BOUNDARY LAYERS FOR PARABOLIC REGULARIZATIONS OF TOTALLY CHARACTERISTIC QUASILINEAR PARABOLIC EQUATIONS

By E. GRENIER

ABSTRACT. – In this paper we study boundary layers of nonlinear characteristic parabolic equations as the viscosity goes to zero. We obtain and justify in small time a complete expansion of the solution with respect to the viscosity.

Key words : Boundary layers, parabolic equations

RESUME. – Dans cet article on étudie les couches limites de systèmes paraboliques non linéaires caractéristiques quand la viscosité tend vers zéro. On obtient et justifie en temps petit un développement asymptotique de la solution par rapport à la viscosité.

Introduction

The aim of this paper is to study the behaviour as the viscosity $\varepsilon$ goes to 0 of solutions $u^\varepsilon$ of the following parabolic equations in a bounded domain $\Omega$

\begin{align}
\frac{\partial}{\partial t} u^\varepsilon + \sum_{i=1}^d A_i(t, x, u^\varepsilon) \frac{\partial}{\partial x_i} u^\varepsilon - \varepsilon \Delta u^\varepsilon &= 0, \\
\varepsilon &= 0 \quad \text{on} \quad \partial \Omega, \\
\varepsilon(0) &= u_0^\varepsilon,
\end{align}

where $A_i$ are smooth symmetric matrices, and $u_0^\varepsilon$ is a family of smooth (in $H^s(\Omega) \cap H^1_0(\Omega)$ with $s$ large enough) initial data. We assume that $\Omega$ is a domain of $\mathbb{R}^d$ with smooth boundaries. As usual in parabolic systems (see [7], [14]), by changes of coordinates, the problem reduces to the study of the case of an half plane $\Omega = \mathbb{R}_+ \times \mathbb{R}^{d-1}$, the case of $\Omega = \mathbb{R}^d$ being straightforward.
This framework covers many different situations and physical phenomena that lead to the use of different mathematical techniques. We refer to [3] for a general introduction and a survey of the various results. Two main problems appear. The first problem is to describe the limit (say in $L^2$, or in any Sobolev space which does not see the boundary values) $u$ of $u^\varepsilon$, and to find an equation for it. More precisely we expect that $u$ satisfies

$$ \partial_t u + \sum_{i=1}^d A_i(t, x, u) \partial_i u = 0, \tag{4} $$

but we have to precise the boundary conditions.

The second problem is to give a complete asymptotic expansion of $u^\varepsilon$ in $\varepsilon$, and in particular to describes the behaviour of $u^\varepsilon$ near $x_1 = 0$. As in the linear case [7], the boundary layer can be of size $\varepsilon$ or $\sqrt{\varepsilon}$, depending on algebraic properties of $A_1$.

These problems have been much studied in the linear and semilinear case, for instance in [7], [10], [13]. This paper deals with the quasilinear case when the boundary layer is of size $\sqrt{\varepsilon}$, and is the first of a series of papers which will study the case of a boundary layer of size $\varepsilon$ (noncharacteristic boundary layer, see [3] and [4]), parabolic regularization of shocks [5] and degenerate cases (partial viscosity for instance). Notice that this paper does not treat inviscid limits of fluid mechanics, since the incompressibility condition does not enter our framework.

In the linear case, to have a boundary layer of size $\sqrt{\varepsilon}$, it is sufficient to have $A_1 = 0$ on $\partial \Omega$. So here, we could think that it is sufficient to inforce $A_1(t, x, u) = 0$ on the boundary, or equivalently $A_1(t, x, 0) = 0$. In fact this is not the case, and we have to inforce the stronger condition

$$ A_1(t, x, w) = 0 \tag{5} $$

for every $t \geq 0$, for every $x \in \partial \Omega$ and for every $w$. By Taylor formula, (5) is equivalent to

$$ A_1(t, x, u) = \phi(x_1) \tilde{A}_1(t, x, u), \tag{6} $$

where $\tilde{A}_1$ is smooth, and

$$ \phi(x_1) = \frac{x_1}{1 + x_1}. \tag{7} $$

Assumption (6) which inforces the characteristic character of (1,2,3) will be assumed in the whole paper. Notice that it is satisfied for instance if $\sum_i A_i(t, x) \partial_i$ is the transport by a given divergence free vector field.

So in the first part, we investigate, in the case where $\Omega$ is the half space $x_1 > 0$, energy estimates of solutions of a linear system linked to (1,2,3), with coefficients $A_i$ depending on $\varepsilon$ and having a boundary type behaviour, and energy estimates on the whole system (1,2,3). This work is related and extends previous works, in particular of O. Guès ([6], [7], [8]). This allows us to prove (Theorem 1.2) that if we can build an approximate solution to (1,2,3), with high order error terms (in $\varepsilon$), then there exists an exact solution of the same system, close to the approximate one.
The second section investigates the boundary layer equations obtained from (1,2,3) by studying $u^\varepsilon$ at scales of order $\sqrt{\varepsilon}$ in $x_1$, and the third section ends the construction of the approximate solution required by Theorem 1.2.

Let us now state the main Theorem in the case of an half plane, in order to avoid technicalities.

**Theorem 0.1.** Let $\Omega = \mathbb{R}_+ \times \mathbb{R}^{d-1}$. Let us assume (6). Let us assume that $u_0^\varepsilon$ has the following asymptotic expansion (at any order $N$ and regularity $\bar{s}$):

$$
\left\| u_0^\varepsilon - \sum_{i=0}^{N} \sqrt{\varepsilon}^i \left[ u_b^i(x_1, \ldots, x_d) + u_b^0 \left( \frac{x_1}{\sqrt{\varepsilon}}, x_2, \ldots, x_d \right) \right] \right\|_{H^2} \leq C \sqrt{\varepsilon}^{N-\bar{s}},
$$

where $u_b^0$ and $u_b^i$ are in $H^\infty(\Omega)$ and $u_b^0$ are rapidly decreasing in $x_1$. Let $u^0$ be a smooth solution of the limit system:

$$
\partial_t u^0 + \sum_{i=1}^{d} A_i(t, x, u^0) \partial_i u^0 = 0
$$
on $[0, T]$ ($u^0 \in H^\infty([0, T] \times \Omega)$, with $s$ large), with initial data $u_0^0$. Then there exists $0 < T' < T$ and functions $u^\varepsilon$ and $u_b^\varepsilon$ (in $H^\infty([0, T'] \times \Omega)$) with $u_b^\varepsilon$ rapidly decreasing in $x_1$; and, for $\varepsilon$ small enough, there exists a family of smooth solutions $u^\varepsilon$ of (1,2,3) on $[0, T']$, such that

$$
\| u^\varepsilon(t, x_1, \ldots, x_d) - u_b^\varepsilon(t, x_1, \ldots, x_d) - u_0^\varepsilon(t, x_1, \ldots, x_d) \|_{H^\varepsilon} \leq C \sqrt{\varepsilon}^{N-\bar{s}}
$$

for every $\varepsilon > 0$. However (12) gives a much more precise description of $u_b^\varepsilon$, and in particular $L^\infty$ convergence at any order in $\varepsilon$.

Some estimates related to linear hyperbolic systems, with coefficients depending on $\varepsilon$ and not bounded in Sobolev spaces as $\varepsilon \to 0$ are discussed in section 4.2. Such estimates could be useful in the study of coupling between a parabolic system with vanishing viscosity and a hyperbolic system. Moreover, it emphasizes that the viscosity is necessary to build approximate solutions and boundary layers, but not to prove stability results.
Notations

In the main part of the paper, $\Omega = \mathbb{R}_+ \times \mathbb{R}^{d-1}$ and $(x_1, ..., x_d)$ is a generic point. $\partial_1$ is the derivative with respect to $x_1$ and to shorten the formulas, $\partial_2^\beta$ will stand for $\partial_{x_2}^\beta_1 ... \partial_{x_d}^\beta_d$, where $\beta$ is a multiindex, whose length $\beta_0 + ... + \beta_d$ is denoted by $|\beta|$. By an abuse of notation, $\beta + 1$ denotes the set of multiindices $(\beta_2 + 1, \beta_3, ..., \beta_d), ..., (\beta_2, ..., \beta_d + 1)$, and by the same abuse of notations, $A_2 \partial_2$ stands for $\sum_{i=1}^d A_i \partial_i$. The index $\epsilon$ will often be omitted, excepted to emphasize the $\epsilon$ dependence of some quantities. $H^\infty$ denotes the intersection of all Sobolev spaces $H^s$.

1. Stability

1.1. Linear case: weighted norms

Let $T > 0$. Let us study the following system

(14) \[ \partial_t v^\epsilon + A_1^\epsilon(t,x) \partial_t \xi + A_2^\epsilon(t,x) \partial_2 v^\epsilon - \epsilon \Delta v^\epsilon = 0 \quad \text{in } \Omega, \]

(15) \[ v^\epsilon = 0 \quad \text{on } \partial \Omega, \]

(16) \[ v^\epsilon(0) = v_0^\epsilon, \]

where $v_0^\epsilon$ are given smooth functions (in $H^s(\Omega) \cap H_0^1(\Omega)$ for some $s > 0$), $\Omega = \mathbb{R}_+ \times \mathbb{R}^{d-1}$ is the half d-dimensional space and $A_1^\epsilon, A_2^\epsilon$ are smooth, symmetric matrices, defined for $0 \leq t \leq T$. In what follows, we will often omit the indices $\epsilon$ in $v^\epsilon, A_1^\epsilon, A_2^\epsilon$.

Theorem 1.1. - Let us assume that for $\alpha + |\beta| + \gamma \leq s$ and all $t, x \in [0, T] \times \Omega$.

