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On the Number of Mendelsohn and Transitive Triple Systems

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1. Introduction

In this note an ordered pair will *always* be an ordered pair (x, y) where $x \neq y$. A *cyclic triple* is a collection of three ordered pairs of the form $\{(a, b), (b, c), (c, a)\}$ and a *transitive triple* is a collection of three ordered pairs of the form $\{(a, b), (a, c), (b, c)\}$. A *Mendelsohn triple system* (MTS) (after N. S. Mendelsohn [5]) is a pair (S, T) where S is a set containing n elements and T is a collection of *cyclic triples* of elements of S such that every ordered pair of distinct elements of S belongs to *exactly one* cyclic triple of T. A *transitive triple system* (TTS) is a pair (S, T) where S is a set containing n elements and T is a collection of *transitive triples* of elements of S such that every ordered pair of distinct elements of S belongs to *exactly one* transitive triple of T. Regardless of whether (S, T) is a MTS or TTS, the number $|S| = n$ is called the *order* of the system and the number of triples is $|T| = n(n-1)/3$. It is well-known that the spectrum for MTSs is the set of all $n \equiv 0$ or 1 (mod 3), *except* $n = 6$ for which no such system exists, and the spectrum for TTSs is the set of all $n \equiv 0$ or 1 (mod 3) without exception.

Not a great deal is known about the *exact* number of nonisomorphic MTSs or TTSs of a given order even for small orders and, in fact, we will not be interested in this problem here. (The interested reader is referred to [1], [2], and [4].) The object of this note is to obtain the *best possible* asymptotic bounds on the number of nonisomorphic MTSs and TTSs. In particular, we show that for *large n* the number $N(n)$ of nonisomorphic MTSs or TTSs of order n is approximately $n^{n^2/3}$ in the sense that

$$\lim_{n\to\infty} (\ln N(n))/(n^2 \ln n) = \tfrac{1}{3}.$$

The similar problem for Steiner triple systems was solved by R. M. Wilson [6]. Wilson proved that if the van der Waerden Conjecture is true (a proof of the van der Waerden Conjecture had not been obtained at the time Wilson wrote his paper), then

$$\lim_{n\to\infty} (\ln S(n))/(n^2 \ln n) = \tfrac{1}{6}$$

where $S(n)$ is the number of nonisomorphic Steiner triple systems of order n.

2. Construction of MTSs and TTSs

In what follows (a, b, c) will denote either the cyclic triple $\{(a, b), (b, c), (c, a)\}$ *or* the transitive triple $\{(a, b), (a, c), (b, c)\}$; the context will make clear which one. Let $(Q, 0_1)$, $(Q, 0_2)$, and $(Q, 0_3)$ be any three *idempotent* quasigroups of order n; i.e., a quasigroup satisfying the identity $x^2 = x$ [equivalent to a Latin square of order n with cell (i, i) occupied by i for every $i = 1, 2, \ldots, n$].

$u = 3n$. Set $S = Q \times \{1, 2, 3\}$ and define a collection of (transitive) cyclic triples T of S as follows:

(1) $((x, 1), (x, 2), (x, 3))$ and $((x, 2), (x, 1), (x, 3))$ belong to T for every $x \in Q$; and

(2) if $x \neq y \in Q$, the six (transitive) cyclic triples
$((x, 1), (y, 1), (x\, 0_1\, y, 2))$, $((y, 1), (x, 1), (y\, 0_1\, x, 2))$,

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$((x, 2), (y, 2), (x \circ_2 y, 3)), ((y, 2), (x, 2), (y \circ_2 x, 3))$,
$((x, 3), (y, 3), (x \circ_3 y, 1))$, and
$((y, 3), (x, 3), (y \circ_3 x, 1))$ belong to T.

$u = 3n + 1$. Set $S = \{\infty\} \cup (Q \times \{1, 2, 3\})$ where ∞ is a symbol not in $Q \times \{1, 2, 3\}$ and define a collection of (transitive) cyclic triples T of S as follows:

(1*) $(\infty, (x, 1), (x, 2))$, $(\infty, (x, 2), (x, 3))$, $(\infty, (x, 3), (x, 1))$, and $((x, 1), (x, 3), (x, 2))$ belong to T for every $x \in Q$; and

(2*) if $x \neq y \in Q$, define six (transitive) cyclic triples precisely as in (2) above.

It is a routine matter to see, regardless of whether $u = 3n$ or $3n + 1$, *that* (S, T) is a (TTS) MTS. Since the three idempotent quasigroups $(Q, \circ_1)$, $(Q, \circ_2)$, and $(Q, \circ_3)$ can be chosen independently, if $I(n)$ denotes the *number* of idempotent quasigroups of order n, then the *number* of nonisomorphic (TTSs) MTSs of order $u = 3n$ or $3n + 1$ is *at least* $(I(n))^3/(u!)$. The main result will now follow if we can show that $\lim_{n\to\infty}(\ln I(n))/(n^2 \ln n) = 1$, which we now proceed to do. The argument is facilitated by using Latin square vernacular.

