Markov processes invariant under a Lie group action

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Abstract

We show that a Markov process in a manifold invariant under the action of a compact Lie group $K$ induces a Lévy process in each $K$-orbit by “forcing” it to run in the orbit.

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1. Introduction

We begin by describing a simple way to obtain a Brownian motion in the unit sphere $S^{n-1}$ in $\mathbb{R}^n$ by forcing the standard Brownian motion $x_t$ in $\mathbb{R}^n$ to run inside $S^{n-1}$. Starting at a point in $S^{n-1}$, run $x_t$ for a small time $\varepsilon$ and radially project paths to $S^{n-1}$; next at the end of each projected path, run $x_t$ again for $\varepsilon$ and radially project paths to $S^{n-1}$. Continue in this way to obtain a process $z_t^\varepsilon$ in $S^{n-1}$. Then when $\varepsilon \to 0$, $z_t^\varepsilon$ converges in distribution to a Brownian motion $z_t$ in $S^{n-1}$. This fact, which may be well known to people in stochastic differential geometry, turns out to have an interesting connection with a Lie group action.

The Brownian motion $x_t$ in $\mathbb{R}^n$ is invariant under the action of the rotation group $SO(n)$ on $\mathbb{R}^n$ in the sense that for $g \in SO(n)$, $g(x_t)$ has the same distribution as the Brownian motion starting from $g(x_0)$. Moreover, the sphere $S^{n-1}$ is an orbit of the compact Lie group $SO(n)$, and the Brownian motion $z_t$ in $S^{n-1}$ is an $SO(n)$-invariant Markov process in $S^{n-1}$.

This leads us to consider a general Markov process $x_t$ in a manifold $X$ invariant under the action of a compact Lie group $K$. As our main result, we will show that by forcing $x_t$ to run inside a $K$-orbit $Z$ by a procedure similar to the one described above, we obtain a $K$-invariant...
Feller process \( z_t \) in \( Z \), which will be called a Lévy process in \( Z \). We will also obtain a simple expression for the generator of \( z_t \) in terms of that of \( x_t \). The case of Brownian motion \( x_t \) in \( \mathbb{R}^n \) with \( Z = S^{n-1} \) follows directly from this result.

More precise definitions are given in the next section. The main result is stated and proved in Section 3. The Peter–Weyl theorem for the representation of compact Lie groups plays an important role in the proof. An application is given to a class of \( K \)-invariant Markov processes in Section 4. Besides forcing \( x_t \) to run in a \( K \)-orbit by an explicit construction, we may also condition \( x_t \) to run in a \( K \)-orbit, but as we will see these two processes are not always equal in distribution.

2. Some definitions

Throughout this paper, let \( X \) be a (smooth) manifold. The space of continuous functions on \( X \) is denoted by \( C(X) \) as usual. Let \( C_b(X), C_c(X) \) and \( C_0(X) \) be the spaces of continuous functions on \( X \) that are respectively bounded, compactly supported and convergent to 0 at infinity in the one-point compactification of \( X \). When a superscript \( \infty \) is added, such as in \( C_c^{\infty}(X) \), it will denote the subspace of smooth functions.

Let \( x_t \) be a Markov process in \( X \) with transition semigroup \( P_t \) and rcll paths (right continuous paths with left limits). Only the simple Markov property will be assumed for \( x_t \) and its lifetime will be allowed to be finite; thus, \( P_t 1 \) may be less than 1. The generator \( L \) of the Markov process \( x_t \) or its transition semigroup \( P_t \) is defined by

\[
Lf = \lim_{t \to 0} (1/t)(P_t f - f)
\]

with domain \( D(L) \) consisting of \( f \in C_0(X) \) for which the above limit exists under the uniform convergence on \( X \). The process \( x_t \) will be called a Feller process if for \( f \in C_0(X), P_t f \in C_0(X) \) and \( P_t f \to f \) uniformly on \( X \) as \( t \to 0 \). In this case, the transition semigroup \( P_t \) is completely determined by its generator \( L \). A continuous Feller process in \( X \) will be called a diffusion process if its generator \( L \) restricted to \( C_c^{\infty}(X) \) is a differential operator with smooth coefficients and no constant term. Then this restriction of \( L \) determines the distribution of the diffusion process. However, the Feller property will not be assumed in our main result.

Let \( G \) be a Lie group acting (smoothly) on \( X \). An operator \( T \) on \( X \) with domain \( D(T) \) (a space of functions on \( X \)) is called \( G \)-invariant if

\[
\forall f \in D(T) \quad \text{and} \quad g \in G, \quad f \circ g \in D(T) \quad \text{and} \quad T(f \circ g) = (Tf) \circ g.
\]

A Markov process \( x_t \) in \( X \) is called \( G \)-invariant if its transition semigroup \( P_t \) with domain \( C_b(X) \) is \( G \)-invariant.

A Feller process \( g_t \) in a Lie group \( G \) with an infinite lifetime that is invariant under the action of \( G \) on itself by left translations will be called a Lévy process in \( G \). Such a process may also be characterized as a rcll process \( g_t \) that possesses independent and stationary increments in the sense that for \( s < t \), \( g_t^{-1} g_s \) is independent of the process up to time \( s \) and its distribution depends only on \( t-s \) (see [8]). This notion extends the usual definition of Lévy processes in \( \mathbb{R}^n \) regarded as an additive group. More generally, for a compact subgroup \( H \) of \( G \), a Feller process \( x_t \) in the homogeneous space \( G/H \) with an infinite lifetime, that is invariant under the natural (left) action of \( G \) on \( G/H \), will also be called a Lévy process in \( G/H \).

