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On sofic actions and equivalence relations

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Abstract

The notion of sofic equivalence relation was introduced by Gabor Elek and Gabor Lippner. Their technics employ some graph theory. Here we define this notion in a more operator algebraic context, starting from Connes' Embedding Problem, and prove the equivalence of these two definitions. We introduce a notion of sofic action for an arbitrary group and prove that an amalgamated product of sofic actions over amenable groups is again sofic. We also prove that an amalgamated product of sofic groups over an amenable subgroup is again sofic.

Keywords: Sofic equivalence relations; Von Neumann algebras; Amalgamated product over amenable subgroups

Contents

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1. Introduction

Sofic groups were first introduced by Gromov, in the context of symbolic dynamics, motivated by the notion of surjunctivity (see [14]). A group is *surjunctive* if for every finite discrete set A the shift on A^G does not contain a proper copy of itself. Gromov showed that every sofic group has this property. The name "sofic" belongs to B. Weiss and was first used in [25].

Examples of sofic groups include amenable and residually finite groups. Elek and Szabo showed in [10] that the class of sofic groups is closed under the following constructions: direct products, subgroups, inverse limits, direct limits, free products, amenable extensions.

Apart from *Gottschalk's Surjunctivity Conjecture* (asserting that every group is surjunctive), there are other conjectures about countable groups known to hold for sofic groups. Elek and Szabo [8] proved that *Kaplansky's Conjecture* is true in the case of sofic groups. For a nice survey on sofic groups (and the related notion of hyperlinear group) see [18] and [19].

In [7], Elek and Lippner introduced the notion of *sofic equivalence relation*. They showed that treeable equivalence relations, as well as equivalence relations arising from Bernoulli actions of sofic groups are sofic. As in the case of groups, some conjectures about equivalence relations are true for sofic equivalence relations. Elek and Lippner proved the *Measure-Theoretic Determinant Conjecture of Lück, Sauer and Wegner* in the case of sofic equivalence relations. We don't know an example of a non-sofic equivalence relation.

The main purpose of this article is to present an operator algebraic motivation for this notion of *soficity*. We shall begin with one of the central open problems of operator algebra theory, namely *Connes' Embedding Problem*. It asserts that every finite, separable von Neumann algebra is embeddable in a tracial ultraproduct of the hyperfinite factor (denoted by R^ω). The study of this conjecture for group algebras led to the notion of *hyperlinear group*, a notion similar to the sofic group. In view of this discussion, it is natural to ask when a crossed product algebra embeds in R^ω. This question suggested the definition of *sofic action*. By investigating its basic properties we are able to present an operator algebraic definition for *sofic equivalence relations*.

1.1. Von Neumann algebras

A *von Neumann algebra* is a $*$-algebra of bounded operators on a Hilbert space that is closed in the weak operator topology and contains the identity operator. This means $\mathcal{B}(H)$ itself is a von Neumann algebra, as are matrix algebras as the particular case when H is finite dimensional.

A von Neumann algebra is a *factor* if its center consists only of scalars $\mathbb{C} \cdot 1$. Factors are the building blocks for all von Neumann algebras, as proven by von Neumann in 1949 (see [6] for a proof). The decomposition of an algebra into factors is essentially unique.

At the other end of the spectrum there are the abelian von Neumann algebras. An algebra of this type is isomorphic to $L^\infty(X)$ for some measure space (X, μ).

We are interested only in algebras that poses a *finite trace*, that is a faithful, positive linear functional $Tr : M \to \mathbb{C}$ such that $Tr(1) = 1$ and $Tr(xy) = Tr(yx)$ for any $x, y \in M$. Examples include matrix algebras and group algebras, $L(G)$ the weak closure in $\mathcal{B}(L^2(G))$ of the algebra generated by λ_g, the left translation operators. The trace on $L(G)$ is determined by $Tr(\lambda_e) = 1$ and $Tr(\lambda_g) = 0$ for $g \neq e$. A factor with such a trace is called a *type II_1 factor*. The group algebra $L(G)$ is a factor iff G has *infinite conjugacy classes* (*ICC*). We shall later see how to associate such an algebra to an action or an equivalence relation.

A von Neumann algebra M is *hyperfinite* if it contains an increasing chain of finite dimensional algebras whose union is weakly dense in M. Murray and von Neumann proved that up to isomorphism there exists only one hyperfinite type II_1 factor. We shall denote this factor by R. In his classic paper [3], Connes proved that the group algebra $L(G)$ is hyperfinite iff G is amenable. S_∞^{fin} is an example of an ICC amenable group, so $R = L(S_\infty^{fin})$:

$$S_\infty^{fin} = \{ f : \mathbb{N} \to \mathbb{N} \mid f \text{ bijective and } \exists k \in \mathbb{N} \text{ such that } f(n) = n \ \forall n > k \}.$$

1.2. Ultraproducts and Connes' Embedding Problem

In order to state Connes' Embedding Problem we need to understand what a tracial ultraproduct is. In a way the first example of an ultraproduct is the construction of the real numbers by Cauchy sequences. The real numbers are the set of Cauchy sequences of rational numbers factor by those sequences that converge to zero. This definition is not suitable for generalizations, as in general is difficult or even impossible to define the notion of Cauchy sequence. We bypass this by using an ultrafilter. Let ω be a free ultrafilter on $\mathbb{N}$. With this new technical tool the real numbers are the set of bounded sequences of rational numbers factor by those sequences convergent to zero w.r.t. ω. We now generalize this to metric groups.

Example 1.1. Let $(G_i, d_i)_{i \in \mathbb{N}}$ be a sequence of groups (not necessary countable) with a biinvariant metric ($d_i(x, y) = d_i(zx, zy) = d_i(xz, yz)$ for all $x, y, z \in G_i$). Define:

$$\mathcal{G} = \Big\{ x \in \Pi_i G_i \colon \sup_i d_i(x_i, e) < \infty \Big\};$$
$$\mathcal{N}_\omega = \Big\{ x \in \mathcal{G} \colon \lim_{i \to \omega} d_i(x_i, e) = 0 \Big\}.$$

Due to the biinvariance property, $\mathcal{N}_\omega$ is a normal subgroup of $\mathcal{G}$ (biinvariance is essential for this to hold, see [19], p. 9, Example 3.2). We define now the ultraproduct of metric groups:

$$\Pi_{i \to \omega}(G_i, d_i) = \mathcal{G}/\mathcal{N}_\omega$$

and the distance $d(x, y) = \lim_{i \to \omega} d(x_i, y_i)$. An easy diagonal argument will show that $(\Pi_{i \to \omega}(G_i, d_i), d)$ is complete.

Example 1.2. Let $(A_i)_{i \in \mathbb{N}}$ a sequence of Banach spaces, or Banach algebras or C^*-algebras. Using the metric induced by the norm, we can construct $\Pi_{i \to \omega} A_i$ exactly as in the previous example. Verifications that the ultraproduct is a Banach space/Banach algebra/C^*-algebra are straightforward, with the exception of completeness that is a diagonal argument.

Example 1.3. For von Neumann algebras things are different and a construction is possible only for finite algebras where we have a trace. Let (M, Tr) be a von Neumann algebra with a finite trace. Besides the operatorial norm, M posses a *Hilbert–Schmidt norm*: $\|x\|_2 = Tr(x^*x)^{1/2}$. For a matrix $(a_{ij}) \in M_n(\mathbb{C})$ this is $\|(a_{ij})\|_2 = (\frac{1}{n}\sum_{ij}|a_{ij}|^2)^{1/2}$. When we construct the ultraproduct we have to take into consideration this norm.

Let (M_i, Tr) a sequence of finite von Neumann algebras with normalized trace. Define:

$$l^\infty(\mathbb{N}, M_i) = \Big\{x \in \Pi_i M_i\colon \sup_i \|x_i\| < \infty\Big\},$$

$$\mathcal{N}_\omega = \Big\{x \in l^\infty(\mathbb{N}, M_i)\colon \lim_{i\to\omega} \|x_i\|_2 = 0\Big\}, \quad \text{and}$$

$$\Pi_{i\to\omega} M_i = l^\infty(\mathbb{N}, M_i)/\mathcal{N}_\omega.$$

The ultraproduct $\Pi_{i\to\omega} M_i$ is a von Neumann algebra, though the proof is a little involved. If $x_i \in M_i$ we shall denote by $\Pi_{i\to\omega} x_i$ the corresponding element in the ultraproduct.

Note that this algebra has a faithful trace, namely $Tr(x) = \lim_{i\to\omega} Tr_{M_i}(x_i)$, where $x = \Pi_{i\to\omega} x_i$. If $M_i = M$ for all i we shall denote $\Pi_{i\to\omega} M_i$ by M^ω (this is called *an ultrapower* of M, its isomorphism class may depend on ω).

The following well-know proposition is a very useful property of ultraproducts.

Proposition 1.4. *Inside the ultraproduct algebra we have*: $\mathcal{U}(\Pi_{i\to\omega} M_i) = \Pi_{i\to\omega}\mathcal{U}(M_i)$. *As group of unitaries* $(\Pi_{i\to\omega}\mathcal{U}(M_i), \|\cdot\|_2) = \Pi_{i\to\omega}(\mathcal{U}(M_i), \|\cdot\|_2)$ (*ultraproduct of metric groups as in Example* 1.1).

Proof. As any sequence of unitaries is bounded, the second equality of the proposition is deduced from definitions. Inclusion $\Pi_{i\to\omega}\mathcal{U}(M_i) \subset \mathcal{U}(\Pi_{i\to\omega} M_i)$ is trivial. Let now $u = \Pi_{i\to\omega} u_i \in \mathcal{U}(\Pi_{i\to\omega} M_i)$. Because M_i is a von Neumann algebra we have the polar decomposition $u_i = v_i|u_i|$, with v_i a partial isometry. Because M_i is a finite von Neumann algebra v_i can be extended to a unitary, denoted also by v_i, such that we still have $u_i = v_i|u_i|$. Now:

$$\big(\Pi_{i\to\omega}|u_i|\big)^2 = \Pi_{i\to\omega} u_i^* u_i = (\Pi_{i\to\omega} u_i)^*(\Pi_{i\to\omega} u_i) = 1,$$

so $\Pi_{i\to\omega}|u_i|$ is a positive element and its square is 1. We are in a C^*-algebra so we can deduce that $\Pi_{i\to\omega}|u_i| = 1$. Then $\Pi_{i\to\omega} u_i = \Pi_{i\to\omega} v_i \in \Pi_{i\to\omega}\mathcal{U}(M_i)$. □

This proposition implies that $\mathcal{U}(\Pi_{i\to\omega} M_i)$ is closed in the Hilbert–Schmidt norm (as any ultraproduct of metric group is complete). Together with some extra machinery we can use this property to show that $\Pi_{i\to\omega} M_i$ is indeed a von Neumann algebra.

In his famous article [3], A. Connes stated the following conjecture. Many definitions and results in this article are motivated by this open problem.

Question 1.5 *(CEP [1976]).* Do all separable type II_1 factors admit a trace preserving embedding in R^ω (ultrapower of the hyperfinite II_1-factor)?

1.3. A trick for diagonal arguments

This article contains plenty of diagonal arguments. Some of this arguments can be bypassed using product ultrafilters. These ideas are from [1].

Definition 1.6. If ω, ϕ are ultrafilters on $\mathbb{N}$, then define the *product ultrafilter* $\omega \otimes \phi$ on $\mathbb{N} \times \mathbb{N}$ by:

$$F \in \omega \otimes \phi \quad \Leftrightarrow \quad \big\{i \in \mathbb{N}\colon \big\{j \in \mathbb{N}\colon (i, j) \in F\big\} \in \phi\big\} \in \omega.$$

Some computations will show $\omega \otimes \phi$ is indeed an ultrafilter. Because $\mathbb{N}$ is in bijection with $\mathbb{N}^2$, $\omega \otimes \phi$ can be still considered as an ultrafilter on $\mathbb{N}$. The following proposition can be easily checked.

