Computers and Mathematics with Applications 60 (2010) 1310–1321

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Convergence and stability of the split-step θ-method for stochastic differential equations☆

Xiaohua Ding *, Qiang Ma, Lei Zhang

Department of Mathematics, Harbin Institute of Technology at Weihai, Weihai 264209, PR China

ARTICLE INFO

Article history:
Received 1 May 2009
Received in revised form 7 June 2010
Accepted 7 June 2010

Keywords:
Stochastic differential equations
Split-step θ-method
Mean-square convergence
Mean-square stability

ABSTRACT

In this paper, we construct a new split-step method for solving stochastic differential equations, namely the split-step θ-method. Under Lipschitz and linear growth conditions, we establish a mean-square convergence theory of split-step θ-approximate solutions. Moreover, the mean-square stability of the method for a linear test equation with real parameters is considered and the real mean-square stability region is plotted. Finally, numerical results are presented to demonstrate the efficiency of the split-step θ-method.

1. Introduction

Stochastic differential equations (SDEs) have been used to model the phenomena arising in many branches of science and industry such as biology, economics, medicine, engineering and finance (see, e.g., [1–4]). So it is valuable to investigate the properties of the solutions of SDEs. For the main theoretical results on SDEs, we refer to [3,4].

As explicit solutions of SDEs can rarely be obtained, the construction of numerical methods for solving SDEs has become an active research area of computational mathematics. For example, Sickenberger [5] analyzed the mean-square convergence of stochastic multi-step methods with variable step size. Wang [6] discussed three-stage stochastic Runge–Kutta methods for solving SDEs. The convergence in probability of the approximate solution to the exact solution was proved in [7]. There are other types of convergence for stochastic numerical methods. Details of these and other concepts on numerical solutions of SDEs can be found in [8,9].

Moreover, the stability of numerical methods for solving SDEs is essential to avoid a possible explosion of numerical solutions. Saito et al. [10] proposed the concept of mean-square stability (MS-stability) of the numerical method for solving scalar SDEs. Higham [11,12] plotted the real MS-stability regions of stochastic θ and semi-implicit Milstein methods for a linear test equation. We can also find other results on the MS-stability of numerical methods (see [6,13–16] and the references therein).

Higham et al. [17] introduced the split-step backward Euler (SSBE) method for solving nonlinear autonomous SDEs. Under the one-sided Lipschitz condition, the authors obtained strong convergence of the SSBE method with order $p = 1/2$. In this paper, we consider the split-step θ-method (SSθ method) for solving nonlinear non-autonomous SDEs. The SSθ method is equivalent to the SSBE method if $\theta = 1$.

This paper is organized as follows. In Section 2, we begin with some preliminary results which are essential for introduction and analysis of the SSθ method. In Section 3, we expound that SSθ approximate solutions are bounded in

☆ This work is supported by the National Natural Science Foundation of China (No. 10671047) and the foundation of HITC (200713).
* Corresponding author. Tel.: +86 631 5672718; fax: +86 631 5672718.
E-mail addresses: mathdxh@126.com, mathdxh@hitwh.edu.cn (X. Ding).

0898-1221/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2010.06.011

the mean-square sense. After that, we analyse the mean-square convergence of continuous-time SSθ approximations. In Section 4, we consider the numerical stability for a linear test equation with real parameters based on some elementary inequalities. Finally, numerical results are given to illustrate the performance of the SSθ method.

2. The split-step θ-method

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\in[0,T]}$. The filtration $\{\mathscr{F}_t\}_{t\in[0,T]}$ is increasing and right continuous, and $\mathscr{F}_0$ contains all P-null sets. Let B_t be a standard one-dimensional Brownian motion defined on $(\Omega, \mathscr{F}, P)$. Let x_0 be an $\mathscr{F}_0$-measurable one-dimensional random variable such that $\mathbb{E}|x_0|^2 < +\infty$. Let $f, g : \mathbb{R} \times [0, T] \mapsto \mathbb{R}$ both be Borel measurable. $\mathscr{L}^p([0, T], \mathbb{R})$ $(p = 1, 2)$ denotes the family of all $\mathbb{R}$-valued measurable $\{\mathscr{F}_t\}$-adapted stochastic processes $f = \{f(t)\}_{t\in[0,T]}$ such that $\int_0^T |f(t)|^p \mathrm{d}t < +\infty$. $\mathscr{M}^2([0, T], \mathbb{R})$ denotes the family of all stochastic processes $f \in \mathscr{L}^2([0, T], \mathbb{R})$ such that $\mathbb{E}\int_0^T |f(t)|^2 \mathrm{d}t < +\infty$.

We consider a one-dimensional stochastic differential equation (SDE) of Itô type,

$$\begin{cases} \mathrm{d}x(t) = f(x(t), t)\mathrm{d}t + g(x(t), t)\mathrm{d}B_t, & t \in [0, T], \\ x(0) = x_0, \end{cases} \tag{2.1}$$

where $0 < T < +\infty$. Assume that f and g satisfy the Lipschitz and linear growth conditions. That is, there exists a $K_1 > 0$ such that

$$|f(x, t) - f(y, t)|^2 \vee |g(x, t) - g(y, t)|^2 \le K_1|x - y|^2 \tag{2.2}$$

for all $x, y \in \mathbb{R}$ and $t \in [0, T]$; and there is, moreover, a $K_2 > 0$ such that

$$|f(x, t)|^2 \vee |g(x, t)|^2 \le K_2(1 + |x|^2) \tag{2.3}$$

for all $(x, t) \in \mathbb{R} \times [0, T]$. The existence and uniqueness of the solution to Eq. (2.1) can be guaranteed by (2.2) and (2.3) (see Theorem 3.1 in Chapter 2 of [3]).

Given a step size $h > 0$, the split-step θ-method (SSθ method) applied to (2.1) computes the approximation $y_k \approx x(t_k)$, where $t_k = kh$, by setting $y_0 = x_0$ and forming

$$y_k^* = y_k + h[(1 - \theta)f(y_k, t_k) + \theta f(y_k^*, t_k)], \tag{2.4a}$$

$$y_{k+1} = y_k^* + g(y_k^*, t_k)\Delta B_k, \tag{2.4b}$$

where $\theta \in [0, 1]$ is a fixed parameter and each $\Delta B_k = B_{t_{k+1}} - B_{t_k}$ is an independent $N(0, h)$-distributed Gaussian random variable.

The choice $\theta = 1$ gives the SSBE method [17]. If $\theta = 0$, the SSθ method is an explicit method. If $0 < \theta \le 1$, (2.4a) is an implicit equation in y_k^* that must be solved in order to obtain the intermediate approximation y_k^*. Having obtained y_k^*, substituting it into (2.4b) produces the next approximation y_{k+1}. Similarly to Lemma 3.4 in [17], using the fixed point theorem, we can prove the following lemma.

Lemma 2.1. *Assume that $f : \mathbb{R} \times [0, T] \mapsto \mathbb{R}$ satisfies (2.2), and let $0 < \theta \le 1$, $0 < h < 1/(\sqrt{K_1}\theta)$. Then, for given $a, b \in \mathbb{R}$, the implicit equation*

$$x = a + h\theta f(x, b)$$

has a unique solution x.

