J. Math. Anal. Appl. 389 (2012) 486–497

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Asymptotic behavior and uniqueness of traveling wave solutions in Ricker competition system ☆

Kun Li [a,*], Xiong Li [b]

[a] School of Mathematical Sciences, Huaibei Normal University, Huaibei Anhui 235000, People's Republic of China
[b] School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People's Republic of China

ARTICLE INFO

Article history:
Received 7 April 2011
Available online 1 December 2011
Submitted by M. Iannelli

Keywords:
Ricker competition system
Traveling wave solutions
Asymptotic behavior
Uniqueness
Sliding method

ABSTRACT

The existence of traveling wave solutions connecting two half-positive equilibria in Ricker competition system can be obtained by the results (B. Li, H.F. Weinberger, M.A. Lewis, Spreading speeds as slowest wave speeds for cooperative systems, Math. Biosci. 196 (2005) 82–98). In this paper we first prove that any nondecreasing traveling wave solutions have the exponential decay asymptotic behavior at the minus/plus infinity by means of Ikehara's Theorem, and then use the strong comparison principle and the sliding method to obtain the uniqueness of the traveling wave solutions for this system.

1. Introduction

For a large class spreading speeds of discrete time and possibly discrete space, recursions of the form

$$u_{n+1} = Q[u_n], \quad n = 1, 2, \ldots, \qquad u_n = \left(u_n^1, \ldots, u_n^k\right) \in \mathbb{R}^k,$$

there are many important results (see [5,8,9,12,11,14–18] and references therein). These results can be applied to the Richer competition system [16]

$$\begin{cases} u_{n+1}(x) = \displaystyle\int_{\mathbb{R}} k_1(x-y)u_n(y)e^{r_1 - u_n(y) - \sigma_1 v_n(y)}\,dy, \\ v_{n+1}(x) = \displaystyle\int_{\mathbb{R}} k_2(x-y)v_n(y)e^{r_2 - v_n(y) - \sigma_2 u_n(y)}\,dy, \end{cases} \tag{1.1}$$

where r_1, r_2, σ_1, σ_2 are all positive constants, $u_n(x)$ and $v_n(x)$ denote the population densities restriction rates of the prey and predator at time n and point x. The re-distribution kernel $g_i(x)$ describes the dispersal of u, v, which it is assumed to depend upon the signed distance $x - y$ connecting the location of "birth" y and the "settlement" location x; $g_i(x)$ is a homogeneous probability kernel that satisfies $\int_{\mathbb{R}} k_i(x)\,dx = 1$ $(i = 1, 2)$.

Recently, Wang et al. [16] considered the spreading speed of system (1.1) with non-cooperative case by carefully constructing monotone functions. For more about results about the spreading speed of non-monotone integro-difference

☆ Supported by NSFC (11031002), the Fundamental Research Funds for the Central Universities and SRF for ROCS, SEM.
* Corresponding author.
E-mail addresses: kli@mail.bnu.edu.cn (K. Li), xli@bnu.edu.cn (X. Li).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.11.055

systems, one can refer to [7,13]. Early results from [5] showed that the traveling wave solutions for a single equation had the exponential decay behavior at the minus infinity, which was determined by the wave speed. It is natural to ask if the traveling wave solutions of system (1.1) have the exponential decay asymptotic behavior. To the best of our knowledge, this problem remains open. In this paper, we will answer this question. For simplicity, we will consider system (1.1) with $k_i(x) = \frac{1}{\sqrt{4\pi d_i}} e^{-\frac{x^2}{4d_i}}$ $(i = 1, 2)$, that is, the following system

$$\begin{cases} u_{n+1}(x) = \int\limits_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(x-y)^2}{4d_1}} u_n(y) e^{r_1 - u_n(y) - \sigma_1 v_n(y)}\, dy, \\ v_{n+1}(x) = \int\limits_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_2}} e^{-\frac{(x-y)^2}{4d_2}} v_n(y) e^{r_2 - v_n(y) - \sigma_2 u_n(y)}\, dy, \end{cases} \tag{1.2}$$

where all the parameters are positive. System (1.2) has four equilibria $(0, 0)$, $(r_1, 0)$, $(0, r_2)$ and $(\frac{r_1-\sigma_1 r_2}{1-\sigma_1\sigma_2}, \frac{r_2-\sigma_2 r_1}{1-\sigma_1\sigma_2})$ (if feasible). In this paper, we are interested in the asymptotic behavior and uniqueness of the traveling wave solutions of system (1.2) connecting $(0, r_2)$ with $(r_1, 0)$ under the assumption

$$\sigma_1 r_2 < r_1 < 1, \quad r_2 < \sigma_2 r_1, \quad r_2 < 1, \tag{1.3}$$

which the existence of traveling wave solutions can be obtained in [12]. We remark that in this case the positive equilibrium does not exist. Similarly, if reversing the inequality (1.3), we also can obtain the asymptotic behavior and uniqueness of the traveling wave solutions connecting $(r_1, 0)$ with $(0, r_2)$. For simplicity, we only consider the case (1.3).

In past few years, we note that many authors used the strong comparison principle and the sliding method (see [1] for the application of the method to a single equation) to investigate the uniqueness of traveling wave solutions for lattice system [4,3,6] and for reaction–diffusion system [10], and used Ikehara's Theorem [2] to study the asymptotic behavior of traveling wave solutions [6]. In this paper we will adopt these methods to deal with our problem.

This paper is organized as follows. In Section 2, by considering the singularity of the bilateral Laplace transform of traveling wave solutions, we obtain the asymptotic behavior of any nondecreasing traveling wave solutions by means of Ikehara's Theorem. In Section 3, we use the strong comparison principle and the sliding method to prove the uniqueness of traveling wave fronts.

2. Asymptotic behavior

In this paper, we use the usual notations for the standard ordering in $\mathbb{R}^2$. In this section, we investigate the asymptotic behavior of traveling wave solutions of system (1.2).

Let $u_n^* = u_n$, $v_n^* = r_2 - v_n$, and drop the star, then the existence of traveling wave solutions of system (1.2) connecting $(0, r_2)$ with $(r_1, 0)$ is equivalent to the existence of traveling wave solutions of system

$$\begin{cases} u_{n+1}(x) = \int\limits_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(x-y)^2}{4d_1}} u_n(y) e^{r_1 - \sigma_1 r_2 - u_n(y) + \sigma_1 v_n(y)}\, dy, \\ v_{n+1}(x) = \int\limits_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_2}} e^{-\frac{(x-y)^2}{4d_2}} \left[r_2 - \left(r_2 - v_n(y)\right) e^{v_n(y) - \sigma_2 u_n(y)}\right] dy \end{cases} \tag{2.1}$$

connecting $(0, 0)$ with (r_1, r_2). A traveling wave solution of (2.1) is a pair of translation invariant solution having the form $(u_n(x), v_n(x)) := (\phi(\xi), \psi(\xi))$, $\xi = x + cn$, $\xi \in \mathbb{R}$, the wave speed $c > 0$. If $(\phi(\xi), \psi(\xi))$ is monotone in $\xi \in \mathbb{R}$, then it is called a traveling wave front. Substituting $(\phi(\xi), \psi(\xi))$ into (2.1), let $\tilde{\xi} = \xi + c$, $\tilde{y} = x - y + c$, and drop the tilde, then system (2.1) has a traveling wave front connecting $(0, 0)$ with (r_1, r_2) if and only if the wave equations

$$\begin{cases} \phi(\xi) = \int\limits_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \phi(\xi - y) e^{r_1 - \sigma_1 r_2 - \phi(\xi - y) + \sigma_1 \psi(\xi - y)}\, dy, \\ \psi(\xi) = \int\limits_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_2}} e^{-\frac{(y-c)^2}{4d_2}} \left[r_2 - \left(r_2 - \psi(\xi - y)\right) e^{\psi(\xi - y) - \sigma_2 \phi(\xi - y)}\right] dy \end{cases} \tag{2.2}$$

with the asymptotic boundary conditions

$$\lim_{\xi \to -\infty} \left(\phi(\xi), \psi(\xi)\right) = \mathbf{0} := (0, 0), \qquad \lim_{\xi \to \infty} \left(\phi(\xi), \psi(\xi)\right) = \mathbf{r} := (r_1, r_2) \tag{2.3}$$

has a pair of monotone solutions on $\mathbb{R}$.