Proposition 1.7. *If* $\{x_i^j\}_{(i,j)\in\mathbb{N}^2}$ *is a bounded sequence of real numbers then*:

$$\lim_{i\to\omega}\Big(\lim_{j\to\phi} x_i^j\Big) = \lim_{(i,j)\to\omega\otimes\phi} x_i^j.$$

A diagonal argument, in general, means selecting in a clever way a subset of $\mathbb{N} \times \mathbb{N}$. The idea of this section is that the product ultrafilter will do the job for us. This is mainly because of its properties contained in previous proposition. A relevant consequence for this proposition is the following result.

Proposition 1.8. *(See Proposition 2.1 from [1].) If* $\{M_i^j\}_{(i,j)\in\mathbb{N}^2}$ *is a sequence of finite von Neumann algebras then*:

$$\Pi_{i\to\omega}\big(\Pi_{j\to\phi} M_i^j\big) = \Pi_{(i,j)\to\omega\otimes\phi} M_i^j.$$

Let us now present an example where a diagonal argument can be bypassed.

Proposition 1.9. *Any type* II_1 *factor embedding in* R^ω *also embeds in an ultraproduct of matrix algebras.*

Proof. Approximating the hyperfinite factor by matrix algebras we can easily see that $R \subset \Pi_{k\to\omega} M_{n_k}$. The proof can be finished using this embedding, the initial embedding $M \subset R^\omega$ and a diagonal argument. Instead we can write: $R^\omega \subset (\Pi_{k\to\omega} M_{n_k})^\omega \simeq \Pi_{(i,k)\to\omega\otimes\omega} M_{m_{(i,k)}}$, where $m_{(i,k)} = n_k$. $\square$

This proposition allows us to work with ultraproducts of matrix algebras instead of R^ω.

1.4. Hyperlinear groups

By studying Connes' Embedding Problem for group algebras, we reach the following definition.

Definition 1.10 *(Rădulescu, 2000).* A countable group is called *hyperlinear* if there exists a trace preserving embedding of $L(G)$ in R^ω.

Needless to say that we don't know a group that is non-hyperlinear, this will solve in negative Connes' Embedding Problem. By using 1.4, 1.9 and the definition of group algebra, we get the following description:

Proposition 1.11. *A group G is hyperlinear iff there exists a sequence $\{n_k\}_k \subset \mathbb{N}$, $\lim_k n_k = \infty$ and a group morphism $\Theta : G \to \Pi_{k\to\omega}\mathcal{U}(n_k)$ such that $Tr(\Theta(g)) = 0$ for any $g \neq e$.*

The numbers n_k don't play a special role here. If such a morphism exists for a sequence, it will exists for any other sequence $\{m_k\}$ as long as $\lim_k m_k = \infty$. The following theorem is due to Florin Rădulescu (see [22], Proposition 2.5) and an earlier work of Eberhard Kirchberg. It contains a very useful tool called *amplification*, that will be used many times in this article.

Theorem 1.12. *A group G is hyperlinear iff there exists an injective group morphism $\Theta : G \to \Pi_{k\to\omega}\mathcal{U}(n_k)$ (we don't need to care about the trace).*

Proof. We shall prove this result when the center of the group is trivial (ICC groups have this property). The proof in general case is not difficult, but it is a little technical and uninteresting to our discussion.

Let $\Theta : G \to \Pi_{k\to\omega}\mathcal{U}(n_k)$ be an injective morphism. Let $\Theta(g) = \Pi_{k\to\omega}u_g^k$ with $u_g^k \in \mathcal{U}(n_k)$. If $|Tr(\Theta(g))| = 1$ then $\|\Theta(g) - \lambda\|_2 = 0$, where $\lambda = Tr(\Theta(g))$. This implies $\Theta(g) = \lambda$, so $\Theta(g)$ commutes with $\Theta(h)$ for any $h \in G$. Because Θ is injective it follows that g is in the center of G, so by our assumption $g = e$. In the end we have $|Tr(\Theta(g))| < 1$ for any $g \neq e$.

Construct $\Theta^{(m)} = \Theta \otimes \Theta \otimes \cdots \otimes \Theta$ (m times tensor product), i.e. $\Theta^{(m)}(g) = \Pi_{k\to\omega}u_g^k \otimes u_g^k \otimes \cdots \otimes u_g^k$. This is a representation of G on $\Pi_{k\to\omega}\mathcal{U}(n_k^m)$. Then $Tr(\Theta^{(m)}(g)) = Tr(\Theta(g))^m$. This means that $Tr(\Theta^{(m)}(g)) \to_{m\to\infty} 0$ for $g \neq e$. A classic diagonal argument will finish the proof.

Alternatively, let us use the methods presented in Section 1.3. The product $\omega \otimes \omega$ is an ultrafilter on $\mathbb{N} \times \mathbb{N}$. Construct $\Phi : G \to \Pi_{(m,k)\to\omega\otimes\omega}\mathcal{U}(n_k^m)$ by $\Phi(g) = \Pi_{m\to\omega}\Theta^{(m)}(g)$. Then by 1.7, $Tr(\Phi(g)) = \lim_{m\to\omega} Tr(\Theta^{(m)}(g)) = 0$. □

1.5. Group actions and crossed product

Apart from group algebras, finite von Neumann algebras arise naturally from group actions on measure spaces. We shall work only with *probability measure preserving* actions. Let (X, μ) be a standard probability space and let $\alpha : G \to Aut(X, \mu)$ be a measure preserving action. The algebraic crossed product is defined by:

$$L^\infty(X) \rtimes_\alpha^{alg} G = \left\{ \sum_{finite} a_g u_g \colon\ a_g \in L^\infty(X),\ g \in G \right\}.$$

The $*$-algebraic structure is defined by:

$$u_g u_h = u_{gh}, \qquad u_g a u_g^* = \alpha(g)(a), \qquad u_g^* = u_{g^{-1}},$$

and multiplication in $L^\infty(X)$ is preserved inside the crossed product. The trace is:

$$Tr\Big(\sum a_g u_g\Big) = \int_X a_e \, d\mu.$$

The von Neumann algebra $L^\infty(X) \rtimes_\alpha G$ will be the weak closure of $L^\infty(X) \rtimes_\alpha^{alg} G$ in the GNS representation of $(L^\infty(X) \rtimes_\alpha^{alg} G, Tr)$.

A crossed product is a copy of an abelian von Neumann algebra $L^\infty(X)$ together with a set of unitaries $\{u_g\colon g \in G\}$ that act on the abelian algebra in the manner prescribed by the action α. This algebra is a factor iff the action is ergodic. It is a hyperlinear algebra iff G is amenable as shown by Connes in [3].

We shall denote by $\mathcal{U}(M)$ the group of unitaries in the algebra M. For a von Neumann algebra inclusion $A \subset M$ define the normalizer $\mathcal{N}_M(A)$:

$$\mathcal{N}_M(A) = \big\{u \in \mathcal{U}(M)\colon uAu^* = A\big\}.$$

By the definition of the crossed product $\{u_g\colon g \in G\}$ is included in the normalizer of $L^\infty(X)$ in $L^\infty(X) \rtimes_\alpha G$. The following elementary example is crucial to our discussion.

Example 1.13. Let $M_n = M_n(\mathbb{C})$ be a matrix algebra. We shall denote by $D_n \subset M_n$ the subalgebra of diagonal matrices and by $P_n \subset M_n$ the subgroup of permutation matrices. Then:

$$\mathcal{N}_{M_n}(D_n) = \mathcal{U}(D_n) \cdot P_n.$$

Notation 1.14. Given an ultraproduct $\Pi_{k\to\omega} M_{n_k}(\mathbb{C})$ we shall denote by $\Pi_{k\to\omega} D_{n_k}$ and $\Pi_{k\to\omega} P_{n_k}$ the corresponding subsets.

By a theorem of Sorin Popa (see [20], Proposition 4.3) $\Pi_{k\to\omega} D_{n_k}$ is a maximal abelian non-separable subalgebra of $\Pi_{k\to\omega} M_{n_k}(\mathbb{C})$. It can be proven that its normalizer is the ultraproduct of normalizers:

$$\mathcal{N}(\Pi_{k\to\omega} D_{n_k}) = \mathcal{U}(\Pi_{k\to\omega} D_{n_k}) \cdot \Pi_{k\to\omega} P_{n_k}. \tag{1}$$

2. Sofic objects

We now begin the study of Connes' Embedding Problem for crossed product algebras. We can easily see that if a crossed product algebra $L^\infty(X) \rtimes_\alpha G$ embeds in an ultraproduct $\Pi_{k\to\omega} M_{n_k}$ then we can construct an embedding $\Theta : L^\infty(X) \rtimes_\alpha G \to \Pi_{k\to\omega} M_{n_k}$ such that $\Theta(L^\infty(X)) \subset \Pi_{k\to\omega} D_{n_k}$. In fact this is a property of hyperfinite algebras.

Proposition 2.1. *Let N be a hyperfinite algebra and let $\Theta_1, \Theta_2 : N \to \Pi_{k\to\omega} M_{n_k}$ be two embeddings. Then there exists a unitary $u \in \mathcal{U}(\Pi_{k\to\omega} M_{n_k})$ such that $\Theta_2 = Ad\,u \circ \Theta_1$.*

Proof *(Sketch).* Let $N = \overline{\bigcup_i N_i}^w$, where N_i are finite dimensional subalgebras. Find a unitary $u_i \in \mathcal{U}(\Pi_{k\to\omega} M_{n_k})$ such that $\Theta_2(x) = u_i \Theta_1(x) u_i^*$ for any $x \in N_i$. By a diagonal argument construct u such that $\Theta_2(x) = u\Theta_1(x)u^*$, for any $x \in \bigcup_i N_i$. □

In [15], Kenley Jung proved the converse of this result, if any two embeddings in R^ω of a von Neumann algebra N are conjugate by a unitary then N is hyperfinite.

Now we come back to our problem of constructing an embedding Θ of $L^\infty(X) \rtimes_\alpha G$ in $\Pi_{k\to\omega} M_{n_k}$. We can safely assume that $\Theta(L^\infty(X)) \subset \Pi_{k\to\omega} D_{n_k}$. It is very difficult to find unitaries in the ultraproduct that are in the normalizer of $\Theta(L^\infty(X))$. Inspired by equality (1) we shall assume that $\Theta(u_g) \in \Pi_{k\to\omega} P_{n_k}$ for any $g \in G$. This is primarily a restriction on the group.

2.1. Sofic groups

Definition 2.2. A group G is called *sofic* if there exists a sequence $\{n_k\}_k \subset \mathbb{N}$, $\lim_k n_k = \infty$ and a group morphism $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$ such that $Tr(\Theta(g)) = 0$ for any $g \neq e$.

The following theorem due to Elek and Szabo [9], is similar to 1.12.

Theorem 2.3. *A group G is sofic iff there exists an injective group morphism $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$.*

Any sofic group is hyperlinear, while the converse implication is unknown. It is important to note that, in the case of permutations, the Hilbert–Schmidt distance used to construct the ultraproduct is related to the normalized Hamming distance.

Definition 2.4. For $\sigma, \tau \in S_n$ define the *normalized Hamming distance* by:

$$d_{hamm}(\sigma, \tau) = \frac{1}{n} Card\big\{i\colon \sigma(i) \neq \tau(i)\big\}.$$

The following definition is sometimes easier to check for a particular example.

Proposition 2.5. *A group G is sofic iff for every finite $F \subset G$ and every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ and $\theta : F \to S_n$ such that*:

- *if $g, h, gh \in F$, $d_{hamm}(\Theta(g)\Theta(h), \Theta(gh)) < \varepsilon$;*
- *if $g \in F$, $g \neq e$, $d_{hamm}(\Theta(g), Id) > 1/2$.*

The value $1/2$ from the definition can be replaced by any real number in $(0, 1)$. With this definition we can easily see that residual finite groups are sofic. This includes free groups. Amenable groups are sofic (use a Folner sequence to construct permutations). As we said in the introduction the class of sofic groups is closed under the following constructions: direct products, subgroups, inverse limits, direct limits, free products, amenable extensions (see [10]). In 2008 A. Thom [24] constructed a hyperlinear group not know to be sofic at the moment. His motivation was to provide an example of a sofic group that is not initially subamenable (every finite part of the multiplication table can be recovered inside an amenable group). In 2009 Cornulier presented in [5] an example of such a sofic group. Thompson groups F, T and V are not known to be hyperlinear.