(17) \[ ||\partial_t^\alpha \partial_2^\beta \partial_1^\gamma A_1(t,x)|| \leq C + C|\sqrt{\epsilon}|^{-\alpha+1} \theta(x_1/\sqrt{\epsilon}). \]

(18) \[ ||\partial_2^\beta \partial_1^\gamma A_1(t,x)|| \leq C \phi(x_1), \]

(19) \[ ||\partial_t^\alpha \partial_2^\beta \partial_1^\gamma A_2(t,x)|| \leq C + C|\sqrt{\epsilon}|^{-\alpha} \theta(x_1/\sqrt{\epsilon}), \]

where $\theta(x_1)$ is a smooth, nonnegative function, such that

(20) \[ |x_1^n \theta(x_1)| \leq C_n \]

for $x_1 \in \mathbb{R}_+$, and for every $n$. Then for $s$ large enough, the system (14,15,16) has a unique solution $v(t,x)$ in $H^s([0, T] \times \Omega)$. Furthermore, there exists two constants $C_0$ and $C_s$ independent on $\epsilon$ such that

(21) \[ \partial_t ||v||_s^2 \leq C_s ||v||_s^2, \]
where

\[ |||v|||_s^2 = \sum_{\alpha \in \mathbb{N}^N, \beta \in \mathbb{N}^{N-1}, \gamma \in \mathbb{N}} C_0^{-\alpha - |\beta| - \gamma} |||v|||_{\alpha, \beta, \gamma}^2, \]

with \( \phi \) defined by (7).

**Remarks**

- The weighted norm \( |||\cdot|||_s \) controls the system. The use of this norm is not new. In particular, it is equivalent to the use of \( x_1 \partial_1 \) instead of \( \partial_1 \) in classical estimates (conormal estimates), as used for instance in [7]. However, it provides a clearer insight in the bounds.

- Existence is not new, however, the estimates obtained here are uniform in \( \varepsilon \), which is not the case in [7].

- At \( t = 0 \), \( |||v^\varepsilon|||_s \) can be formally computed with the help of the equation, and involves \( |||v_0^\varepsilon|||_{H^3s} \). So in order to insure \( |||v^\varepsilon|||_s \) bounded at \( t = 0 \), we must enforce some additional regularity on \( v_0^\varepsilon \), namely, for instance \( v_0^\varepsilon \) uniformly bounded in \( H^3s \). We will not detail this classical point further in this paper.

- It is in fact sufficient to assume

\[ \theta(x) = \frac{C}{1 + x^n} \]

for some large \( n \) (depending on \( s \)).

**Proof.** We will not prove the existence part, and refer to [7] for a detailed study, and will restrict ourselves to the proof of (21).

\[ \partial_t \int |||\phi^\varepsilon(x_1)\partial_1^\varepsilon \partial_2^\varepsilon \partial_3^\varepsilon v|||^2 = \int \phi^{2\alpha} \partial_1^\varepsilon \partial_2^\varepsilon \partial_3^\varepsilon \partial_1^\varepsilon v \cdot \partial_0 \partial_1^\varepsilon \partial_2^\varepsilon \partial_3^\varepsilon v, \]

but

\[ \partial_0 \partial_1^\varepsilon \partial_2^\varepsilon \partial_3^\varepsilon v = -A_1 \partial_1 \partial_2^\varepsilon \partial_3^\varepsilon \partial_1^\varepsilon v - A_2 \partial_2 \partial_1^\varepsilon \partial_3^\varepsilon \partial_1^\varepsilon v + \varepsilon \Delta \partial_1^\varepsilon \partial_2^\varepsilon \partial_3^\varepsilon \partial_1^\varepsilon v + [A_1 \partial_1, \partial_0 \partial_1^\varepsilon \partial_2^\varepsilon \partial_3^\varepsilon \partial_1^\varepsilon v] + [A_2 \partial_2, \partial_0 \partial_2^\varepsilon \partial_3^\varepsilon \partial_1^\varepsilon v], \]

so we have to estimate five integrals, \( I_1, \ldots, I_5 \).

Let us start by \( I_1 \) :

\[ I_1 = - \int \phi^{2\alpha} \partial_1^\varepsilon \partial_2^\varepsilon \partial_3^\varepsilon \partial_1^\varepsilon v A_1 \partial_1^\varepsilon \partial_2^\varepsilon \partial_3^\varepsilon \partial_1^\varepsilon v = \int \phi^{2\alpha} \partial_1 \partial_2^\varepsilon \partial_3^\varepsilon \partial_1^\varepsilon v A_1 \partial_0 \partial_1^\varepsilon \partial_2^\varepsilon \partial_3^\varepsilon \partial_1^\varepsilon v + \int \partial_1^\varepsilon \partial_2^\varepsilon \partial_3^\varepsilon \partial_1^\varepsilon v A_1 \partial_0^\varepsilon \partial_2^\varepsilon \partial_3^\varepsilon \partial_1^\varepsilon v \]

**JOURNAL DE MATHEMATIQUES PURES ET APPLIQUEES**
but if \( \alpha \geq 1 \), \( \phi^{2\alpha} = 0 \) and if \( \alpha = 0 \), \( \partial_2^2 \partial_1^\alpha v = 0 \) on \( \partial \Omega \), so the boundary integral vanishes. As \( A_1 \) is symmetric, the first integral equals \(-I_1\). Thus

\[
2I_1 = \int \partial_2^\alpha \partial_2^2 \partial_1^\alpha v \partial_1 (\phi^{2\alpha} A_1) \partial_1^\alpha \partial_2^2 \partial_1^\alpha v.
\]

From (17), we deduce that

\[
\|\partial_1 A_1\| \leq C,
\]

so

\[
\|\phi^{2\alpha} \partial_1 A_1\| \leq C\phi^{2\alpha}.
\]

On the otherside, \( \|A_1\| \leq C x_1 \) and

\[
|x_1 \partial_1 \phi^{2\alpha}| \leq C\phi^{2\alpha},
\]

so

\[
\|\partial_1 (\phi^{2\alpha} A_1)\| \leq C\phi^{2\alpha}
\]

and

\[
|I_1| \leq C \int \phi^{2\alpha} \|\partial_1^\alpha \partial_2^2 \partial_1^\alpha v\|^2 \leq C \|v\|^2_{\alpha, \beta, \gamma}.
\]

Let us study \( I_2 \).

\[
I_2 = - \int \phi^{2\alpha} \partial_2^\alpha \partial_2^2 \partial_1^\alpha v A_2 \partial_2 \partial_1^\alpha \partial_2^2 \partial_1^\alpha v.
\]

Since \( A_2 \) is symmetric,

\[
I_2 = \frac{1}{2} \int \partial_1^\alpha \partial_2^2 \partial_1^\alpha v \partial_2 (\phi^{2\alpha} A_2) \partial_1^\alpha \partial_2^2 \partial_1^\alpha v,
\]

but \( \phi \) only depends on \( x_1 \), and \( \|\partial_2 A_2\| \leq C \), so

\[
|I_2| \leq C \|v\|^2_{\alpha, \beta, \gamma}.
\]

Let us turn to \( I_3 \).

\[
I_3 = \varepsilon \int \phi^{2\alpha} \partial_1^\alpha \Delta \partial_2^2 \partial_1^\alpha v = \varepsilon \int \phi^{2\alpha} (\partial_1^\alpha + \partial_2^\alpha \partial_2^2 \partial_1^\alpha v.
\]

But if \( \alpha = 0 \), we have:

\[
\int \partial_2^2 \partial_1^\alpha v \partial_2^2 \partial_1^\alpha v = - \int \|\partial_1^\alpha \partial_2^2 \partial_1^\alpha v\|^2 - \int_{\partial \Omega} \partial_2^2 \partial_1^\alpha v \partial_1^\alpha \partial_2^2 \partial_1^\alpha v\quad \text{and} \quad \partial_2^2 \partial_1^\alpha v = 0 \quad \text{on} \quad \partial \Omega,
\]

so the boundary integral vanishes.
If $\alpha \geq 1$,

$$
(37) \quad \epsilon \int \phi^{2\alpha} \partial_{\ell}^{2} \partial_{\ell}^{2} v \partial_{\ell}^{2} v = -\epsilon \int \phi^{2\alpha} \left| \partial_{\ell}^{2} \partial_{\ell}^{2} v \right|^2 
+ \epsilon \int \partial_{\ell} \phi^{2\alpha} \partial_{\ell}^{2} \partial_{\ell}^{2} v \partial_{\ell} \partial_{\ell}^{2} \partial_{\ell}^{2} v 
$$

since the boundary integral vanishes because $\phi = 0$ on $\partial \Omega$. But we have:

$$
(38) \quad |\partial_{\ell} \phi^{2\alpha}| \leq C \phi^{2\alpha-1}
$$

so

$$
(39) \quad \left| \epsilon \int \partial_{\ell} \phi^{2\alpha} \partial_{\ell}^{2} \partial_{\ell}^{2} v \partial_{\ell}^{2} v \partial_{\ell} \partial_{\ell}^{2} \partial_{\ell}^{2} v \right|
\leq \sqrt{\epsilon \int \phi^{2\alpha-2} \left| \partial_{\ell} \partial_{\ell}^{2} \partial_{\ell}^{2} v \right|^2 \left| \epsilon \int \phi^{2\alpha} \left| \partial_{\ell}^{2} \partial_{\ell}^{2} v \right|^2 \right|}
\leq C \epsilon \int \phi^{2\alpha-2} \left| \partial_{\ell}^{2} \partial_{\ell}^{2} \partial_{\ell}^{2} v \right|^2 + \frac{\epsilon}{2} \int \phi^{2\alpha} \left| \partial_{\ell}^{2} \partial_{\ell}^{2} \partial_{\ell}^{2} v \right|^2 .
$$

So adding the term coming from $\partial_{\ell}^{2}$,

$$
(40) \quad I_3 = \frac{\epsilon}{2} \int \phi^{2\alpha} \left| \partial_{\ell}^{2} \partial_{\ell}^{2} \partial_{\ell}^{2} v \right|^2 + \epsilon \int \phi^{2\alpha} \left| \partial_{\ell}^{2} \partial_{\ell}^{2} \partial_{\ell}^{2} \partial_{\ell}^{2} v \right|^2
\leq C \epsilon \int \phi^{2\alpha-2} \left| \partial_{\ell}^{2} \partial_{\ell}^{2} \partial_{\ell}^{2} v \right|^2
$$

(expression which is also valid for $\alpha = 0$, with a zero right hand side).