An explicit formula for the generator of a Lévy process in \( G \) or \( G/H \), restricted to smooth functions with compact supports, in terms of a differential operator and a Lévy measure, is...
obtained by Hunt [7]; see also Heyer [6, Chapter IV] and Liao [8, Chapter 2]. Note that this restriction of the generator determines the distribution of the Lévy process completely. A Lévy process in $G$ may be obtained from a stochastic integral equation driven by a Brownian motion and a Poisson random measure; see Applebaum–Kunita [1]. A Lévy process in $G/H$ can be obtained as the projection of a Lévy process in $G$; see [8, Theorem 2.2].

Let $K$ be a compact Lie group acting on $X$ and let $Y$ be a (smooth) submanifold of $X$, possibly with a (piecewise smooth) boundary, that is transversal to the action of $K$ in the sense that it intersects each orbit of $K$ at exactly one point, that is,

$$\forall y \in Y, \quad (K y) \cap Y = \{y\} \quad \text{and} \quad X = \bigcup_{y \in Y} K y. \quad (3)$$

Let $J: X \to Y$ be the projection map $J(x) = y$ for $x \in K y$, which is continuous because $K$ is compact. Note that $J \circ k = J$ for $k \in K$. It is easy to show (see [9]) that $y_t = J(x_t)$ is a Markov process in $Y$ with transition semigroup $Q_t$ given by $Q_t f (y) = P_t (f \circ J) (y)$ for $y \in Y$ and $f \in C_b(Y)$. The process $y_t$ is called the radial part of $x_t$.

For $x \in X$, $K_x = \{k \in K; k x = x\}$ is a closed subgroup of $K$, called the isotropy subgroup of $K$ at $x$. Let $Y^0$ be $Y$ minus its boundary $\partial Y$. In the rest of this paper, we will assume that $K_y$ is the same closed subgroup $M$ of $K$ as $y$ varies over $Y^0$. This assumption is often satisfied when the transversal submanifold $Y$ is properly chosen. For example, for $X = \mathbb{R}^n$ and $K = SO(n)$, $Y$ may be chosen to be the positive part of the $x_1$-axis with boundary $\partial Y$ containing only the origin, and then $M = \text{diag}(1, \ldots, 0, \ldots, 1)$. See [9] for other examples. Under this assumption, each $K$-orbit $K y$, $y \in Y^0$, is naturally identified with $K/M$.

We will now strengthen the transversality condition (3) by assuming

$$\forall y \in Y^0, \quad T_y X = T_y (K y) \oplus T_y Y \quad \text{(direct sum)}, \quad (4)$$

where $T_y X$ is the tangent space of $X$ at $y$; see Lemma 3.3 in [5, Chapter II]. Then the union of the $K$-orbits through $Y^0$, denoted by $X^0$, is an open dense subset of $X$ that is invariant under the $K$-action, and $X^0 = Y^0 \times (K/M)$ as a product manifold.

Let $\pi: K \to K/M$ be the natural projection and let $Z = K/M$. A Borel measurable map $S: Z \to K$ is called a section map if $\pi \circ S = \text{id}_Z$ (identity map on $Z$). For $y \in Y^0$ and $z \in Z$, $z y = S(z) y$ is well defined (independent of choice for section map $S$), and any $x \in X^0$ may be written uniquely as $x = z y$ for $y \in Y^0$ and $z \in Z$. Let $J_1: X^0 \to Y^0$ be given by $x \mapsto y$ (which is the restriction of $J$ to $X^0$) and $J_2: X^0 \to Z$ by $x \mapsto z$.

3. Main result

For simplicity, a function $h$ on $Y^0$ may be regarded as the function $h \circ J_1$ on $X^0$ and a function $\phi$ on $Z = K/M$ may be regarded as the function $\phi \circ J_2$ on $X^0$. Note that if $h \in C_c^\infty(Y^0)$ and $\phi \in C_c^\infty(Z)$, then $h \phi \in C_c^\infty(X^0) \subset C_c^\infty(X)$.

We will impose the following two conditions on the transition semigroup $P_t$ and the generator $L$ of the Markov process $x_t$ in $X$:

$$\forall y \in Y^0 \quad \text{and} \quad t > 0, \quad P_t (y, \partial X^0) = 0, \quad (5)$$

and

$$C_c^\infty (X) \subset D(L) \quad \text{and} \quad \forall y \in Y^0, \quad L h_n (y) \to 0 \quad \text{as} \quad n \uparrow \infty \quad (6)$$
for any sequence \( h_n \in C^\infty_c(Y^\circ) \) satisfying \( 0 \leq h_n \uparrow 1 \) and \([h_n = 1] \uparrow Y^\circ\), where \([h_n = 1] = \{y \in Y^\circ; h_n(y) = 1\}\). Here for a sequence of sets \( Y_n \subseteq Y^\circ, Y_n \uparrow Y^\circ \) means that \( Y_n \subset Y_{n+1} \) for all \( n > 0 \) and for any compact subset \( Y' \) of \( Y^\circ, Y' \subseteq Y_n \) for some \( n > 0 \).

Condition (5) means that the process \( x_t \) with \( x_0 \in X^\circ \) has zero probability of reaching the boundary \( \partial X^\circ \) at any fixed time \( t \). It is always satisfied if one is willing to replace \( x_t \) by its sub-process in \( X^\circ \) killed upon leaving \( X^\circ \). Condition (6) holds if \( x_t \) is a diffusion process as its generator is a differential operator. It is expected to hold for more general processes because when there is no killing inside the state space, one would expect \( L1 = \lim_{t \to 0}(1/t)(P_1t - 1) = 0 \).