When $t \in [t_k, t_{k+1})$, the above lemma ensures the existence of y_k^* verifying (2.4a), and allows us to define

$$y(t) = y_k + (1 - \theta)(t - t_k)f(y_k, t_k) + \theta(t - t_k)f(y_k^*, t_k) + g(y_k^*, t_k)(B_t - B_{t_k}). \tag{2.5}$$

It is convenient to use a continuous-time approximation. We define two step functions:

$$Z_1(t) = \sum_{k=0}^{N-1} y_k \chi_{[t_k,t_{k+1})}(t) + y_N \chi_{\{t=T\}}(t), \tag{2.6}$$

$$Z_2(t) = \sum_{k=0}^{N-1} y_k^* \chi_{[t_k,t_{k+1})}(t) + y_N^* \chi_{\{t=T\}}(t), \tag{2.7}$$

where $\chi_F(t)$ is the characteristic function of a set F; that is,

$$\chi_F(t) = \begin{cases} 0, & t \notin F, \\ 1, & t \in F. \end{cases} \tag{2.8}$$

Then we obtain

$$y(t) = y_0 + \int_0^t (1-\theta) f(Z_1(s), \tilde{s}) \mathrm{d}s + \int_0^t \theta f(Z_2(s), \tilde{s}) \mathrm{d}s + \int_0^t g(Z_2(s), \tilde{s}) \mathrm{d}B_s, \tag{2.9}$$

with initial value $y(0) = x_0$, $\tilde{s} = [s/h]h$, where $[a]$ is the integer part of a (that is, the largest integer not larger than a). It is straightforward to check that $Z_1(t_k) = y_k = y(t_k)$; that is, $y(t)$ and $Z_1(t)$ coincide with the discrete solutions at the gridpoints. We refer to $y(t)$ as a continuous-time extension of the discrete approximation $\{y_k\}$.

3. Mean-square convergence of the SSθ method

In this section, we prove the mean-square convergence of the SSθ method. The proof is relatively long. Therefore, we divide it into four steps for readability. The following lemma gives the relationship between $\mathbb{E}|y_k|^2$ and $\mathbb{E}|y_k^*|^2$.

Lemma 3.1. *Suppose that $f : \mathbb{R} \times [0, T] \mapsto \mathbb{R}$ satisfies (2.3) and let $0 < \theta \le 1$, $h < min\{1, 1/(4\theta K_2)\}$. Then there exist two positive constants $A = 4(1 + K_2)$ and $B = 4K_2$ such that*

$$\mathbb{E}|y_k^*|^2 \le A\mathbb{E}|y_k|^2 + B,$$

where y_k and y_k^* ($k = 0, 1, 2, \ldots, N$) are produced by (2.4a) and (2.4b).

Proof. Squaring both sides of (2.4a), we find

$$\begin{aligned} |y_k^*|^2 = {} & |y_k|^2 + (1-\theta)^2 h^2 |f(y_k, t_k)|^2 + \theta^2 h^2 |f(y_k^*, t_k)|^2 + 2\theta h y_k f(y_k^*, t_k) \\ & + 2(1-\theta) h y_k f(y_k, t_k) + 2\theta(1-\theta) h^2 f(y_k, t_k) f(y_k^*, t_k). \end{aligned} \tag{3.1}$$

Using the elementary inequality $2ab \le a^2 + b^2$, we obtain

$$\begin{aligned} |y_k^*|^2 \le {} & |y_k|^2 + (1-\theta)^2 h^2 |f(y_k, t_k)|^2 + \theta^2 h^2 |f(y_k^*, t_k)|^2 + (1-\theta) h |y_k|^2 \\ & + (1-\theta) h |f(y_k, t_k)|^2 + \theta h |y_k|^2 + \theta h |f(y_k^*, t_k)|^2 + \theta(1-\theta) h^2 |f(y_k, t_k)|^2 + \theta(1-\theta) h^2 |f(y_k^*, t_k)|^2. \end{aligned} \tag{3.2}$$

Due to $h < 1$ and (2.3), we get

$$\begin{aligned} |y_k^*|^2 \le {} & |y_k|^2 + [(1-\theta)^2 h^2 + (1-\theta) h + \theta(1-\theta) h^2] K_2 (1 + |y_k|^2) + h|y_k|^2 \\ & + [\theta^2 h^2 + \theta h + \theta(1-\theta) h^2] K_2 (1 + |y_k^*|^2) \\ \le {} & |y_k|^2 + 2(1-\theta) K_2 h |y_k|^2 + h|y_k|^2 + 2\theta K_2 h |y_k^*|^2 + K_2(h^2 + h). \end{aligned} \tag{3.3}$$

We can obtain the following estimate derived from (3.3):

$$\mathbb{E}|y_k^*|^2 \le (1 + 2(1-\theta) K_2 h + h) \mathbb{E}|y_k|^2 + 2\theta K_2 h \mathbb{E}|y_k^*|^2 + K_2(h^2 + h). \tag{3.4}$$

Since $2\theta K_2 h < 1/2$, we can derive that

$$\mathbb{E}|y_k^*|^2 \le \frac{1 + 2(1-\theta) K_2 h + h}{1 - 2\theta K_2 h} \mathbb{E}|y_k|^2 + \frac{K_2(h^2 + h)}{1 - 2\theta K_2 h} \le A\mathbb{E}|y_k|^2 + B, \tag{3.5}$$

where $A = 4(1 + K_2)$, $B = 4K_2$. The proof is complete. □

The second lemma shows that the numerical solutions y_k ($k = 1, 2, \ldots, N$) produced by the SSθ method are bounded in the mean-square sense.

Lemma 3.2. *Let y_k and y_k^* ($k = 0, 1, 2, \ldots, N$) be produced by (2.4a) and (2.4b). Assume that $f, g : \mathbb{R} \times [0, T] \mapsto \mathbb{R}$ satisfy (2.2), (2.3), and let $0 < \theta \le 1$, $h < min\{1, 1/(4\theta K_2), 1/(\sqrt{K_1}\theta)\}$. Then we have*

$$\mathbb{E}|y_k|^2 \le F, \qquad \mathbb{E}|y_k^*|^2 \le G,$$

where

$$\begin{aligned} F &= \frac{3(B+1)K_2}{1 + 2K_2 + 3AK_2} (\mathrm{e}^{(1+2K_2+3AK_2)T} - 1) + \mathbb{E}|x_0|^2 \mathrm{e}^{(1+2K_2+3AK_2)T}, \\ G &= AF + B. \end{aligned}$$

Proof. Lemma 2.1 allows us to express the SSθ method (2.4a) and (2.4b) in the form

$$y_{k+1} = y_k + (1-\theta)hf(y_k, t_k) + \theta hf(y_k^*, t_k) + g(y_k^*, t_k)\Delta B_k. \tag{3.6}$$

Squaring both sides of (3.6), it follows that

$$\begin{aligned} |y_{k+1}|^2 = {} & |y_k|^2 + (1-\theta)^2h^2|f(y_k,t_k)|^2 + \theta^2h^2|f(y_k^*,t_k)|^2 + |g(y_k^*,t_k)\Delta B_k|^2 \\ & + 2(1-\theta)hy_kf(y_k,t_k) + 2\theta hy_kf(y_k^*,t_k) + 2y_kg(y_k^*,t_k)\Delta B_k + 2\theta(1-\theta)h^2f(y_k,t_k)f(y_k^*,t_k) \\ & + 2(1-\theta)hf(y_k,t_k)g(y_k^*,t_k)\Delta B_k + 2\theta hf(y_k^*,t_k)g(y_k^*,t_k)\Delta B_k. \end{aligned} \tag{3.7}$$