Let us study $I_4$.

$$
(41) \quad I_4 = \int \phi^{2\alpha} \partial_{\ell}^{2} \partial_{\ell}^{2} \partial_{\ell}^{2} \partial_{\ell}^{2} [A_1 \partial_{\ell}^{2} \partial_{\ell}^{2} \partial_{\ell}^{2} \partial_{\ell}^{2}] v.
$$

But $[A_1 \partial_{\ell}^{2} \partial_{\ell}^{2} \partial_{\ell}^{2} \partial_{\ell}^{2}] v$ is a sum of terms of the form

$$
(42) \quad \partial_{\ell}^{\alpha'} \partial_{\ell}^{\beta'} \partial_{\ell}^{\gamma'} A_1 \partial_{\ell}^{\alpha-\alpha'} \partial_{\ell}^{\beta-\beta'} \partial_{\ell}^{\gamma-\gamma'} v,
$$

where $\alpha' + \beta' + \gamma' \geq 1, 0 \leq \alpha' \leq \alpha, 0 \leq \beta' \leq \beta$ and $0 \leq \gamma' \leq \gamma$. So $I_4$ is a sum of terms of the form

$$
(43) \quad J = \int \phi^{2\alpha} \partial_{\ell}^{\alpha'} \partial_{\ell}^{\beta'} \partial_{\ell}^{\gamma'} v \partial_{\ell}^{\alpha'} \partial_{\ell}^{\beta'} \partial_{\ell}^{\gamma'} A_1 \partial_{\ell}^{\alpha-\alpha'} \partial_{\ell}^{\beta-\beta'} \partial_{\ell}^{\gamma-\gamma'} v.
$$

If $\alpha' = 0$, as $|\partial_{\ell}^{\beta'} \partial_{\ell}^{\gamma'} A_1| \leq C \phi$,

$$
(44) \quad |J| \leq C \int |\phi^{\alpha} \partial_{\ell}^{\alpha} \partial_{\ell}^{\beta} \partial_{\ell}^{\gamma} v| |\phi^{\alpha+1} \partial_{\ell}^{\alpha+1} \partial_{\ell}^{\beta} \partial_{\ell}^{\gamma} v|
\leq C \left| v \right|_{\alpha+|\beta|+\gamma}^2.
$$
But if \( \alpha' \geq 1 \), we obtain:

\[
\| \partial_1^{\alpha'} \partial_2^{\beta'} \partial_1^{\gamma'} A_1 \| \leq C \sqrt{\varepsilon}^{\alpha'-1} \| \theta(x/\sqrt{\varepsilon}) + C
\]

thus

\[
\phi^{2\alpha} \| \partial_1^{\alpha'} \partial_2^{\beta'} \partial_1^{\gamma'} A_1 \| \leq C \phi^{2\alpha-\alpha'+1}.
\]

So

\[
|J| \leq \int |\phi^{\alpha} \partial_1^{\alpha} \partial_2^{\beta} \partial_1^{\gamma} v| |\phi^{\alpha-\alpha'+1} \partial_1^{\alpha-\alpha'+1} \partial_2^{\beta-\beta'} \partial_1^{\gamma'-\gamma} v|
\]

\[
\leq C \|v\|_{H_{\alpha+|\beta|+\gamma}}^2
\]

and

\[
|I_1| \leq C \|v\|_{H_{\alpha+|\beta|+\gamma}}^2.
\]

It remains to study \( I_5 \)

\[
I_5 = \int \phi^{2\alpha} \partial_1^{\alpha} \partial_2^{\beta} \partial_1^{\gamma} [A_2 \partial_2, \partial_1^{\alpha} \partial_2^{\beta} \partial_1^{\gamma}] v.
\]

\( I_5 \) is a sum of terms of the form

\[
J = \int \phi^{2\alpha} \partial_1^{\alpha} \partial_2^{\beta} \partial_1^{\gamma} v \partial_1^{\alpha'} \partial_2^{\beta'} \partial_1^{\gamma'} A_2 \partial_1^{\alpha-\alpha'} \partial_2^{\beta-\beta'+1} \partial_1^{\gamma'-\gamma} v.
\]

But

\[
\phi^{\gamma} |\partial_1^{\alpha} \partial_2^{\beta} \partial_1^{\gamma} A_2| \leq C
\]

so

\[
|J| \leq C \int |\phi^{\alpha} \partial_1^{\alpha} \partial_2^{\beta} \partial_1^{\gamma} v| |\phi^{\alpha-\alpha'} \partial_1^{\alpha-\alpha'} \partial_2^{\beta-\beta'+1} \partial_1^{\gamma'-\gamma} v|
\]

\[
\leq C \|v\|_{H_{\alpha+|\beta|+\gamma}}^2.
\]

So we proved that

\[
\partial_1 \|v\|_{H_{\alpha+|\beta|+\gamma}}^2 + \frac{\varepsilon}{2} \int \phi^{3\alpha} \|\partial_1^{\alpha+1} \partial_2^{\beta} \partial_1^{\gamma} v\|^2 + \varepsilon \int \phi^{3\alpha} \|\partial_1^{2\alpha} \partial_2^{\beta+1} \partial_1^{\gamma} v\|^2
\]

\[
\leq C \|v\|_{H_{\alpha+|\beta|+\gamma}}^2 + C \varepsilon \int \phi^{2\alpha-2} \|\partial_1^{\alpha} \partial_2^{\beta} \partial_1^{\gamma} v\|^2,
\]

for all \( \alpha, \beta \) and \( \gamma \). Choosing \( C_0 \) in (22) larger than two times the largest of the constants \( C \) which appear and summing these inequalities, the \( C \varepsilon \) terms on the right are absorbed by the \( \varepsilon/2 \) term on the left, yielding

\[
\partial_1 \|v\|_{s}^2 \leq C_s \|v\|_{s}^2,
\]

which ends the proof.
1.2. Stability

Let us study the following non-linear equation

\begin{align}
\partial_t v + A_1(t, x, v)\partial_t v + A_2(t, x, v)\partial_x v - \varepsilon \Delta v &= 0 \quad \text{in } \Omega, \\
v(t) &= 0 \quad \text{on } \partial \Omega, \\
v(0) &= v_0^\varepsilon,
\end{align}

where $A_1$ and $A_2$ are smooth symmetric matrices. We will also assume

\begin{align}
A(t, x, v) &= \phi(x_1)\hat{A}(t, x, v),
\end{align}

where $\hat{A}$ is smooth. The following Theorem is a stability result:

**Theorem 1.2.** Let $N$ be an integer, large enough, and $T > 0$. Let us assume that $u^\varepsilon$ is a sequence of approximate solutions of (59, 60, 61), such that $B_1(t, x) = A_1(t, x, u^\varepsilon)$ and $B_2(t, x) = A_2(t, x, u^\varepsilon)$ satisfy (17, 18, 19) and (62), for $\alpha + |\beta| + \gamma \leq s$, with $s$ large enough on $0 \leq t \leq T$. Let us assume in addition that:

\begin{align}
&u^\varepsilon = 0 \quad \text{on } \partial \Omega, \\
&\sqrt{\varepsilon}^{-N} \|u^\varepsilon(0) - v_0^\varepsilon\|_{H^s} \leq C, \\
&\sqrt{\varepsilon}^{-N} \|\partial_t u^\varepsilon + A_1(t, x, u^\varepsilon)\partial_t u^\varepsilon + A_2(t, x, u^\varepsilon)\partial_x u^\varepsilon - \varepsilon \Delta u^\varepsilon\|_{H^s} \leq C
\end{align}

and

\begin{align}
\|\partial_1^\alpha \partial_2^\beta \partial_3^\gamma u^\varepsilon\|_{L^\infty} \leq C \varepsilon^{-\alpha}, \\
\|\phi^\alpha \partial_1^\alpha \partial_2^\beta \partial_3^\gamma u^\varepsilon\|_{L^\infty} \leq C
\end{align}

for $\alpha + |\beta| + \gamma \leq s$. Then there exists, for $\varepsilon$ small enough, a solution $v^\varepsilon$ of (59, 60, 61), on $0 \leq t \leq T$, such that

\begin{align}
\varepsilon^{-\sigma'} \|u^\varepsilon - v^\varepsilon\|_{H^s} \to 0, \quad \text{as } \varepsilon \to 0,
\end{align}

for some $s'$ and $\sigma'$ depending on $s$ and $N$ and going to $\infty$ as $s$ and $N$ go to $\infty$.

The construction of an approximate solution $u^\varepsilon$ will be treated in section 3.
Proof. - Let $w^\varepsilon = v^\varepsilon - u^\varepsilon$. We will estimate $||w^\varepsilon||_s$, for $s$ large enough (depending on $d$). This function solves (dropping the $\varepsilon$)

$$
\partial_t w + B_1 \partial_t w + B_2 \partial_2 w + \tilde{A}_1(w) \partial_1 w + \tilde{A}_2(w) \partial_2 w + \phi(x_1) A_\varepsilon(w + u) \partial_1 u + \tilde{A}_2(w + u) \partial_2 u - \varepsilon \Delta w = \sqrt{\varepsilon} \sigma^s,
$$

where

$$
\tilde{A}_1(w) = A_1(w + u) - A_1(u),
$$

$$
\tilde{A}_2(w) = A_2(w + u) - A_2(u),
$$

$$
A_\varepsilon(w) = A(w + u) - A(u)
$$

and

$$
\sigma^s = -\sqrt{\varepsilon} \left( \partial_t u^\varepsilon + A_1(t, x, u^\varepsilon) \partial_1 u^\varepsilon + A_2(t, x, u^\varepsilon) \partial_2 u^\varepsilon - \varepsilon \Delta u^\varepsilon \right).
$$

The terms $B_1 \partial_t w, B_2 \partial_2 w$ and $\varepsilon \Delta w$ can be estimated as in the linear case. Seven integrals appear:

$$
I_1 = - \int \phi^{2\alpha} \partial_t^\alpha \partial_2^2 \partial_1^2 w \tilde{A}_1(w) \partial_1 \partial_2^\alpha \partial_1^2 \partial_1^2 w
$$

$$
+ \int \phi^{2\alpha} \partial_t^\alpha \partial_2^\alpha \partial_1^2 \partial_1^2 w \tilde{A}_1(w) \partial_1^2 \partial_2 \partial_1^2 \partial_1^2 w
$$

$$
+ \int \phi^{2\alpha} \partial_t^\alpha \partial_2^\alpha \partial_1^2 \partial_1^2 w \tilde{A}_1(w) \partial_1^2 \partial_2 \partial_1^2 \partial_1^2 w.
$$

