)	·	·	·	·	·	·	·	x	·	·	·	Oak P.	295	425	~1638	≥4
Portrait of Maria Trip	(2022)	x	·	·	·	x	·	·	·	·	·	·	Teak W.	1070	820	1639	1
Rembrandt's Mother	(2024 A1)	x	·	x	·	x	·	·	·	·	·	·	Oak P.	600	480	1631	1
Peacocks	(2024 A2)	·	·	·	·	·	·	·	·	x	·	·	Canvas	1450	1335	~1639	1
S. Peter's Denial	(2024 A3)	·	·	·	·	·	·	·	·	·	·	x	Canvas	1540	1690	1660	10
Titus in a Monk's Habit	(2024 A4)	x	·	x	x	·	·	·	·	·	·	·	Canvas	795	677	1660	1
Jeremiah	(2024 A5)	·	·	·	·	·	·	·	·	·	x	·	Oak P.	580	460	1630	1
Oriental Potentate	(2024 A6)	x	·	·	x	·	·	·	·	·	·	·	Mahogany	720	545	1635	1
Joseph Telling His Dreams	(2024 A7)	·	·	·	·	·	·	·	·	·	x	·	Oiled Paper	510	390	~1637	12
Dr. Ephraim Bueno	(2024 A8)	x	·	·	x	·	·	·	·	·	·	·	Oak P.	190	150	~1647	1
Self-Portrait	(2024 A10)	x	x	x	x	·	·	·	·	·	·	·	Oak P.	710	570	~1645	1
Portrait of Titus	(2024 A11)	x	·	x	x	·	·	·	·	·	·	·	Canvas	720	560	~1660	1
Tobit and Anna with a Kid	(2024 A12)	·	·	·	·	·	·	·	·	·	x	·	Oak P.	395	300	1626	3
Portr. of Himself as S. Paul	(2024 A13)	x	x	x	x	·	·	·	·	·	·	x	Canvas	910	770	1661	1
Study of His Own Face	(2024 A14)	x	x	x	x	·	·	·	·	·	·	·	Oak P.	412	338		1

Figure 20. Conceptual knowledge system of Rembrandt's painting in the Rijksmuseum Amsterdam.

a one-dimensional ordinal scale for the attribute "No. of Persons." The knowledge about the structure of the scales determining U may be coded in a conceptual knowledge system of U by term descriptors. For instance, the term μ_U(Material: Canvas) $\sqcap\, ^{\neg}\, \mu_U$ (Material: Paper) as descriptor for the (one-valued) attribute "Material: Canvas" yields the information about the nominal scale that an object having "Material: Canvas" has never "Material: Paper." Of course, an implementation of our conceptual knowledge system may not explicitly code such term descriptors, but store the many-valued attributes with the corresponding scales. Nevertheless, the understanding of our conceptual knowledge system as an instance of the general model is important because it opens for the system the general methods of representation, inference, acquisition, and communication.