The characteristic equation of the linearization equation of the first equation in (2.2) at **0** is $\Delta_1(\lambda, c) = 0$, where

$$\Delta_1(\lambda, c) := 1 - \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1} - \lambda y + r_1 - \sigma_1 r_2}\, dy = 1 - e^{d_1\lambda^2 - c\lambda + r_1 - \sigma_1 r_2}. \tag{2.4}$$

Then $\Delta_1(\lambda, c) = 0$ has two real roots

$$\lambda_1 = \frac{c - \sqrt{c^2 - 4d_1(r_1 - \sigma_1 r_2)}}{2d_1} > 0, \qquad \lambda_2 = \frac{c + \sqrt{c^2 - 4d_1(r_1 - \sigma_1 r_2)}}{2d_1} > 0$$

for $c > c_0 := 2\sqrt{d_1(r_1 - \sigma_1 r_2)}$.

Let

$$\Delta_2(\lambda, c) := 1 - \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_2}} e^{-\frac{(y-c)^2}{4d_2} - \lambda y + \ln(1-r_2)}\, dy = 1 - e^{d_2\lambda^2 - c\lambda + \ln(1-r_2)}. \tag{2.5}$$

Then, for any $c > 0$, $\Delta_2(\lambda, c) = 0$ has a unique positive root

$$\lambda_3 = \frac{c + \sqrt{c^2 - 4d_2 \ln(1 - r_2)}}{2d_2} > 0 \quad (\text{since } r_2 < 1)$$

and

$$\Delta_2(\lambda, c) < 0, \quad \lambda \in (0, \lambda_3).$$

From Theorem 3.1 in [12], one can easily get the existence result of traveling wave front of system (2.1). We list it as a lemma.

Lemma 2.1. *System* (2.1) *has a traveling wave front connecting* **0** *with* **r** *for the wave speed* $c \geqslant c_0$.

In this paper we are interesting in the asymptotic behavior and uniqueness of traveling wave solutions of system (2.1). First we prove that any nonnegative solutions of (2.2) and (2.3) cannot reach the equilibria **0** and **r** at any finite time, which will be used later.

Lemma 2.2. *Assume that* $(\phi(\xi), \psi(\xi))$ *is any nonnegative solution of* (2.2) *and* (2.3). *If* $\mathbf{0} \leqslant (\phi(\xi), \psi(\xi)) \leqslant \mathbf{r}$, $\forall \xi \in \mathbb{R}$, *then* $0 < \phi < r_1$ *and* $0 < \psi < r_2$.

Proof. We first prove $\phi, \psi > 0$. For ϕ, it is obvious that $\phi(\xi) \equiv 0$ if there exists ξ_0 such that $\phi(\xi_0) = 0$. For ψ, assume that there exists ξ_0 such that $\psi(\xi_0) = 0$, since

$$-\sigma_2 r_1 \leqslant \psi - \sigma_2\phi \leqslant r_2 \quad \text{and} \quad e^x \leqslant \frac{1}{1-x}, \quad x \in (-\infty, 1) \supset [-\sigma_2 r_1, r_2], \tag{2.6}$$

we have

$$r_2 - (r_2 - \psi)e^{\psi - \sigma_2\phi} \geqslant r_2 - \frac{r_2 - \psi}{1 - \psi + \sigma_2\phi} = \frac{(1 - r_2)\psi + \sigma_2 r_2\phi}{1 - \psi + \sigma_2\phi} \geqslant \frac{(1 - r_2)\psi + \sigma_2 r_2\phi}{1 + \sigma_2 r_1} \geqslant 0,$$

which implies that $\psi(\xi) \equiv 0$. This is a contradiction. Similarly, $\psi > 0$. Let $\phi^* = r_1 - \phi$, $\psi^* = r_2 - \psi$, and drop the star, substituting them into (2.2), one can prove that $\phi < r_1$ and $\psi < r_2$. The proof is completed. □

Now we first consider the asymptotic behavior of any nondecreasing solution $(\phi(\xi), \psi(\xi))$ of (2.2) and (2.3) at the minus infinity. Give the continuous function $\varphi : \mathbb{R} \to \mathbb{R}$, define the bilateral Laplace transform

$$L(\lambda, \varphi) = \int_{-\infty}^{\infty} \varphi(\xi) e^{-\lambda\xi}\, d\xi.$$

Then we have the following lemma.

Lemma 2.3. *Assume that* (1.3) *holds and* $(\phi(\xi), \psi(\xi))$ *is any nondecreasing solution of* (2.2) *and* (2.3) *with the wave speed* $c \geqslant c_0$. *Then the followings are true*:

(i) $L(\lambda, \phi) < \infty$, $\lambda \in (0, \Lambda)$ *and* $L(\lambda, \phi) = \infty$, $\lambda \in \mathbb{R} \setminus (0, \Lambda)$, *where* $\Lambda \in \{\lambda_1, \lambda_2\}$;
(ii) $L(\lambda, \psi) < \infty$, $\lambda \in (0, \gamma)$ *and* $L(\lambda, \psi) = \infty$, $\lambda \in \mathbb{R} \setminus (0, \gamma)$, *where* $\gamma = \min\{\Lambda, \lambda_3\}$.

Proof. We divide this proof into two steps.

Step 1. We first prove the following facts:

(1) there exists $\lambda' > 0$ such that $L(\lambda, \phi) < \infty$, $\lambda \in (0, \lambda')$;
(2) there exists $\sigma > 0$ such that $L(\lambda, \psi) < \infty$, $\lambda \in (0, \sigma)$.

We first show (1). To do this, we will show that there exists $\lambda' > 0$ such that $\sup_{\xi\in\mathbb{R}} \phi(\xi)e^{-\lambda'\xi} < \infty$. Take ν satisfying $0 < \nu < r_1 - \sigma_1 r_2 < 1$ and $(1-\nu)e^{r_1-\sigma_1 r_2-\nu} > 1$. Since

$$\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}}\,dy = 1 \quad \text{and} \quad \lim_{\xi\to-\infty}\big(\phi(\xi), \psi(\xi)\big) = \mathbf{0},$$

there exists $y_0 = y_0(\nu) > 0$ large enough such that

$$\int_{-\infty}^{y_0} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}}\,dy \geqslant 1-\nu \quad \text{and} \quad \phi(\xi - y_0) \leqslant \nu, \quad \forall \xi < 0.$$

By the first equation of (2.2), the monotonicity of $(\phi(\xi), \psi(\xi))$ and

$$xe^{-x} \text{ is nondecreasing in } [0,1], \tag{2.7}$$

we have

$$\begin{aligned}
\phi(\xi) &= \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \phi(\xi-y) e^{r_1-\sigma_1 r_2-\phi(\xi-y)+\sigma_1\psi(\xi-y)}\,dy \\
&\geqslant \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \phi(\xi-y) e^{r_1-\sigma_1 r_2-\phi(\xi-y)}\,dy \\
&\geqslant \int_{-\infty}^{y_0} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \phi(\xi-y) e^{r_1-\sigma_1 r_2-\phi(\xi-y)}\,dy \\
&\geqslant \int_{-\infty}^{y_0} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \phi(\xi-y_0) e^{r_1-\sigma_1 r_2-\phi(\xi-y_0)}\,dy \\
&\geqslant (1-\nu)e^{r_1-\sigma_1 r_2-\nu}\phi(\xi-y_0)
\end{aligned}$$

for any $\xi < 0$. Let $h(\xi) = \phi(\xi)e^{-\lambda'\xi}$ with $\lambda' = \frac{1}{y_0}\ln[(1-\nu)e^{r_1-\sigma_1 r_2-\nu}] > 0$ (since $(1-\nu)e^{r_1-\sigma_1 r_2-\nu} > 1$), then

$$h(\xi - y_0) \leqslant h(\xi), \quad \forall \xi < 0. \tag{2.8}$$

Since $h(\xi)$ is bounded on the bounded closed interval $[-y_0, 0]$, then (2.8) implies that $h(\xi)$ is bound on the interval $(-\infty, 0]$. Hence,

$$0 < \sup_{\xi\in\mathbb{R}} \phi(\xi)e^{-\lambda'\xi} < \infty$$

by $\lim_{\xi\to\infty}\phi(\xi) = r_1$, which implies that $L(\lambda, \phi) < \infty$, $\lambda \in (0, \lambda')$.