The last integral vanishes, since $\tilde{A}_1 = 0$ on $\partial \Omega$. So, as $\tilde{A}_1$ is symmetric,

$$
2I_1 = \int \phi^{2\alpha} \partial_t^\alpha \partial_2^2 \partial_1^2 w \partial_1 (\phi^{2\alpha} A_1(w)) \partial_1^2 \partial_2 \partial_1^2 \partial_1^2 w.
$$

But

$$
\partial_t [\tilde{A}_1(w)] = \partial_t A_1(t, x, w + u)(\partial_1 w + \partial_1 u) - \partial_t A_1(t, x, u) \partial_1 u + \partial_1 \tilde{A}_1(t, x, w)
$$

so

$$
|\partial_t [\tilde{A}_1(w)]| \leq C(|w|_{L^\infty}) \left( \frac{|w|_{L^\infty}}{\sqrt{\varepsilon}} + |\nabla w|_{L^\infty} \right)
$$

and $|\tilde{A}_1| \leq C(|w|_{L^\infty}) |w|$, so as

$$
|w| \leq C|\nabla w|_{L^\infty} x_1.
$$
Let us turn to \( I_2 \).

\[
I_2 = - \int \phi^{2\alpha} \partial_2^{\alpha} \partial_2^{\beta} w \ A_2(w) \ \partial_1^\alpha \partial_2^\beta \partial_i^\gamma w.
\]

Similarly,

\[
|I_2| \leq C(|w|_{L^\infty}) \left( |\nabla w|_{L^\infty} + \frac{|w|_{L^\infty}}{\sqrt{\varepsilon}} \right) |||w|||^2_{\alpha,\beta,\gamma}.
\]

Now we have to deal with the commutators \( I_3 \) and \( I_4 \).

\[
I_3 = \int \phi^{2\alpha} \partial_1^{\alpha} \partial_2^{\beta} \partial_2^{\gamma} w \ \left[ \hat{A}_1(w) \partial_1^\gamma + \partial_1^\alpha \partial_2^\beta \partial_2^\gamma \right] w
\]

is the sum of terms of the form:

\[
J = \int \phi^{2\alpha} \partial_1^{\alpha} \partial_2^{\beta} \partial_2^{\gamma} w \ \partial_1^\gamma + \partial_1^\alpha \partial_2^\beta \partial_2^\gamma \ [\hat{A}_1(w)] \partial_1^{\alpha-1} \partial_2^{\beta+1} \partial_2^{-\gamma-w},
\]

with \( \alpha + \beta + \gamma \geq 1 \). But \( \partial_1^\gamma + \partial_1^\alpha \partial_2^\beta \partial_2^\gamma \ [A_1(u + w)] \) is the sum of terms of the form:

\[
[\partial_1^{\alpha_1} \partial_2^{\beta_1} \partial_2^{\gamma_1} A_1](u + w) \partial_1^{\alpha_1} \partial_2^{\beta_2} \partial_2^{\gamma_2} (u + w) \partial_1^{\alpha_2} \partial_2^{\beta_3} \partial_2^{\gamma_3} (u + w),
\]

where \( \alpha + \alpha_1 + \ldots + \alpha_n = \alpha_1 + \beta_1 + \ldots + \beta_n = \beta_1 + \gamma_1 + \ldots + \gamma_n = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 \) and \( \gamma + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 \geq 0 \), and similarly for \( \partial_2^\beta \partial_2^\gamma \ [A_1(t, x, u)] \). So \( \partial_1^\gamma + \partial_1^\alpha \partial_2^\beta \partial_2^\gamma \ [A(u + w)] \) is bounded by a sum of terms of the form

\[
||[\partial_1^{\alpha_1} \partial_2^{\beta_1} \partial_2^{\gamma_1} A_1](u + w) - [\partial_1^{\alpha_1} \partial_2^{\beta_1} \partial_2^{\gamma_1} A_1](u)\]
\times [\partial_1^{\alpha_1} \partial_2^{\beta_1} \partial_2^{\gamma_1} (u + w) - \partial_1^{\alpha_1} \partial_2^{\beta_1} \partial_2^{\gamma_1} (u + w) - \partial_1^{\alpha_1} \partial_2^{\beta_1} \partial_2^{\gamma_1} (u + w) - \partial_1^{\alpha_1} \partial_2^{\beta_1} \partial_2^{\gamma_1} u].
\]

and

\[
||[\partial_1^{\alpha_1} \partial_2^{\beta_1} \partial_2^{\gamma_1} A_1](u)\]
\times [\partial_1^{\alpha_1} \partial_2^{\beta_1} \partial_2^{\gamma_1} (u + w) - \partial_1^{\alpha_1} \partial_2^{\beta_1} \partial_2^{\gamma_1} u].
\]

The first kind of term is controlled by

\[
C(|w|_{L^\infty}) |w|_{L^\infty} \left( \frac{1}{\sqrt{\varepsilon}} \partial_1^{\alpha_1} \partial_2^{\beta_1} \partial_2^{\gamma_1} w \right) \ldots \left( \frac{1}{\sqrt{\varepsilon}} \partial_1^{\alpha_1} \partial_2^{\beta_1} \partial_2^{\gamma_1} w \right)
\]
since

\[ |\partial_1^{\alpha_1} \partial_2^{\beta_1} \partial_1^{\gamma_1} u|_{L^\infty} \leq C \sqrt{\varepsilon} \]

and the second kind by

\[ C \sqrt{\varepsilon}^{-\alpha_1 \cdots -\alpha_n} \left( (1 + |\partial_1^{\alpha_1} \partial_2^{\beta_1} \partial_1^{\gamma_1} w|) \cdots (1 + |\partial_1^{\alpha_n} \partial_2^{\beta_n} \partial_1^{\gamma_n} w|) - 1 \right). \]

So we have to bound terms of the form:

\[ K = \int |\partial_1^{\alpha_1} \partial_2^{\beta_1} \partial_1^{\gamma_1} w| |\partial_1^{\alpha_2} \partial_2^{\beta_2} \partial_1^{\gamma_2} w| \cdots |\partial_1^{\alpha_n} \partial_2^{\beta_n} \partial_1^{\gamma_n} w| |\partial_1^{\alpha_1-1} \partial_2^{\beta_1-1} \partial_1^{\gamma_1-1} w|. \]

Let us make an important (and classical) remark. Assume that

\[ (91) \quad |w|_{L^\infty} \leq C |w|_{L^\infty}. \]

By recurrence, one can control \(|w|_{L^\infty}\) by \(C \varepsilon^{-s} \|w\|_{L^\infty}\), and thus \(|\partial_1^{\alpha_1} \partial_2^{\beta_1} \partial_1^{\gamma_1} w|_{L^\infty}\) by \(C \varepsilon^{-s} \|w\|_{L^\infty}\), for \(\alpha + |\beta| + \gamma < s/2\). If \(s > d\), now, with standard arguments (to be done at the end of the proof), we see that if we prove that

\[ \|w\|_{L^\infty} \leq C \varepsilon^{s+1}. \]

we are able to propagate assumption (91), which will be satisfied at \(t = 0\) (uniformly in \(\varepsilon\)).

Let us return to (90). As \(\alpha + \beta + \gamma \leq s\), we can not have more than two sums \(\alpha + \beta + \gamma\), \(\alpha_1 + \beta_1 + \gamma_1, \ldots, \alpha_n + \beta_n + \gamma_n\), \(\alpha - \alpha' + 1 + \beta - \beta' + \gamma - \gamma'\) larger than \(s/2\). So, we bound two factors by \(\|w\|_{L^\infty,s+|\beta|+\gamma}\), and the other (at most \(s\) of them since \(\alpha + |\beta| + \gamma \leq s\)), by \(C \varepsilon^{-s} \|w\|_{L^\infty, s}\). This leads to

\[ |K| \leq C(\varepsilon^{-s} \|w\|_{L^\infty,s})^s \|w\|_{L^\infty,s}^2 \]

and to

\[ |I_3| \leq C(\|w\|_{L^\infty,s}(1 + \varepsilon^{-s} |w|_{L^\infty,s} + \varepsilon^{-s} \|w\|_{L^\infty,s})(1 + \varepsilon^{-s} |w|_{L^\infty,s})^s \|w\|_{L^\infty,s}^2). \]

Let us turn to \(I_4\), sum of terms of the form

\[ \int_\phi^2 \partial_1^{\alpha_1} \partial_2^{\beta_1} \partial_1^{\gamma_1} w \partial_1^{\alpha_2} \partial_2^{\beta_2} \partial_1^{\gamma_2} [\hat{A}_2(w)] \partial_1^{\alpha_1-1} \partial_2^{\beta_1-1} \partial_1^{\gamma_1-1} w. \]

Similar calculations lead to a similar bound for \(I_4\).

Let us turn to \(I_5\) and \(I_6\).

\[ I_5 = \int_\phi^3 \partial_1^{\alpha_1} \partial_2^{\beta_1} \partial_1^{\gamma_1} w \partial_1^{\alpha_2} \partial_2^{\beta_2} \partial_1^{\gamma_2} \left( \phi(x_1) |\hat{A}(t, x, u + w) - \hat{A}(t, x, u)| \partial_1 u \right). \]

But we have:

\[ \phi^3 \partial_1^{\alpha_1} \partial_2^{\beta_1} \partial_1^{\gamma_1}(\phi(x_1) \partial_1 u) \leq C. \]
so, as for $I_3$,

$$|I_3| \leq C(|w|_{L^\infty})(1 + \varepsilon^{-2\beta}|||w|||_s)^{\alpha}||w||_s^2,$$ \hfill (98)

The case of $I_6$

$$I_6 = \int \phi^{2\alpha} \partial_1^\alpha \partial_2^\beta \partial_3^\gamma w \ \partial_1^\alpha \partial_2^\beta \partial_3^\gamma ([A_2(u + w) - A_2(u)]\partial_2 u)$$ \hfill (99)

is similar.