Applying the elementary inequality $2ab \le a^2 + b^2$, we get that

$$\begin{aligned} |y_{k+1}|^2 \le {} & |y_k|^2 + (1-\theta)^2h^2|f(y_k,t_k)|^2 + \theta^2h^2|f(y_k^*,t_k)|^2 + |g(y_k^*,t_k)\Delta B_k|^2 + (1-\theta)h|y_k|^2 \\ & + \theta(1-\theta)h^2|f(y_k^*,t_k)|^2 + \theta h|y_k|^2 + \theta h|f(y_k^*,t_k)|^2 + 2y_kg(y_k^*,t_k)\Delta B_k + \theta(1-\theta)h^2|f(y_k,t_k)|^2 \\ & + (1-\theta)h|f(y_k,t_k)|^2 + 2(1-\theta)hf(y_k,t_k)g(y_k^*,t_k)\Delta B_k + 2\theta hf(y_k^*,t_k)g(y_k^*,t_k)\Delta B_k \\ = {} & |y_k|^2 + (1-\theta)(h^2+h)|f(y_k,t_k)|^2 + h|y_k|^2 + \theta(h^2+h)|f(y_k^*,t_k)|^2 \\ & + |g(y_k^*,t_k)\Delta B_k|^2 + 2y_kg(y_k^*,t_k)\Delta B_k + 2\theta hf(y_k^*,t_k)g(y_k^*,t_k)\Delta B_k + 2(1-\theta)hf(y_k,t_k)g(y_k^*,t_k)\Delta B_k. \end{aligned}$$

Next, note that, by $h < 1$, (2.3) and the above inequality,

$$\begin{aligned} |y_{k+1}|^2 \le {} & |y_k|^2 + (1-\theta)(h^2+h)K_2(1+|y_k|^2) + \theta(h^2+h)K_2(1+|y_k^*|^2) + h|y_k|^2 + K_2(1+|y_k^*|^2)|\Delta B_k|^2 \\ & + 2y_kg(y_k^*,t_k)\Delta B_k + 2(1-\theta)hf(y_k,t_k)g(y_k^*,t_k)\Delta B_k + 2\theta hf(y_k^*,t_k)g(y_k^*,t_k)\Delta B_k. \end{aligned} \tag{3.8}$$

Taking mathematical expectation on both sides of (3.8), noting that $\mathbb{E}(\Delta B_k) = 0$ and $\mathbb{E}|\Delta B_k|^2 = h$, from Lemma 3.1, we deduce that

$$\begin{aligned} \mathbb{E}|y_{k+1}|^2 & \le \mathbb{E}|y_k|^2 + (2K_2+1)h\mathbb{E}|y_k|^2 + 3K_2h\mathbb{E}|y_k^*|^2 + 3K_2h \\ & \le (1+Ch)\mathbb{E}|y_k|^2 + Dh, \end{aligned} \tag{3.9}$$

where $C = 1 + 2K_2 + 3AK_2$ and $D = 3(B+1)K_2$.

In view of the Gronwall lemma (see Theorem 1.1.12 in Chapter 1 of [18]), we see that

$$\begin{aligned} \mathbb{E}|y_k|^2 & \le \frac{Dh}{1-(1+Ch)}(1-(1+Ch)^k) + \mathbb{E}|y_0|^2(1+Ch)^k \\ & \le \frac{D}{C}\left(\left(1+\frac{CT}{N}\right)^N - 1\right) + \mathbb{E}|y_0|^2\left(1+\frac{CT}{N}\right)^N \\ & \le \frac{D}{C}(e^{CT}-1) + \mathbb{E}|y_0|^2e^{CT}. \end{aligned} \tag{3.10}$$

Since $x_0 = y_0$, we have $\mathbb{E}|y_k|^2 \le F$, where $F = D(e^{CT}-1)/C + \mathbb{E}|x_0|^2e^{CT}$. Thus, Lemma 3.1 implies that $\mathbb{E}|y_k^*|^2 \le G$, where $G = AF + B$. The proof is complete. □

Now, we show that the continuous-time approximation $y(t)$ in (2.5) remains close to the two step functions $Z_1(t)$ and $Z_2(t)$ in the mean-square sense.

Lemma 3.3. *Suppose that $f, g : \mathbb{R} \times [0,T] \mapsto \mathbb{R}$ satisfy (2.2), (2.3), and let $0 < \theta \le 1$, $h < \min\{1, 1/(4\theta K_2), 1/(\sqrt{K_1}\theta)\}$. Then there exist two positive constants $H = (3F + 6G + 9)K_2$ and $I = (10F + 16G + 26)K_2$ such that*

$$\mathbb{E}|y(t) - Z_1(t)|^2 \le Hh, \qquad \mathbb{E}|y(t) - Z_2(t)|^2 \le Ih,$$

where $y(t)$, $Z_1(t)$, $Z_2(t)$ are defined by (2.5), (2.6), (2.7), respectively.

Proof. For $t \in [0,T]$, there exists a nonnegative integer k such that $t \in [kh, (k+1)h)$. By virtue of (2.5) and (2.6), we have

$$y(t) - Z_1(t) = (1-\theta)(t-t_k)f(y_k,t_k) + \theta(t-t_k)f(y_k^*,t_k) + g(y_k^*,t_k)(B_t - B_{t_k}). \tag{3.11}$$

Using $(a+b+c)^2 \le 3a^2 + 3b^2 + 3c^2$, (2.3) and Lemma 3.2, we obtain

$$\begin{aligned} \mathbb{E}|y(t) - Z_1(t)|^2 & \le 3(1-\theta)^2(t-t_k)^2K_2(1+\mathbb{E}|y_k|^2) + 3\theta^2(t-t_k)^2K_2(1+\mathbb{E}|y_k^*|^2) + 3K_2(1+\mathbb{E}|y_k^*|^2)\mathbb{E}|(B_t - B_{t_k})|^2 \\ & \le Hh, \end{aligned} \tag{3.12}$$

where $H = (3F + 6G + 9)K_2$.

By (2.4a), we know that

$$Z_1(t) - Z_2(t) = y_k - y_k^* = -(1-\theta)hf(y_k,t_k) - \theta hf(y_k^*,t_k). \tag{3.13}$$

Similarly to the proof of (3.12), we can show that

$$\mathbb{E}|Z_1(t) - Z_2(t)|^2 \leq (2F + 2G + 4)K_2 h. \tag{3.14}$$

Since $(a+b)^2 \leq 2a^2 + 2b^2$, we find

$$\mathbb{E}|y(t) - Z_2(t)|^2 \leq 2\mathbb{E}|y(t) - Z_1(t)|^2 + 2\mathbb{E}|Z_1(t) - Z_2(t)|^2 \leq Ih, \tag{3.15}$$

where $I = (10F + 16G + 26)K_2$. The proof is complete. □

Next, we use the above lemmas to prove a strong convergent result.

Theorem 3.1. *Let $x(t)$ be the exact solution of Eq. (2.1) and $y(t)$ be defined by (2.9). Suppose that $f, g : \mathbb{R} \times [0, T] \mapsto \mathbb{R}$ satisfy (2.2), (2.3), and let $0 < \theta \leq 1$, $h < \min\{1, 1/(4\theta K_2), 1/(\sqrt{K_1}\theta)\}$. Assume that there exists a positive constant K_3 such that*

$$|f(x,s) - f(x,t)|^2 \vee |g(x,s) - g(x,t)|^2 \leq K_3(1 + |x|^2)|s - t| \tag{3.16}$$

for all $s, t \in [0, T]$, $x \in \mathbb{R}$. Then there exists a positive constant M such that

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} |x(t) - y(t)|^2\Big) \leq Mh.$$