Next we show (2). Multiplying the second equation of (2.2) by $e^{-\lambda\xi}$ with $\lambda \in (0, \frac{c}{d_2})$, integrating from $-\infty$ to ∞, we have

$$\begin{aligned}
0 &\leqslant \big(1 - e^{d_2\lambda^2 - c\lambda}\big)L(\lambda, \psi) \\
&\leqslant \big(1 - e^{d_2\lambda^2 - c\lambda}\big)L(\lambda, \psi) + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_2}} e^{-\frac{(y-c)^2}{4d_2}} \big[r_2 - \psi(\xi-y)\big]\big(e^{\psi(\xi-y)} - 1\big)e^{-\lambda\xi}\,dy\,d\xi \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_2}} e^{-\frac{(y-c)^2}{4d_2}} \big[r_2 - \psi(\xi-y)\big]\big(1 - e^{-\sigma_2\phi(\xi-y)}\big)e^{\psi(\xi-y)}e^{-\lambda\xi}\,dy\,d\xi
\end{aligned}$$

$$
\begin{aligned}
&\leqslant r_2 e^{r_2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_2}} e^{-\frac{(y-c)^2}{4d_2}} \left(1-e^{-\sigma_2\phi(\xi-y)}\right) e^{-\lambda\xi}\,dy\,d\xi \\
&\leqslant \sigma_2 r_2 e^{r_2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_2}} e^{-\frac{(y-c)^2}{4d_2}} \phi(\xi-y) e^{-\lambda\xi}\,dy\,d\xi \\
&= \sigma_2 r_2 e^{r_2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_2}} e^{-\frac{(y-c)^2}{4d_2}-\lambda y} \phi(\tilde\xi) e^{-\lambda\tilde\xi}\,d\tilde\xi\,dy \\
&= \sigma_2 r_2 e^{r_2} e^{d_2\lambda^2-c\lambda} L(\lambda,\phi)
\end{aligned} \tag{2.9}
$$

with $\tilde\xi = \xi - y$, where the fourth inequality holds since

$$
e^x \geqslant 1 + x, \quad \forall x \in \mathbb{R}. \tag{2.10}
$$

Since $d_2\lambda^2 - c\lambda < 0$ for $\lambda \in (0, \frac{c}{d_2})$, it follows that there exists $\sigma \in (0, \min\{\lambda', \frac{c}{d_2}\})$ such that $L(\lambda,\psi)$ is well defined for $\lambda \in (0,\sigma)$.

Step 2. We prove: (1) $\max\lambda' = \Lambda$, (2) $\max\sigma = \gamma$.

For (1), multiplying the first equation of (2.2) by $e^{-\lambda\xi}$ with $\lambda > 0$ and integrating from $-\infty$ to ∞, we obtain that

$$
\begin{aligned}
\Delta_1(\lambda,c)L(\lambda,\phi) &= e^{r_1-\sigma_1 r_2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \phi(\xi-y)\left(e^{-\phi(\xi-y)+\sigma_1\psi(\xi-y)}-1\right) e^{-\lambda\xi}\,dy\,d\xi \\
&= e^{r_1-\sigma_1 r_2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}-\lambda y} \phi(\tilde\xi)\left(e^{-\phi(\tilde\xi)+\sigma_1\psi(\tilde\xi)}-1\right) e^{-\lambda\tilde\xi}\,d\tilde\xi\,dy \\
&= e^{r_1-\sigma_1 r_2} e^{d_1\lambda^2-c\lambda} \int_{-\infty}^{\infty} \phi(\tilde\xi)\left(e^{-\phi(\tilde\xi)+\sigma_1\psi(\tilde\xi)}-1\right) e^{-\lambda\tilde\xi}\,d\tilde\xi
\end{aligned} \tag{2.11}
$$

with $\tilde\xi = \xi - y$. We first claim that $\max\lambda' < \infty$. Otherwise, if $\max\lambda' = \infty$, we can choose large enough $\lambda > \lambda_2$ such that $(1-r_1)e^{d_1\lambda^2-c\lambda+r_1-\sigma_1 r_2} > 1$, that is, $\Delta_1(\lambda,c) < -r_1 e^{d_1\lambda^2-c\lambda+r_1-\sigma_1 r_2}$. By $\phi(\xi) \leqslant r_1$ and (2.10), we have

$$
\begin{aligned}
\int_{-\infty}^{\infty} \phi(\tilde\xi)\left(e^{-\phi(\tilde\xi)+\sigma_1\psi(\tilde\xi)}-1\right) e^{-\lambda\tilde\xi}\,d\tilde\xi &\geqslant \int_{-\infty}^{\infty} \phi(\tilde\xi)\left[-\phi(\tilde\xi)+\sigma_1\psi(\tilde\xi)\right] e^{-\lambda\tilde\xi}\,d\tilde\xi \\
&\geqslant -\int_{-\infty}^{\infty} \phi^2(\tilde\xi) e^{-\lambda\tilde\xi}\,d\tilde\xi \\
&\geqslant -r_1 \int_{-\infty}^{\infty} \phi(\tilde\xi) e^{-\lambda\tilde\xi}\,d\tilde\xi \\
&= -r_1 L(\lambda,\phi),
\end{aligned}
$$

which is a contradiction by $0 < L(\lambda,\phi) < \infty$ and (2.11). Thus the left side of (2.11) is well defined for $\lambda \in (0,\lambda')$ and the right side of (2.11) is well defined for $\lambda \in (0, \min\{2\lambda', \lambda'+\sigma\})$. It follows that the singularity of $L(\lambda,\phi)$ only happens at the zeros of $\Delta_1(\lambda,c)$. If not, $\max\lambda' = \infty$, which is a contradiction.

For (2), multiplying the second equation of (2.2) by $e^{-\lambda\xi}$ with $\lambda > 0$ and integrating from $-\infty$ to ∞, we obtain that

$$
\begin{aligned}
\Delta_2(\lambda,c)L(\lambda,\psi) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_2}} e^{-\frac{(y-c)^2}{4d_2}} \left[r_2-(1-r_2)\psi(\xi-y)-\left(r_2-\psi(\xi-y)\right)e^{\psi(\xi-y)-\sigma_2\phi(\xi-y)}\right] e^{-\lambda\xi}\,dy\,d\xi \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_2}} e^{-\frac{(y-c)^2}{4d_2}-\lambda y} \left[r_2-(1-r_2)\psi(\tilde\xi)-\left(r_2-\psi(\tilde\xi)\right)e^{\psi(\tilde\xi)-\sigma_2\phi(\tilde\xi)}\right] e^{-\lambda\tilde\xi}\,d\tilde\xi\,dy
\end{aligned}
$$

$$= e^{d_2\lambda^2 - c\lambda} \int_{-\infty}^{\infty} \left[r_2 - (1-r_2)\psi(\tilde{\xi}) - \left(r_2 - \psi(\tilde{\xi})\right)e^{\psi(\tilde{\xi}) - \sigma_2\phi(\tilde{\xi})}\right] e^{-\lambda\tilde{\xi}}\, d\tilde{\xi} \tag{2.12}$$

with $\tilde{\xi} = \xi - y$.

On one hand, by (2.10), we have

$$\begin{aligned} r_2 - (1-r_2)\psi - (r_2-\psi)e^{\psi - \sigma_2\phi} &\leqslant r_2 - (1-r_2)\psi - (r_2-\psi)(1+\psi-\sigma_2\phi) \\ &= \psi^2 + \sigma_2\phi(r_2-\psi), \end{aligned} \tag{2.13}$$

on the other hand, by (2.6), we have

$$\begin{aligned} r_2 - (1-r_2)\psi - (r_2-\psi)e^{\psi - \sigma_2\phi} &\geqslant r_2 - (1-r_2)\psi - \frac{r_2-\psi}{1-\psi+\sigma_2\phi} \\ &= \frac{(1-r_2)\psi^2 + \sigma_2\phi(r_2-\psi) + \sigma_2 r_2\phi\psi}{1-\psi+\sigma_2\phi} \\ &= \frac{1}{1+\sigma_2 r_1}\left[(1-r_2)\psi^2 + \sigma_2\phi(r_2-\psi)\right]. \end{aligned} \tag{2.14}$$

Then the right side of (2.12), $\int_{-\infty}^{\infty}[\psi^2(\tilde{\xi}) + \sigma_2\phi(\tilde{\xi})(r_2-\psi(\tilde{\xi}))]\,d\tilde{\xi}$ and $\int_{-\infty}^{\infty}[(1-r_2)\psi^2(\tilde{\xi}) + \sigma_2\phi(\tilde{\xi})(r_2-\psi(\tilde{\xi}))]\,d\tilde{\xi}$ are both positive and have the same singularity. Since $\int_{-\infty}^{\infty}[\psi^2(\tilde{\xi}) + \sigma_2\phi(\tilde{\xi})(r_2-\psi(\tilde{\xi}))]\,d\tilde{\xi}$ is well defined for $\lambda \in (0, \min\{2\max\sigma, \Lambda\})$, it follows that $\max\sigma \leqslant \Lambda$ from (2.12), (2.13) and (2.14). We claim that $\max\sigma \leqslant \lambda_3$. Otherwise, if $\max\sigma > \lambda_3$, then $L(\lambda_3, \psi) < \infty$. Taking $\lambda = \lambda_3$ in (2.12), the left-hand side of (2.12) equals to 0 by $\Delta_2(\lambda_3, c) = 0$ and the right-hand side of (2.12) is always positive by $\psi \leqslant r_2$, which is a contradiction. Also it follows easily from (2.12) that $\gamma = \Lambda$ if $\Lambda < \lambda_3$ and $\gamma = \lambda_3$ if $\Lambda \geqslant \lambda_3$. Furthermore, if $\Lambda \geqslant \lambda_3$, $\lim_{\lambda \to \lambda_3^-} L(\lambda, \phi)(\lambda_3 - \lambda)$ exists. The proof is completed. □

In order to study the asymptotic behavior $(\phi(\xi), \psi(\xi))$ at the minus infinity, we need the following modified version of Ikehara's Theorem [2].