**Proposition 1.** For \( h_n \) in (6) and \( \phi \in C^\infty(Z) \), \( L(h_n\phi)(zy) \) converges uniformly for \((y, z) \in Y' \times Z\) for any compact subset \( Y' \) of \( Y^\circ \), and hence the limit \( \lim_{n \to \infty} L(h_n\phi)(zy) \) is continuous in \((y, z) \in Y^\circ \times Z\).

**Proof.** By (6), \( Lh_n(y) \to 0 \). For each fixed \( y \in Y^\circ \), when \( n \) is sufficiently large, \( h_n(y) = 1 \) and \( Lh_n(y) = \lim_{t \to 0}(1/t)[P_t h_n(y) - 1] \uparrow 0 \) as \( n \uparrow \infty \). Since \( Lh_n \) is continuous, the convergence is uniform on compact subsets of \( X^\circ \). We may assume \( 0 \leq \phi \leq 1 \). For \( z \in Z, L(h_n\phi)(zy) = \lim_{t \to 0}(1/t)[P_t (h_n\phi)(zy) - \phi(z)] \) increases as \( n \uparrow \infty \). Similarly, \( L(h_n(1 - \phi))(zy) \) increases as \( n \uparrow \infty \). Because \( Lh_n = L(h_n\phi) + L(h_n(1 - \phi)) \), this implies that \( \lim_{n \to \infty} L(h_n\phi)(zy) \) is finite and is l.s.c. (lower semi-continuous) in \((y, z) \in Y' \times Z\) for any compact subset \( Y' \) of \( Y^\circ \). Since the same holds for \( \lim_{n \to \infty} L(h_n(1 - \phi)) \), it follows that \( \lim_{n \to \infty} L(h_n\phi)(zy) \) is continuous in \((y, z) \in Y' \times Z\). As an increasing limit, the convergence is also uniform in \((y, z) \in Y' \times Z\). \( \square \)

A measure \( \mu \) on \( Z = K/M \) is called \( M \)-invariant if \( m\mu = \mu \) for \( m \in M \), where \( m\mu \) is the measure defined by the left action of \( M \) on \( Z \) as \( m\mu(f) = \mu(f \circ m) \) for \( f \in C_b(Z) \) and \( \mu(f) = \int f d\mu \). The convolution of two \( M \)-invariant measures \( \mu \) and \( \nu \) is the \( M \)-invariant measure \( \mu \ast \nu \) on \( Z \), defined by

\[
\mu \ast \nu(f) = \int f(S(x)y)\mu(dx)\nu(dy),
\]

where \( S: K/M \to K \) is a section map, the choice of which will not affect the definition of \( \mu \ast \nu \). When \( M = \{e\} \), this becomes the usual definition of the convolution on the Lie group \( K \). Note that the convolution is associative, that is, \( \mu \ast (\nu \ast \gamma) = (\mu \ast \nu) \ast \gamma \).

Let \( e \) be the identity element of \( K \) and let \( o = eM \) be the origin of \( Z = K/M \). A family of \( M \)-invariant probability measures \( \mu_t \) on \( Z, t \in \mathbb{R}_+ = [0, \infty) \), is called a convolution semigroup if \( \mu_{t+s} = \mu_t \ast \mu_s \) and \( \mu_0 = \delta_o \), the unit point mass at \( o \). It is called continuous if \( \mu_t \to \mu_0 \) weakly as \( t \to 0 \). Let \( z_t \) be a Lévy process in \( Z = K/M \) with transition semigroup \( P_t \). Then \( \mu_t = P_t(o, \cdot) \) is a continuous convolution semigroup of \( M \)-invariant probability measures on \( Z \). Conversely, given such a convolution semigroup \( \mu_t \), there is a Lévy process \( z_t \) in \( Z \) with \( z_0 = o \), unique in distribution, and so associated with \( \mu_t \); see [8, Section 2.2].

For a measure \( \mu \) on \( X^\circ \), \( \mathcal{J}_2 \mu \) is the measure on \( Z \) given by \( \mathcal{J}_2 \mu_f = \mu_f(o) \) for \( f \in C_b(Z) \). If \( \mu \) is \( M \)-invariant on \( X^\circ \), then \( \mathcal{J}_2 \mu \) is \( M \)-invariant on \( Z \). The \( K \)-invariance of \( P_t \) implies that for \( y \in Y^\circ, P_t(y, \cdot) \) is \( M \)-invariant on \( X^\circ \) and hence \( \mathcal{J}_2 P_t(y, \cdot) \) is \( M \)-invariant on \( Z \).

Note that for \( z_1, z_2 \in Z \), the product \( z_1z_2 = S(z_1)z_2 \) may depend on the choice of the section map \( S \), but for an \( M \)-invariant measure \( \mu \) on \( Z \), the integral \( \int f(z_1z_2)\mu(dz_2) \) is independent of \( S \). By the \( K \)-invariance of \( P_t \), for \( f \in C(Z) \),
\[ \iint \phi(z_2) [J_2 P_t(y, \cdot)](dz_1) P_s(z_1 y, dy_2 \times dz_2) \]

\[ = \iint \phi(z_1 z_2) [J_2 P_t(y, \cdot)](dz_1) [J_2 P_s(y, \cdot)](dz_2) \]

\[ = [J_2 P_t(y, \cdot)] * [J_2 P_s(y, \cdot)](\phi). \] (7)

