Characterization of the Repetitive Commutative Semigroups

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Let $X^*$ (resp., $XX^* = X^* \setminus \{1\}$; resp., $X^+$) denote the free monoid (resp., semigroup; resp., commutative monoid) generated by the finite set $X$. Throughout, $\mathbb{Z}$ (resp., $\mathbb{N}$; resp., $\mathbb{P}$), the set of integers (resp., nonnegative integers; resp., positive integers), will be considered as an additive group (resp., semigroup). Recall [2, 3] the

**Definition.** A semigroup $S$ is repetitive iff for any $R \in \mathbb{P}$ and morphism $\varphi : XX^* \to S$ there exists $L = L_\varphi(k) \in \mathbb{P}$ such that any word $f \in X^*$ of length $|f| \geq L$ has at least one factorization $f = f'gf''$ ($f', f'' \in X^*$) where $g = g_1g_2 \cdots g_k$ ($g_1, g_2, \ldots, g_k \in XX^*$) with $\varphi g_1 = \varphi g_2 = \cdots = \varphi g_k$, or, as we shall say, where $g$ is a $k$-power mod $\varphi$.

In [1], Evdokimov constructed an infinite sequence on an alphabet of 25 letters without two consecutive segments containing the same number of each letter. This implies that $\mathbb{N}^{25}$ is not repetitive, letting $\mathbb{N}^n$ denote the direct product of $n$ copies of $\mathbb{N}$.

We prove here the

**Theorem.** A commutative semigroup $A$ is repetitive iff it contains no subsemigroup isomorphic to the free commutative semigroup on two generators.

According to [2], this statement can be considered as a generalization of van der Waerden's theorem on arithmetic progressions.

**Proof of the "only if" part.** We shall construct arbitrary long words on the alphabet $X = \{x, y\}$ containing as a factor no $5$-power mod $\alpha$ where $\alpha$ is the natural morphism: $X^* \to X^+$. If $w = x_1x_2 \cdots x_n$ ($x_i \in X$) is a word of $X^*$, introduce the notation $w[i, j]$ for the factor $x_ix_{i+1} \cdots x_j$ ($1 \leq i \leq j \leq n$).

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Inspired by [1], consider the morphism \( \varphi : X^* \to X^* \) given by

\[
\varphi x = xxxxy, \\
\varphi y = xyyyy.
\]

Let \( g \in XX^* \) and \( f = \varphi g \). Suppose that \( f \) contains a 5-power mod \( \alpha \), say \( f_1f_2f_3f_4f_5 \) with \( f_u = f[i_u + 1, i_{u+1}] \).

Let \( i_u = 5r_u + j_u \) \( 0 \leq j_u < 5 \) and let \( x_u \) be the \((r_u + 1)\)-th letter of \( g \) \((1 \leq u \leq 6)\). Write \( S_u = \varphi x_u \) and \( W_u = S_u[1, j_u] \) (with \( W_u = 1 \) if \( j_u = 0 \)).

Now consider the morphism \( \theta : X^* \to \mathbb{Z}/5\mathbb{Z} = \{0, 1, 2\} \) given by \( \theta x = 1 \), \( \theta y = 2 \), and write \( d_u = \theta W_u \). Remarking that \( \theta \varphi x = \theta \varphi y = 0 \), we have

\[
\begin{align*}
\sin u + v - \sin u - v & \equiv j_u - j_u \pmod{5}, \\
(1 \leq u \leq 4), \\
\end{align*}
\]

\[
\begin{align*}
\sin u + v - \sin u - v & \equiv d_u - d_u \pmod{3}, \\
(1 \leq u \leq 4).
\end{align*}
\]

If \( j_u + 1 \neq j_u \), we go from \( j_u \) to \( j_u + 1 \) by some power of the cycle \((0, 1, 2, 3, 4)\).

Simple verifications lead to impossibility. If all the \( j_u \)'s are equal to some \( j \), there are only four possibilities: \( j = 0 \), \( j = 1 \), \( j = 4 \), or all the \( x_u \)'s are equal.

In all the cases, for \( 1 \leq u \leq 6 \), let \( i_u' = i_u + t \) with \( t = -j \) if \( j \neq 4 \), or \( t = 1 \) if \( j = 4 \), and \( f_u' = f[i_u' + 1, i_{u+1}'] \). Thus \( f_1'f_2' \cdots f_5' \) is a 5-power mod \( \alpha \) and, as \( i_u' = 0 \pmod{5} \), we have \( f_u' = \varphi q_u \), where the \( g_u \)'s are consecutive factors of \( g \). But then, the factor \( g_1g_2 \cdots g_5 \) of \( g \) is a 5-power mod \( \alpha \) because \( \alpha m \) determines in a unique way \( \alpha m \) for any \( m \in X^* \).