Lemma 2.4 *(Ikehara's Theorem). Let φ be a positive nondecreasing function on $\mathbb{R}$, and define $F(\lambda) := \int_{-\infty}^{0} \varphi(\xi)e^{-\lambda\xi}\,d\xi$. Assume that F can be written as $F(\lambda) = H(\lambda)/(\alpha-\lambda)^{\nu+1}$, where $\nu > -1$, $\alpha > 0$, and H is analytic in the strip $0 < Re\,\lambda \leqslant \alpha$, then*

$$\lim_{\xi \to -\infty} \frac{\varphi(\xi)}{|\xi|^{\nu} e^{\alpha\xi}} = \frac{H(\alpha)}{\Gamma(\alpha+1)}.$$

We have the following the exponential asymptotic behavior of $(\phi(\xi), \psi(\xi))$ at the minus infinity.

Theorem 2.1. *Assume that* (1.3) *holds and $(\phi(\xi), \psi(\xi))$ is any nondecreasing solution of* (2.2) *and* (2.3) *with the wave speed $c \geqslant c_0$. Then*

(i) *There exists $\theta_i = \theta_i(\phi, \psi)$ $(i = 1, 2)$ such that*

$$\lim_{\xi \to -\infty} \frac{\phi(\xi+\theta_1)}{e^{\Lambda\xi}} = 1, \quad \text{if } c > c_0,$$

$$\lim_{\xi \to -\infty} \frac{\phi(\xi+\theta_2)}{|\xi|^{\mu} e^{\Lambda\xi}} = 1, \quad \text{if } c = c_0.$$

(ii) *For $c > c_0$, there exists $\theta_i = \theta_i(\phi, \psi)$ $(i = 3, 4, 5)$ such that*

$$\lim_{\xi \to -\infty} \frac{\psi(\xi+\theta_3)}{e^{\Lambda\xi}} = 1, \quad \text{if } \lambda_3 > \Lambda,$$

$$\lim_{\xi \to -\infty} \frac{\psi(\xi+\theta_4)}{|\xi| e^{\Lambda\xi}} = 1, \quad \text{if } \lambda_3 = \Lambda,$$

$$\lim_{\xi \to -\infty} \frac{\psi(\xi+\theta_5)}{e^{\lambda_3\xi}} = 1, \quad \text{if } \lambda_3 < \Lambda.$$

(iii) *For $c = c_0$, there exists $\theta_i = \theta_i(\phi, \psi)$ $(i = 6, 7, 8)$ such that*

$$\lim_{\xi\to-\infty} \frac{\psi(\xi+\theta_6)}{|\xi|^{\mu} e^{\Lambda\xi}} = 1, \quad \text{if } \lambda_3 > \Lambda,$$
$$\lim_{\xi\to-\infty} \frac{\psi(\xi+\theta_7)}{|\xi|^{\mu+1} e^{\Lambda\xi}} = 1, \quad \text{if } \lambda_3 = \Lambda,$$
$$\lim_{\xi\to-\infty} \frac{\psi(\xi+\theta_8)}{e^{\lambda_3\xi}} = 1, \quad \text{if } \lambda_3 < \Lambda,$$

where

$$\mu = 1, \quad \text{if } \int_{-\infty}^{\infty} \phi(\xi)\big(e^{-\phi(\xi)+\sigma_1\psi(\xi)} - 1\big)e^{-\Lambda\xi}\, d\xi \neq 0,$$
$$\mu = 0, \quad \text{if } \int_{-\infty}^{\infty} \phi(\xi)\big(e^{-\phi(\xi)+\sigma_1\psi(\xi)} - 1\big)e^{-\Lambda\xi}\, d\xi = 0.$$

Proof. From Lemma 2.3, $L(\lambda, \phi)$ and $L(\lambda, \psi)$ are also well defined for $\lambda \in \mathbb{C}$ with $Re\,\lambda \in (0, \Lambda)$ and $Re\,\lambda \in (0, \gamma)$, respectively. It follows from (2.2), (2.11) and (2.12) that

$$\Delta_1(\lambda, c) \int_{-\infty}^{\infty} \phi(\xi)e^{-\lambda\xi}\, d\xi = e^{r_1-\sigma_1 r_2} e^{d_1\lambda^2-c\lambda} \int_{-\infty}^{\infty} \phi(\xi)\big(e^{-\phi(\xi)+\sigma_1\psi(\xi)} - 1\big)e^{-\lambda\xi}\, d\xi \tag{2.15}$$

for $\lambda \in \mathbb{C}$ with $0 < Re\,\lambda < \Lambda$ and

$$\begin{aligned}\Delta_2(\lambda, c) &\int_{-\infty}^{\infty} \psi(\xi)e^{-\lambda\xi}\, d\xi \\ &= \sigma_2 r_2 e^{r_1-\sigma_1 r_2} e^{2d_1\lambda^2-2c\lambda} \frac{1}{\Delta_1(\lambda, c)} \int_{-\infty}^{\infty} \phi(\xi)\big(e^{-\phi(\xi)+\sigma_1\psi(\xi)} - 1\big)e^{-\lambda\xi}\, d\xi \\ &\quad + e^{d_2\lambda^2-c\lambda} \int_{-\infty}^{\infty} \big[r_2 - (1-r_2)\psi(\xi) - \big(r_2 - \psi(\xi)\big)e^{\psi(\xi)-\sigma_2\phi(\xi)} - \sigma_2 r_2\phi(\xi)\big]e^{-\lambda\xi}\, d\xi\end{aligned} \tag{2.16}$$

for $\lambda \in \mathbb{C}$ with $0 < Re\,\lambda < \gamma$ by (2.15). The following conclusions are obvious:

(1) $\lambda = \Lambda$ is a unique root with $Re\,\lambda = \Lambda$ of $\Delta_1(\lambda, c) = 0$ and $\lambda = \lambda_3$ is a unique root with $Re\,\lambda = \lambda_3$ of $\Delta_2(\lambda, c) = 0$.
(2) The function

$$\int_{-\infty}^{\infty} \phi(\xi)\big(e^{-\phi(\xi)+\sigma_1\psi(\xi)} - 1\big)e^{-\lambda\xi}\, d\xi$$

is analytic in the strip $0 < Re\,\lambda < \Lambda + \gamma$ by $e^x \leqslant \frac{1}{1-x}$, $x \in (-\infty, 1) \supset [-r_1, \sigma_1 r_2]$ and

$$\int_{-\infty}^{\infty} \big[r_2 - (1-r_2)\psi(\xi) - \big(r_2 - \psi(\xi)\big)e^{\psi(\xi)-\sigma_2\phi(\xi)} - \sigma_2 r_2\phi(\xi)\big]e^{-\lambda\xi}\, d\xi$$

is analytic in the strip $0 < Re\,\lambda < 2\gamma$ by Lemma 2.3, (2.13) and (2.14).