2.2. Definitions of hyperlinear and sofic actions

We now introduce a notion of hyperlinearity and soficity for actions.

Definition 2.6. An action α of a countable group G on a standard probability space (X, μ) is called *hyperlinear* if the crossed product $L^\infty(X) \rtimes_\alpha G$ admits a trace preserving embedding in R^ω.

Definition 2.7. An action α of a countable group G on a standard probability space (X, μ) is called *sofic* if there exists a trace preserving embedding $\Theta : L^\infty(X) \rtimes_\alpha G \to \Pi_{k\to\omega} M_{n_k}(\mathbb{C})$ such that $\Theta(L^\infty(X)) \subset \Pi_{k\to\omega} D_{n_k}(\mathbb{C})$ and $\Theta(u_g) \in \Pi_{k\to\omega} P_{n_k}(\mathbb{C})$ for all $g \in G$.

The property of hyperlinear/sofic action is invariant under orbit equivalence. This is actually a property of the crossed product. Recall that two free actions $\alpha : G \to Aut(X, \mu)$ and $\beta : H \to Aut(X, \mu)$ on the same measure space are *orbit equivalent* if $\alpha(G)(x) = \beta(H)(x)$ for almost all $x \in X$.

Theorem 2.8. *(See Singer [23].) Let $\alpha : G \to Aut(X, \mu)$ and $\beta : H \to Aut(X, \mu)$ be two free actions on the same probability space. Then α and β are orbit equivalent iff there exists a von Neumann algebra isomorphism $\Psi : L^\infty(X) \rtimes_\alpha G \to L^\infty(X) \rtimes_\beta H$ such that Ψ is the identity on $L^\infty(X)$.*

Proof *(Sketch of the direct implication).* Let $\{u_g\colon g \in G\}$ and $\{v_h\colon h \in H\}$ be the unitaries in the crossed product that implement the actions α and β respectively. Let $p_g^h \in L^\infty(X)$ be the projection onto the set $\{x \in X\colon \alpha(g)(x) = \beta(h)(x)\}$. Because of the orbit equivalence we deduce that $\sum_g p_g^h = 1\ \forall h$ and $\sum_h p_g^h = 1\ \forall g$. Define now $\Psi : L^\infty(X) \rtimes_\alpha G \to L^\infty(X) \rtimes_\beta H$ by

$$\Psi(a) = a \quad \forall a \in L^\infty(X);$$
$$\Psi(u_g) = \sum_h p_g^h v_h \quad \forall g \in G.$$

Notice that $\Psi(p_g^h u_g) = p_g^h v_h$ so $\Psi(\sum_h p_g^h u_g) = v_h$. It follows that Ψ is an isomorphism. □

The next theorem hints very clearly that being sofic is a property of the orbit equivalence relation, rather than of the action itself.

Theorem 2.9. *Let α and β be two free orbit equivalent actions. If α is hyperlinear* (*sofic*) *then also β is hyperlinear* (*sofic*).

Proof. Let $\Psi : L^\infty(X) \rtimes_\alpha G \to L^\infty(X) \rtimes_\beta H$ be the isomorphism constructed in the previous proposition. The existence of such an isomorphism is enough to deduce the hyperlinear part of the theorem. Consider now $\Theta : L^\infty(X) \rtimes_\beta H \to \Pi_{k\to\omega} M_{n_k}$ an embedding like in Definition 2.7. We shall prove that $\Theta \circ \Psi$ is the required embedding for $L^\infty(X) \rtimes_\alpha G$.

Because Ψ equals identity on $L^\infty(X)$ we deduce that $\Theta \circ \Psi(L^\infty(X)) \subset \Pi_{k\to\omega} D_{n_k}$. Using the same notations as in the previous theorem we have $\Theta \circ \Psi(u_g) = \sum_h \Theta(p_g^h)\Theta(v_h)$. By hypothesis $\Theta(v_h) \in \Pi_{k\to\omega} P_{n_k}$ and $\Theta(p_g^h)$ is a projection in $\Pi_{k\to\omega} D_{n_k}$. By the next lemma, $\Theta \circ \Psi(u_g) \in \Pi_{k\to\omega} P_{n_k}$ and we are done. □

The following lemma is not difficult, but it is essential to our discussion. We shall use this lemma many times. It provides elements in $\Pi_{k\to\omega} P_{n_k}$ which are needed if we want to prove the soficity of a certain object.

Lemma 2.10. *Let $\{e_i \mid i \in \mathbb{N}\}$ be projections in $\Pi_{k\to\omega} D_{n_k}$ such that $\sum_i e_i = 1$. Let $\{u_i \mid i \in \mathbb{N}\}$ be unitary elements in $\Pi_{k\to\omega} P_{n_k}$ such that $v = \sum_i e_i u_i$ is a unitary. Then $v \in \Pi_{k\to\omega} P_{n_k}$.*

Proof. We should first visualize this result inside the algebra $M_n(\mathbb{C})$. Let $\{e_i \mid i \in \mathbb{N}\}$ be projections in D_n such that $\sum_i e_i = 1$ (only a finite number of projections will be nonzero in this case). Let $\{u_i \mid i \in \mathbb{N}\} \subset P_n$ such that $v = \sum_i e_i u_i$ is a unitary. The matrix v has only 0 and 1 entries and exactly one entry of 1 on each row. On each column there has to be a nonzero entry, otherwise v cannot be a unitary. This is enough to deduce that v is a permutation matrix.

Now back to the general case. Using the equation $\sum_i e_i = 1$ we can construct projections $e_i^k \in D_{n_k}$ such that:

1. $e_i = \Pi_{k\to\omega} e_i^k$;
2. $\sum_i e_i^k = 1_{n_k}$.

By hypothesis we have $u_i = \Pi_{k\to\omega} u_i^k$ where $u_i^k \in P_{n_k}$. If $v^k = \sum_i e_i^k u_i^k$ then $v = \Pi_{k\to\omega} v^k$, but v^k are not necessary unitary matrices. However v^k is still a matrix only with 0 and 1 entries and exactly one entry of 1 on each row.

In order to prove that $v \in \Pi_{k\to\omega} P_{n_k}$ we shall construct $w^k \in P_{n_k}$ such that $\lim_{k\to\omega} \|v^k - w^k\|_2 = 0$. For this we need to estimate the number of columns in v^k having only 0 entries. Denote this number by r_k. Then $v^{k*}v^k$ is a diagonal matrix having r_k entries of 0 on the diagonal. This implies:

$$\left\|v^{k*}v^k - Id\right\|_2^2 \geqslant \frac{r_k}{n_k}.$$

Because $\Pi_{k\to\omega} v^{k*}v^k = 1$ we have $r_k/n_k \to_{k\to\omega} 0$. This relation represents the upper bound of r_k that we need. We now construct w^k as follows. The matrix v^k has $n_k - r_k$ columns with at least one nonzero entry. For each such column j chose a row i such that $v^k(i,j) = 1$. Let $w^k(i,j) = 1$. In this way we have $n_k - r_k$ nonzero entries in w^k, all of them distributed on different rows and different columns. Choose a bijection between the remaining r_k rows and r_k columns and complete w^k to a permutation matrix. Then:

$$\left\|v^k - w^k\right\|_2^2 = \frac{2r_k}{n_k}.$$

Combined with $r_k/n_k \to_{k\to\omega} 0$ we get $v = \Pi_{k\to\omega} w^k$. This will prove the lemma. □

Theorem 2.9 obliges us to define the notion of sofic equivalence relation. Definition 2.7 together with this theorem provides a well-define notion of soficity for equivalence relations that are generated by a free action. Unfortunately not all equivalence relations have this property. In order to provide a definition for sofic equivalence relation we need a different construction than crossed product. We need a construction that associates a von Neumann algebra to an equivalence relation. This is the topic of the next section.

2.3. The Feldman–Moore construction

Let us recall some things from [12]. We shall ignore the cocycle that is needed for the Feldman–Moore construction in its full generality. Let $(X, \mathcal{B}, \mu)$ be a probability space as usual. Let $E \subset X^2$ be an equivalence relation on X such that $E \in \mathcal{B} \times \mathcal{B}$. We shall work only with equivalence relations that are countable, i.e. every equivalence class is countable, and μ-invariant. Before we recall what this means we introduce some notation.

Denote by $[E]$ the *full group of the relation* E, i.e. set of all isomorphism with graph in E and by $[[E]]$ set of all partial isomorphism with graph in E:

$$[E] = \{\theta : X \to X\colon\ \theta \text{ bijection},\ graph\,\theta \subset E\};$$
$$\big[[E]\big] = \{\phi : A \to B\colon\ A, B \subset X,\ \phi \text{ bijection},\ graph\,\phi \subset E\}.$$

If X is reducible to a finite space of cardinality n and $E = X^2$ then $[E]$ is just the symmetric group S_n.

Definition 2.11. Let E an equivalence relation on (X, μ). Then E is called μ-invariant if for any $\phi : A \to B$, $\phi \in [[E]]$ we have $\mu(A) = \mu(B)$.

Now we can construct the algebra $M(E)$ associated to an equivalence relation.

Definition 2.12. A measurable function $a : E \to \mathbb{C}$ is called *finite* if a is bounded and there is a natural number n such that:

$$Card\big(\big\{x\colon\ a(x, y) \neq 0\big\}\big) \leqslant n \quad \forall y \in X;$$
$$Card\big(\big\{y\colon\ a(x, y) \neq 0\big\}\big) \leqslant n \quad \forall x \in X.$$

A finite function (matrix) is a bounded function with finite number of nonzero entries on each line and column (having also a global margin). We shall multiply this functions as general matrices and the definition of finite function guarantees we get a $*$-algebra. Define:

$$M_0(E) = \{a : E \to \mathbb{C}\colon\ a \text{ finite function}\};$$
$$a \cdot b(x, z) = \sum_y a(x, y)b(y, z);$$
$$a^*(x, y) = \overline{a(y, x)}.$$

It is easy to check that this is indeed a $*$-algebra. The trace is defined in a similar way as in the case of matrices:

$$Tr(a) = \int_X a(x, x)\, d\mu.$$

The algebra $M(E)$ will be the weak closure of $M_0(E)$ in the GNS representation of $(M_0(E), Tr)$. By general theory of von Neumann algebras, using the cyclic separating vector of the GNS representation, we can still see elements of $M(E)$ as measurable functions $a : E \to \mathbb{C}$. This algebra is a factor iff E is an ergodic equivalence relation. By a famous theorem of Connes, Feldman, and Weiss [4], $M(E)$ is hyperlinear iff E is a *hyperfinite equivalence relation*, that is up to a set of measure 0, E is the union of an ascending sequence of finite equivalence relations.

Let $\Delta = \{(x, x)\colon x \in X\}$ be the diagonal in E. Define the subalgebra of diagonal matrices:

$$A = \{a \in M(E)\colon supp(a) \subset \Delta\}.$$

We shall denote by δ_x^y the Kronecker delta function, i.e. $\delta_x^y = 1$ iff $x = y$; otherwise $\delta_x^y = 0$. Notation χ_A stands for the characteristic function of A.

Definition 2.13. For $\theta \in [E]$ define $u_\theta \in M(E)$ by: $u_\theta(x, y) = \delta_x^{\theta(y)}$. For $\phi \in [[E]]$, define $v_\phi(x, y) = \chi_{dom(\phi)}(y) \cdot \delta_x^{\phi(y)}$.

It is not hard to see that $u_\theta \in \mathcal{N}(A)$ for any $\theta \in [E]$. Instead v_ϕ is a partial isometry that belongs to a set called the *normalizing pseudogroup*:

$$\mathcal{GN}_M(A) = \{v \in M \text{ partial isometry: } vv^*, v^*v \in A,\ vAv^* = vv^*A\}.$$

Unitaries in $\mathcal{GN}_M(A)$ are actually elements in $\mathcal{N}_M(A)$, the same way as an element of $[[E]]$ defined on all X is an element of $[E]$. More general, any element $v \in \mathcal{GN}_M(A)$ is of the form $p \cdot u$, where p is a projection in A and $u \in \mathcal{N}_M(A)$.