Consequently, the iteration of \( h_{n+1} = \varphi h_n \), starting with \( h_1 = x \), gives the result.

**Proof of the “if” part.** It suffices to prove that any finitely generated commutative monoid such that there exists a nontrivial relation between any two generators is repetitive.\(^1\)

Consider \( B = \{b_1, b_2, \ldots, b_n\} \) and a congruence \( \Sigma \) in \( B^+ \) generated by the relations \( \Sigma_{ij} \) \((1 \leq i < j \leq n)\):

\[
r_{ij}b_i + r_{ji}b_j \sim s_{ij}b_i + s_{ji}b_j,
\]

where \((r_{ij}, r_{ji}) \neq (s_{ij}, s_{ji})\) are in \( \mathbb{N}^2 \).

We assume \( n \geq 2 \) minimal such that \( A = B^+ / \Sigma \) is not repetitive and consider two cases:

(1) Suppose that there exist an element of \( B \), say \( b_1, f \in B_1^+ \), where \( B_1 = B \setminus \{b_1\} \) and \( p, q \in \mathbb{N} \) with \( p < q \) such that

\[
f + pb_1 \sim f + qb_1 \pmod{\Sigma}.
\]

\(^1\) For simplicity, we let aside the algebraic study of this type of monoids.
Let $\Theta$ be the congruence in $B^+_{1+}$ generated by $\{\Sigma_{ij} \mid 2 \leq i < j \leq n\}$ and $\Delta$ be the congruence in $B^+$ generated by $\Theta$ together with the relation

$$f + pb_1 \sim f + qb_1.$$ 

Let $\sigma$ (resp., $\theta, \delta$) be the natural morphisms associated with $\Sigma$ (resp., $\Theta, \Delta$). We remark that two elements of $f + B^+ := \{f + g \mid g \in B^+\}$ are congruent modulo $\Delta$ iff they can be written $f + g + xb_1$ and $f + g' + x'b_1$, where $g, g' \in B^+_{1+}$, $x, x' \in N$ with $f + g \sim f + g'$ (mod $\Theta$) and $x \sim x'$ (mod $\Omega$), where $\Omega$ is the congruence in $N$ generated by $p \sim q$. Consequently, $E = \delta(f + B^+)$ is isomorphic to the direct product of $\theta(f + B^+_{1+})$ by the finite cyclic monoid $N/\Omega$. But $B^+_{1+}$ is repetitive by the minimality of $n$ and so is $\Theta(f + B^+_{1})$ as an homomorphic image of a subsemigroup. Thus $E$ is repetitive by the first theorem of [3] and so is $F = \sigma(f + B^+)$ because $\Delta \subseteq \Sigma$.

Now consider a morphism $\varphi : X^* \to A$. We shall prove that an infinite sequence $s = x_1x_2x_3 \cdots$ (where $x_i \in X$) contains as a factor a $k$-power mod $\varphi$ for arbitrary $k$. Choose a morphism $\psi : X^* \to B^+$ such that $\sigma\psi x = \varphi x$ (\forall x \in X). If there exists a proper subset $B_2$ of $B$ such that $\psi B_2 \subsetneq B^+$ for some arbitrarily long factors $g$ of $s$, we are finished because $\sigma B_2$ is repetitive in view of the minimality of $n$. Thus we may assume that $\psi g \in f + B^+$ for every sufficiently long factor $g$ of $s$. But then we may write $s = g_1g_2g_3 \cdots$ where $G = \{g_i \mid i \in \mathbb{P}\}$ is a finite subset of $XX^*$ such that $\varphi G \subsetneq F$. Then, we are finished because $F$ is repetitive.

(2) If we are not in the first case, let $d_{ij} = s_{ij} - r_{ij}$ for any $1 \leq i, j \leq n$ and $t = \sup\{|d_{ij}|\}$. We have $d_{ij} \neq 0$. Define a morphism $\theta : B^+ \to \mathbb{Z}$ by

$$\theta b_1 = t!,$$

$$\theta b_j = -\frac{d_{ij} t!}{d_{1j}}, \quad (j > 1).$$

It is easily seen that $\Sigma \subseteq \Theta$ where $\Theta$ is the congruence of $\theta$. Moreover, the classes of $A$ modulo the morphism $\theta^{-1} : A \to \mathbb{Z}$ have bounded cardinal numbers since any $a \in A$ can be written $a = \sum_{u=1}^{n} \lambda_u a_u$, where $\lambda_u \in \mathbb{N}$, and for all $u$'s, except possibly one, $\lambda_u < \sup\{x_{ij}, t_{ij}\}$. Since $\mathbb{Z}$ is repetitive [2] so is $A$ by [3].

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REFERENCES