Let

$$\begin{aligned}F(\lambda) &:= \int_{-\infty}^{0} \phi(\xi)e^{-\lambda\xi}\, d\xi \\ &= \frac{e^{r_1-\sigma_1 r_2} e^{d_1\lambda^2-c\lambda} \int_{-\infty}^{\infty} \phi(\xi)(e^{-\phi(\xi)+\sigma_1\psi(\xi)} - 1)e^{-\lambda\xi}\, d\xi}{\Delta_1(\lambda, c)} - \int_{0}^{\infty} \phi(\xi)e^{-\lambda\xi}\, d\xi \quad \big(\text{by (2.15)}\big)\end{aligned} \tag{2.17}$$

and

$$H(\lambda) := \frac{e^{r_1-\sigma_1 r_2} e^{d_1\lambda^2-c\lambda} \int_{-\infty}^{\infty} \phi(\xi)(e^{-\phi(\xi)+\sigma_1\psi(\xi)}-1)e^{-\lambda\xi}\,d\xi}{\Delta_1(\lambda,c)/(\Lambda-\lambda)^{\nu+1}} - (\Lambda-\lambda)^{\nu+1}\int_0^{\infty} \phi(\xi)e^{-\lambda\xi}\,d\xi, \tag{2.18}$$

where $\nu = 0$ if $c > c_0$ and $\nu = \mu$ if $c = c_0$. By the relation between $F(\lambda)$ and $H(\lambda)$, we can get $H(\lambda)$ is analytic in the strip $0 < Re\,\lambda < \Lambda$. And, by the facts (1), (2) and the presentation of $H(\lambda)$, we also get $H(\lambda)$ is analytic in the strip $\{\lambda \mid Re\,\lambda = \Lambda\}$. Therefore, $H(\lambda)$ is analytic in the strip $0 < Re\,\lambda \leqslant \Lambda$. Since $\phi(\xi)$ is nondecreasing in $\mathbb{R}$, then, by Lemmas 2.2 and 2.4, it follows that

$$\lim_{\xi\to-\infty} \frac{\phi(\xi)}{|\xi|^{\nu}e^{\Lambda\xi}} = \frac{H(\Lambda)}{\Gamma(\Lambda+1)},$$

where $\nu = 0$ if $c > c_0$ and $\nu = \mu$ if $c = c_0$. If $H(\Lambda) \neq 0$, then (i) holds. Hence we only need to prove $H(\Lambda) \neq 0$.

If $c > c_0$, since Λ is a simple root of $\Delta_1(\lambda, c)$ and $\nu = 0$, it follows that the denominator of the first term of the right-hand side of (2.18) does not equal to zero for $0 < Re\,\lambda < \gamma + \Lambda$. We claim that

$$\int_{-\infty}^{\infty} \phi(\xi)\big(e^{-\phi(\xi)+\sigma_1\psi(\xi)}-1\big)e^{-\Lambda\xi}\,d\xi \neq 0. \tag{2.19}$$

In fact, if not, we obtain that $L(\Lambda, \phi)$ exists by (2.17), which contradicts Lemma 2.3. Thus $H(\Lambda) \neq 0$ by (2.18).

For $c = c_0$, then Λ is a double root of $\Delta_1(\lambda, c)$. If (2.19) holds, we can take $\mu = 1$ such that $H(\Lambda) \neq 0$ by (2.18). If (2.19) does not hold, then

$$\lim_{\lambda\to\Lambda} \frac{\int_{-\infty}^{\infty} \phi(\xi)(e^{-\phi(\xi)+\sigma_1\psi(\xi)}-1)e^{-\lambda\xi}\,d\xi}{(\lambda-\Lambda)^2}$$

does not exist. Indeed, if it exists, $L(\Lambda, \psi)$ exists by (2.17), which contradicts Lemma 2.3. Thus we can take $\mu = 0$ such that $H(\Lambda) \neq 0$ by (2.18).

Next we only need to prove (ii) since the proof of (iii) is similar. Let

$$\begin{aligned} F(\lambda) &:= \int_{-\infty}^{0} \psi(\xi)e^{-\lambda\xi}\,d\xi \\ &= -\int_0^{\infty} \psi(\xi)e^{-\lambda\xi}\,d\xi + \frac{\sigma_2 r_2 e^{r_1-\sigma_1 r_2} e^{2d_1\lambda^2-2c\lambda} \int_{-\infty}^{\infty} \phi(\xi)(e^{-\phi(\xi)+\sigma_1\psi(\xi)}-1)e^{-\lambda\xi}\,d\xi}{\Delta_1(\lambda,c)\Delta_2(\lambda,c)} \\ &\quad + \frac{e^{d_2\lambda^2-c\lambda}}{\Delta_2(\lambda,c)} \int_{-\infty}^{\infty} \big[r_2-(1-r_2)\psi(\xi) - \big(r_2-\psi(\xi)\big)e^{\psi(\xi)-\sigma_2\phi(\xi)} - \sigma_2 r_2\phi(\xi)\big]e^{-\lambda\xi}\,d\xi \quad \big(\text{by (2.16)}\big) \end{aligned} \tag{2.20}$$

and

$$H(\lambda) := (\gamma-\lambda)^{\nu+1}F(\lambda) \tag{2.21}$$

in the strip $0 < Re\,\lambda \leqslant \gamma$, where $\nu = 0$ if $\lambda_3 \neq \Lambda$, $\nu = 1$ if $\lambda_3 = \Lambda$. By using a similar argument as (i), $H(\lambda)$ is analytic in the strip $0 < Re\,\lambda \leqslant \gamma$. Since $\phi(\xi)$ is nondecreasing in $\mathbb{R}$, by Lemmas 2.2 and 2.4, it follows that

$$\lim_{\xi\to-\infty} \frac{\psi(\xi)}{|\xi|^{\nu}e^{\gamma\xi}} = \frac{H(\gamma)}{\Gamma(\gamma+1)},$$

where $\nu = 0$ if $\lambda_3 \neq \Lambda$, $\nu = 1$ if $\lambda_3 = \Lambda$. Next we only need to prove $H(\gamma) \neq 0$.

If $\lambda_3 \geqslant \Lambda$, then $\gamma = \Lambda$. Combining (2.20) with (2.21), it follows easily that $H(\gamma) \neq 0$ by (2.19). If $\lambda_3 < \Lambda$, then $\gamma = \lambda_3$. Since

$$H(\lambda) = \frac{e^{d_2\lambda^2-c\lambda} \int_{-\infty}^{\infty} [r_2-(1-r_2)\psi(\xi) - (r_2-\psi(\xi))e^{\psi(\xi)-\sigma_2\phi(\xi)}]e^{-\lambda\xi}\,d\xi}{\Delta_2(\lambda,c)/(\lambda_3-\lambda)} - (\lambda_3-\lambda)\int_0^{\infty} \psi(\xi)e^{-\lambda\xi}\,d\xi,$$

$H(\lambda_3) \neq 0$. Indeed, if $H(\lambda_3) = 0$, then

$$\int_{-\infty}^{\infty} \big[r_2-(1-r_2)\psi(\xi) - \big(r_2-\psi(\xi)\big)e^{\psi(\xi)-\sigma_2\phi(\xi)}\big]e^{-\lambda\xi}\,d\xi = 0,$$

furthermore, $\phi(\xi) \equiv \psi(\xi) \equiv 0$ by (2.14) and $\mathbf{0} \leqslant (\phi(\xi), \psi(\xi)) \leqslant \mathbf{r}$, which is a contradiction. The proof is completed. □

Next we investigate the asymptotic behavior of $(\phi(\xi), \psi(\xi))$ at the plus infinity. For convenience, let $\tilde{\phi} = r_1 - \phi$, $\tilde{\psi} = r_2 - \psi$, substituting $\tilde{\phi}, \tilde{\psi}$ into (2.2), we have

$$\begin{cases} \tilde{\phi}(\xi) = \displaystyle\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \big[r_1 - \big(r_1 - \tilde{\phi}(\xi - y)\big) e^{\tilde{\phi}(\xi - y) - \sigma_1 \tilde{\psi}(\xi - y)}\big]\, dy, \\ \tilde{\psi}(\xi) = \displaystyle\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_2}} e^{-\frac{(y-c)^2}{4d_2}} \tilde{\psi}(\xi - y) e^{r_2 - \sigma_2 r_1 - \tilde{\psi}(\xi - y) + \sigma_2 \tilde{\phi}(\xi - y)}\, dy \end{cases} \tag{2.22}$$

satisfying

$$\lim_{\xi \to -\infty} \big(\tilde{\phi}(\xi), \tilde{\psi}(\xi)\big) = \mathbf{r}, \qquad \lim_{\xi \to \infty} \big(\tilde{\phi}(\xi), \tilde{\psi}(\xi)\big) = \mathbf{0}. \tag{2.23}$$

Let

$$\Delta_3(\lambda, c) := 1 - \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1} - \lambda y + \ln(1 - r_1)}\, dy = 1 - e^{d_1 \lambda^2 - c\lambda + \ln(1 - r_1)}.$$

Then $\Delta_3(\lambda, c) = 0$ has a unique negative root $\lambda_4 = \frac{c - \sqrt{c^2 - 4d_1 \ln(1 - r_1)}}{2d_1} < 0$ (since $r_1 < 1$).

The characteristic equation of the second equation of (2.22) at $\mathbf{0}$ is $\Delta_4(\lambda, c) = 0$, where

$$\Delta_4(\lambda, c) := 1 - \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_2}} e^{-\frac{(y-c)^2}{4d_2} - \lambda y + r_2 - \sigma_2 r_1}\, dy = 1 - e^{d_2 \lambda^2 - c\lambda + r_2 - \sigma_2 r_1}.$$

Then $\Delta_4(\lambda, c) = 0$ has a unique negative root $\lambda_5 = \frac{c - \sqrt{c^2 - 4d_2(r_2 - \sigma_2 r_1)}}{2d_2} < 0$ by (1.3) and $\Delta_4(\lambda, c) < 0, \lambda \in (\lambda_5, 0)$.