Fix \( y \in Y^0 \) and a small \( \varepsilon > 0 \). Let \( z_t^{y, \varepsilon} \) be the process in \( Z \) obtained by forcing \( x_t \) to run inside the orbit \( Ky \). With the identification of \( Ky \) with \( Z = K/M \) via the map \( ky \mapsto J_2(ky) \), this may be described as follows. Starting at \( y \in Y^0 \), run the process \( x_t \) for a small time \( \varepsilon > 0 \) and project its paths to \( Z \) via \( J_2 \), ignoring the paths that exit \( X^0 \) or reach the lifetime by time \( \varepsilon \); then at the end of each projected path, run \( x_t \) again for time \( \varepsilon \) and project the paths to \( Z \) via \( J_2 \), and continue in this way to obtain a process \( z_t^{y, \varepsilon} \) in \( Z \). Note that \( z_0^{y, \varepsilon} = o \). By (7), the distribution of \( z_t^{y, \varepsilon} \) is given by

\[ \mu_t^{y, \varepsilon} = [J_2 P_t(y, \cdot)]^{*n} * [J_2 P_{t-n\varepsilon}(y, \cdot)], \] (8)

where \( n = [t/\varepsilon] \) is the integer part of \( t/\varepsilon \) and the superscript \( *n \) denotes an \( n \)-fold convolution. The process \( z_t^{y, \varepsilon} \) will be called an induced process in \( Z \) obtained by forcing \( x_t \) to run in \( Ky \). Note that it may have a finite lifetime because \( \mu_t^{y, \varepsilon}(1) \) may be less than 1.

As usual, a sequence of processes \( z^n_t \) is said to converge to a process \( z_t \) in finite dimensional distributions if for any \( t_1 < t_2 < \cdots < t_k \), the joint distribution of \( z^n_{t_1}, z^n_{t_2}, \ldots, z^n_{t_k} \) converges weakly to that of \( z_{t_1}, z_{t_2}, \ldots, z_{t_k} \).

**Theorem 1.** Let \( x_t \) be a \( K \)-invariant Markov process in \( X \) with transition semigroup \( P_t \) and generator \( L \) satisfying (5) and (6). Then for any \( y \in Y^0 \), there is a Lévy process \( z_t^y \) in \( Z = K/M \) such that \( z_t^{y, \varepsilon} \) defined above converges to \( z_t^y \) in finite dimensional distributions as \( \varepsilon \to 0 \). Moreover, the generator \( L^y_Z \) of \( z_t^y \) is given by

\[ L^y_Z \phi(z) = \lim_{n \to \infty} L(h_n \phi)(zy) \quad \text{for} \quad \phi \in C^\infty(Z), \] (9)

where the limit is given in Proposition 1.

**Proof.** Let \( \mu_t^{y, \varepsilon} \) be the \( M \)-invariant sub-probability measure on \( Z \) given in (8) and let \( \rho_M \) be the normalized Haar measure on \( M \). For \( \phi \in C^\infty(K) \), let \( \hat{\phi}(k) = \int_M \phi(km) \rho_M(dm) \) for \( k \in M \). Then \( \hat{\phi} \) is smooth and right \( M \)-invariant, and is called the right \( M \)-invariant regularization of \( \phi \). Note that \( \hat{\phi} = \phi \) if \( \phi \) is right \( M \)-invariant. Any right \( M \)-invariant smooth function \( \phi \) on \( K \) may be regarded as a smooth function on \( Z = K/M \) by setting \( \phi(kM) = \phi(k) \), which does not depend on the choice of \( k \in K \), to represent the coset \( kM \).

Let \( h = h_n \) in (6). We may assume \( h(y) = 1 \) by choosing a sufficiently large \( n \). In the following expression, letter \( n \) has a different meaning. Let

\[ \mu_t^{y, \varepsilon, h} = [J_2(h P_{t}(y, \cdot))]^{*n} * [J_2(h P_{t-n\varepsilon}(y, \cdot))], \] (10)

where \( n = [t/\varepsilon] \) and \( J_2(h P_{t}(y, \cdot)) \) is the projection of the measure \( h(x) P_{t}(y, dx) \) on \( X^0 \) to \( Z \) via \( J_2 \). Then \( \mu_t^{y, \varepsilon, h} \) is an \( M \)-invariant sub-probability measure on \( Z \) and \( \mu_t^{y, \varepsilon, h} \uparrow \mu_t^{y, \varepsilon} \) as \( h \uparrow 1 \).

We now briefly describe the basic theory of Fourier analysis on a compact Lie group \( K \); see [3] for more details. A Lie group homomorphism \( u \) from \( K \) into the group \( U(n) \) of \( n \times n \) unitary matrices is called a unitary representation of \( K \), which may be regarded as a linear action of \( K \) on \( \mathbb{C}^n \). It is called nontrivial if \( u \neq I \) (identity matrix) and irreducible if it has no nontrivial invariant
subspace of \( \mathbb{C}^n \). Two representations \( u_1 \) and \( u_2 \) are called equivalent if \( u_2(k) = bu_1(k)b^{-1} \) for some invertible matrix \( b \) and all \( k \in K \), that is, if they differ only by a change of basis on \( \mathbb{C}^n \). The set \( \hat{K} \) of equivalence classes of non-trivial irreducible unitary representations of a compact Lie group \( K \) is countable.

For \( \delta \in \hat{K} \), let \( u_\delta \) be a representation in class \( \delta \), which is a unitary matrix-valued smooth function on \( K \) satisfying \( u_\delta(k_1k_2) = u_\delta(k_1)u_\delta(k_2) \) for \( k_1, k_2 \in K \). By the Peter–Weyl Theorem, the linear span of the matrix elements of \( u_\delta \), \( \delta \in \hat{K} \), is dense in \( C(K) \) under the supremum norm. It follows that the linear span of matrix elements of \( \hat{u}_\delta \), \( \delta \in \hat{K} \), is dense in \( C(Z) \), where \( \hat{u}_\delta \) are right \( M \)-invariant regularizations of \( u_\delta \) and are regarded as functions on \( Z \).