3. Idempotent Latin Squares

In what follows we will denote by $I(n)$ the number of idempotent Latin squares of order n, by $I_5(n)$ the number of idempotent Latin squares of order n containing a subsquare of order 5, and by $L(n)$ the number of Latin squares of order n.

The recent proof of the van der Waerden Conjecture gives at least $(n!)^{2n}/n^{n^2}$ distinct Latin squares of order n and since $n! > e^{-n} \cdot n^n$ it follows that

$$\lim_{n\to\infty}\left(\frac{\ln L(n)}{n^2 \ln n}\right) = 1.$$

We will use this limit to show that $\lim_{n\to\infty}(\ln I(n))/(n^2 \ln n) = 1$. The argument is based on the following singular direct product.

Let Q_1, Q_2, and Q_3 be any three Latin squares, based on $\{1, 2, 3, \ldots, n\}$, each containing the subsquare P, based on $\{1, 2, \ldots, p\}$, in the upper left hand corner. Further, let A_1, A_2, A_3, A_4, A_5, and A_6 be any six Latin squares based on $\{p+1, p+2, \ldots, n\}$ and define a Latin square of order $3(n-p)+p$ based on $\{1, 2, 3, \ldots, p\} \cup (\{p+1, p+2, \ldots, n\} \times \{1, 2, 3\})$ as follows:

P	R_1	R_2	R_3
C_1	B_1	$A_1 \times \{3\}$	$A_2 \times \{2\}$
C_2	$A_3 \times \{3\}$	B_2	$A_4 \times \{1\}$
C_3	$A_5 \times \{2\}$	$A_6 \times \{1\}$	B_3

where $A_i \times \{k\}$ is the Latin square obtained from A_i by replacing each symbol x with the ordered pair (x, k); and the subsquare consisting of P, R_i, C_i, and B_i is the Latin square Q_i with each of the symbols $x \in \{p+1, p+2, \ldots, n\}$ replaced by (x, i).

It is immediately apparent that the Latin square constructed above is idempotent if and only if *each* of the Latin squares Q_1, Q_2, and Q_3 is idempotent. It is also apparent that the Latin squares Q_1, Q_2, Q_3; A_1, A_2, A_3, A_4, A_5, and A_6 can be chosen independently.

Theorem 1. $\lim_{n\to\infty}(\ln I(n))/(n^2 \ln n) = 1$.

PROOF. To begin with, if each of Q_1, Q_2, and Q_3 is an idempotent Latin square of order m each containing the same subsquare of order 5 in the upper left hand corner, then Q_1, Q_2, and Q_3 can be used in the above construction to construct idempotent Latin squares of orders $3m$, $3(m-1)+1$, and $3(m-5)+5$ *each* of which contains a subsquare of order 5 in the upper left hand corner. The following inequalities are immediate.

$$I_5(3n) \geq I_5(n)^3 \cdot L(n)^6$$
$$I_5(3n+1) \geq I_5(n+1)^3 \cdot L(n)^6$$
$$I_5(3n+5) \geq I_5(n+5)^3 \cdot L(n)^6.$$

If $\lim_{n\to\infty}(\ln I_5(n))/(n^2 \ln n) \geq \alpha$, then the above inequalities imply $\lim_{n\to\infty}(\ln I_5(n))/(n^2 \ln n) \geq \frac{1}{3} \cdot \alpha + \frac{2}{3}$. It is a trivial fact that an idempotent Latin square of order m containing a subsquare of order 5 in the upper left hand corner exists for every $m \geq 11$. As a consequence, the above inequalities give $\lim_{n\to\infty}(\ln I_5(n))/(n^2 \ln n) \geq \frac{2}{3}$ and so iteration improves this bound from $\frac{2}{3}$ to $\frac{8}{9}$ to $\frac{26}{27}$ to *eventually* 1. Since $I(n)$ is *greater than* $I_5(n)$ we have $\lim_{n\to\infty}(\ln I(n))/(n^2 \ln n) \geq 1$, which, of course, implies equality.

4. Asymptotic Bounds

Let, as before, $N(n)$ denote the number of nonisomorphic (TTSs) MTSs of order n. The asymptotic lower bound of $N(n)$ follows almost immediately from the previously established lower bound of the number of idemptotent quasigroups.

LEMMA 2. $\lim_{n\to\infty}(\ln N(n))/(n^2 \ln n) \geq \frac{1}{3}$.