Lemma 2.5. *Assume that* (1.3) *holds and* $(\phi(\xi), \psi(\xi))$ *is any nondecreasing solution of* (2.2) *and* (2.3) *with the wave speed* $c \geqslant c_0$. *Then the following is true*:

(i) $L(\lambda, \tilde{\psi}) < \infty, \lambda \in (\lambda_5, 0)$ *and* $L(\lambda, \tilde{\psi}) = \infty, \lambda \in \mathbb{R} \setminus (\lambda_5, 0)$;
(ii) $L(\lambda, \tilde{\phi}) < \infty, \lambda \in (\gamma_1, 0)$ *and* $L(\lambda, \tilde{\phi}) = \infty, \lambda \in \mathbb{R} \setminus (\gamma_1, 0)$, *where* $\gamma_1 = \max\{\lambda_4, \lambda_5\} < 0$.

Proof. Similar to Lemma 2.3, one can show that there exists $\lambda' < 0$ such that $L(\lambda, \tilde{\psi}) < \infty$, $\lambda \in (\lambda', 0)$. Next we show that there exists $\sigma < 0$ such that

$$L(\lambda, \tilde{\phi}) < \infty, \quad \lambda \in (\sigma, 0). \tag{2.24}$$

Since

$$d_1 \lambda^2 - c\lambda + \frac{1}{2} \ln(1 - r_1) = 0$$

has only two real roots: one is $\Lambda_* = \frac{c - \sqrt{c^2 - 2d_1 \ln(1 - r_1)}}{2d_1} < 0$, the other is positive. Then

$$d_1 \lambda^2 - c\lambda + \frac{1}{2} \ln(1 - r_1) < 0, \quad \lambda \in (\Lambda_*, 0). \tag{2.25}$$

Since $\tilde{\phi}(\xi) \to 0$ as $\xi \to \infty$, there exists ξ_0 large enough such that $\tilde{\phi}(\xi) \leqslant \sqrt{1 - r_1}$ for $\xi \geqslant \xi_0$. From (2.10), it follows that

$$\begin{aligned} r_1 - (1 - r_1)\tilde{\phi} - (r_1 - \tilde{\phi}) e^{\tilde{\phi} - \sigma_1 \tilde{\psi}} &\leqslant r_1 - (1 - r_1)\tilde{\phi} - (r_1 - \tilde{\phi})(1 + \tilde{\phi} - \sigma_1 \tilde{\psi}) \\ &= \tilde{\phi}^2 + \sigma_1 \tilde{\psi}(r_1 - \tilde{\phi}). \end{aligned} \tag{2.26}$$

Multiplying the second equation of (2.22) by $e^{-\lambda\xi}$ with $\lambda \in (\Lambda_*, 0)$ and integrating from $-\infty$ to ∞, by (2.26), we have

$$\begin{aligned} \sigma_1 r_1 e^{d_1 \lambda^2 - c\lambda} L(\lambda, \tilde{\psi}) &= \sigma_1 r_1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1} - \lambda y} \tilde{\psi}(\tilde{\xi}) e^{-\lambda\tilde{\xi}}\, d\tilde{\xi}\, dy \\ &= \sigma_1 r_1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \tilde{\psi}(\xi - y) e^{-\lambda\xi}\, dy\, d\xi \end{aligned}$$

$$
\begin{aligned}
&= \Delta_3(\lambda,c)L(\lambda,\tilde{\phi}) - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \\
&\quad \times \big[r_1 - (1-r_1)\tilde{\phi}(\xi-y) - \big(r_1 - \tilde{\phi}(\xi-y)\big)e^{\tilde{\phi}(\xi-y)-\sigma_1\tilde{\psi}(\xi-y)} - \sigma_1 r_1\tilde{\psi}(\xi-y)\big]e^{-\lambda\xi}\,dy\,d\xi \\
&\geqslant \Delta_3(\lambda,c)L(\lambda,\tilde{\phi}) - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \tilde{\phi}^2(\xi-y)e^{-\lambda\xi}\,dy\,d\xi \quad \big(\text{by (2.26)}\big) \\
&= \Delta_3(\lambda,c)L(\lambda,\tilde{\phi}) - e^{d_1\lambda^2-c\lambda}\int_{-\infty}^{\infty}\tilde{\phi}^2(\tilde{\xi})e^{-\lambda\tilde{\xi}}\,d\tilde{\xi} \\
&= \Delta_3(\lambda,c)\int_{-\infty}^{\xi_0}\tilde{\phi}(\tilde{\xi})e^{-\lambda\tilde{\xi}}\,d\tilde{\xi} - e^{d_1\lambda^2-c\lambda}\int_{-\infty}^{\xi_0}\tilde{\phi}^2(\tilde{\xi})e^{-\lambda\tilde{\xi}}\,d\tilde{\xi} \\
&\quad + \Delta_3(\lambda,c)\int_{\xi_0}^{\infty}\tilde{\phi}(\tilde{\xi})e^{-\lambda\tilde{\xi}}\,d\tilde{\xi} - e^{d_1\lambda^2-c\lambda}\int_{\xi_0}^{\infty}\tilde{\phi}^2(\tilde{\xi})e^{-\lambda\tilde{\xi}}\,d\tilde{\xi} \\
&\geqslant \Delta_3(\lambda,c)\int_{-\infty}^{\xi_0}\tilde{\phi}(\tilde{\xi})e^{-\lambda\tilde{\xi}}\,d\tilde{\xi} - e^{d_1\lambda^2-c\lambda}\int_{-\infty}^{\xi_0}\tilde{\phi}^2(\tilde{\xi})e^{-\lambda\tilde{\xi}}\,d\tilde{\xi} \\
&\quad + \big(1 - e^{d_1\lambda^2-c\lambda+\frac{1}{2}\ln(1-r_1)}\big)\int_{\xi_0}^{\infty}\tilde{\phi}(\tilde{\xi})e^{-\lambda\tilde{\xi}}\,d\tilde{\xi}
\end{aligned} \tag{2.27}
$$

with $\tilde{\xi} = \xi - y$. Since the singularity of $L(\lambda,\tilde{\phi})$ only happens at the plus infinity when $\lambda < 0$, the singularity of the last inequality of (2.27) is equivalent to that of the third term. Since the first and the second terms of the last inequality of (2.27) are finite, by Lemma 2.5, (2.25) and $\tilde{\phi}(\xi) \geqslant 0$, it follows that there exists $\sigma \in (\max\{\lambda', \Lambda_*\}, 0)$ such that $L(\lambda,\tilde{\phi})$ is well defined for $\lambda \in (\sigma, 0)$. The proofs of $\min\lambda' = \lambda_5$ and $\min\sigma = \gamma_1$ are similar to those of Lemma 2.3. The proof is completed. □

Using a similar argument as Theorem 2.1, we have the following the exponential asymptotic behavior of $(\tilde{\phi}(\xi), \tilde{\psi}(\xi)) = (r_1 - \phi(\xi), r_2 - \psi(\xi))$ at the plus infinity. We omit the proof here.

Theorem 2.2. *Assume that* (1.3) *holds and* $(\phi(\xi), \psi(\xi))$ *is any nondecreasing solution of* (2.2) *and* (2.3) *with the wave speed* $c \geqslant c_0$. *Then*

(i) *There exists* $\theta_9 = \theta_9(\phi,\psi)$ *such that* $\lim_{\xi\to\infty}\frac{r_2-\psi(\xi+\theta_9)}{e^{\lambda_5\xi}} = 1$.
(ii) *There exists* $\theta_i = \theta_i(\phi,\psi)$ $(i = 10, 11, 12)$ *such that*

$$
\begin{aligned}
&\lim_{\xi\to\infty}\frac{r_1-\phi(\xi+\theta_{10})}{e^{\lambda_5\xi}} = 1, \quad \text{if } \lambda_5 > \lambda_4, \\
&\lim_{\xi\to\infty}\frac{r_1-\phi(\xi+\theta_{11})}{\xi e^{\lambda_5\xi}} = 1, \quad \text{if } \lambda_5 = \lambda_4, \\
&\lim_{\xi\to\infty}\frac{r_1-\phi(\xi+\theta_{12})}{e^{\lambda_4\xi}} = 1, \quad \text{if } \lambda_5 < \lambda_4.
\end{aligned}
$$

3. Uniqueness

In this section, we adopt the strong comparison principle and the sliding method to prove the uniqueness of traveling wave fronts of (2.1). We first give the strong comparison principle.