For \( M \)-invariant measures \( \mu \) and \( \nu \) on \( Z \),

\[
\mu \ast \nu(\hat{u}_\delta) = \mu(\hat{u}_\delta) \nu(\hat{u}_\delta),
\]

(11)

because

\[
\mu \ast \nu(\hat{u}_\delta) = \iiint u_\delta(S(x)S(y)m)\mu(dx)\nu(dy)\rho_M(dm)
= \iiint u_\delta(S(x)m_1S(y)m_2)\mu(dx)\nu(dy)\rho_M(dm_1)\rho_M(dm_2)
\quad \text{(by } M \text{-invariance of } \nu) \nonumber
= \iiint u_\delta(S(x)m_1)\mu(dx)\rho_M(dm_1) \iiint u_\delta(S(y)m_2)\nu(dy)\rho_M(dm_2)
= \mu(\hat{u}_\delta) \nu(\hat{u}_\delta),
\]

and

\[
\mu(\hat{u}_\delta) = \rho_M(u_\delta)\mu(\hat{u}_\delta) = \mu(\hat{u}_\delta)\rho_M(u_\delta),
\]

(12)

because

\[
\rho_M(u_\delta)\mu(\hat{u}_\delta) = \iiint u_\delta(m)u_\delta(S(x)m')\rho_M(dm)\mu(dx)\rho_M(dm')
= \iiint u_\delta(ms(x)m')\rho_M(dm)\mu(dx)\rho_M(dm')
= \iiint u_\delta(S(x)m')\mu(dx)\rho_M(dm') \quad \text{(by } M \text{-invariance of } \mu) \nonumber
= \mu(\hat{u}_\delta) = \iiint u_\delta(S(x)m'm)\mu(dx)\rho_M(dm')\rho_M(dm) = \mu(\hat{u}_\delta)\rho_M(u_\delta).
\]

Recall that \( Lf \) is defined to be the uniform convergence limit of \( (1/t)[P_t f - f] \) as \( t \rightarrow 0 \). Because \( h(y) = 1 \), \( P_0(zy, h\hat{u}_\delta) = h(y)\hat{u}_\delta(z) = \hat{u}_\delta(z) \). By (6), \( C_c^\infty(X) \subset D(L) \) and hence

\[
P_\varepsilon(zy, h\hat{u}_\delta) = P_0(zy, h\hat{u}_\delta) + \varepsilon \frac{P_\varepsilon(zy, h\hat{u}_\delta) - P_0(zy, h\hat{u}_\delta)}{\varepsilon}
= \hat{u}_\delta(z) + \varepsilon L(h\hat{u}_\delta)(zy) + o(\varepsilon),
\]

(13)

where \( o(\varepsilon)/\varepsilon \rightarrow 0 \) uniformly for \( z \in Z \) as \( \varepsilon \rightarrow 0 \). Note that \( P_0(y, h\hat{u}_\delta) = \hat{u}_\delta(o) = \rho_M(u_\delta) \) by the definition of \( \hat{u}_\delta \). Because \( P_\varepsilon(y, h\hat{u}_\delta) = \{ J_2[hP_\varepsilon(y, \cdot)](\hat{u}_\delta) \} \), where as before \( J_2[hP_\varepsilon(y, \cdot)] \) is the projection of the measure \( h(x)P_\varepsilon(y, dx) \) on \( \mathcal{X}^0 \) to \( Z \) via \( J_2 \) and is \( M \)-invariant, by (12),

\[
P_\varepsilon(y, h\hat{u}_\delta) = \rho_M(u_\delta)P_\varepsilon(y, h\hat{u}_\delta) = P_\varepsilon(y, h\hat{u}_\delta)\rho_M(u_\delta).
\]
The same holds with \( P_\varepsilon(y, h\hat{\delta}) \) replaced by \( L(h\hat{\delta})(y) \) or by \( o(\varepsilon) \) in (13) when \( z = o \). Then with \( n = [t/\varepsilon] \),

\[
\{J_2[hP_\varepsilon(y, \cdot)]^n(\hat{u}_\delta)\} = \{[J_2[hP_\varepsilon(y, \cdot)](\hat{u}_\delta)]^n = P_\varepsilon(y, h\hat{\delta})^n
\]

\[
= [\rho_M(u_\delta) + \varepsilon L(h\hat{\delta})(y) + o(\varepsilon)]^n = [\rho_M(u_\delta)[1 + \varepsilon L(h\hat{\delta})(y) + o(\varepsilon)]]^n
\]

\[
= [\rho_M(u_\delta) \exp[\varepsilon L(h\hat{\delta})(y) + o(\varepsilon)]]^n \quad \text{(a different } o(\varepsilon) \text{ but still } o(\varepsilon)/\varepsilon \to 0)
\]

\[
= \rho_M(u_\delta) \exp[n\varepsilon L(h\hat{\delta})(y) + no(\varepsilon)] \quad \text{(because } \rho_M(u_\delta) \text{ commutes with } L(h\hat{\delta})(y) \text{ and } o(\varepsilon) \text{, and } \rho_M(u_\delta)^2 = \rho_M(u_\delta))
\]

\[
\to \rho_M(u_\delta) \exp[tL(h\hat{\delta})(y)] \quad \text{as } \varepsilon \to 0 \quad (n \to \infty \text{ and } n\varepsilon \to t).
\]

Because \( P_\varepsilon(y, \cdot) \to \delta_y \) weakly as \( \varepsilon \to 0 \), by (10), \( \mu_{t,\varepsilon}^{y,e,h}(\hat{u}_\delta) \to \rho_M(u_\delta) \exp[tL(h\hat{\delta})(y)] \) as \( \varepsilon \to 0 \). Since the matrix elements of \( \hat{u}_\delta \), as \( \delta \) varies over \( \hat{K} \), span a dense subspace of \( C(Z) \), it follows that for any \( t \geq 0 \), \( \mu_{t,\varepsilon}^{y,e,h} \to \mu_{t}^{y,0,h} \) weakly as \( \varepsilon \to 0 \) for some \( M \)-invariant subprobability measure \( \mu_{t}^{y,0,h} \) on \( Z \) with \( \mu_{t}^{y,0,h}(\hat{u}_\delta) = \rho_M(u_\delta) \exp[tL(h\hat{\delta})(y)] \).