Inside a matrix algebra we have $\mathcal{N}_{M_n}(D_n) = \mathcal{U}(D_n) \cdot P_n$. Something similar is true for the Feldman–Moore construction. Any $u \in \mathcal{N}_{M(E)}(A)$ is of the form $a \cdot u_\theta$, where $a \in \mathcal{U}(A)$ and $\theta \in [E]$. Also $u_\theta u_\psi = u_{\theta\circ\psi}$ for $\theta, \psi \in [E]$. This provides a group isomorphism between the Weyl group $\mathcal{N}(A)/\mathcal{U}(A)$ and $[E]$. This is the analog of the isomorphism between the group of permutation matrices and the symmetric group.

The algebra A is maximal abelian in $M(E)$. Also $\mathcal{N}(A)'' = M(E)$. This properties make A a Cartan subalgebra of $M(E)$. We shall call $A \subset M(E)$ a Cartan pair.

Motivation of Feldman–Moore construction was the invariance of crossed product up to orbit equivalent actions. The next example shows this is indeed the right construction.

Example 2.14. Let $\alpha : G \to Aut(X, \mu)$ a free action. Denote by E_α the orbit equivalence relation generated by α on X. Then:

$$L^\infty(X) \rtimes_\alpha G \simeq M(E_\alpha).$$

2.4. Definition of sofic equivalence relations

The notion of sofic equivalence relation was introduced by Gabor Elek and Gabor Lippner (see [7]). We shall provide a different definition here and prove in Section 3.4 the equivalence of the two definitions.

Definition 2.15. An equivalence relation E is called *sofic* if there is an embedding of $M(E)$ in some $\Pi_{k\to\omega} M_{n_k}$ such that $A \subset \Pi_{k\to\omega} D_{n_k}$ and $\mathcal{N}(A) \subset \mathcal{U}(A) \cdot \Pi_{k\to\omega} P_{n_k}$.

This has the advantage of being a compact definition, but in practice we shall need the following type of embeddings.

Definition 2.16. Let E an equivalence relation and $A \subset M(E)$ the Cartan pair associated to E. We call an embedding $\Theta : M(E) \to \Pi_{k\to\omega} M_{n_k}$ *sofic* if $\Theta(A) \subset \Pi_{k\to\omega} D_{n_k}$ and $\Theta(u_\theta) \subset \Pi_{k\to\omega} P_{n_k}$ for any $\theta \in [E]$.

Proposition 2.17. *An equivalence relation E is a sofic if and only if its Cartan pair $A \subset M(E)$ admits a sofic embedding.*

Proof. Let $\Theta : M(E) \to \Pi_{k\to\omega} M_{n_k}$ an embedding such that $\Theta(A) \subset \Pi_{k\to\omega} D_{n_k}$ and $\Theta(\mathcal{N}(A)) \subset \Theta(\mathcal{U}(A)) \cdot \Pi_{k\to\omega} P_{n_k}$.

For $\varphi \in [E]$ we have a unique decomposition $\Theta(u_\varphi) = \Theta(f_\varphi)v_\varphi$, where $f_\varphi \in \mathcal{U}(A)$ and $v_\varphi \in \Pi_{k\to\omega} P_{n_k}$. Then:

$$\Theta\big(f_\psi \circ \varphi^{-1}\big) = \Theta(u_\varphi)\Theta(f_\psi)\Theta\big(u_\varphi^*\big) = \Theta(f_\varphi)\big(v_\varphi\Theta(f_\psi)v_\varphi^*\big)\Theta\big(f_\varphi^*\big) = v_\varphi\Theta(f_\psi)v_\varphi^*.$$

Because of the uniqueness of the decomposition of $\Theta(u_{\varphi\psi})$ we have $f_{\varphi\psi} = f_\varphi(f_\psi \circ \varphi^{-1})$. If χ_φ denotes the projection with support $\{x \in X\colon\ \varphi(x) = x\}$, one has $\chi_\varphi u_\varphi = \chi_\varphi$ and hence:

$$\Theta\big(f_\varphi^*\chi_\varphi\big) = \Theta\big(f_\varphi^*\chi_\varphi u_\varphi\big) = \Theta(\chi_\varphi)v_\varphi.$$

The conditional expectation of v_φ on $\Pi_{k\to\omega} D_{n_k}$ is a projection. Thus, taking the conditional expectation on $\Theta(A)$ it follows that $f_\varphi^*\chi_\varphi$ is positive and hence equal to χ_φ. So, for all $\varphi \in [E]$ we have $f_\varphi\chi_\varphi = \chi_\varphi$. Altogether, it follows that the formula:

$$\alpha(u_\varphi) = f_\varphi^* u_\varphi \quad \text{for all } \varphi \in [E],$$
$$\alpha(a) = a \quad \text{for all } a \in A$$

provides a well-defined automorphism of $M(E)$. The composition of Θ and α is the required sofic embedding of $M(E)$. □

As a consequence of this proposition and Lemma 2.10, we have the following result.

Proposition 2.18. *Let α be a free action. Then E_α is a sofic equivalence relation if and only if α is a sofic action.*

Observation 2.19. Let $\Theta = \Pi_{k\to\omega}\Theta_k$ be a sofic embedding of some von Neumann algebra M in $\Pi_{k\to\omega} M_{n_k}$. Consider also $\{r_k\}_k$ a sequence of natural numbers. Then $\Theta \otimes 1 = \Pi_{k\to\omega}\Theta_k \otimes 1_{r_k}$ is again a sofic embedding of M in $\Pi_{k\to\omega} M_{n_k} \otimes M_{r_k} = \Pi_{k\to\omega} M_{n_k r_k}$.

This trick will be used when we need to embed two algebras in the same $\Pi_{k\to\omega} M_{n_k}$ (that is the same matrix dimension at each step).

3. Results

3.1. Sofic embeddings of hyperfinite Cartan pairs

The goal of this section is to prove that if we have a Cartan pair $A \subset M$ and M is hyperfinite then any two sofic embeddings of $A \subset M$ are conjugate by a permutation. The starting point for the proof is the sketch from 2.1. However we first need to conjugate embeddings in $\Pi_{k\to\omega}D_{n_k}$ by permutations (Lemma 3.3).

Lemma 3.1. *Let e, f be two projections in $\Pi_{k\to\omega}D_{n_k}$ such that $Tr(e) = Tr(f)$. Then there is a unitary $u \in \Pi_{k\to\omega}P_{n_k}$ such that $f = ueu^*$.*

Proof. Let $e = \Pi_{k\to\omega}e^k$ and $f = \Pi_{k\to\omega}f^k$ such that e^k and f^k are projections in D_{n_k}. Assume e^k has t_k entries of 1 and f^k has s_k entries of 1, so $\lim_{k\to\omega} t_k/n_k = Tr(e) = Tr(f) = \lim_{k\to\omega} s_k/n_k$. Choose $p_1^k \in P_{n_k}$ such that $p_1^k e^k p_1^{k*}$ has the first t_k entries of 1 on the diagonal. In the same way choose p_2^k such that $p_2^k f^k p_2^{k*}$ has the first s_k entries of 1 on the diagonal. Define $p_i = \Pi_{k\to\omega}p_i^k$ for $i = 1, 2$. Our constructions guarantee that $Tr(|p_1ep_1^* - p_2fp_2^*|) = \lim_{k\to\omega} |t_k - s_k|/n_k = 0$. Then $p_1ep_1^* = p_2fp_2^*$ so define $u = p_2^*p_1$. □

Lemma 3.2. *Let $\{e_i\}_{i=1}^m$ and $\{f_i\}_{i=1}^m$ be two sequences of projections in $\Pi_{k\to\omega}D_{n_k}$ such that $\sum_{i=1}^m e_i = 1 = \sum_{i=1}^m f_i$ and $Tr(e_i) = Tr(f_i)$ for each $i = 1, \ldots, m$. Then there is a unitary $u \in \Pi_{k\to\omega}P_{n_k}$ such that $f_i = ue_iu^*$ for all $i = 1, \ldots, m$.*

Proof. Apply the previous lemma for each $i = 1, \ldots, m$ to get elements $u_i \in \Pi_{k\to\omega}P_{n_k}$ such that $u_ie_iu_i^* = f_i$. Define $u = \sum_{i=1}^m u_ie_i$. Then by Lemma 2.10 we know that $u \in \Pi_{k\to\omega}P_{n_k}$. Also $ue_iu^* = u_ie_iu_i^* = f_i$. □

Proposition 3.3. *Let Θ_1, Θ_2 be two embeddings of $L^\infty(X)$ in $\Pi_{k\to\omega}D_{n_k}$. Then there exists a unitary $u \in \Pi_{k\to\omega}P_{n_k}$ such that $\Theta_2(a) = u\Theta_1(a)u^*$ for every $a \in L^\infty(X)$.*

Proof. Let A_m be an increasing sequence of commutative finite dimensional subalgebras such that $L^\infty(X) = (\bigcup_m A_m)''$. By the previous lemma there exists a unitary $u_m \in \Pi_{k\to\omega}P_{n_k}$ such that $\Theta_2(a) = Adu_m \circ \Theta_1(a)$ for $a \in A_m$. We shall construct $u \in \Pi_{k\to\omega}P_{n_k}$ using a diagonal argument. Let $u_m = \Pi_{k\to\omega}u_m^k$ with $u_m^k \in P_{n_k}$ and $\Theta_i(a) = \Pi_{k\to\omega}\Theta_i(a)^k$ with $\Theta_i(a)^k \in D_{n_k}$.

Inductively choose smaller $F_m \in \omega$, $m \in \mathbb{N}$ such that $\|u_m^k\Theta_1(a)^ku_m^{k*} - \Theta_2(a)^k\|_2 < 1/m$ for any $a \in (A_m)_1$, $k \in F_m$. Define $u^k = u_m^k$ for $k \in F_m \setminus F_{m+1}$ and set $u = \Pi_{k\to\omega}u^k$. □

We are now ready to prove an analog of 2.1 for sofic embeddings.

Proposition 3.4. *Let E be a hyperfinite equivalence relation and $A \subset M(E)$ the Cartan pair associated to E. Let Θ_1, Θ_2 two sofic embeddings of $M(E)$ in $\Pi_{k\to\omega}M_{n_k}$. Then there exists a unitary $u \in \Pi_{k\to\omega}P_{n_k}$ such that $\Theta_2(x) = u\Theta_1(x)u^*$ for every $x \in M(E)$.*

Proof. The algebra $M(E)$ is hyperlinear. If we write $M(E) = \overline{\bigcup_m N_m}^w$ as in the proof of 2.1, we have no control on the algebras N_m and we cannot use the hypothesis of sofic embedding. Instead we have to use the properties of the equivalence relation. The hyperfinite property of

$M(E)$ implies that E is a hyperfinite equivalence relation. So, up to a set of measure 0, E is the union of an ascending sequence of finite equivalence relations.

Using the previous proposition we can assume Θ_1 and Θ_2 coincide on A. We shall first prove this result in case of ergodic equivalence relations, i.e. $M(E)$ is the hyperfinite factor. By the definition of hyperfinite equivalence relation and Feldman–Moore construction (see also proof of 4.1 from [21]) there exists an increasing sequence of matrix algebras $\{N_m\}_{m\geqslant 1}$ of $M(E)$ each of them with a set of matrix units $\{e^m_{ij}\}$ such that:

1. $M(E)$ is the weak closure of $\bigcup_m N_m$;
2. $e^m_{ii} \in A$ and $\sum_i e^m_{ii} = 1$;
3. e^m_{ij} are of the form v_θ with $\theta \in [[E]]$;
4. every e^p_{rs}, for $p \leqslant m$, is the sum of some e^m_{ij}.