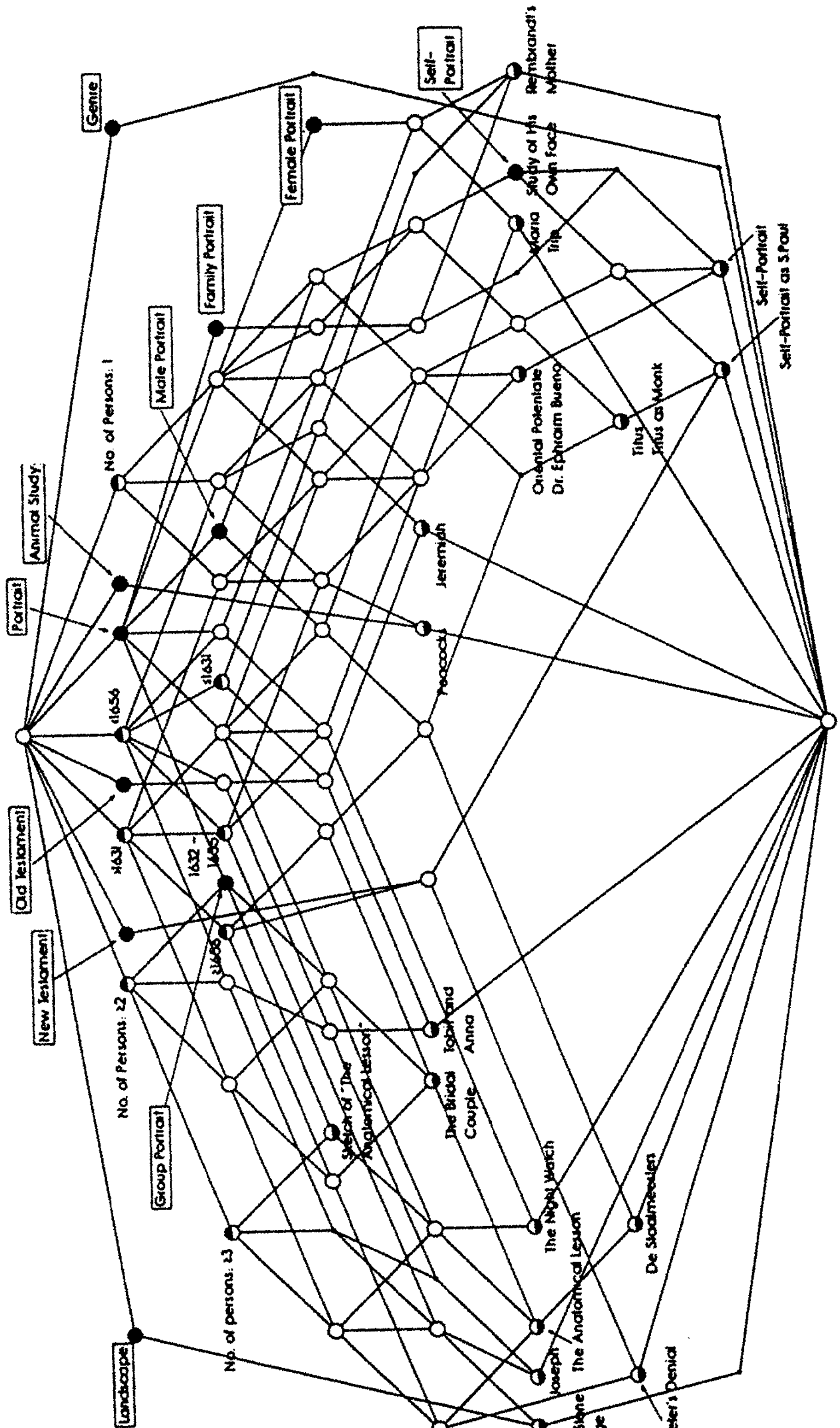

Figure 21. Concept lattice derived from Figure 20.