Because \( \mu_{t,\varepsilon}^{y,e,h} \) increases as \( h \uparrow 1 \), \( \mu_{t}^{y,0,h} \uparrow \mu_{t}^{y} \) when \( h = h_n \uparrow 1 \) as in (6) for some \( M \)-invariant subprobability measure \( \mu_{t}^{y} \) such that

\[
\mu_{t}^{y}(\hat{u}_\delta) = \rho_M(u_\delta) \exp[t \lim_{h \uparrow 1} L(h\hat{\delta})(y)],
\]

where the limit \( \lim_{h \uparrow 1} L(h\hat{\delta})(y) \) is given in Proposition 1. For the trivial representation \( \delta \) of \( K \), \( u_\delta = 1 \), then by (15) and (6), \( \mu_{t}^{y}(1) = \rho_M(1)e^{0} = 1 \). Thus, \( \mu_{t}^{y} \) is a probability measure on \( Z \).

By (11), \( \mu_{t}^{y} \ast \mu_{t}^{y}(\hat{u}_\delta) = \mu_{t}^{y}(\hat{u}_\delta) \mu_{t}^{y}(\hat{u}_\delta) \), and by (15), this is equal to \( \mu_{t}^{y}(\hat{u}_\delta) \). This shows that \( \mu_{t}^{y}, t \geq 0 \), form a convolution semigroup. Its continuity also follows from (15).

Because \( \mu_{t}^{y,e} \geq \mu_{t}^{y,e,h} \), \( \mu_{t}^{y,e} \geq \mu_{t}^{y,0,h} \). It follows that for \( \phi \in C(Z) \) with \( 0 \leq \phi \leq 1 \),

\[
\lim_{\varepsilon \to 0} \mu_{t}^{y,e}(\phi) \geq \mu_{t}^{y}(\phi).
\]

On the other hand, \( \lim_{\varepsilon \to 0} \mu_{t}^{y,e}(1 - \phi) \geq \mu_{t}^{y}(1 - \phi) \). Because \( \mu_{t}^{y,e}(1) \leq 1 \) and \( \mu_{t}^{y}(1) = 1 \), we have \( 1 - \lim_{\varepsilon \to 0} \mu_{t}^{y,e}(\phi) \geq 1 - \mu_{t}^{y}(\phi) \). This implies that \( \lim_{\varepsilon \to 0} \mu_{t}^{y,e}(\phi) \leq \mu_{t}^{y}(\phi) \) and hence \( \lim_{\varepsilon \to 0} \mu_{t}^{y,e}(\phi) = \mu_{t}^{y}(\phi) \) for \( \phi \in C(K) \). We have proved that \( \mu_{t}^{y,e} \to \mu_{t}^{y} \) weakly as \( \varepsilon \to 0 \) for some continuous convolution semigroup \( \mu_{t}^{y} \) of \( M \)-invariant probability measures on \( Z \).

Let \( z_{i}^{\varepsilon} \) be the Lévy process in \( Z \) associated with \( \mu_{t}^{y} \) with \( z_{i}^{\varepsilon} = 0 \). Then for \( s < t \) and \( f \in C(Z^{3}) \), with \( m = [s/\varepsilon] \) and \( n = [t/\varepsilon] \),

\[
E[f(z_{i}^{\varepsilon,y},z_{i}^{\varepsilon,y})] = \iint f(z_{1}z_{2}, z_{1}z_{2}z_{3}) [J_{2}P_{\varepsilon}(y, \cdot)]^{n} m(dz_{1}) P_{s-m}e(y,dy_{2} \times dz_{2})
\]

\[
\times \{[J_{2}P_{\varepsilon(y, \cdot)}]^{m}[(y_{2}, \cdot)]^{n} \cdot [J_{2}P_{\varepsilon(y, \cdot)}]^{m} n(dy_{2} \times dz_{2})
\]

\[
\rightarrow \iint f(z_{1}, z_{1}z_{3}) \mu_{s}^{y}(dz_{1}) \mu_{t-s}^{y}(dz_{3}) \quad \text{as } \varepsilon \to 0
\]

(because \( P_{\varepsilon}(y, \cdot) \to \delta_{y} \), \( [J_{2}P_{\varepsilon}(y, \cdot)]^{m} \to \mu_{s}^{y} \) and

\[
[J_{2}P_{\varepsilon}(y, \cdot)]^{n} \to \mu_{t-s}^{y} \quad \text{weakly})
\]

\[
= E[f(z_{i}^{\varepsilon,y},z_{i}^{\varepsilon,y})].
\]

This proves that the distribution of \( (z_{i}^{\varepsilon,y}, z_{i}^{\varepsilon,y}) \) converges weakly to that of \( (z_{i}^{\varepsilon,y}, z_{i}^{\varepsilon,y}) \). Similarly one can show the weak convergence of the distributions at more than two time points, and hence the convergence in finite dimensions of \( z_{i}^{\varepsilon,y} \) to \( z_{i}^{\varepsilon} \) as \( \varepsilon \to 0 \).
We now show that for \( h = h_n \) as before and \( \phi \in C^\infty(Z) \),

\[
L(h \phi)(y) = \lim_{t \to 0} (1/t)[\mu^y_{i,t}(h \phi) - \phi(o)].
\]