Elements v_θ are of the form $e \cdot u_\phi$, where e is a projection in A and $\phi \in [E]$. Combined with Θ_l is sofic, we get that $\Theta_l(e^m_{ij})$ is an ultraproduct of permutations cut with a projection in $\Pi_{k\to\omega} D_{n_k}$. Define

$$p_m = \sum_j \Theta_2\big(e^m_{j1}\big)\Theta_1\big(e^m_{1j}\big).$$

Then

$$\begin{aligned} p_m p_m^* &= \sum_{i,j} \Theta_2\big(e^m_{i1}\big)\Theta_1\big(e^m_{1i}\big)\Theta_1\big(e^m_{j1}\big)\Theta_2\big(e^m_{1j}\big) \\ &= \sum_j \Theta_2\big(e^m_{j1}\big)\Theta_1\big(e^m_{11}\big)\Theta_2\big(e^m_{1j}\big) = \sum_j \Theta_2\big(e^m_{jj}\big) = 1, \end{aligned}$$

so p_m is a unitary. Using 2.10 we have $p_m \in \Pi_{k\to\omega} P_{n_k}$. Moreover:

$$\begin{aligned} p_m\Theta_1\big(e^m_{rs}\big)p_m^* &= \sum_{i,j} \Theta_2\big(e^m_{i1}\big)\Theta_1\big(e^m_{1i}\big)\Theta_1\big(e^m_{rs}\big)\Theta_1\big(e^m_{j1}\big)\Theta_2\big(e^m_{1j}\big) \\ &= \Theta_2\big(e^m_{r1}\big)\Theta_1\big(e^m_{11}\big)\Theta_2\big(e^m_{1s}\big) = \Theta_2\big(e^m_{rs}\big). \end{aligned}$$

We obtained $p_m\Theta_1(x)p_m^* = \Theta_2(x)$ for $x \in N_m$. Employing another diagonal argument we construct a permutation $p \in \Pi_{k\to\omega} P_{n_k}$ such that $p\Theta_1(x)p^* = \Theta_2(x)$ for $x \in \bigcup_m N_m$. Using 1 we are done.

The proof in general case works the same. The only difference is that $\{N_m\}_{m\geqslant 1}$ are finite dimensional algebras instead of matrix algebras, so we need to be more careful when defining p_m. Assume that $N_m = N^1_m \oplus N^2_m \oplus \cdots \oplus N^t_m$, with N^v_m factors for $v = 1, \ldots, t$. Let $\{e^m_{ij;v}\}$ a set of matrix units for N^v_m. Then define:

$$p_m = \sum_{v=1}^{t} \sum_j \Theta_2\big(e^m_{j1;v}\big)\Theta_1\big(e^m_{1j;v}\big).$$

Computations that p_m is a unitary and $p_m\Theta_1(e^m_{rs})p_m^* = \Theta_2(e^m_{rs})$ are the same. □

3.2. Bernoulli shifts

In [7] Elek and Lippner proved that equivalence relations generated by Bernoulli shifts of sofic groups are sofic. We present here the nice proof of Narutaka Ozawa from [17].

Theorem 3.5 *(Elek–Lippner). Equivalence relations generated by Bernoulli shifts of sofic groups are sofic.*

Proof *(Ozawa).* Let G be a sofic group. Every Bernoulli shift is a free action. Using 2.18 we just need to prove that each Bernoulli shift of G is a sofic action.

Let $X = \{0,1\}^G = \{f\colon G \to \{0,1\}\}$. For distinct $g_1, g_2, \ldots, g_m$, define the cylinder set:

$$c^{i_1,i_2,\ldots,i_m}_{g_1,g_2,\ldots,g_m} = \bigl\{f \in X\colon f(g_j) = i_j \ \forall j = 1,\ldots,m\bigr\},$$

and let $Q^{i_1,i_2,\ldots,i_m}_{g_1,g_2,\ldots,g_m}$ be the projection onto this set. Then β is the action of G on X such that $\beta(g)c^{i_1,i_2,\ldots,i_m}_{g_1,g_2,\ldots,g_m} = c^{i_1,i_2,\ldots,i_m}_{gg_1,gg_2,\ldots,gg_m}$.

Let $\Theta_0 : G \to \Pi_{k\to\omega} P_{n_k}$ be a sofic embedding of G with $Tr(\Theta_0(g)) = 0$ for each $g \neq e$. Write $\Theta_0(g) = \Pi_{k\to\omega} p_{g;k}$ such that $p_{g;k} \in P_{n_k}$. Define $\Theta : G \to \Pi_{k\to\omega} M_{n_k} \otimes M_{2^{n_k}}$ by $\Theta = \Theta_0 \otimes 1$. Let Y_k a set with n_k elements and identify D_{n_k} with $L^\infty(Y_k)$. Also let $Z_k = \{\eta\colon Y_k \to \{0,1\}\}$ and identify $D_{2^{n_k}}$ with $L^\infty(Z_k)$. Define now:

$$c^{i_1,i_2,\ldots,i_m}_{g_1,g_2,\ldots,g_m;k} = \bigl\{(\xi,\eta) \in Y_{n_k} \times Z_{n_k}\colon \eta\bigl(p^{-1}_{g_j;k}(\xi)\bigr) = i_j,\ j = 1,\ldots,m\bigr\}.$$

Let $Q^{i_1,i_2,\ldots,i_m}_{g_1,g_2,\ldots,g_m;k} \in D_{n_k} \otimes D_{2^{n_k}}$ be the characteristic function of $c^{i_1,i_2,\ldots,i_m}_{g_1,g_2,\ldots,g_m;k}$. Define now $\Theta(Q^{i_1,i_2,\ldots,i_m}_{g_1,g_2,\ldots,g_m}) = \Pi_{k\to\omega} Q^{i_1,i_2,\ldots,i_m}_{g_1,g_2,\ldots,g_m;k}$. Then:

$$\begin{aligned}
\Theta(g)\Theta\bigl(Q^{i_1,i_2,\ldots,i_m}_{g_1,g_2,\ldots,g_m}\bigr)\Theta(g)^* &= \Pi_{k\to\omega}(p_{g;k}\otimes 1)\,Q^{i_1,i_2,\ldots,i_m}_{g_1,g_2,\ldots,g_m;k}\bigl(p^{-1}_{g;k}\otimes 1\bigr)\\
&= \Pi_{k\to\omega}\chi_{\{(\xi,\eta)\colon\ (p^{-1}_{g;k}\otimes 1)(\xi,\eta)\in c^{i_1,i_2,\ldots,i_m}_{g_1,g_2,\ldots,g_m;k}\}}\\
&= \Pi_{k\to\omega}\chi_{\{(\xi,\eta)\colon\ \eta(p^{-1}_{g_j;k}p^{-1}_{g;k}(\xi))=i_j,\ j=1,\ldots,m\}}\\
&=_{not} \Pi_{k\to\omega}\chi_{T_k},\\
\Theta\bigl(Q^{i_1,i_2,\ldots,i_m}_{gg_1,gg_2,\ldots,gg_m}\bigr) &= \Pi_{k\to\omega}\chi_{\{(\xi,\eta)\colon\ \eta(p^{-1}_{gg_j;k}(\xi))=i_j,\ j=1,\ldots,m\}}\\
&=_{not} \Pi_{k\to\omega}\chi_{S_k}.
\end{aligned}$$

If $(\xi,\eta) \in T_k \Delta S_k$ then for some $j = 1,\ldots,m$ we have $p^{-1}_{g_j;k}p^{-1}_{g;k}(\xi) \neq p^{-1}_{gg_j;k}(\xi)$. Given the fact that Θ_0 is a sofic embedding it follows that $\Pi_{k\to\omega}\chi_{T_k} = \Pi_{k\to\omega}\chi_{S_k}$.

The only thing left is to compute the trace of $\Theta(Q^{i_1,i_2,\ldots,i_m}_{g_1,g_2,\ldots,g_m})$. For this, let $A_k = \{\xi \in Y_k\colon p^{-1}_{g_j;k}(\xi)$ are different for $j = 1,\ldots,m\}$. Because $Tr(\Theta_0(g)) = 0$ for $g \neq e$ we have $\lim_{k\to\omega} Card(A_k)/n_k = 1$. Then:

$$Tr\bigl(\Theta\bigl(Q^{i_1,i_2,\ldots,i_m}_{g_1,g_2,\ldots,g_m}\bigr)\bigr) = \lim_{k\to\omega} Tr\bigl(Q^{i_1,i_2,\ldots,i_m}_{g_1,g_2,\ldots,g_m;k}\bigr) = \lim_{k\to\omega}\frac{1}{n_k 2^{n_k}}\Bigl(\sum_{\xi\in A_k} 2^{n_k-m} + \sum_{\xi\notin A_k} v_\xi\Bigr) = \frac{1}{2^m}.$$

This will prove that Θ is an embedding of $L^\infty(X) \rtimes_\beta G$, proving the soficity of the action β.

The proof can be adapted to work for any Bernoulli shift. For a finite uniform Bernoulli shift the proof works the same. A diagonal argument will prove the theorem in case $X = [0,1]^G$ (with product of Lebesgue measure). Any other Bernoulli shift will yield a subalgebra of $L^\infty([0,1]^G) \rtimes G$. □

The next easy proposition will be used in the proof of Corollary 3.7.

Proposition 3.6. *Let G act freely on a countable set I. Then the generalized Bernoulli shift of G on $\{0,1\}^I$ is sofic.*

Proof. If G acts freely on I then I is of the form $G \times I'$ and the action is a shift on the first component. The generalized Bernoulli shift on $\{0,1\}^I$ is a classical Bernoulli shift on X^G where $X = \{0,1\}^{I'}$. □

A formally weaker version of the following result was first obtain by Benoit Collins and Ken Dykema (see [2]). Independently, Elek and Szabo proved this theorem using different methods (see [11]).

Corollary 3.7. *Amalgamated products of sofic groups over amenable groups are sofic.*

Proof. Let G_1, G_2 be two sofic groups with a common amenable subgroup H. Let $X = \{0,1\}^{G_1 *_H G_2}$ equipped with product measure. Then G_1 and G_2 act on X as generalized Bernoulli shifts and this actions coincide on H. Using the above proposition (and 2.19) we can construct sofic embeddings $\Theta_i : L^\infty(X) \rtimes G_i \to \Pi_{k\to\omega} M_{n_k}$ for $i = 1,2$. By proposition (3.4) we can assume $\Theta_1 = \Theta_2$ on $L^\infty(X) \rtimes H$ (here we use H amenable and the classic result from [4]). Note that now Θ_1 acts on $\Theta_i(L^\infty(X))$ by shifting with G_1 and Θ_2 acts on the same space by shifting with G_2. This will provide a representation Θ of $G_1 *_H G_2$ on $\Pi_{k\to\omega} P_{n_k}$. Also, Θ acts on $\Theta_i(L^\infty(X))$ as a classic Bernoulli shift. This implies Θ is faithful, so $G_1 *_H G_2$ is sofic. □

Corollary 3.8. *Let H be an abelian group and G a sofic group. Then $H \wr G$ (wreath product) is sofic.*

Proof. The wreath product $H \wr G$ is the semidirect product of G and H^G by the shift action of G. We shall work with the following presentation $\langle S|R\rangle$ of the wreath product:

$$\begin{aligned}
S &= \{f_g^h, u_g\colon \text{ for every } h \in H \text{ and } g \in G\};\\
R &= \{f_g^e = e\colon \forall g \in G\} \cup \{f_g^{h_1} f_g^{h_2} = f_g^{h_1 h_2}\colon \forall g \in G,\ \forall h_1, h_2 \in H\}\\
&\quad \cup \{f_{g_1}^{h_1} f_{g_2}^{h_2} = f_{g_2}^{h_2} f_{g_1}^{h_1}\colon \forall g_1, g_2 \in G,\ g_1 \neq g_2,\ \forall h_1, h_2 \in H\}\\
&\quad \cup \{u_{g_1} u_{g_2} = u_{g_1 g_2}\colon \forall g_1, g_2 \in G\}\\
&\quad \cup \{u_{g_1} f_{g_2}^h u_{g_1}^{-1} = f_{g_1 g_2}^h\colon \forall g_1, g_2 \in G,\ h \in H\}.
\end{aligned}$$

Consider first the case of $\mathbb{Z}_2 \wr G$. Apply Elek–Lippner result to embed $L(\mathbb{Z}_2^G) \rtimes_\beta G \simeq L(\mathbb{Z}_2^G \rtimes G) = L(\mathbb{Z}_2 \wr G)$ in some $\Pi_{k\to\omega} M_{n_k}$. Generators u_g will be ultraproduct of permutations. Instead, elements of the type f_g^h are unitaries in $\Pi_{k\to\omega} D_{n_k}$ with ± 1 entries. Construct a sofic representation of $\mathbb{Z}_2 \wr G$ in $\Pi_{k\to\omega}P_{2n_k}$ by replacing a 1 entry with I_2 and a -1 entry with: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Consider now the general case. Let $\Theta : L^\infty(\{0,1\}^G) \rtimes G \to \Pi_{k\to\omega}M_{n_k}$ the sofic embedding constructed in the last proof. Also let $\Lambda : H \to P_{m_k}$ be a sofic embedding of H. We shall construct $\Phi : H \wr G \to \Pi_{k\to\omega}P_{n_k} \otimes P_{m_k}$ as follows:

$$\Phi(u_g) = \Theta(g) \otimes 1;$$
$$\Phi\bigl(f_g^h\bigr) = c_g^0 \otimes 1 + c_g^1 \otimes \Lambda(h).$$

Relations in the set R are easy to check (one needs H abelian for $f_{g_1}^{h_1} f_{g_2}^{h_2} = f_{g_2}^{h_2} f_{g_1}^{h_1}$). Also $Tr(\Phi(u_g)) = 0$ and $Tr(\Phi(f_g^h)) = 1/2$. In order to finish the proof we need to see that Φ is injective.