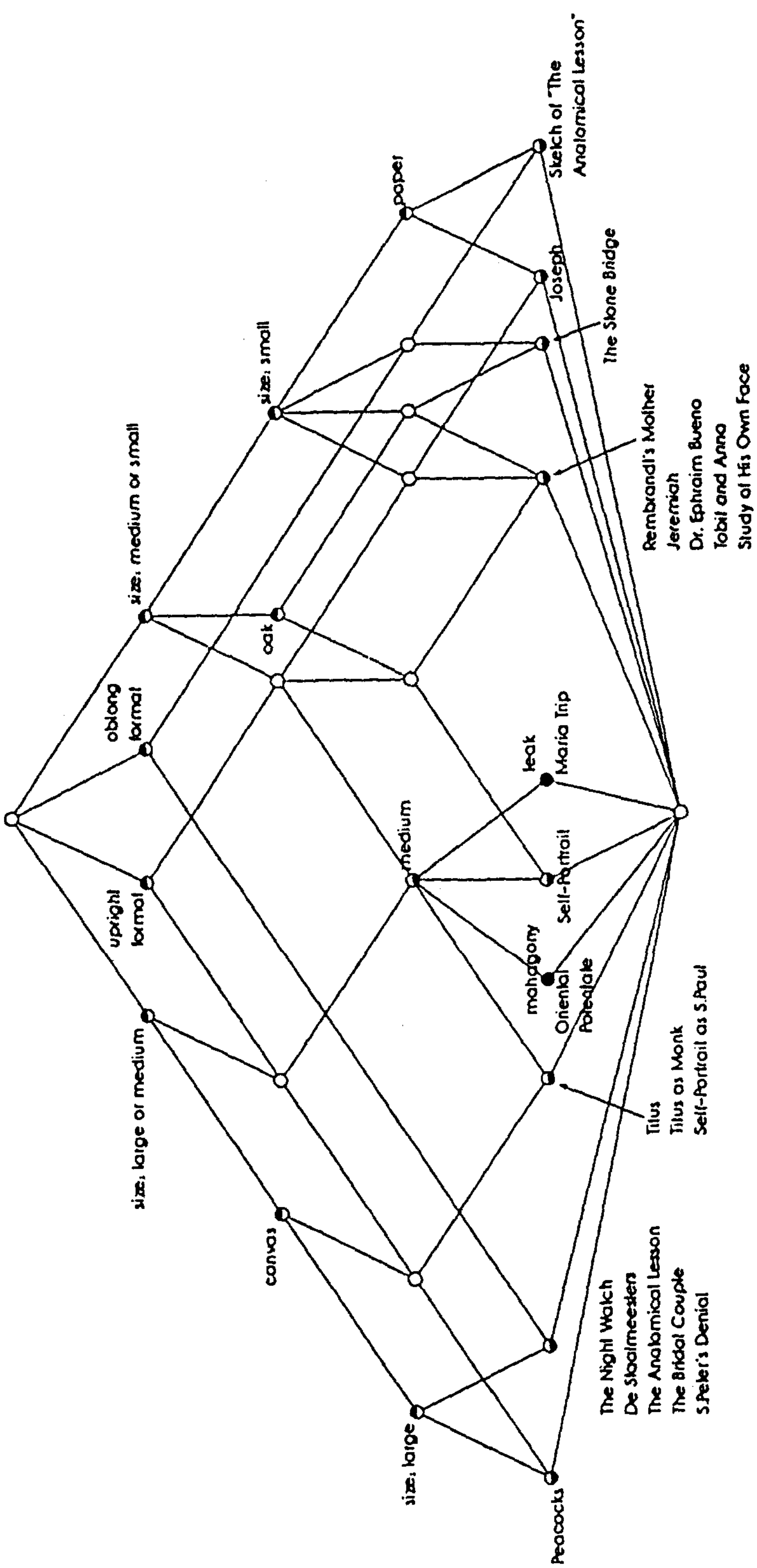

Figure 22. Concept lattice derived from Figure 20.

In Figure 20, incomplete knowledge [42] is indicated in different ways. First of all, empty cells signify missing data. For instance, the catalogue [49] does not report the date of the painting "Study of His Own Face." The full information for the cell (Portrait, Date) would be the exact time period in which Rembrandt has painted portraits. A time period as partial information is given for the "The Bridal Couple" by the dating "> 1665." Also the information "~1638" for "The Stone Bridge" might be understood as a given time interval, for instance 1637–1639. All such types of incomplete information does not prevent the application of the conceptual scaling method; in [31], this method is also described for incomplete many-valued contexts. Thus, we can derive from Figure 20 a formal context and a concept lattice for Rembrandt's paintings in the Rijksmuseum. Instead of representing the total concept lattice of our knowledge system, we show in Figure 21 and 22 smaller concept lattices of the paintings determined by some derived attributes.

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