Proof. By virtue of (2.1) and (2.9), we can obtain for $t \in [0, T]$

$$\begin{aligned} x(t) - y(t) = {} & (1-\theta)\int_0^t (f(x(s),s) - f(Z_1(s),s))\mathrm{d}s + (1-\theta)\int_0^t (f(Z_1(s),s) - f(Z_1(s),\tilde{s}))\mathrm{d}s \\ & + \theta\int_0^t (f(x(s),s) - f(Z_2(s),s))\mathrm{d}s + \theta\int_0^t (f(Z_2(s),s) - f(Z_2(s),\tilde{s}))\mathrm{d}s \\ & + \int_0^t (g(x(s),s) - g(Z_2(s),s))\mathrm{d}B_s + \int_0^t (g(Z_2(s),s) - g(Z_2(s),\tilde{s}))\mathrm{d}B_s. \end{aligned} \tag{3.17}$$

In view of the elementary inequality $(a+b+c+d+e+f)^2 \leq 6a^2 + 6b^2 + 6c^2 + 6d^2 + 6e^2 + 6f^2$, we see that

$$\begin{aligned} |x(t) - y(t)|^2 \leq {} & 6(1-\theta)^2\left|\int_0^t (f(x(s),s) - f(Z_1(s),s))\mathrm{d}s\right|^2 + 6(1-\theta)^2\left|\int_0^t (f(Z_1(s),s) - f(Z_1(s),\tilde{s}))\mathrm{d}s\right|^2 \\ & + 6\theta^2\left|\int_0^t (f(x(s),s) - f(Z_2(s),s))\mathrm{d}s\right|^2 + 6\theta^2\left|\int_0^t (f(Z_2(s),s) - f(Z_2(s),\tilde{s}))\mathrm{d}s\right|^2 \\ & + 6\left|\int_0^t (g(x(s),s) - g(Z_2(s),s))\mathrm{d}B_s\right|^2 + 6\left|\int_0^t (g(Z_2(s),s) - g(Z_2(s),\tilde{s}))\mathrm{d}B_s\right|^2. \end{aligned}$$

Next, note that, by the Hölder inequality (see Theorem 3.5 in Chapter 3 of [19]),

$$\begin{aligned} |x(t) - y(t)|^2 \leq {} & 6T(1-\theta)^2\int_0^t |f(x(s),s) - f(Z_1(s),s)|^2\mathrm{d}s + 6T(1-\theta)^2\int_0^t |f(Z_1(s),s) - f(Z_1(s),\tilde{s})|^2\mathrm{d}s \\ & + 6T\theta^2\int_0^t |f(x(s),s) - f(Z_2(s),s)|^2\mathrm{d}s + 6T\theta^2\int_0^t |f(Z_2(s),s) - f(Z_2(s),\tilde{s})|^2\mathrm{d}s \\ & + 6\left|\int_0^t (g(x(s),s) - g(Z_2(s),s))\mathrm{d}B_s\right|^2 + 6\left|\int_0^t (g(Z_2(s),s) - g(Z_2(s),\tilde{s}))\mathrm{d}B_s\right|^2. \end{aligned}$$

Using (2.2), (3.16), $(a+b)^2 \leq 2a^2 + 2b^2$ and the above inequality, we observe that

$$\begin{aligned} |x(t) - y(t)|^2 \leq {} & 24TK_1\int_0^t |x(s) - y(s)|^2\mathrm{d}s + 12TK_1\int_0^t |y(s) - Z_1(s)|^2\mathrm{d}s + 6TK_3h\int_0^t |Z_1(s)|^2\mathrm{d}s \\ & + 6TK_3h\int_0^t |Z_2(s)|^2\mathrm{d}s + 12T^2K_3h + 12TK_1\int_0^t |y(s) - Z_2(s)|^2\mathrm{d}s \\ & + 6\left|\int_0^t (g(x(s),s) - g(Z_2(s),s))\mathrm{d}B_s\right|^2 + 6\left|\int_0^t (g(Z_2(s),s) - g(Z_2(s),\tilde{s}))\mathrm{d}B_s\right|^2. \\ \leq {} & 24TK_1\int_0^t \Big(\sup_{0\leq r\leq s} |x(r) - y(r)|^2\Big)\mathrm{d}s + 12TK_1\int_0^t |y(s) - Z_1(s)|^2\mathrm{d}s + 6TK_3h\int_0^t |Z_1(s)|^2\mathrm{d}s \end{aligned}$$

$$+6TK_3h\int_0^t |Z_2(s)|^2\mathrm{d}s + 12T^2K_3h + 12TK_1\int_0^t |y(s)-Z_2(s)|^2\mathrm{d}s$$
$$+6\sup_{0\le r\le t}\left|\int_0^r (g(x(s),s)-g(Z_2(s),s))\mathrm{d}B_s\right|^2 + 6\sup_{0\le r\le t}\left|\int_0^r (g(Z_2(s),s)-g(Z_2(s),\tilde{s}))\mathrm{d}B_s\right|^2.$$

Then it follows that

$$\sup_{0\le s\le t}|x(s)-y(s)|^2 \le 24TK_1\int_0^t\Big(\sup_{0\le r\le s}|x(r)-y(r)|^2\Big)\mathrm{d}s + 12TK_1\int_0^t |y(s)-Z_1(s)|^2\mathrm{d}s$$
$$+6TK_3h\int_0^t |Z_1(s)|^2\mathrm{d}s + 6TK_3h\int_0^t |Z_2(s)|^2\mathrm{d}s + 12T^2K_3h + 12TK_1\int_0^t |y(s)-Z_2(s)|^2\mathrm{d}s$$
$$+6\sup_{0\le r\le t}\left|\int_0^r (g(x(s),s)-g(Z_2(s),s))\mathrm{d}B_s\right|^2 + 6\sup_{0\le r\le t}\left|\int_0^r (g(Z_2(s),s)-g(Z_2(s),\tilde{s}))\mathrm{d}B_s\right|^2. \quad (3.18)$$

Taking mathematical expectation on both sides of (3.18), we know that

$$\mathbb{E}\Big(\sup_{0\le s\le t}|x(s)-y(s)|^2\Big) \le 24TK_1\int_0^t \mathbb{E}\Big(\sup_{0\le r\le s}|x(r)-y(r)|^2\Big)\mathrm{d}s + 12TK_1\int_0^t \mathbb{E}|y(s)-Z_1(s)|^2\mathrm{d}s$$
$$+6TK_3h\int_0^t \mathbb{E}|Z_1(s)|^2\mathrm{d}s + 6TK_3h\int_0^t \mathbb{E}|Z_2(s)|^2\mathrm{d}s + 12T^2K_3h + 12TK_1\int_0^t \mathbb{E}|y(s)-Z_2(s)|^2\mathrm{d}s$$
$$+6\mathbb{E}\Big(\sup_{0\le r\le t}|\int_0^r (g(x(s),s)-g(Z_2(s),s))\mathrm{d}B_s|^2\Big) + 6\mathbb{E}\Big(\sup_{0\le r\le t}\left|\int_0^r (g(Z_2(s),s)-g(Z_2(s),\tilde{s}))\mathrm{d}B_s\right|^2\Big). \quad (3.19)$$

In view of the Burkholder–Davis–Gundy inequality (see Theorem 7.3 in Chapter 1 of [3]), we can show that

$$6\mathbb{E}\left(\sup_{0\le r\le t}\left|\int_0^r (g(x(s),s)-g(Z_2(s),s))\mathrm{d}B_s\right|^2\right) \le 24\mathbb{E}\left(\int_0^t |g(x(s),s)-g(Z_2(s),s)|^2\mathrm{d}s\right)$$
$$\le 24K_1\mathbb{E}\left(\int_0^t |x(s)-Z_2(s)|^2\mathrm{d}s\right)$$
$$\le 48K_1\int_0^t \mathbb{E}\Big(\sup_{0\le r\le s}|x(r)-y(r)|^2\Big)\mathrm{d}s + 48K_1\int_0^t \mathbb{E}|y(s)-Z_2(s)|^2\mathrm{d}s \quad (3.20)$$

and

$$6\mathbb{E}\left(\sup_{0\le r\le t}\left|\int_0^r (g(Z_2(s),s)-g(Z_2(s),\tilde{s}))\mathrm{d}B_s\right|^2\right) \le 24\mathbb{E}\left(\int_0^t |g(Z_2(s),s)-g(Z_2(s),\tilde{s})|^2\mathrm{d}s\right)$$
$$\le 24K_3h\mathbb{E}\left(\int_0^t (1+|Z_2(s)|^2)\mathrm{d}s\right)$$
$$= 24K_3h\int_0^t \mathbb{E}|Z_2(s)|^2\mathrm{d}s + 24TK_3h. \quad (3.21)$$