It remains to study

$$I_7 = \int \phi^{2\alpha} \partial_1^\alpha \partial_2^\beta \partial_3^\gamma w \ \partial_1^\alpha \partial_2^\beta \partial_3^\gamma \varepsilon^N \sigma.$$ \hfill (100)

We immediately have

$$|I_7| \leq \varepsilon^N ||w||_s |||\sigma||_s.$$ \hfill (101)

So we obtain:

$$\partial_t ||w||_s^2 \leq C(|w|_{L^\infty})(1 + \varepsilon^{-2\beta}|||w|||_s)^{\alpha + 1}||w||_s^2$$

$$+ C|||w|||_s^2 + \varepsilon^{N}|||w|||_s |||\sigma||_s.$$ \hfill (102)

Let

$$z - \varepsilon^{-N}w.$$ \hfill (103)

If $N > 2\beta$, we have

$$\partial_t ||z||_s^2 \leq C(\varepsilon^N |z|_{L^\infty})(1 + ||z||_s)^{\alpha + 1}||z||_s^2$$

$$+ C|||z|||_s^2 + ||w||_s |||\sigma||_s.$$ \hfill (104)

So we can control $z$, and thus $w$, on a time interval which is independent on $\varepsilon$. This ends the proof of Theorem 1.2 (we skip the existence part, which can be proved by using classical iterative schemes). \hfill \Box

2. Boundary layer equations

2.1. The linear case

Let us consider the following system, called boundary layer equations

$$\partial_t w + x_1 B_1 \partial_1 w + B_2 \partial_2 w - \partial_2^2 w = 0,$$ \hfill (105)

$$w = w(t, x_2, ..., x_d) \text{ on } \partial \Omega,$$ \hfill (106)
where $w_0$ and $w_1$ are given function such that $w_0 = w_1$ for $t = 0$ on $\partial \Omega$. We will assume that $B_1$ and $B_2$ are coefficients satisfying

$$\sup_{x_1 \geq 0} |x_1^\alpha \partial_1^\theta \partial_2^2 \partial_1^\gamma B_i| < +\infty,$$

for $i = 1, 2$.

This system has been studied in [7]. However, we will study it again, since it is the first step to handle the nonlinear case. Recall that we have to take care of $x_1 B_1 \partial_1$ which is a large term. So the Sobolev norms are not enough to control the solutions, and we have to study weighted norms. We will also control the time derivatives, which will be useful in the next section.

First, let us treat the boundary condition at $x_1 = 0$. Let $\zeta$ be a smooth, $C^\infty$, decreasing, positive function, with

$$\zeta(x_1) = 1 \text{ for } x_1 \leq 1,$$

$$0 \text{ for } x_1 \geq 2.$$

Let

$$\dot{w}(t, x_1, \ldots, x_d) = \zeta(x_1) w_1(t, x_2, \ldots, x_d)$$

and let

$$z = w - \dot{w}.$$

It satisfies

$$\partial_t z + x_1 B_1 \partial_1 z + B_2 \partial_2 z - \partial_1^2 z = \sigma,$$

where

$$\sigma = -\partial_t \dot{w} - x_1 B_1 \partial_1 \dot{w} - B_2 \partial_2 \dot{w} + \partial_1^2 \dot{w}$$

for

$$\zeta \partial_t \dot{w}_1 - x_1 B_1 \zeta' \dot{w}_1 - \zeta B_2 \partial_2 \dot{w}_1 + \zeta'' \dot{w}_1,$$

$$z = 0 \text{ on } \partial \Omega,$$

$$z(0) = w(0) - \dot{w},$$
(117) \[ z \to 0 \quad \text{as} \quad x_1 \to +\infty \]

(the last boundary condition must be understood in a Sobolev sense). Notice that if \( w_I \in H^s([0, T] \times \mathbb{R}^{d-1}) \), then \( \sigma \in H^{s-1}([0, T] \times \Omega) \) and

\[
\int x_1^{2n} |\partial_1^n \partial_1^m \partial_2^3 \sigma|^2 \leq C_{\alpha, n} \|w_I\|_{H^s}
\]

for \( 0 \leq t \leq T \), for all \( \alpha \) and all \( n \), and for \( |\beta| + \gamma \leq s - 1 \).

We introduce, as in [7],

(118) \[ \|z\|_{s, \alpha, \gamma}^2 = \int x_1^{2n+2\alpha} |\partial_1^n \partial_2^3 \partial_1^\gamma z|^2 \]

and

(119) \[ \|z\|_{s, \alpha, \gamma}^2 = \sum_{|\beta| + \gamma \leq s} C_0^{-n-|\beta|-\gamma} \|z\|_{s, \alpha, \gamma}^2 \]

where \( n \geq 0 \) is fixed.

**Theorem 2.1.** - Let \( n \geq 0 \). There exists a unique solution \( w \in H^{s-1}([0, T] \times \Omega) \) of system (105, 106, 107, 108) if \( w_I \in H^s([0, T] \times \mathbb{R}^{d-1}) \) and

(120) \[ \|w_0\|_{s-1} < +\infty. \]

Moreover, there exists two constants \( C_0 \) and \( C_s \), the latest depending only on the norm of \( w_I \), such that

(121) \[ \partial_t \|z\|_{s, \alpha, \gamma}^2 \leq C_s (\|w_I\|_{H^s})(\|z\|_{s-1}^2 + 1) \]

on \([0, T] \).

**Remark.** - In this Theorem, \( n \) is arbitrarily large, but \( T \) may depend on \( n \).

**Proof.** - As usual, we will estimate

(122) \[ \|z\|_{s, \alpha, \gamma}^2 = \sum_{\alpha + |\beta| + \gamma \leq s} C_0^{-n-|\beta|-\gamma} \|z\|_{s, \alpha, \gamma}^2 \]

which can be split in six integrals \( I_1, \ldots, I_6 \).

(123) \[ \partial_t \int x_1^{2n+2\alpha} |\partial_1^n \partial_2^3 \partial_1^\gamma z|^2 \]

since \( x_1 B_1 = 0 \) on \( \partial \Omega \), and \( B_1 \) is symmetric. As \( x_1 \partial_1 B_1 \) is bounded,

(124) \[ I_1 = - \int x_1^{2n+2\alpha} \partial_1^n \partial_2^3 \partial_1^\gamma z \left[ x_1 B_1 \right] \partial_1^n \partial_2^3 \partial_1^\gamma z \]

(125) \[ = \frac{1}{2} \int \partial_1^n \partial_2^3 \partial_1^\gamma z \partial_t \left[ x_1^{2n+2\alpha+1} B_1 \right] \partial_1^n \partial_2^3 \partial_1^\gamma z, \]

\[ |I_1| \leq C \|z\|_{s, \alpha, \gamma}^2. \]
Let us turn to $I_2$.

\begin{equation}
I_2 = - \int x_1^{2\alpha + 2n} \partial_1^{\gamma} \partial_2^{\beta} \partial_3^{\gamma} z \ [B_2] \ \partial_2 \partial_1^{\alpha} \partial_3^{\beta} \partial_3^{\gamma} z
\end{equation}

\begin{equation}
= \frac{1}{2} \int x_1^{2\alpha + 2\alpha} \partial_1^{\gamma} \partial_2^{\beta} \partial_3^{\gamma} z \partial_2 B_2 \partial_1^{\alpha} \partial_3^{\beta} \partial_3^{\gamma} z.
\end{equation}

so, as $|\partial_2 B_2| \leq C$,

\begin{equation}
|I_2| \leq C M^2 \beta \alpha \gamma.
\end{equation}

The Laplace term leads to:

\begin{equation}
I_3 = \int x_1^{2\alpha + 2n} \partial_1^{\gamma} \partial_2^{\beta} \partial_3^{\gamma} z \partial_2 \partial_1^{\alpha} \partial_3^{\beta} \partial_3^{\gamma} z.
\end{equation}

The case $\alpha = 0$ is obvious. If $\alpha \geq 1$.

\begin{equation}
I_3 = - \int x_1^{2\alpha + 2n} |\partial_1^{\gamma} \partial_2^{\beta} \partial_3^{\gamma} z|^2 - (2\alpha + 2n) \int x_1^{2\alpha + 2n} \partial_1^{\gamma} \partial_2^{\beta} \partial_3^{\gamma} z \partial_1 \partial_2^{\alpha} \partial_3^{\beta} \partial_3^{\gamma} z
\end{equation}

since $x_1^{2\alpha + 2n}$ is on $\partial \Omega$. But we have:

\begin{equation}
(2\alpha + 2n) \int x_1^{2\alpha + 2n} |\partial_1^{\gamma} \partial_2^{\beta} \partial_3^{\gamma} z |^2 \partial_1 \partial_2^{\alpha} \partial_3^{\beta} \partial_3^{\gamma} z
\end{equation}

\begin{equation}
\leq (2\alpha + 2n) \int x_1^{2\alpha + 2n} |\partial_1^{\gamma} \partial_2^{\beta} \partial_3^{\gamma} z |^2 \partial_1 \partial_2^{\alpha} \partial_3^{\beta} \partial_3^{\gamma} z
\end{equation}

\begin{equation}
\leq C \int x_1^{2\alpha + 2n} |\partial_1^{\gamma} \partial_2^{\beta} \partial_3^{\gamma} z |^2 + \frac{1}{2} \int x_1^{2\alpha + 2n} |\partial_1 \partial_2^{\alpha} \partial_3^{\beta} \partial_3^{\gamma} z |^2.
\end{equation}