Let \( t = n \varepsilon \) for an integer \( n > 0 \) and a small \( \varepsilon > 0 \). Then

\[
\mu^y_{i,t,h}(\phi) = [J_2[h P_\varepsilon(y, \cdot)]]^n(\phi) = \int_Z [J_2[h P_\varepsilon(y, \cdot)]]^{(n-1)}(dz) P_\varepsilon(z, y, h \phi)
\]

(by (13) with \( \hat{\mu} \), replaced by \( \phi \))

\[
= [J_2[h P_\varepsilon(y, \cdot)]]^{(n-1)}(\phi) + \varepsilon [J_2[h P_\varepsilon(y, \cdot)]]^{(n-1)}[L(h \phi)(\cdot)] + r_1(\varepsilon)
\]

(\( r_1(\varepsilon) \) is \( [J_2[h P_\varepsilon(y, \cdot)]]^{(n-1)}[o(\varepsilon)] \) controlled by \( o(\varepsilon) \))

\[
= \ldots
\]

where the remainders \( r_1(\varepsilon), \ldots, r_n(\varepsilon) \) are controlled by the same \( o(\varepsilon) \) with \( o(\varepsilon)/\varepsilon \to 0 \)

\( \varepsilon \to 0 \).

By (14), \( [J_2[h P_\varepsilon(y, \cdot)]]^{n} \to \delta_o \) weakly as \( n \varepsilon \to 0 \). Then \( [J_2[h P_\varepsilon(y, \cdot)]]^{*k}[L(h \phi)(\cdot)] \to L(h \phi)(y) \) as \( t = n \varepsilon \to 0 \) uniformly for \( k = 1, 2, \ldots, n \) and

\[
(1/t)[\mu^y_{i,t,h}(\phi) - \phi(o)]
\]

\[
= (1/n)[L(h \phi)(y) + [J_2[h P_\varepsilon(y, \cdot)]][L(h \phi)(\cdot)] + \ldots +
\]

\[
+ [J_2[h P_\varepsilon(y, \cdot)]]^{(n-1)}[L(h \phi)(\cdot)]]
\]

\[
+ (r_1 + \ldots + r_n)/t \quad \text{(note that } |r_1 + \ldots + r_n| \leq no(\varepsilon) = to(\varepsilon)/\varepsilon \text{)}
\]

\[
\to L(h \phi)(y) \quad \text{as } t = n \varepsilon \to 0.
\]

This proves (16).

Let \( L^Y_Z \) be the generator of \( z^Y_i \). For \( \phi \in C^\infty(Z) \) with \( 0 \leq \phi \leq 1 \),

\[
L^Y_Z \phi(o) = \lim_{t \to 0} (1/t)[\mu^Y_{i,t}(\phi) - \phi(o)] \geq \lim_{t \to 0} (1/t)[\mu^{y,0,h}_{i,t}(\phi) - \phi(o)] = L(h \phi)(y).
\]

Replacing \( \phi \) by \( 1 - \phi \), we get \( L^Y_Z(1 - \phi)(o) \geq L(h(1 - \phi))(y) \). Because \( L^Y_Z 1 = 0 \) and \( Lh \to 0 \) as \( h = h_n \uparrow 1 \), it follows that (9) holds at \( z = o \). By the \( K \)-invariance of \( z^Y_i \), (9) must hold also at any \( z \in Z = K/M \). \( \square \)

For the Brownian motion \( x_t \) in \( X = \mathbb{R}^n \) under the action of \( K = SO(n) \), we may take \( Y \) to be the positive part of \( x_1 \)-axis and identify it with \( \mathbb{R}_+ \). Then \( M = \text{diag}[1, \text{SO}(n-1)] \) and \( K/M \) is identified with the unit sphere \( S^{n-1} \), and \( J_2 \) is the radial projection to \( S^{n-1} \). Because the generator \( L \) of \( x_t \) is the one-half Laplacian on \( \mathbb{R}^n \), using the spherical coordinates, it is easy to show from (9) that for \( r \in \mathbb{R}_+ \), \( L^Y_Z = (1/2)r^{-2} \Delta_Z \), where \( \Delta_Z \) is the spherical Laplacian on \( S^{n-1} \). It follows that \( z^Y_i \) is a time-scaled Brownian motion in \( S^{n-1} \) corresponding to the spherical Brownian motion in a sphere of radius \( r \).

4. A class of \( K \)-invariant Markov processes

We now consider a \( K \)-invariant Markov process \( x_t \) in \( X \) which is obtained from a \( K \)-invariant diffusion process interlaced with jumps. See Applebaum [2] for a similar construction on Lie
groups. The generator of the induced Lévy process $z^y_t$ in each $K$-orbit, given in Theorem 1, will be determined explicitly.