PROOF. As already noted, the number of nonisomorphic (TTSs) MTSs of order $u = 3n$ or $3n+1$ is at least $(I(n))^3/(u!)$. The arguments are virtually identical and so we handle the case $u = 3n$ only. So,

$$N(3n) \geq I(n)^3/(3n)! \quad \text{gives}$$

$$\lim_{n\to\infty} \frac{\ln N(3n)}{(3n)^2 \ln(3n)} \geq \lim_{n\to\infty} \frac{3(\ln I(n)) - 3n(\ln(3n))}{(3n)^2 \ln(3n)}$$
$$\geq \lim_{n\to\infty}\left(\frac{3(\ln I(n))}{9n^2(\ln n + \ln 3)} - \frac{1}{3n}\right)$$
$$\geq \frac{1}{3}\left(\lim_{n\to\infty} \frac{\ln I(n)}{n^2 \ln n}\right) - \left(\lim_{n\to\infty} \frac{1}{3n}\right) \geq \frac{1}{3}.$$

To show that this limit is best possible, we must also show that $\lim_{n\to\infty}(\ln N(n))/(n^2 \ln n) \leq \frac{1}{3}$. This is not difficult.

LEMMA 3. $\lim_{n\to\infty}(\ln N(n))/(n^2 \ln n) \leq \frac{1}{3}$.

PROOF. To begin with a *partial* (TTS) MTS is a pair (S, t) where t is a collection of (transitive) cyclic triples such that every ordered pair of distinct elements of S belongs to *at most* one (transitive) cyclic triple of t. The following proof is for MTSs, the proof for TTSs being a trivial modification. So, let $S = \{1, 2, 3, \ldots, n\}$ and construct a sequence of partial MTSs $T_1, T_2, \ldots, T_{n(n-1)/3}$ as follows:

(1) T_1 consists of the $n-2$ partial MTSs $(S, \{(1, 2, 3)\}), (S, \{(1, 2, 4)\}), \ldots, (S, \{(1, 2, n)\})$; and

(2) for *each* partial MTS (S, t) belonging to T_k choose *any* ordered pair (a, b) not

belonging to a triple of t and place $(S, t \cup \{(a, b, x)\})$ in T_{k+1} for each $x \notin \{a, b\}$ for which $(S, t \cup \{(a, b, x\})$ is a partial MTS. If for each $x \notin \{a, b\}$ each $(S, t \cup \{(a, b, x)\})$ fails to be a partial MTS do *not* place (S, t) in T_{k+1}.

Claim. $T_{n(n-1)/3}$ is precisely the set of *all* MTSs defined on S. This is quite easy to see. Let (S, t) by *any* MTS and let (S, t^*) be any partial subsystem of (S, t) such that $|t^*| = k$ *and* $(S, t^*) \in T_k$. Now by construction, for some ordered pair (a, b) not in a triple of t^*, $(S, t^* \cup \{(a, b, x)\})$ belongs to T_{k+1} whenever $(S, t^* \cup \{(a, b, x)\})$ is a partial MTS. Since $t^* \subseteq t$, if $(a, b, c) \in t$, then not only does $(S, t^* \cup \{(a, b, c)\}) \in T_{k+1}$, but $t^* \cup \{(a, b, c)\} \subseteq t$ as well. Hence, the existence of a partial subsystem of (S, t) consisting of k triples belong to T_k implies the existence of a partial subsystem of (S, t) consisting of $k+1$ triples belonging to T_{k+1}. Therefore, starting with the cyclic triple of (S, t) of the form $(1, 2, x)$ guarantees that (S, t) belongs to $T_{n(n-1)/3}$. Trivially $|T_{n(n-1)/3}| \leq (n-2)^{n(n-1)/3} \leq n^{n(n-1)/3} \leq n^{n^2/3}$ from which the statement of the lemma follows. If we modify the above construction by defining T_1 to be the $3(n-2)$ transitive triples of the form $(1, 2, x)$, $(1, x, 2)$, and $(x, 1, 2)$ for all $x = 3, 4, \ldots, n$ then an analogous argument shows that $T_{n(n-1)/3}$ consists of all TTSs defined on S. In this case, $|T_{n(n-1)/3}| \leq (3(n-2))^{n(n-1)/3} \leq (3n)^{n^2/3}$ and the statement of the lemma follows.

Now combining Lemmas 2 and 3 gives the following theorem which is, of course, best possible.

Theorem 4. *If $M(n)$ and $T(n)$ denote the number of non-isomorphic MTSs and TTSs (respectively) of order n, then*

$$\lim_{n\to\infty} (\ln M(n))/(n^2 \ln n) = \lim_{n\to\infty} (\ln T(n))/(n^2 \ln n) = \tfrac{1}{3}.$$

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