The commutators lead to:

\begin{equation}
I_4 = \int x_1^{2\alpha + 2n} \partial_1^{\gamma} \partial_2^{\beta} \partial_3^{\gamma} z \ [x_1 B_1 \partial_1, \partial_1^{\gamma} \partial_2^{\beta} \partial_3^{\gamma} z]
\end{equation}

which is a sum of terms of the form

\begin{equation}
J = \int x_1^{2\alpha + 2n} \partial_1^{\gamma} \partial_2^{\beta} \partial_3^{\gamma} z \partial_1^{\alpha'} \partial_2^{\beta'} \partial_3^{\gamma'} [x_1 B_1] \ \partial_2 \partial_1^{\alpha + \beta + \gamma} z
\end{equation}

where $\alpha' + \beta' + \gamma' \geq 1$. If $\alpha' \geq 1$, then

\begin{equation}
|J| \leq \int x_1^{\alpha + n} |\partial_1^{\alpha + \beta + \gamma} z | [x_1^{\alpha + n} - 1] \partial_1^{\alpha'} \partial_2^{\beta'} \partial_3^{\gamma'} [x_1 B_1] |x_1^{\alpha + n - \alpha + \beta + \gamma} z | |x_1^{\alpha + n - \alpha + \beta + \gamma} z |}
\end{equation}
(137) \[ \leq C \left( \int x_1^{2\alpha + 2n} |\partial_1^{\alpha} \partial_2^{\beta} \partial_1^{\gamma} z|^2 \right)^{1/2} \left( \int x_1^{2\alpha + 2n - 2\alpha' + 2} |\partial_1^{\alpha' - \alpha} + 1 \partial_2^{\beta - \beta'} \partial_1^{\gamma'} z|^2 \right)^{1/2} \]

(138) \[ \leq C |||z|||_{\alpha + |\beta| + \gamma}^2 \]

since \[ |x_1^{\alpha'} \partial_1^{\alpha'} \partial_2^{\beta'} \partial_1^{\gamma'} z| [x_1 B_1] \leq C. \]

If \[ \alpha' = 0, \] we have:

(139) \[ |J| \leq \int x_1^{\alpha + n} |\partial_1^{\alpha} \partial_2^{\beta} \partial_1^{\gamma} z| |\partial_2^{\beta'} \partial_1^{\gamma'} B_1| x_1^{\alpha + n + 1} |\partial_1^{\alpha + 1} \partial_2^{\beta - \beta'} \partial_1^{\gamma'} z| \]

(140) \[ \leq C |||z|||_{\alpha + |\beta| + \gamma}^2 \]

Let us turn to

(141) \[ I_5 = \int x_1^{2\alpha + 2n} \partial_1^{\alpha} \partial_2^{\beta} \partial_1^{\gamma} z \left[ B_2 \partial_2, \partial_1^{\alpha} \partial_2^{\beta} \partial_1^{\gamma} \right] z \]

sum of terms of the form

(142) \[ J = \int x_1^{2\alpha + 2n} \partial_1^{\alpha} \partial_2^{\beta} \partial_1^{\gamma} z \partial_1^{\alpha'} \partial_2^{\beta'} \partial_1^{\gamma'} B_2 \partial_1^{\alpha - \alpha'} \partial_2^{\beta - \beta'} \partial_1^{\gamma - \gamma'} z, \]

(143) \[ |J| \leq \int \left| x_1^{\alpha + n} \partial_1^{\alpha} \partial_2^{\beta} \partial_1^{\gamma} z \right| \left| x_1^{\alpha + n} \partial_1^{\alpha'} \partial_2^{\beta'} \partial_1^{\gamma'} B_2 \right| \left| x_1^{\alpha + n + 1} \partial_1^{\alpha - \alpha'} \partial_2^{\beta - \beta'} \partial_1^{\gamma - \gamma'} z \right| \]

(144) \[ \leq C |||z|||_{\alpha + |\beta| + \gamma}^2 \]

since \[ |x_1^{\alpha'} \partial_1^{\alpha'} \partial_2^{\beta'} \partial_1^{\gamma'} B_2| \leq C. \]

It remains

(145) \[ I_6 = \int x_1^{2\alpha + 2n} \partial_1^{\alpha} \partial_2^{\beta} \partial_1^{\gamma} z \partial_1^{\alpha} \partial_2^{\beta} \partial_1^{\gamma} \sigma. \]

We have

(146) \[ |I_6| \leq \left( \int x_1^{2\alpha + 2n} |\partial_1^{\alpha} \partial_2^{\beta} \partial_1^{\gamma} z|^2 \right)^{1/2} \left( \int x_1^{2\alpha + 2n} |\partial_1^{\alpha} \partial_2^{\beta} \partial_1^{\gamma} \sigma|^2 \right)^{1/2} \]

(147) \[ \leq C (|||\sigma|||_{\alpha + |\beta| + \gamma}) |||z|||_{\alpha + |\beta| + \gamma}. \]

So we obtain:

(148) \[ \partial_t \int x_1^{2\alpha + 2n} |\partial_1^{\alpha} \partial_2^{\beta} \partial_1^{\gamma} z|^2 + \frac{1}{2} \int x_1^{2\alpha + 2n} |\partial_1^{\alpha + 1} \partial_2^{\beta} \partial_1^{\gamma} z|^2 \]

\[ \leq C (|||\sigma|||_{\alpha + |\beta| + \gamma})(|||z|||_{\alpha + |\beta| + \gamma} + 1) + C \int x_1^{2\alpha + 2n - 2} |\partial_1^{\alpha} \partial_2^{\beta} \partial_1^{\gamma} z|^2, \]

for some constant \( C \) which depends on \( |||\nu|||_{\alpha + |\beta| + \gamma + 1} \). As for (58), choosing \( C_0 \) large enough and summing, we can absorb the last right hand side term in the last left hand side term, which ends the proof of Theorem (2.1).
2.2. Nonlinear case

Let us consider:

\[ \partial_t w + x_1 B_1(t, x, w) \partial_{x_1} w + B_2(t, x, w) \partial_{x_2} w - \partial_{x_2}^2 w = 0, \]

\[ w = w(t, x_2, ...) \text{ on } \partial \Omega, \]

\[ w(t, +\infty, x_2, ...) = 0. \]

We introduce the function \( z(t, x_1, ..., x_d) \) as in the previous section. Let \( n \) be an integer, large enough.

**Theorem 2.2.** Assume that \( B_1 \) and \( B_2 \) are smooth functions of \( (t, x, w) \) and that

\[ \sup_{x_1 > 0, 0 \leq t \leq \pi} |x_1^{\alpha} \partial_1^{\alpha} \partial_2^{\beta} \partial_x^{\gamma} \partial_\nu B(t, x, w)| \leq C(|w|) \]

for every \( \alpha, \beta, \gamma, \delta \), with \( \alpha + |\beta| + \gamma + \delta \leq s \), with \( s \) large enough, where \( C \) is a non-decreasing function. Then, if \( \tilde{w} \in H^s([0, T] \times \mathbb{R}^d) \), there exists \( 0 < T' \leq T \) and a unique solution \( w \in \tilde{H}^{s-1}([0, T'] \times \Omega) \) of system (149,150,151).

Moreover, there exists a constant \( C_{s,n} \) depending only on the norm of \( w_1 \), such that

\[ \partial_t ||z||_{s-1}^2 \leq C_{s,n}(1 + ||z||_{s-1}^2)^2. \]

**Proof.** The proof is very similar to that of Theorems 1.1 and 1.2 and we will only sketch it. The main problem is to recover some control on Sobolev norms from \( ||z||_s \).

For that we will need the following Lemma (used in [7]).

**Lemma 2.3.** For every smooth function \( f \), if \( s' > (d+1)/2 \),

\[ ||f||_{L^\infty} \leq C \left( \sum_{|\beta| + |\gamma| \leq s'} \int_0^T ||\partial_2^{\beta} \partial_1^{\gamma} f||_{L^2}^2, \sum_{|\beta| + |\gamma| \leq s'} \int_0^T ||\partial_1 \partial_2^{\beta} \partial_1^{\gamma} f||_{L^2}^2 \right). \]

(To prove this inequality just write \( ||f||_{L^\infty} \leq \int |\mathcal{F} f| \) where \( \mathcal{F} \) is the Fourier transform and use a Cauchy-Schwartz inequality with weight \( (1 + |\eta|^{2s'} + \zeta^2(1 + |\eta|^{2s'}))^{1/2} \) where \( (\zeta, \eta) \) is the dual Fourier variable of \( (x_1, x_2) \).)

Let us notice, by integrating the energy estimate, that if we control \( ||z||_s \) on \([0, T] \), we control \( \int_0^T ||z||_{s'}^2 \) and \( \int_0^T ||\partial_1 z||_{s'}^2 \). Using the Lemma, we get bounds on \( ||x_1^{\alpha} \partial_1^{\beta} \partial_2^{\gamma} \partial_1^{\gamma} z||_{L^\infty} \) for \( \alpha + \beta + \gamma \leq s - d/2 - 1 \) on this time interval. To get other bounds, we use the equation to express \( \partial_2^2 z \). Applying \( x_1^{\alpha} \partial_1^{\beta} \partial_2^{\gamma} \partial_1^{\gamma} \) to the equation gives bounds on the \( L^2 \) norms of \( x_1^{\alpha} \partial_1^{\beta+2} \partial_2 \partial_1^{\gamma} z \) for \( \alpha + |\beta| + \gamma \leq s - d/2 - 2 \). By recurrence, one then controls, for \( s \) large enough, \( ||\partial_1 \partial_2^2 \partial_1^{\gamma} z||_{L^\infty} \) for \( \alpha + |\beta| + \gamma \leq s/2 \).
The difficult terms in the energy estimates are:

\begin{align}
I_4 &= \int x_1^{2a+2n} \partial_1^a \partial_2^b \partial_1^c z \left[ x_1 B_1 \partial_1, \partial_1^a \partial_2^b \partial_1^c \right] z \\
\text{and} \\
I_5 &= \int x_1^{2a+2n} \partial_1^a \partial_2^b \partial_1^c z \left[ B_2 \partial_2, \partial_1^a \partial_2^b \partial_1^c \right] z.
\end{align}

Let us study \( I_4 \). It is the sum of terms of the form

\begin{align}
J &= \int x_1^{2a+2n} \partial_1^a \partial_2^b \partial_1^c z \partial_1^{a'} \partial_2^{b'} \partial_1^{c'} \left[ x_1 B_1 \right] \partial_1^{-\alpha' + 1} \partial_2^{-\beta'} \partial_1^{-\gamma'} z.
\end{align}