Lemma 3.1. *Let* (ϕ_1,ψ_1) *and* (ϕ_2,ψ_2) *be two any nonnegative solutions of* (2.2) *and* (2.3) *with the wave speed* $c \geqslant c_0$ *satisfying* $\phi_1 \geqslant \phi_2$ *and* $\psi_1 \geqslant \psi_2$ *in* $\mathbb{R}$. *Then either* $\phi_1 > \phi_2$ *and* $\psi_1 > \psi_2$ *in* $\mathbb{R}$ *or* $\phi_1 \equiv \phi_2$ *and* $\psi_1 \equiv \psi_2$ *in* $\mathbb{R}$.

Proof. Let

$$f_1(t,s) = te^{r_1-\sigma_1 r_2 - t + \sigma_1 s} \quad \text{and} \quad f_2(t,s) = r_2 - (r_2 - s)e^{s-\sigma_2 t}.$$

By (2.7), we have

$$\begin{aligned} f_1(\phi_1,\psi_1) - f_1(\phi_2,\psi_2) &= \phi_1 e^{r_1-\sigma_1 r_2-\phi_1+\sigma_1\psi_1} - \phi_2 e^{r_1-\sigma_1 r_2-\phi_2+\sigma_1\psi_2} \\ &= \left(\phi_1 e^{-\phi_1} - \phi_2 e^{-\phi_2}\right)e^{r_1-\sigma_1 r_2+\sigma_1\psi_1} + \left(e^{\sigma_1\psi_1} - e^{\sigma_1\psi_2}\right)\phi_2 e^{r_1-\sigma_1 r_2-\phi_2} \geqslant 0, \end{aligned}$$

and by $(x-r_2)e^x$ is nondecreasing in $[r_2-1,\infty) \supset [0,\infty)$, we have

$$\begin{aligned} f_2(\phi_1,\psi_1) - f_2(\phi_2,\psi_2) &= (r_2-\psi_2)e^{\psi_2-\sigma_2\phi_2} - (r_2-\psi_1)e^{\psi_1-\sigma_2\phi_1} \\ &= \left[(\psi_1-r_2)e^{\psi_1} - (\psi_2-r_2)e^{\psi_2}\right]e^{-\sigma_2\phi_1} + (r_2-\psi_2)\left(e^{-\sigma_2\phi_2} - e^{-\sigma_2\phi_1}\right)e^{\psi_2} \geqslant 0. \end{aligned}$$

If these exists $\xi_0 \in \mathbb{R}$ such that $\phi_1(\xi_0) = \phi_2(\xi_0)$, then

$$\begin{aligned} 0 &= \phi_1(\xi_0) - \phi_2(\xi_0) \\ &= \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \left[f_1\left(\phi_1(\xi_0-y),\psi_1(\xi_0-y)\right) - f_1\left(\phi_2(\xi_0-y),\psi_2(\xi_0-y)\right)\right] dy \\ &\geqslant \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \times \left\{\left[\phi_1(\xi_0-y)e^{-\phi_1(\xi_0-y)} - \phi_2(\xi_0-y)e^{-\phi_2(\xi_0-y)}\right]e^{r_1-\sigma_1 r_2+\sigma_1\psi_1(\xi_0-y)} \right. \\ &\quad \left. + \left(e^{\sigma_1\psi_1(\xi_0-y)} - e^{\sigma_1\psi_2(\xi_0-y)}\right)\phi_2(\xi_0-y)e^{r_1-\sigma_1 r_2-\phi_2(\xi_0-y)}\right\} dy \\ &\geqslant 0, \end{aligned}$$

which implies that $\phi_1(\xi) \equiv \phi_2(\xi), \psi_1(\xi) \equiv \psi_2(\xi)$ by $\phi_1 \geqslant \phi_2, \psi_1 \geqslant \psi_2$. Similarly, if these exists $\xi_0 \in \mathbb{R}$ such that $\psi_1(\xi_0) = \psi_2(\xi_0)$, we also have $\phi_1(\xi) \equiv \phi_2(\xi), \psi_1(\xi) \equiv \psi_2(\xi)$. The proof is completed. □

Theorem 3.1. *Assume that* (1.3) *holds and $d_1 \geqslant d_2$. Then for every two traveling wave fronts $(\phi_1(\xi),\psi_1(\xi))$ and $(\phi_2(\xi),\psi_2(\xi))$ of* (2.1) *with the wave speed $c \geqslant c_0$ which connects* **0** *with* **r***, there exists $\xi_0 \in \mathbb{R}$ such that $(\phi_1(\xi+\xi_0),\psi_1(\xi+\xi_0)) \equiv (\phi_2(\xi),\psi_2(\xi))$.*

Proof. Since $\Lambda = \lambda_1$ or λ_2, by Theorem 2.1, there exists $\eta_i = \eta_i(\phi_i,\psi_i)$ $(i=1,2)$, such that one of the following is at least true:

(i) $\displaystyle\lim_{\xi\to-\infty} \frac{\phi_i(\xi+\eta_i)}{|\xi|^\omega e^{\lambda_1\xi}} = 1, \quad i=1,2;$

(ii) $\displaystyle\lim_{\xi\to-\infty} \frac{\phi_i(\xi+\eta_i)}{|\xi|^\omega e^{\lambda_2\xi}} = 1, \quad i=1,2;$

(iii) $\displaystyle\lim_{\xi\to-\infty} \frac{\phi_1(\xi+\eta_1)}{|\xi|^\omega e^{\lambda_1\xi}} = 1, \qquad \lim_{\xi\to-\infty} \frac{\phi_2(\xi+\eta_2)}{|\xi|^\omega e^{\lambda_2\xi}} = 1;$

(iv) $\displaystyle\lim_{\xi\to-\infty} \frac{\phi_1(\xi+\eta_1)}{|\xi|^\omega e^{\lambda_2\xi}} = 1, \qquad \lim_{\xi\to-\infty} \frac{\phi_2(\xi+\eta_2)}{|\xi|^\omega e^{\lambda_1\xi}} = 1,$

where $\omega = 0$ if $c > c_0$ and $\omega = \mu$ if $c = c_0$.

Therefore there exists $\theta_1 = \theta_1(\phi_1,\psi_1,\phi_2,\psi_2)$ such that one of the following is true, which corresponds to the cases as the above (i)–(iv):

(i) $\lim_{\xi\to-\infty}\phi_1(\xi+\theta_1)/\phi_2(\xi) = 1$, then $\lim_{\xi\to-\infty}\phi_1(\xi+\bar{\xi})/\phi_2(\xi) = e^{\lambda_1(\bar{\xi}-\theta_1)} > 1$ for all $\bar{\xi} > \max\{\theta_1, 0\}$;
(ii) $\lim_{\xi\to-\infty}\phi_1(\xi+\theta_1)/\phi_2(\xi) = 1$, then $\lim_{\xi\to-\infty}\phi_1(\xi+\bar{\xi})/\phi_2(\xi) = e^{\lambda_2(\bar{\xi}-\theta_1)} > 1$ for all $\bar{\xi} > \max\{\theta_1, 0\}$;
(iii) $\lim_{\xi\to-\infty}\phi_1(\xi+\theta_1)/\phi_2(\xi) = 1$ or ∞, then $\lim_{\xi\to-\infty}\phi_1(\xi+\bar{\xi})/\phi_2(\xi) > 1$ or $=\infty$ for all $\bar{\xi} > \max\{\theta_1, 0\}$ (since $\lambda_1 = \lambda_2$ or $\lambda_1 < \lambda_2$);
(iv) $\lim_{\xi\to-\infty}\phi_1(\xi)/\phi_2(\xi+\theta_1) = 1$ or 0, then $\lim_{\xi\to-\infty}\phi_1(\xi)/\phi_2(\xi+\bar{\xi}) < 1$ or $=0$ for all $\bar{\xi} < \min\{\theta_1, 0\}$ (since $\lambda_1 = \lambda_2$ or $\lambda_1 < \lambda_2$).