Inserting estimates (3.20) and (3.21) into (3.19), we deduce that

$$\mathbb{E}\Big(\sup_{0\le s\le t}|x(s)-y(s)|^2\Big) \le (12THK_1 + 6TFK_3 + (6T+24)GK_3 + (12T+24)K_3$$
$$+(12T+48)IK_1)Th + (24T+48)K_1\int_0^t \mathbb{E}\Big(\sup_{0\le r\le s}|x(r)-y(r)|^2\Big)\mathrm{d}s. \quad (3.22)$$

Using the Gronwall inequality (see Theorem 8.1 in Chapter 1 of [3]), we have

$$\mathbb{E}\Big(\sup_{0\le s\le t}|x(s)-y(s)|^2\Big) \le Mh,$$

where $M = [12THK_1 + 6TFK_3 + (6T+24)GK_3 + (12T+24)K_3 + (12T+48)IK_1]T\mathrm{e}^{(24T+48)K_1T}$.

Table 5.1
The endpoint mean-square errors of the SSθ method for Eq. (5.1) with $\theta = 0.1$.

Step size	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}
Errors	0.0004270	0.0002321	0.0001202	0.0000563	0.0000264

The assertion follows since $t \in [0, T]$ is arbitrary; that is,

$$\mathbb{E}\Big(\sup_{0\le t\le T} |x(t) - y(t)|^2\Big) \le Mh.$$

The proof is complete. □

Remark 3.1. In the case of $\theta = 0$, similarly to Theorem 3.1, we can prove the mean-square convergence of the SSθ method. The proof is rather similar, so is omitted.

4. Mean-square stability of the SSθ method

In order to study the stability property of the SSθ method, we focus on a linear scalar SDE of Itô type:

$$\begin{cases} \mathrm{d}x(t) = ax(t)\mathrm{d}t + bx(t)\mathrm{d}B_t, \quad t \ge 0, \\ x(0) = x_0, \end{cases} \tag{4.1}$$

where $a, b \in \mathbb{R}$ are constants. The zero solution to Eq. (4.1) is said to be mean-square stable if $\lim_{t\to\infty} \mathbb{E}|x(t)|^2 = 0$ (see [12,13]). It is known [10,12,13] that the mean-square stability for Eq. (4.1) is equivalent to

$$a < -\frac{1}{2}b^2. \tag{4.2}$$

Applying the SSθ method to Eq. (4.1), we can obtain the following discrete schemes:

$$y_k^* = y_k + h[(1-\theta)ay_k + \theta a y_k^*], \tag{4.3a}$$

$$y_{k+1} = y_k^* + b y_k^* \Delta B_k. \tag{4.3b}$$

Assuming that $1 - \theta ah \neq 0$, we have

$$y_k^* = \frac{1 + (1-\theta)ah}{1 - \theta ah} y_k. \tag{4.4}$$

Substituting (4.4) into (4.3b) yields

$$\begin{aligned} y_{k+1} &= \frac{1 + (1-\theta)ah}{1 - \theta ah} y_k + \frac{(1 + (1-\theta)ah)b}{1 - \theta ah} y_k \Delta B_k \\ &= \frac{1 + (1-\theta)ah}{1 - \theta ah}(1 + b\Delta B_k) y_k. \end{aligned} \tag{4.5}$$

Now we investigate the mean-square stability of the SSθ method.

Definition 4.1 (*[12]*)**.** A numerical method is said to be mean-square stable (MS-stable) for a particular a, b, h if

$$\lim_{k\to\infty} \mathbb{E}|y_k|^2 = 0,$$

where y_k $(k = 1, 2, \ldots)$ are numerical solutions produced by the numerical method.

Theorem 4.1. *Suppose that condition (4.2) holds; then we have the following statements.*

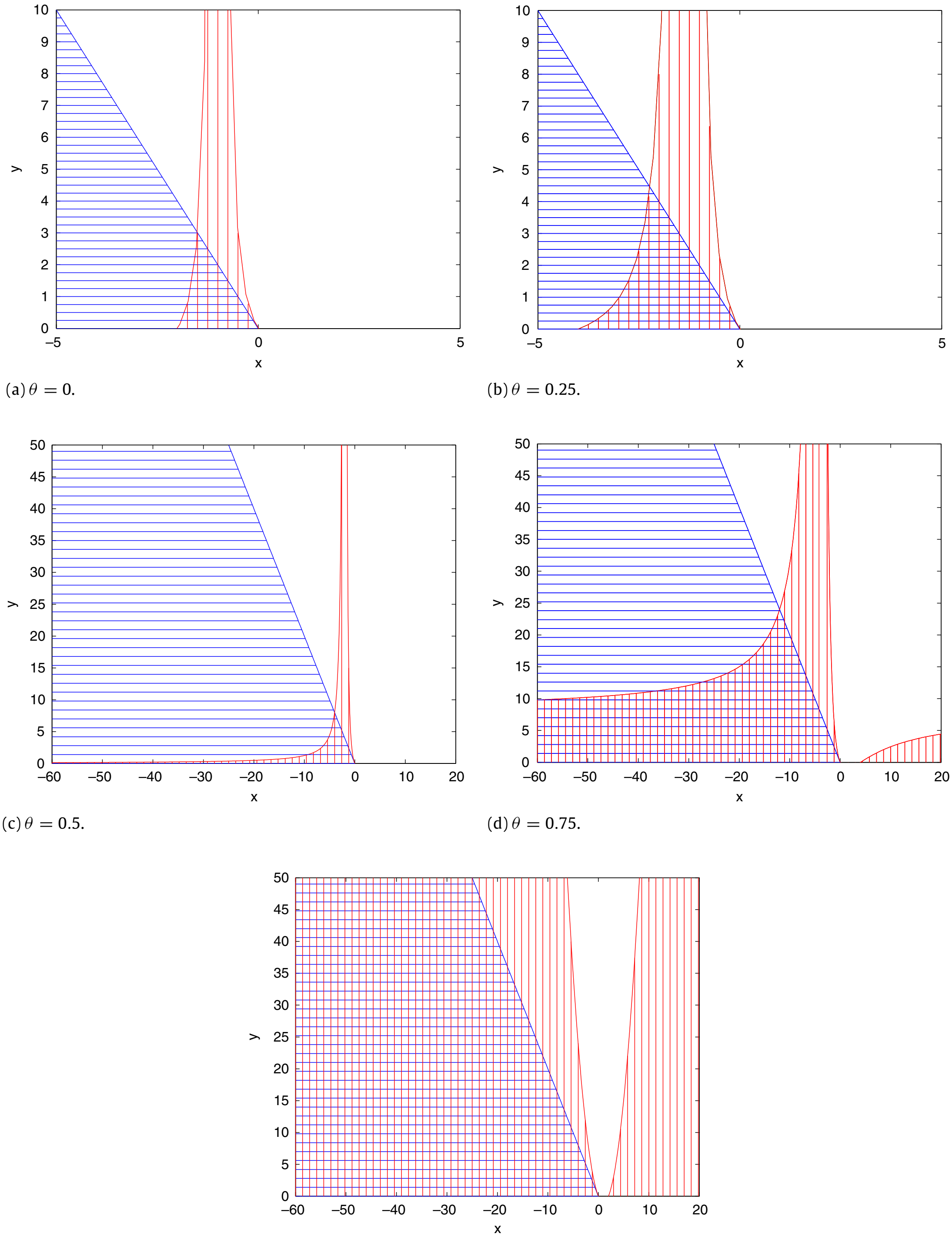
(1) *For given a, b satisfying (4.2), when $\theta = 1$, the SSθ method (4.5) is MS-stable for all $h > 0$.*