The generic element of $H \wr G$ is $s = f_{g_1}^{h_1} f_{g_2}^{h_2} \ldots f_{g_n}^{h_n} u_g$ with $g_1, g_2, \ldots, g_n$ distinct. Then:

$$\Phi(s) = \biggl(\sum_{(i_1,\ldots,i_n)\in\{0,1\}^n} c_{g_1,g_2,\ldots,g_n}^{i_1,i_2,\ldots,i_n} \otimes \Lambda(\Pi_{i_k=1}h_k)\biggr)\bigl(\Theta(g) \otimes 1\bigr).$$

Assume $\Phi(s) = 1$. Then for any $(i_1, \ldots, i_n) \in \{0,1\}^n$, $\Lambda(\Pi_{i_k=1}h_k) = 1$. This force $h_k = e$ for any k. Then $\Phi(u_g) = 1$, so $u_g = e$. It follows that $s = e$. □

3.3. Sofic actions

The goal would be to prove that every (free) action of a sofic group is sofic. While this remains open we shall prove this fact for a family of groups. Let's first solve this ambiguity: free or general actions.

Theorem 3.9. *Let G be a group such that every free action is sofic. Then every action of G is sofic.*

Proof. Let α be an action of G on X. Let $\beta : G \to Aut(Y)$ be a free action (e.g. Bernoulli shift). Define $\alpha \otimes \beta : G \to Aut(X \times Y)$ by $g(x, y) = (gx, gy)$. With this definition $\alpha \otimes \beta$ is a free action of G, so it is sofic. We can embed $L^\infty(X \times Y) \rtimes_{\alpha\otimes\beta} G$ in some $\Pi_{k\to\omega}M_{n_k}$ satisfying the requirements of sofic action. The space $L^\infty(X)$ can be embedded in $L^\infty(X \times Y)$ by $id \otimes 1$. This embedding can be extended to an embedding of $L^\infty(X) \rtimes_\alpha G$ in $L^\infty(X \times Y) \rtimes_{\alpha\otimes\beta} G$. This will prove α is sofic. □

Definition 3.10. Denote by $\mathcal{S}$ the class of groups for which every action is sofic.

While we cannot prove that every sofic group is in $\mathcal{S}$, we will provide some examples. First goal is to deal with amenable groups.

Proposition 3.11. *Each action of the integers admits a sofic embedding.*

Proof. Let $\alpha : L^\infty(X) \to L^\infty(X)$ the automorphism that generates the action. Choose $\Theta : L^\infty(X) \to \Pi_{k\to\omega}D_{n_k}$ an embedding. Apply Proposition 3.3 to Θ and $\Theta \circ \alpha$ to get a unitary $u \in \Pi_{k\to\omega}P_{n_k}$ such that $Adu \circ \Theta = \Theta \circ \alpha$.

As powers of permutation matrices are still permutation matrices, we have $u^m \in \Pi_{k\to\omega} P_{n_k}$. Also $u^m \Theta(f)(u^m)^* = \Theta(\alpha^m(f))$ for any $f \in L^\infty(X)$. Now we have an embedding Θ of the algebraic crossed product $L^\infty(X) \rtimes_\alpha^{alg} \mathbb{Z}$. In order to have an embedding of the crossed product we need the relation $Tr(u^m) = 0$ for any $m \in \mathbb{Z}^*$.

Let Λ be an embedding of $\mathbb{Z}$ in some $\Pi_{k\to\omega} P_{r_k}$ using only elements of trace 0. Define the embedding $\Theta \otimes \Lambda$ of $L^\infty(X) \rtimes_\alpha \mathbb{Z}$ in $\Pi_{k\to\omega} M_{n_k \cdot r_k}$ by:

$$\Theta \otimes \Lambda(T) = \Theta(T) \otimes 1 \quad \text{for } T \in L^\infty(X);$$

$$\Theta \otimes \Lambda(u_g) = \Theta(u_g) \otimes \Lambda(u_g) \quad \text{for } g \in \mathbb{Z}.$$

This embedding $\Theta \otimes \Lambda$ of the algebraic crossed product respects the trace of the von Neumann crossed product. Using the unique feature of the type II case the closure of its imagine will be the crossed product. □

Proposition 3.12. *Amenable groups are in $\mathcal{S}$.*

Proof. Let G be an amenable group and let $\alpha : G \to Aut(X, \mu)$ be a free action. Then E_α is amenable. By [4] E_α is generated by an action β of $\mathbb{Z}$. By the previous proposition $beta$ is sofic. Because almost all equivalence classes of E_α are non-finite, β is free. Using Proposition 2.9 we deduce α is sofic. Combined with Theorem 3.9, we get $G \in \mathcal{S}$. □

The next proposition will enlarge the class of groups for which such results hold.

Theorem 3.13. *Let α_1 and α_2 be two sofic actions of G_1 and G_2 on the same space X. Consider H, a common amenable subgroup of G_1 and G_2. Assume α_1 and α_2 coincide on H, and this action of H is free. Then the action $\alpha_1 *_H \alpha_2$ of $G_1 *_H G_2$ is sofic.*

Proof. Using 2.19 we can construct sofic embeddings of the two crossed products in the same ultraproduct. So let $\Theta_i : L^\infty(X) \rtimes G_i \to \Pi_{k\to\omega} M_{n_k}$, $i = 1, 2$. By 3.4 we can assume $\Theta_1 = \Theta_2$ on $L^\infty(X) \rtimes H$ (using the freeness of this action). Now we can construct a representation Θ of the algebraic crossed product $L^\infty(X) \rtimes (G_1 *_H G_2)$ on $\Pi_{k\to\omega} M_{n_k}$. In order to embed the von Neumann crossed product the trace of each nontrivial u_g, $g \in G_1 *_H G_2$ must be equal to 0. The groups G_1 and G_2 must be sofic, as only sofic groups can admit sofic actions. Then $G_1 *_H G_2$ is sofic (see 3.7). There exists an embedding Λ of $G_1 *_H G_2$ in some $\Pi_{k\to\omega} P_{r_k}$ using only elements of trace 0. Define the embedding $\Theta \otimes \Lambda$ of $L^\infty(X) \rtimes_{\alpha_1 *_H \alpha_2} G_1 *_H G_2$ like in 3.11. □

Adapting the same methods we can prove this result for a countable family of actions.

Proposition 3.14. *Let $\{\alpha_i\}_{i\in\mathbb{N}}$ be a family of sofic actions of $\{G_i\}_{i\in\mathbb{N}}$ on the same space. Assume H is an amenable common subgroup of G_i and the actions α_i coincide on H. Then $*_H \alpha_i$ is sofic.*

Corollary 3.15. *Each action of a free group, including $\mathbb{F}_\infty$ is sofic.*

Proof. Corollary of 3.11 and 3.14. □

We now recover with our methods the result of Elek and Lippner that any treeable equivalence relation is sofic. A good reference for treeable equivalence relation is [16].

Proposition 3.16. *Every treeable equivalence relation is sofic.*

Proof. Well, treeable is some kind of freeness and freeness in general goes well with soficity.

Let E be a treeable equivalence relation on (X,μ). Fix a treeing of E, i.e. a countable set of partial Borel isomorphism $\{\phi_i\}_{i\in\mathbb{N}^*}\subset[[E]]$. For each i we have $\phi_i=a_i\lambda_i$, where a_i is a projection in $L^\infty(X)$ and $\lambda_i\in[E]$.

Define an action α of $\mathbb{F}_\infty$ on X such that $\alpha(\gamma_i)=\lambda_i$ (where $\{\gamma_i\}_i$ are the generators of $\mathbb{F}_\infty$). Being an action of $\mathbb{F}_\infty$, α is sofic.

The von Neumann subalgebra of $L^\infty(X)\rtimes_\alpha\mathbb{F}_\infty$ generated by $a_iu_{\gamma_i}$ is naturally isomorphic to $M(E)$. Hence every sofic embedding of $L^\infty(X)\rtimes_\alpha\mathbb{F}_\infty$ can be restricted to a sofic embedding of $M(E)\subset L^\infty(X)\rtimes_\alpha\mathbb{F}_\infty$. □

We end this section with the following theorem.

Theorem 3.17. *Class $\mathcal{S}$ is closed under amalgamated product over amenable groups. It is strictly larger than the class of treeable groups.*

Proof. First part of the theorem is 3.13 and 3.9. By 3.16 (and again 3.9) every treeable group is in $\mathcal{S}$.

Consider now the group $G=\mathbb{Z}*_{(2,3)\mathbb{Z}}\mathbb{Z}$. It is not treeable but $G\in\mathcal{S}$. This example is from [13].

Relation $G\in\mathcal{S}$ is just 3.11 and 3.13. By general theory of Gaboriau, the cost of G is $1+1-1=1$. If amalgamation is done with good morphism (multiplication by 2 and 3) then G is not amenable. This implies G is not treeable. □

3.4. *Sofic equivalence relations*

Now we shall present from [7] the original definition of Elek and Lippner of soficity for actions and equivalence relations.

Definition 3.18. We call a *basic sequence of projections* for $L^\infty(X)$ a collection $\{e_{i,m}\}_{1\leqslant i\leqslant 2^m,\,m\geqslant 0}\subset L^\infty(X)$ with the following properties:

1. $\overline{span}^w\{e_{i,m}\}_{i,m}=L^\infty(X)$;
2. $\mu(e_{i,m})=2^{-m}$, $1\leqslant i\leqslant 2^m$, $m\geqslant 0$;
3. $e_{2i-1,m}+e_{2i,m}=e_{i,m-1}$, $m\geqslant 1$.

Let $\mathbb{F}_\infty=\langle\gamma_1,\gamma_2,\ldots\rangle$. For any $r\in\mathbb{N}$ denote by W_r the subset of reduced words of length at most r containing only the first r generators and their inverses. We have $W_0\subset W_1\subset\cdots$ and $\mathbb{F}_\infty=\bigcup_{r\geqslant 0}W_r$.

Let $\alpha:\mathbb{F}_\infty\curvearrowright X$ be a Borel action and fix $\{e_{i,r}\}_{1\leqslant i\leqslant 2^r}$ a basic sequence of projections for $L^\infty(X)$. The following definition will allow us to keep track of the position of a point $x\in X$ relative to sets $\{e_{i,r}\}_{1\leqslant i\leqslant 2^r}$ under the action of $W_r\subset\mathbb{F}_\infty$.