Assume that (i) happens. Since $d_1 \geqslant d_2$, then $\lambda_3 > \lambda_2 \geqslant \lambda_1$, and thus $\lambda_3 > \Lambda$. By Theorem 2.1 again, there exists $\theta_2 = \theta_2(\phi_1,\psi_1,\phi_2,\psi_2)$ such that $\lim_{\xi\to-\infty}\psi_1(\xi+\theta_2)/\psi_2(\xi) = 1$, then $\lim_{\xi\to-\infty}\psi_1(\xi+\bar{\xi})/\psi_2(\xi) = e^{\lambda_1(\bar{\xi}-\theta_2)} > 1$ for all $\bar{\xi} > \max\{\theta_2, 0\}$. Thus, choosing $\bar{\xi}_1 > \max\{\theta_1,\theta_2,0\}$, then there exists $N_1 \gg 1$ large enough such that $\phi_1(\xi+\bar{\xi}_2) \geqslant \phi_2(\xi)$ and $\psi_1(\xi+\bar{\xi}_1) \geqslant \psi_2(\xi)$, $\forall\xi \in (-\infty,-N_1]$. By the monotonicity of $(\phi_i(\xi),\psi_i(\xi))$ $(i=1,2)$, for any $\bar{\xi} \geqslant \bar{\xi}_1$, $\phi_1(\xi+\bar{\xi}) \geqslant \phi_2(\xi)$ and $\psi_1(\xi+\bar{\xi}) \geqslant \psi_2(\xi)$, $\forall\xi \in (-\infty,-N_1]$.

By Theorem 2.2, there exists $\theta_3 = \theta_3(\phi_1, \psi_1, \phi_2, \psi_2)$ such that $\lim_{\xi\to\infty}(r_2 - \psi_1(\xi + \theta_3))/(r_2 - \psi_2(\xi)) = 1$, then $\lim_{\xi\to\infty}(r_2 - \psi_1(\xi + \bar{\xi}))/(r_2 - \psi_2(\xi)) = e^{\lambda_5(\bar{\xi}-\theta_3)} < 1$ for all $\bar{\xi} > \max\{\theta_3, 0\}$. By Theorem 2.2 again, there exists $\theta_4 = \theta_4(\phi_1, \psi_1, \phi_2, \psi_2)$ such that $\lim_{\xi\to\infty}(r_1 - \phi_1(\xi + \theta_4))/(r_1 - \phi_2(\xi)) = 1$, then $\lim_{\xi\to\infty}(r_1 - \phi_1(\xi + \bar{\xi}))/(r_1 - \phi_2(\xi)) = e^{\lambda_*(\bar{\xi}-\theta_4)} < 1$ for all $\bar{\xi} > \max\{\theta_4, 0\}$, where $\lambda_* = \lambda_4$ or λ_5. Thus, choosing $\bar{\xi}_2 > \max\{\theta_3, \theta_4, 0\}$, then there exists $N_2 \gg 1$ large enough such that $\phi_1(\xi + \bar{\xi}_2) \geqslant \phi_2(\xi)$ and $\psi_1(\xi + \bar{\xi}_2) \geqslant \psi_2(\xi)$, $\forall\xi \in [N_2, \infty)$. By the monotonicity of $(\phi_i(\xi), \psi_i(\xi))$ $(i = 1, 2)$ for any $\bar{\xi} \geqslant \bar{\xi}_2$, $\phi_1(\xi + \bar{\xi}) \geqslant \phi_2(\xi)$ and $\psi_1(\xi + \bar{\xi}) \geqslant \psi_2(\xi), \forall\xi \in [N_2, \infty)$.

Take $N = \max\{N_1, N_2\}$, by the monotonicity of $(\phi_i(\xi), \psi_i(\xi))$ $(i = 1, 2)$, we can choose $\bar{\xi}_0 > \max\{\bar{\xi}_1, \bar{\xi}_2\}$ suitably large such that $\phi_1(\xi + \bar{\xi}_0) \geqslant \phi_2(\xi)$ and $\psi_1(\xi + \bar{\xi}_0) \geqslant \psi_2(\xi), \forall\xi \in [-N, N]$.

Therefore, from the above, $\phi_1(\xi + \bar{\xi}_0) \geqslant \phi_2(\xi)$ and $\psi_1(\xi + \bar{\xi}_0) \geqslant \psi_2(\xi), \forall\xi \in \mathbb{R}$. Then there exists $\xi_0 \leqslant \bar{\xi}_0$ (by translation) such that at least one of the following is true

(1) $\phi_1(\tilde{\xi} + \xi_0) = \phi_2(\tilde{\xi})$ for some $\tilde{\xi} \in \mathbb{R}, \phi_1(\xi + \xi_0) \geqslant \phi_2(\xi)$ and $\psi_1(\xi + \xi_0) \geqslant \psi_2(\xi), \xi \in \mathbb{R}$,
(2) $\psi_1(\tilde{\xi} + \xi_0) = \psi_2(\tilde{\xi})$ for some $\tilde{\xi} \in \mathbb{R}, \phi_1(\xi + t_0) \geqslant \phi_2(\xi)$ and $\psi_1(\xi + \xi_0) \geqslant \psi_2(\xi), \xi \in \mathbb{R}$.

Without loss of generality, we assume that (1) is true, since the traveling wave fronts of (2.1) are translation invariant solutions, $(\phi_1(\xi + \xi_0), \psi_1(\xi + \xi_0))$ is also a traveling wave front of (2.1). By Lemma 3.1, $\phi_1(\xi + \xi_0) \equiv \phi_2(\xi)$ and $\psi_1(\xi + \xi_0) \equiv \psi_2(\xi)$.

For the case (ii), the proof is completely similar.

We claim that the cases (iii) and (iv) cannot happen. Otherwise, if (iii) happens, similar argument to the above, there exists $\xi_0 \in \mathbb{R}$ such that $(\phi_1(\xi + \xi_0), \psi_1(\xi + \xi_0)) = (\phi_2(\xi), \psi_2(\xi))$, $\xi \in \mathbb{R}$, which contradicts the asymptotic behavior $\psi_1(\xi)$ and $\psi_2(\xi)$ at the minus infinity. Similarly, (iv) cannot hold. The proof is completed. □

References

[1] H. Berestycki, B. Nicolaenko, B. Scheurer, Travelling wave solutions to combustion models and their singular limits, SIAM J. Math. Anal. 16 (1985) 1207–1242.
[2] J. Carr, A. Chmaj, Uniqueness of travelling waves for nonlocal monostable equations, Proc. Amer. Math. Soc. 132 (2004) 2433–2439.
[3] X. Chen, J.S. Guo, Uniqueness and existence of travelling waves for discrete quasilinear monostable dynamics, Math. Ann. 326 (2003) 123–146.
[4] X. Chen, S.C. Fu, J.S. Guo, Uniqueness and asymptotics of traveling waves of monostable dynamics on lattices, SIAM J. Math. Anal. 38 (2006) 233–258.
[5] O. Diekmann, H.G. Kaper, On the bounded solutions of a nonlinear convolution equation, Nonlinear Anal. TMA 2 (1978) 721–737.
[6] J.S. Guo, C.H. Wu, Traveling wave front for a two-component lattice dynamical system arising in competition models, preprint.
[7] S. Hsu, X.Q. Zhao, Spreading speeds and traveling waves for nonmonotone integrodifference equations, SIAM J. Math. Anal. 40 (2008) 776–789.
[8] M.A. Lewis, Spread rate for a nonlinear stochastic invasion, J. Math. Biol. 41 (2000) 430–454.
[9] M.A. Lewis, B. Li, H.F. Weinberger, Spreading speed and linear determinacy for two-species competition models, J. Math. Biol. 45 (2002) 219–233.
[10] A.W. Leung, X.J. Hou, Y. Li, Exclusive traveling waves for competitive reaction–diffusion systems and their stabilities, J. Math. Anal. Appl. 338 (2008) 902–924.
[11] B. Li, Some remarks on traveling wave solutions in competition models, Discrete Contin. Dyn. Syst. Ser. B 12 (2009) 389–399.
[12] B. Li, H.F. Weinberger, M.A. Lewis, Spreading speeds as slowest wave speeds for cooperative systems, Math. Biosci. 196 (2005) 82–98.
[13] B. Li, M. Lewis, H. Weinberger, Existence of traveling waves for integral recursions with nonmonotone growth functions, J. Math. Biol. 58 (2009) 323–338.
[14] R. Lui, Biological growth and spread modeled by systems of recursions, I Mathematical theory, Math. Biosci. 93 (1989) 269–295.
[15] R. Lui, Biological growth and spread modeled by systems of recursions, II Biological theory, Math. Biosci. 93 (1989) 297–312.
[16] H. Wang, C. Castillo-Chavez, Spreading speeds and traveling waves for non-cooperative integro-difference systems, arXiv:1003.1600v2 [q-bio.PE].
[17] H.F. Weinberger, Asymptotic behavior of a model in population genetics, in: J.M. Chadam (Ed.), Nonlinear Partial Differential Equations and Applications, in: Lecture Notes in Math., vol. 648, Springer-Verlag, Berlin, 1978, pp. 47–96.
[18] H.F. Weinberger, Long-time behavior of a class of biological models, SIAM J. Math. Anal. 13 (1982) 353–396.