If \( \alpha' \geq 1 \), then

\begin{align}
|J| &\leq \int x_1^{a+n} |\partial_1^{a} \partial_2^{b} \partial_1^{c} z| x_1^{a' - 1} \left[ \partial_1^{a'} \partial_2^{b'} \partial_1^{c'} x_1 B_1 \right] |x_1^{\alpha - \alpha' + 1} \partial_1^{\alpha' - \beta'} \partial_1^{-\gamma'} z|,
\end{align}

\begin{align}
x_1^{\alpha' - 1} \partial_1^{a'} \partial_2^{b'} \partial_1^{c'} (x_1 B_1) &= x_1^{\alpha'} \partial_1^{a'} \partial_2^{b'} \partial_1^{c'} B_1 + x_1^{\alpha' - 1} \partial_1^{-\alpha' - 1} \partial_2^b \partial_1^{c'} B_1.
\end{align}

The most difficult term is \( x_1^{a'} \partial_1^{a'} \partial_2^{b'} \partial_1^{c'} [B_1(w)] \), which is a sum of terms of the form:

\begin{align}
x_1^{a'} [\partial_1^{a'} \partial_2^{b'} \partial_1^{c'} \partial_1^{a'' + \beta'' + \gamma'' - \alpha'' - \beta'' - \gamma''} B_1](w) &\partial_1^{a''} \partial_2^{b''} \partial_1^{c''} z \partial_1^{a'' + \beta'' + \gamma'' - \alpha'' - \beta'' - \gamma''} B_1,
\end{align}

with \( \alpha'' + \alpha + \ldots + \alpha_n = \alpha', \beta'' + \beta_1 + \ldots + \beta_n = \beta', \) and \( \gamma'' + \gamma_1 + \ldots + \gamma_n = \gamma' \), so

\begin{align}
|x_1^{a'} \partial_1^{a'} \partial_2^b \partial_1^{c'} B_1| &\leq x_1^{a''} [\partial_1^{a''} \partial_2^b \partial_1^{c''} \partial_1^{a'' + \beta'' + \gamma'' - \alpha'' - \beta'' - \gamma''} B_1] \\
&\times |x_1^{a''} \partial_1^{a''} \partial_2^{b''} \partial_1^{c''} z| \ldots |x_1^{a_n} \partial_1^{a_n} \partial_2^{b_n} \partial_1^{c_n} z|.
\end{align}

But \( x_1^{a''} \partial_1^{a''} \partial_2^{b''} \partial_1^{c''} \partial_1^{a'' + \beta'' + \gamma'' - \alpha'' - \beta'' - \gamma''} B_1 \) is bounded by \( C(|w|_{L^\infty}) \) and therefore by \( C(||w||_{L^\infty}) \).

Now, as \( \alpha + \beta + \gamma \leq s \), at most two sums \( \alpha + \beta + \gamma, \alpha' + \beta' + \gamma', \alpha_1 + \beta_1 + \gamma_1, \ldots, \alpha_n + \beta_n + \gamma_n \), are greater or equal \( s/2 + 1 \). The terms for which the sum does not exceed \( s/2 \) can be bounded in \( L^\infty \) norm by \( ||z||_s \) by Sobolev injections in the following way.

For \( x_1 \leq 1 \), we use the control provided by \( |\partial_1 \partial_2 \partial_1^c z|_{L^\infty} \).

For \( x_1 > 1 \), let us study for instance, if \( \alpha_1 + \beta_1 + \gamma_1 \leq s/2 \),

\begin{align}
|x_1^{\alpha_1} \partial_1^{a_1} \partial_2^b \partial_1^{c_1} z| &\leq x_1^{\alpha_1} |\partial_1^{a_1} \partial_2^b \partial_1^{c_1} z|_{L^\infty([x_1, +\infty[ \times \mathbb{R}^{d-1})},
\end{align}

\begin{align}
&\leq C x_1^{\alpha_1} \sum_{\alpha + \beta + \gamma \leq s/2} |||\partial_1^{a_1} \partial_2^b \partial_1^{c_1} \partial_1^{a_1} \partial_2^b \partial_1^{c_1} z|||_{L^2([x_1, +\infty[ \times \mathbb{R}^{d-1})}}.
\end{align}
for $s$ large enough ($s > d$).

If $\alpha' = 0$, then

\begin{equation}
|J| \leq \int x^{n+1} \partial^\alpha \partial^\beta \partial^\gamma \partial^{\alpha'} B_2 |x^{n+1} \partial^\alpha \partial^\beta \partial^\gamma \partial^{\alpha'}|
\end{equation}

with $|\beta'| + \gamma' \geq 1$, and the proof is similar, so

\begin{equation}
|J_4| \leq C(||z||_s)
\end{equation}

and similarly for $J_5$.

\[\square\]

**Remark.** — In fact, we get

\begin{equation}
|\partial_1^\alpha \partial_2^\beta \partial_t^\gamma w|_{L^\infty} \leq \frac{C}{1 + x_1^n},
\end{equation}

for $n$, and $\alpha + |\beta| + \gamma$ bounded (the bound depending on $s$).

### 3. Construction of an approximate solution

In this section we will construct an approximate solution of the nonlinear problem (1,2,3), at any order in $\varepsilon$. We can then deduce the existence of an exact solution by using Theorem 1.2, which will end the proof of Theorem 0.1 of the introduction.

#### 3.1. Inner behaviour

We assume that we have a smooth solution (in $L^\infty([0,T],H^s)$ for $s$ large enough) $u^0$ of

\begin{equation}
\partial_t u^0 + A_1(t,x,u^0)\partial_t u^0 + A_2(t,x,u^0)\partial_\gamma u^0 = 0,
\end{equation}

\begin{equation}
u^0(0) = n_0^0.
\end{equation}

The existence of such a solution can be obtained by various means, and we will not investigate this problem. For instance we can use the results of [8].

In what follows we will assume that $s$ is large enough and we will not try to get a lower bound on $s$.

**Remark.** — We do not have to impose a boundary condition on (169,170) since $\partial \Omega$ is characteristic:

\begin{equation}
\partial_t u^0 + A_2(t,x,u^0)\partial_\gamma u^0 = 0, \quad \text{on } \partial \Omega.
\end{equation}

Notice that we do not have necessary $u^0 = 0$ on $\partial \Omega$.
3.2. Boundary layer behaviour

This section mainly consists in formal calculations. We are looking for an approximate solution \( v^0 \) of the form:

\[
u^0(t, x_1, \ldots, x_d) = u^0(t, x_1, \ldots, x_d) + u_b^0(t, \sqrt{\varepsilon}^{-1} x_1, \ldots, x_d),\]

where \( u_b^0 \) describes the boundary layer structure and

\[
u_b^0(t, 0, x_2, \ldots, x_d) = -u^0(t, 0, x_2, \ldots, x_d),\]

\[
u_b^0(t, +\infty, x_2, \ldots, x_d) = 0.\]

Let

\[
u_{0b}(t, x_2, \ldots, x_d) = u^0(t, 0, x_2, \ldots, x_d),\]

and \( X = x_1/\sqrt{\varepsilon} \). Formally,

\[egin{align*}
\partial_t v^0 + \partial_t u_b^0 + A_1(t, x, u^0 + u_b^0)\partial_t u^0 + A_1(t, x, u^0 + u_b^0)\partial_X \frac{u_b^0}{\sqrt{\varepsilon}} + A_2(t, x, u^0 + u_b^0)\partial_2 u_b^0 - \varepsilon \Delta u^0 - \partial^2_X u_b^0 = 0.
\end{align*}
\]

So, we have:

\[egin{align*}
\partial_t u^0 + \partial_t u_b^0 + A_1(t, x, u^0 + u_b^0)\partial_t u^0 + A_1(t, x, u^0 + u_b^0)\partial_X \frac{u_b^0}{\sqrt{\varepsilon}} + A_2(t, x, u^0 + u_b^0)\partial_2 u_b^0 + \varepsilon \Delta u^0 + \varepsilon \partial^2_X u_b^0.
\end{align*}
\]

As \( A_1 = \phi(x_1) A_1 \) by (6),

\[
A_1(t, x, u^0 + u_b^0)\partial_X \frac{u_b^0}{\sqrt{\varepsilon}} = \frac{\phi(x_1)}{\sqrt{\varepsilon}} A_1(t, x, u^0 + u_b^0)\partial_X u_b^0,
\]

so looking at scales of order \( \sqrt{\varepsilon} \) in \( x_1 \) by setting \( X = x_1/\sqrt{\varepsilon} \) and \( y = (x_2, \ldots, x_d) \), dropping higher order terms like \( \varepsilon \Delta u^0 + \varepsilon \partial^2_X u_b^0 \) and replacing \( u^0 \) by \( u_{0b} \) in (177) (since these two functions only differ by \( O(\sqrt{\varepsilon}) \) for finite \( X \)) gives:

\[egin{align*}
\partial_t u_b^0 + X \dot{A}_1(t, 0, y, u_{0b}^0 + u_b^0)\partial_X u_b^0 + A_2(t, 0, y, u_{0b}^0 + u_b^0)\partial_2 u_b^0 - \partial^2_X u_b^0 + [A_1(t, 0, y, u_{0b}^0) - A_3(t, 0, y, u_{0b}^0 + u_b^0)]\partial_t u^0(t, 0, y)
\end{align*}
\]

\[
+ [A_2(t, 0, y, u_{0b}^0) - A_3(t, 0, y, u_{0b}^0 + u_b^0)]\partial_2 u^0(t, 0, y).
\]
We complete this equation with the following boundary conditions

\[ u_b^0 = -u_{\partial\Omega}^0 \quad \text{on} \quad \partial\Omega, \]

\[ u_b^0 = 0 \quad \text{when} \quad X \to +\infty, \]

\[ u_b^0(t = 0) = u_{b,0}, \]

where \( u_{b,0}^0 \) is a given smooth function.

We then get the existence of a smooth solution \( u_b^0 \) of (179,180,181,182), on a time interval \([0, T']\) with \( 0 < T' \leq T \) by using Theorem 2.2 (the right hand side of (179) is easy to handle is the framework of 2.2). We can bound, in terms of the Sobolev norms of \( u^0 \), the following norms on \( u_b^0 \):

\[ |\partial_t^a \partial_x^\beta \partial_y^\gamma u_b^0|_{L^\infty} \leq \frac{C}{1 + X^\alpha}, \]

for all \( n \) and all \( \alpha, \beta \) and \( \gamma \), suitably bounded by \( s \) (see 168).