Definition 3.19. Let $r\in\mathbb{N}$. A *r-labeled, r-neighborhood* is a finite oriented multi-graph containing:

1. a root vertex such that any vertex is connected to the root by a path of length at most r;
2. ever vertex has a label from the set $\{1,\dots,2^r\}$;
3. out-edges of every vertex have different colors from the set $\{\gamma_1,\gamma_1^{-1},\dots,\gamma_r,\gamma_r^{-1}\}$;
4. if edge xy is colored with γ_i then yx is colored by γ_i^{-1}.

Isomorphism classes of such objects form a finite set that we shall denote by $U^{r,r}$.

For $G\in U^{r,r}$, denote by R_G the root vertex in G. For $\gamma\in W_r$ let γR_G be the vertex in G obtained by starting from R_G and following the path given by γ (if such a path exists). Finally, let $l(\gamma R_G)$ be the label of the vertex γR_G in the set $\{1,2,\dots,2^r\}$.

Let X be a space together with a basic sequence of projections. For an action $\alpha:\mathbb{F}_\infty\curvearrowright X$ and $x\in X$ we can define $B_r^r(x)\in U^{r,r}$ by taking the imagines of x under W_r and their labels with respect to $\{e_{i,r}\}_{1\leqslant i\leqslant 2^r}$. For $G\in U^{r,r}$ let $T(\alpha,G)=\{x\in X\colon B_r^r(x)\equiv G\}$. Define also $p_G(\alpha)=\mu(T(\alpha,G))$.

If α is an action on a finite space Y (having the normalized cardinal measure) we have the same definitions provided that we still have some subsets $\{e_{i,r}\}_{1\leqslant i\leqslant 2^r,r\geqslant 0}$ of Y satisfying the same summation relations. This are needed to give labels to our vertices. Finite spaces with this kind of partitions are called *X-sets*. We are now ready to give the definition.

Definition 3.20. An action α of $\mathbb{F}_\infty$ is called *sofic* (in Elek–Lippner sense) if there exists a sequence of actions α_k of $\mathbb{F}_\infty$ on X-sets such that for any $r\geqslant 1$, for any $G\in U^{r,r}$ we have $\lim_{k\to\infty}p_G(\alpha_k)=p_G(\alpha)$.

Definition 3.21. An equivalence relation is called *sofic* if it is generated by a sofic action of $\mathbb{F}_\infty$ (all this in Elek–Lippner sense).

For actions of $\mathbb{F}_\infty$ the two notions of soficity are different. With our definition every action of $\mathbb{F}_\infty$ is sofic (see 3.15). Instead for equivalence relations the two notions are the same. This is what we shall prove now.

Proposition 3.22. *Let $E\subset X^2$ be an equivalence relation. Then E is sofic in sense of Elek–Lippner if and only if E is sofic ($M(E)$ admits a sofic embedding in some $\Pi_{k\to\omega}M_{n_k}$).*

Proof. Let $\alpha:\mathbb{F}_\infty\curvearrowright(X,\mu)$ be a sofic action in the sense of Elek–Lippner such that $E=E_\alpha$. Let α_k a sequence of actions on X-sets Y_k such that $\lim_{k\to\infty}p_G(\alpha_k)=p_G(\alpha)$. Finally let n_k be the cardinal of Y_k. We shall embed $M(E)$ in $\Pi_{k\to\omega}M_{n_k}$ in a sofic way. For this we need:

1. an embedding $L^\infty(X)\subset\Pi_{k\to\omega}D_{n_k}$;
2. a representation Θ of $\mathbb{F}_\infty$ on $\Pi_{k\to\omega}P_{n_k}$;
3. the formula $Tr(f\Theta(\gamma))=\int_{X_\gamma}f\,d\mu$ for every $f\in L^\infty(X)$ and $\gamma\in\mathbb{F}_\infty$, where $X_\gamma=\{x\in X\colon \gamma x=x\}$.

By hypothesis Y_k are X-sets, so they come together with projections $\{e_{i,r}^k\}_{i,r}$. Construct $e_{i,r}=\Pi_{k\to\omega}e_{i,r}^k$. We claim that $\{e_{i,r}\}_{1\leqslant i\leqslant 2^r,r\geqslant 0}$ form a basic sequence of projections for the algebra they generate. Relations $e_{i,r}=e_{2i-1,r+1}+e_{2i,r+1}$ are automatic, we just need to prove that $Tr(e_{i,r})=2^{-r}$.

Let $\{f_{i,r}\}_{1\leqslant i\leqslant 2^r, r\geqslant 0} \subset L^\infty(X)$ be the basic sequence of projections used in the construction of numbers $p_G(\alpha)$. Fix i and r. Let $U_i^{r,r} = \{G \in U^{r,r}\colon l(R_G) = i\}$, i.e. graphs such that the root has label i. Then $T(\alpha, G) \subset f_{i,r}$ for each $G \in U_i^{r,r}$. Moreover: $f_{i,r} = \bigsqcup_{G\in U_i^{r,r}} T(\alpha, G)$. In the same way we have $e^k_{i,r} = \bigsqcup_{G\in U_i^{r,r}} T(\alpha_k, G)$. Because $\lim_{k\to\infty} p_G(\alpha_k) = p_G(\alpha)$ we have $Tr(e_{i,r}) = \lim_{k\to\omega} Tr(e^k_{i,r}) = Tr(f_{i,r}) = 2^{-r}$.

By identifying $e_{i,r}$ with $f_{i,r}$ we get an embedding of $L^\infty(X)$. Now we construct the representation Θ of $\mathbb{F}_\infty$ in $\Pi_{k\to\omega} P_{n_k}$. We identified set Y_k with diagonal D_{n_k} and we have actions α_k of $\mathbb{F}_\infty$ that are defined on Y_k. This will construct a representation. We need to make sure Θ acts the same way as α.

Let γ be one of the generators of $\mathbb{F}_\infty$. Fix i, j and r. Let $U^{r,r}_{i,\gamma j} = \{G \in U^{r,r}\colon l(R_G) = i, l(\gamma R_G) = j\}$, the set of graphs such that the root has label i and the vertex connected with the root by the γ edge has label j (the existence of such an edge is a requirement we ask now for G). It is easy to see that $f_{i,r} \cap \alpha(\gamma^{-1})(f_{j,r}) = \bigsqcup_{G\in U^{r,r}_{i,\gamma j}} T(\alpha, G)$. In the same way $e^k_{i,r} \cap \alpha_k(\gamma^{-1})(e^k_{j,r}) = \bigsqcup_{G\in U^{r,r}_{i,\gamma j}} T(\alpha_k, G)$. Using the hypothesis we get $Tr(e_{i,r} \cdot \Theta(\gamma^{-1})(e_{j,r})) = \mu(f_{i,r} \cap \alpha(\gamma^{-1})(f_{j,r}))$. This is enough to deduce that the action that Θ induce on our embedding of $L^\infty(X)$ is equal to α.

For the third requirement let now $\gamma \in \mathbb{F}_\infty$ be an arbitrary element. It is of course sufficient to assume that f is one of the projections $e_{i,r}$. We need to prove that $Tr(e_{i,r}\Theta(\gamma)) = \mu(X_\gamma \cap e_{i,r})$. Lets say that in our construction we have $\Theta(\gamma) = \Pi_{k\to\omega}\gamma_k$. Then $Tr(e_{i,r}\Theta(\gamma)) = \lim_{k\to\infty} Tr(e^k_{i,r}\gamma_k)$. Let $U^{r,r}_{i,\gamma} = \{G \in U^{r,r}\colon l(R_G) = i,\ \gamma R_G = R_G\}$, i.e. the set of $G \in U^{r,r}$ such that the root has label i and the path in G described by γ, starting from the root, returns to the root. Then $X_\gamma \cap e_{i,r} = \bigsqcup_{G\in U^{r,r}_{i,\gamma}} T(\alpha, G)$. A similar formula with fixed points of γ_k and $e^k_{i,r}$ takes place. By $\lim_{k\to\infty} p_G(\alpha_k) = p_G(\alpha)$ we get $Tr(e_{i,r}\Theta(\gamma)) = \mu(X_\gamma \cap e_{i,r})$ and we are done.

For the reverse implication we shall assume that $M(E)$ embeds in some $\Pi_{k\to\omega} M_{n_k}$. We want to prove that E is also sofic in the sense of Elek–Lippner.

By 2.17 we have a sofic embedding $M(E) \subset \Pi_{k\to\omega} M_{n_k}$ such that $L^\infty(X) = A \subset \Pi_{k\to\omega} D_{n_k}$ and $u_\theta \subset \Pi_{k\to\omega} P_{n_k}$ for any $\theta \in [E]$.

We shall denote by d the normalized Hamming distance on P_{n_k}. In general γ, δ will denote elements in $\mathbb{F}_\infty$ and γ_i, δ_i will denote generators of $\mathbb{F}_\infty$. Let $\alpha : \mathbb{F}_\infty \curvearrowright (X, \mu)$ an action that generates the equivalence relation E on X. For any element $\gamma \in \mathbb{F}_\infty$, $\alpha(\gamma)$ induce an element $u_\gamma \in \mathcal{N}(A)$ and $u_\gamma u_\delta = u_{\gamma\delta}$. We shall write $u_\gamma = \Pi_{k\to\omega} u^k_\gamma \in \Pi_{k\to\omega} P_{n_k}$.

Let Y_k be a set with n_k elements and identify algebra D_{n_k} with $L^\infty(Y_k)$. For any generator γ_i of $\mathbb{F}_\infty$, $u^k_{\gamma_i} \in P_{n_k}$ induce an automorphism of Y_k. Denote it by $\alpha_k(\gamma_i)$ and extend α_k by multiplicity to an action of $\mathbb{F}_\infty$.

Let $\{e_{i,m}\}_{1\leqslant i\leqslant 2^m, m\geqslant 0} \subset L^\infty(X)$ a basic sequence of projections. Use it in order to construct sets $T(\alpha, G)$. Write $e_{i,m} = \Pi_{k\to\omega} e^k_{i,m}$ such that $\{e^k_{i,m}\}_{i,m}$ respect the same summation relations. Now elements in Y_k are labeled by projections $\{e^k_{i,m}\}_{i,m}$ so we have ingredients for constructing sets $T(\alpha_k, G)$.

We need to show that out of this actions we can find a subsequence satisfying the definition of soficity, namely that $\lim_{k\to\infty} \mu_{n_k}(T(\alpha_k, G)) = \mu(T(\alpha, G))$ for any $r \in \mathbb{N}$ and any $G \in U^{r,r}$ (denote by μ_{n_k} the normalized cardinal measure on a set with n_k elements). The subsequence is just to get rid of the ultrafilter and obtain classical limit for the countable set of objects that we are working with.

Fix $r \in \mathbb{N}$ and $\varepsilon > 0$. Let us see that it is enough to find $k \in \mathbb{N}$ such that $|\mu_{n_k}(T(\alpha_k, G)) - \mu(T(\alpha, G))| < \varepsilon$ for any $G \in U^{r,r}$. The phenomena here is that, if we fix a $G \in U^{r,r}$, when we pass from step r to $r+1$ we have $T(\alpha, G) = \bigcup_{G' \in U^{r+1,r+1}; G < G'} T(\alpha, G')$ (relation $G < G'$ is defined in an obvious way). So $\mu(T(\alpha_{k'}, G))$ is a sum of other $\mu(T(\alpha_{k'}, G'))$, but a finite sum. When we choose our sequence $\{\varepsilon_r\}$ we have to make sure that it compensates this growth.

Sets $\{T(\alpha, G)\colon G \in U^{r,r}\}$ form a partition of X. Let $T(\alpha, G) = \Pi_{k\to\omega} T(\alpha, G)^k$ such that $\{T(\alpha, G)^k\colon G \in U^{r,r}\}$ is a partition of Y_k. We also have in D_{n_k} projections $T(\alpha_k, G)$. We know that $\mu_{n_k}(T(\alpha, G)^k) \to_k \mu(T(\alpha, G))$ and we want to show that $\mu_{n_k}(T(\alpha_k, G)) \to_k \mu(T(\alpha, G))$.