(2) *For given $a < 0$ and $b = 0$, when $\theta \in [0, 1/2)$, the SSθ method (4.5) is MS-stable if $h \in (0, -2/(a(1-2\theta)))$; and when $\theta \in [1/2, 1)$, the SSθ method (4.5) is MS-stable for all $h > 0$.*

(3) *For given a, b satisfying (4.2) and $ab \neq 0$, when $\theta \in [0, 1)$, the SSθ method (4.5) is MS-stable if $h \in \big(0, h_0(a, b, \theta)\big)$, where*

$$h_0(a, b, \theta) = \big(-(a^2(1-2\theta) + 2ab^2(1-\theta)) + \sqrt{\Delta}\big)/\big(2a^2b^2(1-\theta)^2\big)$$

$$\Delta = (a^2(1-2\theta) + 2ab^2(1-\theta))^2 - 4a^2b^2(1-\theta)^2(b^2 + 2a).$$



(a) $\theta = 0$.
(b) $\theta = 0.25$.
(c) $\theta = 0.5$.
(d) $\theta = 0.75$.
(e) $\theta = 1$.

Fig. 4.1. Real MS-stability regions for different values of θ of the SSθ method (vertical hashing) and Eq. (4.1) (horizontal hashing).

Proof. Under (4.2), $\theta \in [0, 1]$ and $h > 0$, it is easy to see that $1 - \theta ah \neq 0$. Squaring both sides of (4.5), we can obtain

$$|y_{k+1}|^2 = \left(\frac{1 + (1-\theta)ah}{1 - \theta ah}\right)^2 (1 + 2b\Delta B_k + b^2(\Delta B_k)^2)|y_k|^2. \tag{4.6}$$

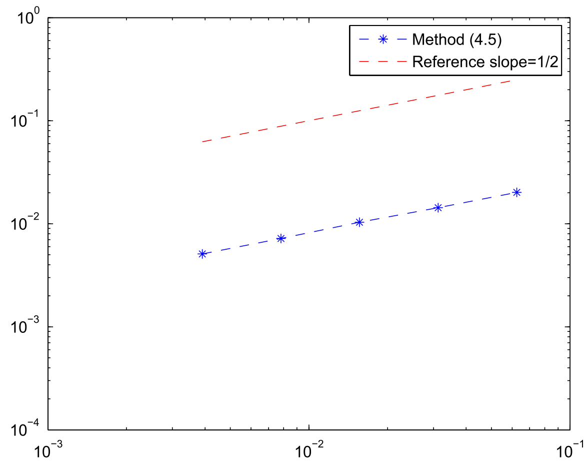
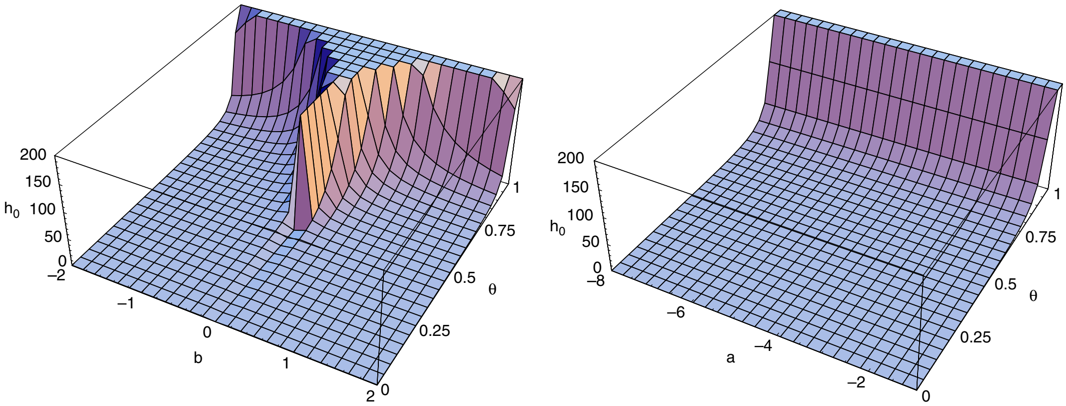


Fig. 5.1. The convergence rate of the SSθ method with fixed $\theta = 0.1$ for (5.1).



(a) Simulations with fixed $a = -2$.

(b) Simulations with fixed $b = 1$.

Fig. 5.2. Simulations for $h_0(a, b, \theta)$.

Taking mathematical expectation on both sides of (4.6) yields

$$\mathbb{E}|y_{k+1}|^2 = \left(\frac{1+(1-\theta)ah}{1-\theta ah}\right)^2 (1 + 2b\mathbb{E}\Delta B_k + b^2\mathbb{E}(\Delta B_k)^2)\mathbb{E}|y_k|^2$$
$$= \left(\frac{1+(1-\theta)ah}{1-\theta ah}\right)^2 (1 + b^2 h)\mathbb{E}|y_k|^2.$$

By recursive calculation, we conclude that $\lim_{k\to\infty} \mathbb{E}|y_k|^2 = 0$ if

$$\left(\frac{1+(1-\theta)ah}{1-\theta ah}\right)^2 (1 + b^2 h) < 1, \tag{4.7}$$

which is equivalent to

$$\varphi(h) := (ab - \theta ab)^2 h^2 + (a^2 - 2a^2\theta + 2ab^2 - 2\theta ab^2)h + b^2 + 2a < 0. \tag{4.8}$$

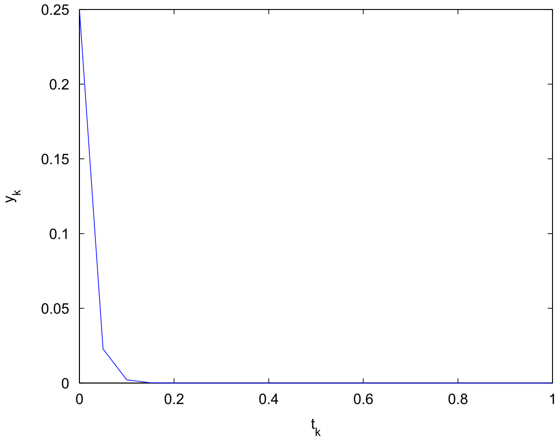
We divide the following proof into three cases. Noting that $b^2 + 2a < 0$, we have the following.

Case 1. For given a, b satisfying (4.2), when $\theta = 1$, then $\varphi(h) = -a^2 h + b^2 + 2a < 0$ holds. By (4.8), the SSθ method (4.5) is MS-stable for all $h > 0$.