### 3.3. Error estimates : first order

The error made by putting \( v^0 \) in (1) equals

\[ E^0 = E_1 + E_2 + E_3 + E_4 \]

where

\[ E_1 = [A_1(t, x, u^0) - A_1(t, x, u^0 + u_b^0) - A_1(t, 0, y, u_{\partial\Omega}^0) + A_1(t, 0, y, u_{\partial\Omega}^0 + u_b^0)]u_b^0, \]

\[ E_2 = [A_2(t, x, u^0) - A_2(t, x, u^0 + u_b^0) - A_2(t, 0, y, u_{\partial\Omega}^0) + A_2(t, 0, y, u_{\partial\Omega}^0 + u_b^0)]u_b^0, \]

\[ E_3 = \varepsilon \Delta u^0 + \varepsilon \partial_b^2 u_b^0, \]

\[ E_4 = \frac{x_1}{\sqrt{\varepsilon}} A_1(t, 0, y, u_{\partial\Omega}^0 + u_b^0) \partial_x u_b^0 - \frac{\phi(x_1)}{\sqrt{\varepsilon}} A_1(t, x, u^0 + u_b^0) \partial_x u_b^0. \]

Let us begin by \( E_1 \).

\[ I_1 = A_1(t, x, u^0) - A_1(t, x, u^0 + u_b^0) - A_3(t, x, u_{\partial\Omega}^0) + A_1(t, x, u_{\partial\Omega}^0 + u_b^0) \]

\[ = - \int_0^1 \partial_x A_1(t, x, u^0 + tu_b^0) u_b^0 dt + \int_0^1 \partial_x A_1(t, x, u_{\partial\Omega}^0 + tu_b^0) u_b^0 dt \]

\[ = \int_0^1 \left( \partial_x A_1(t, x, u_{\partial\Omega}^0 + tu_b^0) - \partial_x A_1(t, x, u^0 + tu_b^0) \right) u_b^0 dt. \]
But
\[ |u^0 - u^0_{\partial \Omega}| \leq C x_1 \]
and
\[ |u_b^0| \leq \frac{C}{1 + (x_1/\sqrt{\varepsilon})^n}, \]
so we get
\[ |I_1| \leq \frac{C \sqrt{\varepsilon}}{1 + (x_1/\sqrt{\varepsilon})^{n-1}}. \]
We then bound the derivatives of \( I_1 \) in a similar way, to get
\[ |\partial^\alpha_x \partial^\beta_t I_1| \leq \frac{\sqrt{\varepsilon}}{1 + x^n} \]
for all \( n, \alpha \) and \( \beta \) (suitably bounded by \( s \)). Now
\[ I_2 = [A_1(t, x, u^0_{\partial \Omega}) - A_1(t, x, u^0_{\partial \Omega} + u_b^0) - A_1(t, 0, y, u^0_{\partial \Omega}) + A_1(t, 0, y, u^0_{\partial \Omega} + u_b^0)] \partial_t u_b^0 \]
can be treated in a similar way, and we get:
\[ |\partial^\alpha_x \partial^\beta_t \partial^\gamma_t E_1| \leq \frac{\sqrt{\varepsilon}}{1 + x^n}; \]
\( E_2 \) can be treated with the same method. \( E_3 \) is straightforward. It remains to study \( E_4 \), which is the sum of
\[ E_4^1 = \left( \frac{x_1}{\sqrt{\varepsilon}} - \frac{\phi(x_1)}{\sqrt{\varepsilon}} \right) \hat{A}_1(t, 0, y, u^0_{\partial \Omega} + u_b^0) \partial_X u_b^0 \]
which is easy to bound with the help of the fast decay of \( u_b^0 \) and of
\[ E_4^2 = \frac{\phi(x_1)}{\sqrt{\varepsilon}} \left( \hat{A}_1(t, 0, y, u^0_{\partial \Omega} + u_b^0) - \hat{A}_1(t, x, u^0 + u_b^0) \right) \partial_X u_b^0, \]
which can be bounded as \( E_1 \).
More precisely, we proved that \( E_4 = \sqrt{\varepsilon} E_4^0 + \sqrt{\varepsilon} E_4^1 \), where
\[ E_4^0(t, X, ..., x_d) = \sqrt{\varepsilon}^{-1} (E_1 + E_2 + E_4)(t, X, ..., x_d) + \varepsilon \partial^2_x u_b^0 \]
has a boundary layer behaviour, namely
\[ |\partial^\alpha_x \partial^\beta_t \partial^\gamma_t E_4^0| \leq \frac{C}{1 + x^n}, \]
for every \( n \), and
\[ E_i^0(t, x, ..., x_d) - \sqrt{\varepsilon} \Delta u^0 \]
is bounded in usual Sobolev spaces (here, in fact, \( E_i^0 \) is bounded by \( C \sqrt{\varepsilon} \)). Moreover, \( E_{i,0}^0 \) and \( E_{i,1}^0 \) have asymptotic expansions in powers of \( \sqrt{\varepsilon} \), the first terms being \( E_{i,0}^0 \) and \( E_{i,0}^0 \) (which in fact vanishes).
3.4. Higher order approximation

Let

\[ v^0(t, x_1, \ldots, x_d) = u^0(t, x_1, \ldots, x_d) + u_k^0(t, \sqrt{\varepsilon^{-1}} x_1, x_2, \ldots, x_d). \]

The approximate solution \( v^0 \) satisfies the boundary condition

\[ v^0 = 0 \quad \text{on } \partial \Omega \]

but not the complete equation (1). However, we have:

\[ \frac{\partial v^0}{\partial t} + A_1(t, x, v^0) \frac{\partial v^0}{\partial t} + A_2(t, x, v^0) \frac{\partial^2 v^0}{\partial t^2} - \varepsilon \Delta v^0 = \sqrt{\varepsilon} E_k^0 \left( t, \frac{x_1}{\sqrt{\varepsilon}}, \ldots, x_d \right) \]

We then look for a smooth solution \( u^1 \) of the following linear equation:

\[ \frac{\partial u^1}{\partial t} + A_1(t, x, u^0) \frac{\partial u^1}{\partial t} + A_2(t, x, u^0) \frac{\partial^2 u^1}{\partial t^2} + \nabla_x A_1(t, x, u^0) \cdot u^1 \frac{\partial u^0}{\partial t} + \nabla_x A_2(t, x, u^0) \cdot u^1 \frac{\partial^2 u^0}{\partial t^2} = E_{t,0}^0, \]

\[ u^1 = 0 \quad \text{on } \partial \Omega, \]

\[ u^1(t = 0) = u_0^1, \]

where \( u_0^1 \) is a given smooth function. The existence of the solution is given by energy estimates similar to Theorem 1.1.

It remains to construct the boundary layer term \( u_k^1 \), solution of the linearized system of (179) which we will not write down here, with source term \( E_{k,0}^0 \). The solution of such a system is given by Theorem 2.1. Now let

\[ v^1(t, x_1, \ldots, x_d) = u^0(t, x_1, \ldots, x_d) + \sqrt{\varepsilon} u^1(t, x_1, \ldots, x_d) + u_k^0(t, x_1/\sqrt{\varepsilon}, \ldots, x_d) + \sqrt{\varepsilon} u_k^1(t, x_1/\sqrt{\varepsilon}, \ldots, x_d) \]

which is an approximate solution of order 2 in \( \sqrt{\varepsilon} \). Similarly we construct an approximate solution \( v^n \) at any order \( n > 0 \). Theorem 1.2 then ends the proof of Theorem 0.1.

4. Remarks and extensions

In the previous sections, we have proven Theorem 0.1. It is of course possible to get better results. We will here give some hints.
4.1. Bounded domains

In the previous sections, we worked on the half-space $\mathbb{R}_+ \times \mathbb{R}^{d-1}$. However the same methods can be applied for more general domains $\Omega$ with smooth boundary, by using changes of coordinates. We refer to [14] for more details.

4.2. Hyperbolic systems

The energy method of the first section can be extended to hyperbolic systems of the form

\begin{align}
\partial_t u^\varepsilon + A^\varepsilon_1 \partial_1 u^\varepsilon + A^\varepsilon_2 \partial_2 u^\varepsilon &= 0, \\
u^\varepsilon(0) &= u_0^\varepsilon,
\end{align}

where $u_0^\varepsilon$, $A^\varepsilon_1$ and $A^\varepsilon_2$ depend on $\varepsilon$.

Remark. - On $\partial \Omega$, $A_1 = 0$, so

\begin{equation}
\partial_1 u = -A_2 \partial_2 u.
\end{equation}

In particular, if at $t = 0$, $u = 0$ on $\partial \Omega$, then for every positive time, $u = 0$ on the boundary.

PROPOSITION 4.1. - Under the assumptions of Theorem 1.1 (and in particular, under assumptions (17, 18, 19)), system (208,209) has a unique solution $u \in L^\infty([0,T], H^s(\Omega))$ for every $T > 0$. Furthermore there exists a constant $C_s$ depending on the various constants of (17, 18, 19), but independent on $\varepsilon$, such that

\begin{equation}
\partial_t \|u\|_s^2 \leq C_s \|u\|_s^2,
\end{equation}

where $\|\cdot\|_s$ is defined as in Theorem 1.1.

The proof of this Proposition is similar to that of Theorem 1.1, excepted that we do not have to take care of the Laplace term.

Remarks

• This describes a regime of the system (208,209), where the coefficients have very sharp variations and tend to be discontinuous, to have a shock type behaviour. However, this shock has special properties (assumptions (17,18,19)), so the solutions remain bounded within times of order one, as $\varepsilon$ goes to 0.

• A typical case of application is the transport of a scalar quantity by an incompressible viscous flow, with boundary layer type behaviour near $\partial \Omega$. The incompressibility then naturally leads to (17,18,19).

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES
REFERENCES


(Manuscript received June 1996.)

E. Grenier
Laboratoire d’Analyse Numérique,
CNRS - URA 189,
Université Paris 6, 4 place Jussieu,
75252 Paris Cedex 05.