Now fix $G \in U^{r,r}$. We need to understand equations that describe points in $T(\alpha, G)$. Remember that R_G is the root vertex in G; for $\gamma \in W_r$, γR_G is the vertex in G obtained by starting from R_G and following the path given by γ (if such a path exists). Finally, $l(\gamma R_G)$ is the label of the vertex γR_G in the set $\{1, 2, \ldots, 2^r\}$. We can now state our characterization of $T(\alpha, G)$.

A point $x \in X$ is an element of the set $T(\alpha, G)$ iff:

1. $\alpha(\gamma)(x) \in e_{l(\gamma R_G),r}$ for any $\gamma \in W_r$ for which γR_G exists;
2. $\alpha(\gamma)(x) = \alpha(\delta)(x)$ $\forall \gamma, \delta \in W_r,\ \gamma R_G = \delta R_G$;
3. $\alpha(\gamma)(x) \neq \alpha(\delta)(x)$ $\forall \gamma, \delta \in W_r,\ \gamma R_G \neq \delta R_G$.

First condition gives the coloring of vertices. The other two give the structure of the graph G. Let $\varepsilon_1 > 0$ such that $2|U^{r,r}|(|W_r| + 2|W_r|^2)\varepsilon_1 < \varepsilon$. We want to find $k \in \mathbb{N}$ such that for any $G \in U^{r,r}$ we have:

$$\mu_{n_k}\bigl(\alpha_k(\gamma)\bigl(T(\alpha, G)^k\bigr) \setminus e^k_{l(\gamma R_G),r}\bigr) < \varepsilon_1 \quad \forall \gamma \in W_r; \tag{2}$$

$$\mu_{n_k}\bigl(T(\alpha, G)^k \setminus \bigl\{y \in Y_k\colon \alpha_k(\gamma)(y) = \alpha_k(\delta)(y)\bigr\}\bigr) < \varepsilon_1 \quad \forall \gamma, \delta \in W_r,\ \gamma R_G = \delta R_G; \tag{3}$$

$$\mu_{n_k}\bigl(T(\alpha, G)^k \setminus \bigl\{y \in Y_k\colon \alpha_k(\gamma)(y) \neq \alpha_k(\delta)(y)\bigr\}\bigr) < \varepsilon_1 \quad \forall \gamma, \delta \in W_r,\ \gamma R_G \neq \delta R_G; \tag{4}$$

$$\bigl|\mu_{n_k}\bigl(T(\alpha, G)^k\bigr) - \mu\bigl(T(\alpha, G)\bigr)\bigr| < \varepsilon/2. \tag{5}$$

First we shall prove that this four conditions are enough to guarantee $|\mu_{n_k}(T(\alpha_k, G)) - \mu(T(\alpha, G))| < \varepsilon$ for every $G \in U^{r,r}$. Using (5) we just need to prove $|\mu_{n_k}(T(\alpha_k, G)) - \mu_{n_k}(T(\alpha, G)^k)| < \varepsilon/2$.

Take $x \in (T(\alpha, G)^k \setminus T(\alpha_k, G))$ for some $G \in U^{r,r}$. Following our characterization of $T(\alpha, G)$, we have:

1. $\exists \gamma \in W_r$ such that $\alpha_k(\gamma)(x)$ does not have the right label, namely $l(\gamma R_G)$;
2. or $\exists \gamma, \delta \in W_r$ such that $\gamma R_G = \delta R_G$ and $\alpha_k(\gamma)(x) \neq \alpha_k(\delta)(x)$;
3. or $\exists \gamma, \delta \in W_r$ such that $\gamma R_G \neq \delta R_G$ and $\alpha_k(\gamma)(x) = \alpha_k(\delta)(x)$.

Using (2)–(4) we get:

$$\mu_{n_k}\bigl(T(\alpha, G)^k \setminus T(\alpha_k, G)\bigr) < |W_r|\varepsilon_1 + 2|W_r|^2\varepsilon_1.$$

Because both $\{T(\alpha_k, G)\}_G$ and $\{T(\alpha, G)^k\}_G$ are partitions of Y_k and the above formula holds for any $G \in U^{r,r}$ we have:

$$\bigl|\mu_{n_k}\bigl(T(\alpha_k, G)\bigr) - \mu_{n_k}\bigl(T(\alpha, G)^k\bigr)\bigr| < \bigl|U^{r,r}\bigr|\bigl(|W_r|\varepsilon_1 + 2|W_r|^2\varepsilon_1\bigr) < \varepsilon/2.$$

Now back to the choice of k. Let $\gamma = \gamma_{i_1}\gamma_{i_2}\dots\gamma_{i_s} \in W_r$. We should in fact take $\gamma = \gamma_{i_1}^{\zeta_1}\gamma_{i_2}^{\zeta_2}\dots\gamma_{i_s}^{\zeta_s}$, where $\zeta_j \in \{\pm 1\}$. The inverses will change nothing in our arguments and will only overload our notations. Due to Feldman–Moore construction we know that $u_\gamma = u_{\gamma_{i_1}}u_{\gamma_{i_2}}\dots u_{\gamma_{i_s}}$. Next, combine $\alpha(\gamma)(T(\alpha, G)) \subset e_{l(\gamma R),r}$ and $\alpha(\gamma)(T(\alpha, G)) = u_\gamma T(\alpha, G)u_\gamma^*$ to get $u_\gamma T(\alpha, G)u_\gamma^* \subset e_{l(\gamma R),r}$.

Consider now $\gamma, \delta \in W_r$. If $\gamma R_G = \delta R_G$ then $\alpha(\gamma)|_{T(\alpha,G)} = \alpha(\delta)|_{T(\alpha,G)}$ so $Tr(T(\alpha, G)u_\delta^* u_\gamma) = \mu(T(\alpha, G))$ (here we consider $T(\alpha, G)$ to be a projection of $L^\infty(X) \subset \Pi_{k\to\omega} M_{n_k}$). If $\gamma R_G \neq \delta R_G$ then $Tr(T(\alpha, G)u_\delta^* u_\gamma) = 0$. Find $k \in \mathbb{N}$ such that (5) holds and:

$$\left\|u_\gamma^k - u_{\gamma_{i_1}}^k u_{\gamma_{i_2}}^k \dots u_{\gamma_{i_s}}^k\right\|_2 < \varepsilon_1/4 \quad \forall \gamma = \gamma_{i_1}\gamma_{i_2}\dots\gamma_{i_s} \in W_r; \tag{6}$$

$$\mu_{n_k}\left(u_\gamma^k T(\alpha, G)^k u_\gamma^{k*} \setminus e_{l(\gamma R_G),r}^k\right) < \varepsilon_1/2 \quad \forall \gamma \in W_r,\ \forall G \in U^{r,r}; \tag{7}$$

$$\mu_{n_k}\left(T(\alpha, G)^k\right) - Tr\left(T(\alpha, G)^k u_\delta^{k*} u_\gamma^k\right) < \varepsilon_1/2 \quad \forall G \in U^{r,r},\ \forall \gamma, \delta \in W_r,\ \gamma R_G = \delta R_G; \tag{8}$$

$$Tr\left(T(\alpha, G)^k u_\delta^{k*} u_\gamma^k\right) < \varepsilon_1/2 \quad \forall G \in U^{r,r},\ \forall \gamma, \delta \in W_r,\ \gamma R_G \neq \delta R_G. \tag{9}$$

By definition $\alpha_k(\gamma) = \alpha_k(\gamma_{i_1})\alpha_k(\gamma_{i_2})\dots\alpha_k(\gamma_{i_r})$ and $\alpha_k(\gamma_{i_j})(P) = u_{\gamma_{i_j}}^k P u_{\gamma_{i_j}}^{k*}$ for any projection $P \in D_{n_k}$. Then:

$$\alpha_k(\gamma)\left(T(\alpha, G)^k\right) = \left(u_{\gamma_{i_1}}^k u_{\gamma_{i_2}}^k \dots u_{\gamma_{i_r}}^k\right) T(\alpha, G)^k \left(u_{\gamma_{i_1}}^k u_{\gamma_{i_2}}^k \dots u_{\gamma_{i_r}}^k\right)^*. \tag{10}$$

Both u_γ^k and $u_{\gamma_{i_1}}^k u_{\gamma_{i_2}}^k \dots u_{\gamma_{i_s}}^k$ are elements in P_{n_k} so by (6):

$$d\left(u_\gamma^k, u_{\gamma_{i_1}}^k u_{\gamma_{i_2}}^k \dots u_{\gamma_{i_s}}^k\right) < \varepsilon_1/4.$$

Combined with (10), we have $\mu_{n_k}(\alpha_k(\gamma)(T(\alpha, G)^k) \setminus u_\gamma^k T(\alpha, G)^k u_\gamma^{k*}) < \varepsilon_1/4$. Use (7) to get $\mu_{n_k}(\alpha_k(\gamma)(T(\alpha, G)^k) \setminus e_{l(\gamma R),r}^k) < \varepsilon_1/4 + \varepsilon_1/2 < \varepsilon_1$, so we have (2).

Let now $\gamma = \gamma_{i_1}\gamma_{i_2}\dots\gamma_{i_s}$ and $\delta = \delta_{j_1}\delta_{j_2}\dots\delta_{j_t}$ such that $\gamma R_G = \delta R_G$. Use (6) both for γ and δ to get:

$$\left\|u_\delta^{k*} u_\gamma^k - \left(u_{\delta_{j_1}}^k u_{\delta_{j_2}}^k \dots u_{\delta_{j_t}}^k\right)^* \left(u_{\gamma_{i_1}}^k u_{\gamma_{i_2}}^k \dots u_{\gamma_{i_r}}^k\right)\right\|_2 < \varepsilon_1.$$

As before, $u_\delta^{k*} u_\gamma^k$ and $(u_{\delta_{j_1}}^k \dots u_{\delta_{j_t}}^k)^*(u_{\gamma_{i_1}}^k \dots u_{\gamma_{i_r}}^k)$ are elements in P_{n_k}, so:

$$d\left(u_\delta^{k*} u_\gamma^k, \left(u_{\delta_{j_1}}^k u_{\delta_{j_2}}^k \dots u_{\delta_{j_t}}^k\right)^* \left(u_{\gamma_{i_1}}^k u_{\gamma_{i_2}}^k \dots u_{\gamma_{i_r}}^k\right)\right) < \varepsilon_1/2.$$

Restricting this inequality just to fixed points in set $T(\alpha, G)^k$, we get:

$$\left|Tr\left(T(\alpha, G)^k u_\delta^{k*} u_\gamma^k\right) - Tr\left(T(\alpha, G)^k \left(u_{\delta_{j_1}}^k u_{\delta_{j_2}}^k \dots u_{\delta_{j_t}}^k\right)^* \left(u_{\gamma_{i_1}}^k u_{\gamma_{i_2}}^k \dots u_{\gamma_{i_r}}^k\right)\right)\right| < \varepsilon_1/2. \tag{11}$$

Apply (10) for γ and δ to get:

$$\begin{aligned}
&\mu_{n_k}\left(T(\alpha, G)^k \setminus \left\{y \in Y_k\colon \alpha_k(\gamma)(y) = \alpha_k(\delta)(y)\right\}\right) \\
&\quad = \mu_{n_k}\left(T(\alpha, G)^k\right) - Tr\left(T(\alpha, G)^k \left(u_{\delta_{j_1}}^k u_{\delta_{j_2}}^k \dots u_{\delta_{j_t}}^k\right)^* \left(u_{\gamma_{i_1}}^k u_{\gamma_{i_2}}^k \dots u_{\gamma_{i_r}}^k\right)\right).
\end{aligned}$$

This combined with (11) and (8) yields (3).

Assume now that $\gamma R_G \neq \delta R_G$. Then:

$$\mu_{n_k}\big(T(\alpha, G)^k \setminus \{y \in Y_k\colon \alpha_k(\gamma)(y) \neq \alpha_k(\delta)(y)\}\big)$$
$$= Tr\big(T(\alpha, G)^k \big(u^k_{\delta_{j_1}} u^k_{\delta_{j_2}} \ldots u^k_{\delta_{j_t}}\big)^* \big(u^k_{\gamma_{i_1}} u^k_{\gamma_{i_2}} \ldots u^k_{\gamma_{i_r}}\big)\big).$$

Inequalities (11) and (9) will imply (4). □

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