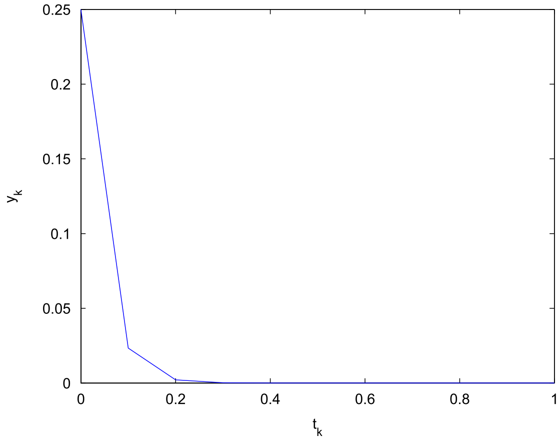
Case 2. For given a, b satisfying (4.2), obviously, $ab = 0$ is equivalent to $a < 0$ and $b = 0$ as a, b satisfy (4.2). Then we have $\varphi(h) = a^2(1 - 2\theta)h + 2a$. For $\theta \in [0, 1/2)$, by (4.8), the SSθ method (4.5) is MS-stable if $h \in (0, -2/(a(1 - 2\theta)))$. When $\theta \in [1/2, 1)$, by (4.8), the SSθ method (4.5) is MS-stable for all $h > 0$.

Case 3. For given a, b satisfying (4.2) and $ab \neq 0$, when $\theta \in [0, 1)$, $\varphi(h)$ is a quadratic function with respect to h. Let

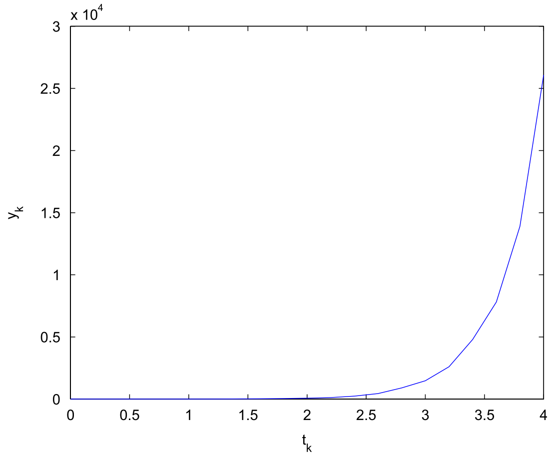
$$\Delta = (a^2(1 - 2\theta) + 2ab^2(1 - \theta))^2 - 4a^2b^2(1 - \theta)^2(b^2 + 2a)$$



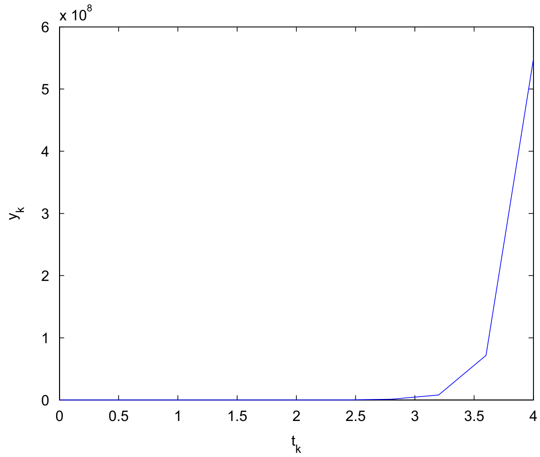
(a) $h = 0.05 < h_0(-15, 1, 0.1) = 0.1593$.



(b) $h = 0.1 < h_0(-15, 1, 0.1) = 0.1593$.



(c) $h = 0.2 > h_0(-15, 1, 0.1) = 0.1593$.



(d) $h = 0.4 > h_0(-15, 1, 0.1) = 0.1593$.

Fig. 5.3. Simulations with fixed parameter $\theta = 0.1$ for $a = -15$, $b = 1$ of (4.1).

and

$$h_0(a, b, \theta) = \frac{-(a^2(1-2\theta) + 2ab^2(1-\theta)) + \sqrt{\Delta}}{2a^2b^2(1-\theta)^2}.$$

Under (4.2), it is easy to verify that $\Delta > 0$ and $h_0(a, b, \theta) > 0$.

From (4.2), $\varphi(h) = 0$ has two different real roots, h_0 and h_1, with $h_1 < 0 < h_0$, where

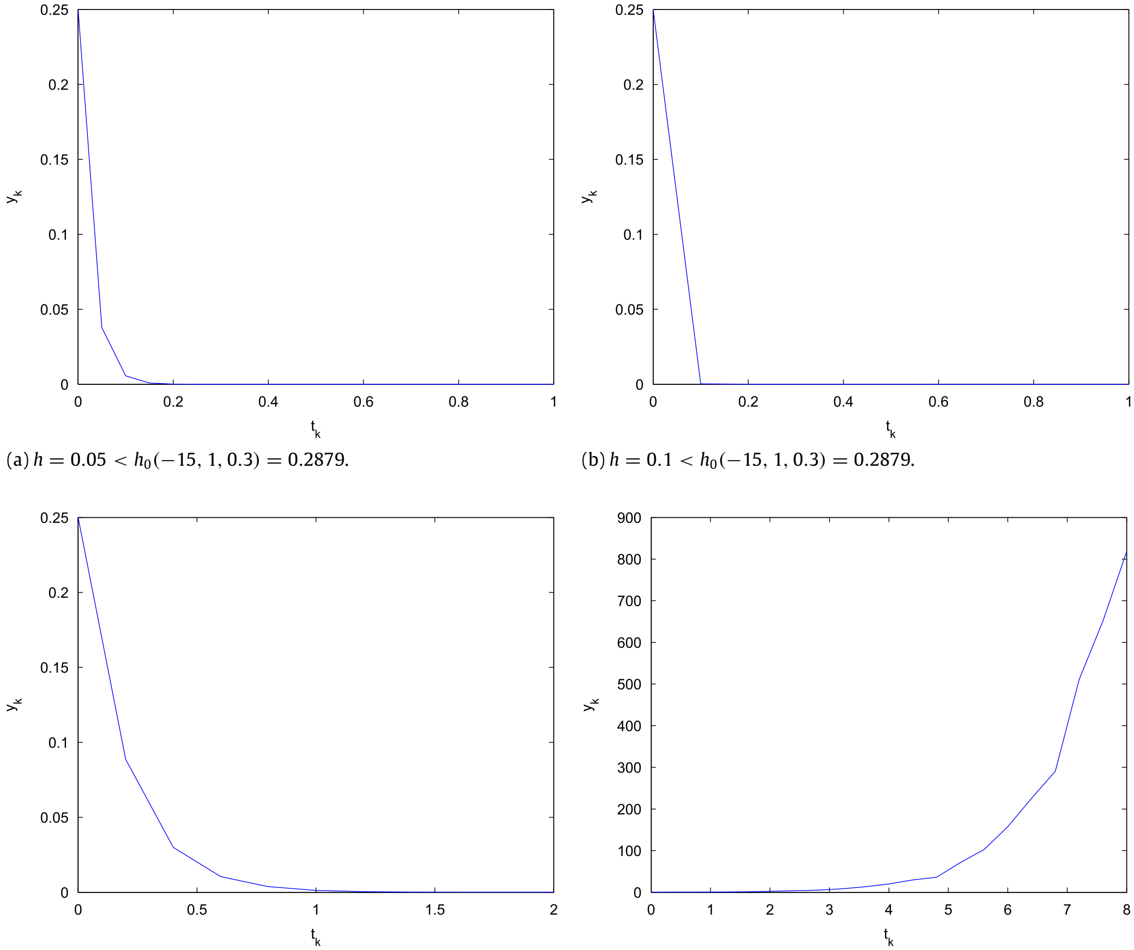
$$h_0 = \frac{-(a^2 - 2a^2\theta + 2ab^2 - 2\theta ab^2) + \sqrt{\Delta}}{2(ab - \theta ab)^2},$$

$$h_1 = \frac{-(a^2 - 2a^2\theta + 2ab^2 - 2\theta ab^2) - \sqrt{\Delta}}{2(ab - \theta ab)^2}.$$

So $\varphi(h) < 0$ holds when $h \in (0, h_0(a, b, \theta))$. According to (4.8), the SSθ method (4.5) is MS-stable if $h \in \big(0, h_0(a, b, \theta)\big)$. The proof is complete. □

Clearly, the mean-square stability of the SSθ method depends on ah and b^2h. Following [12], we discuss the real MS-stability regions in the x–y plane, where $x = ah$ and $y = b^2h$. In this way, for given parameters a and b of Eq. (4.1), varying h corresponds to moving along a ray that passes through the origin and (a, b^2). The following result is immediate from (4.7).

Corollary 4.1. *Suppose that $a, b \in \mathbb{R}$ and let $x = ah$, $y = b^2h$. The SSθ method is mean-square stable if $y < \big((2\theta - 1)x^2 - 2x\big)/(1 + (1-\theta)x)^2$.*



(a) $h = 0.05 < h_0(-15, 1, 0.3) = 0.2879$.

(b) $h = 0.1 < h_0(-15, 1, 0.3) = 0.2879$.

(c) $h = 0.2 < h_0(-15, 1, 0.3) = 0.2879$.

(d) $h = 0.4 > h_0(-15, 1, 0.3) = 0.2879$.

Fig. 5.4. Simulations with fixed parameter $\theta = 0.3$ for $a = -15$, $b = 1$ of (4.1).

Fig. 4.1 illustrates how the real MS-stability region varies with θ. The horizontal hashing marks the region $y < -2x$ where the solution of Eq. (4.1) is MS-stable. The vertical hashing superimposes the real MS-stability region for $\theta = 0, 0.25, 0.5, 0.75, 1$, respectively.

5. Numerical experiments

We first apply the SSθ method to solve the following linear SDE:

$$\begin{cases} dx(t) = -\frac{1}{2}x(t)dt + \frac{1}{2}x(t)dB_t, \quad t \in [0, 1], \\ x(0) = 1. \end{cases} \tag{5.1}$$

The exact solution of Eq. (5.1) is given by

$$x(t) = \exp\left(-\frac{5}{8}t + \frac{1}{2}B_t\right). \tag{5.2}$$

In order to clearly demonstrate the convergence rate of the SSθ method, we present the average sample errors at terminal time. 1000 different discretized Brownian paths over [0,1] will be computed with step size 2^{-9}. For each path, the SSθ method is applied with five different step sizes: $h = 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}$. We present mean-square errors at the terminal time 1 (i.e., $(\sum_{i=1}^{1000} |x(1, \omega_i) - y_N(\omega_i)|^2)/1000$) for the SS$\theta$ method with $\theta = 0.1$ in Table 5.1. Fig. 5.1 shows the results of Table 5.1 in a log–log plot.

We present the values of $h_0(a, b, \theta)$ that are calculated from different values of a, b, θ. Fig. 5.2 shows a three-dimensional figure of $h_0(a, b, \theta)$ values based on different values of b and θ but a fixed value of $a = -2$, and another three-dimensional figure of $h_0(a, b, \theta)$ values based on different values of a and θ but a fixed value of $b = 1$. It is exhibited that stochastic perturbation makes an impact on the restriction of step size for MS-stability. From the figure we also find that the values of $h_0(a, b, \theta)$ increase according to the increase of θ for fixed values a and b.

Next, we study several illustrative numerical examples of applying the SSθ method to Eq. (4.1). The data used in the following figures is obtained by the mean-square of data from 500 trajectories; that is, $\omega_i : 1 \le i \le 500$, $y_k = (\sum_{i=1}^{500} |y_k(\omega_i)|^2)/500$. Each t_k denotes the gridpoint.

We choose the coefficients of Eq. (4.1) as $a = -15$ and $b = 1$ with initial value $x_0 = 0.5$. For $\theta = 0.1$ and 0.3, we obtain $h_0(-15, 1, 0.1) = 0.1593$ and $h_0(-15, 1, 0.3) = 0.2879$. We first fix the parameter $\theta = 0.1$ and change the step size h; see Fig. 5.3. We then fix the parameter $\theta = 0.3$ and change step size h; see Fig. 5.4. It is shown that the SSθ method is MS-stable if $h \in (0, h_0(a, b, \theta))$.

6. Conclusions

In this paper, we construct the SSθ method for solving SDEs of Itô type and prove that the SSθ approximate solution is mean-square convergent with order $p = 1/2$. In addition, we establish criteria for the MS-stability of the SSθ method and plot the real MS-stability regions. Numerical results show that the SSθ method is valid for SDEs.

Acknowledgements

The authors would like to express their appreciation to the anonymous referees for their many valuable suggestions and for carefully correcting the preliminary version of the manuscript.

References

[1] L.P. Blenman, R.S. Cantrell, R.E. Fennell, D.F. Parker, J.A. Reneke, L.F.S. Wang, N.K. Womer, An alternative approach to stochastic calculus for economic and financial models, J. Econ. Dyn. Con. 19 (1995) 553–568.
[2] F.G. Ball, O.D. Lyne, Optimal vaccination policies for stochastic epidemics among a population of households, Math. Biosci. 177 & 178 (2002) 333–354.
[3] X. Mao, Stochastic Differential Equations and their Applications, Harwood, New York, 1997.
[4] B. Øksendal, Stochastic Differential Equations, Springer, Berlin, 2005.
[5] T. Sickenberger, Mean-square convergence of stochastic multi-step methods with variable step-size, J. Comput. Appl. Math. 212 (2008) 300–319.
[6] P. Wang, Three-stage stochastic Runge–Kutta methods for stochastic differential equations, J. Comput. Appl. Math. 222 (2008) 324–332.
[7] P. Bernard, G. Fleury, Convergence of numerical schemes for stochastic differential equations, Monte Carlo Methods Appl. 7 (2001) 35–44.
[8] P.E. Kloeden, E. Platen, Numerical Solution of Stochastic Differential Equations, Springer, Berlin, 1999.
[9] G.N. Milstein, Numerical Integration of Stochastic Differential Equations, Kluwer Academic Publishers, Dordrecht, 1995.
[10] Y. Saito, T. Mitsui, Stability analysis of numerical schemes for stochastic differential equations, SIAM J. Numer. Anal. 33 (1996) 2254–2267.
[11] D.J. Higham, Mean-square and asymptotic stability of the stochastic theta method, SIAM J. Numer. Anal. 38 (2000) 753–769.
[12] D.J. Higham, A-stability and stochastic mean-square stability, BIT 40 (2) (2000) 404–409.
[13] Y. Komori, Y. Saito, T. Mitsui, Some issues in discrete approximate solution for stochastic differential equation, Comput. Math. Appl. 28 (1994) 269–278.
[14] A. Tocino, Mean-square stability of second-order Runge–Kutta methods for stochastic differential equations, J. Comput. Appl. Math. 175 (2005) 355–367.
[15] A. Rathinasamy, K. Balachandran, Mean-square stability of second-order Runge–Kutta methods for multi-dimensional linear stochastic differential systems, J. Comput. Appl. Math. 219 (2008) 170–197.
[16] N. Hofmann, E. Platen, Stability of weak numerical schemes for stochastic differential equations, Comput. Math. Appl. 28 (1994) 45–57.
[17] D.J. Higham, X. Mao, A.M. Stuart, Strong convergence of Euler-type methods for nonlinear stochastic differential equations, SIAM J. Numer. Anal. 40 (2002) 1041–1063.
[18] A.M. Stuart, A.R. Humphries, Dynamical Systems and Numerical Analysis, Cambridge University Press, Cambridge, 1996.
[19] W. Rudin, Real and Complex Analysis, third ed., China Machine Press, Beijing, 2007.