Let $G$ be a topological group acting continuously on $X$ and containing $K$ as a topological subgroup. A measure $\mu$ on $G$ is called $K$-conjugate invariant if $c_k\mu = \mu$ for $k \in K$, where $c_k: g \mapsto kgk^{-1}$ is the conjugation map. Let $x^0_t$ be a $K$-invariant diffusion process in $X$ with generator $L^0$ and let $N$ be a Poisson random measure on $\mathbb{R}_+ \times G$ of intensity measure $d\eta(dg)$, where $\eta$ is a finite $K$-conjugate invariant measure on $G$. Our process $x_t$ is just $x^0_t$ with jumps added at points $(t, \sigma)$ of $N$, that is, $x_t = \sigma x_{t-}$. See [9] for more details. It is not hard to show that $x_t$ is a Feller process in $X$ with generator $L$, restricted to $f \in C_c^\infty(X)$, given by

$$\forall x \in X \quad Lf(x) = L^0f(x) + \int_G [f(\sigma x) - f(x)]\eta(d\sigma).$$

Moreover, the differential operator $L^0$ may be described more explicitly as follows. At $x = zy \in X^\circ$, we may choose local coordinates $y_1, \ldots, y_q$ in $X^\circ$ around $y$ and $z_1, \ldots, z_p$ in $Z = K/M$ around $z$ so that they together form local coordinates in $X$ around $x$. Let $\partial^y_i = \partial/\partial y_i$ and $\partial_z^j = \partial/\partial z_j$. Then by the $K$-invariance of $L^0$ (see [9]),

$$L^0 = L^{0,Y} + L^{0,YZ} + L^{0,Z},$$

where $L^{0,Y} = (1/2)\sum_{i,j=1}^q a_{ij}(y)\partial^y_i\partial^y_j + \sum_{i=1}^q a_i(y)\partial^Y_i$ is the generator of the radial part of $x^0_t$, $L^{0,YZ} = \sum_{i=1}^q \sum_{j=1}^p b_{ij}(y,z)\partial^Z_i\partial^y_j$ is the sum of the mixed partials in $L^0$, and $L^{0,Z} = (1/2)\sum_{i,j=1}^p c_{ij}(y,z)\partial^Z_i\partial^Z_j + \sum_{i=1}^p c_i(y,z)\partial^Z_i$ is a $K$-invariant second-order differential operator on $Z$ possibly depending on $y$, with smooth coefficients $a_{ij}(y), a_i(y), b_{ij}(y,z), c_{ij}(y,z)$ and $c_i(y,z)$. It can be shown that the operators $L^{0,Y}, L^{0,YZ}$ and $L^{0,Z}$ are uniquely determined and are independent of the choice of local coordinates chosen as above. We may write $L^{0,Z}(y)$ for $L^{0,Z}$ to indicate its dependence on $y \in Y^\circ$.

For $x \in X$, let $i_x: G \ni g \mapsto gx \in X$. Then $\eta(x) = i_x\eta$ is a finite measure on $X$ given by $\eta(x)(f) = \eta(f \circ i_x)$ for $f \in C_b(X)$. We will assume that for $y \in Y^\circ$, $\eta(y)$ does not charge the boundary $\partial X^\circ$ of $X^\circ$, that is,

$$\eta(y)(\partial X^\circ) = 0 \quad \text{for } y \in Y^\circ.$$ (19)

It is then easy to see from (17) that condition (6) holds. Replacing $x^0_t$ by its sub-process in $X^\circ$ if necessary, noting this will not change its generator $L^0$ restricted on $C_c^\infty(X^\circ)$, we may assume that $x^0_t$ never reaches $\partial X^\circ$. Then (5) also holds, and hence the assumptions of Theorem 1 are satisfied.

The $K$-conjugate invariance of $\eta$ implies that $k\eta(x) = \eta(kx)$ for $k \in K$, and hence for $y \in Y^\circ$, $\eta(y)$ is $M$-invariant. Therefore, $J_2\eta(y)$ is an $M$-invariant finite measure on $Z$.

For $x \in X^\circ$, let $(x)_Y = y \in Y^\circ$ and $(x)_Z = z \in Z$ be defined by $x = zy$. For $x = zy \in X^\circ$ with $z = kM$ and $f = h\phi$, where $h \in C_c^\infty(Y^\circ)$ with $h(y) = 1$ and $\phi \in C_c^\infty(Z)$, by the $K$-conjugate invariance of $\eta$, the integral term in (17) may be written as

$$\int_G [\phi(k(\sigma y)_Z)h((\sigma y)_Y) - \phi(z)]\eta(d\sigma) \rightarrow \int_G [\phi(k(\sigma y)_Z) - \phi(z)]\eta(d\sigma) \quad \text{as } h \uparrow 1$$

$$= \int_Z [\phi(zz') - \phi(z)][J_2\eta(y)](dz').$$
By (18), the generator of the induced Lévy process \( z_t^y \) in \( Z = K/M \) given in Theorem 1 can now be written as

\[
L_Z^y \phi(z) = L^{0,Z}(y)\phi(z) + \int_Z [\phi(zz') - \phi(z)][J_2 \eta(y)](dz') \quad \text{for } \phi \in C^\infty(Z).
\] (20)

It is shown in [9] that given a radial path \( y_t = J_1(x_t) \), the conditioned angular process \( z_t = J_2(x_t) \) is a nonhomogeneous Lévy process in \( Z \). We briefly recall the definition of nonhomogeneous Lévy processes in a homogeneous space \( Z = K/M \); see [9] for more details.

First a nonhomogeneous Lévy process \( x_t \) in a Lie group \( K \) is a process with rcll paths and infinite lifetime that possesses independent but not necessarily stationary increments, that is, for \( s < t, k^{-1}x_t \) is independent of the process up to time \( s \), but its distribution \( \mu_{s,t} \) may depend on more than the length of the interval \( (s, t] \). The distributions of increments, \( \mu_{s,t} \) for \( s < t \), form a two-parameter convolution semigroup in the sense that \( \mu_{s,t} * \mu_{t,r} = \mu_{s,r} \) for \( s < t < r \), which is continuous in the sense that \( \mu_{s,t} \to \delta_x \) weakly as \( t \downarrow s \). In fact, a nonhomogeneous Lévy process \( k_t \) in \( K \) may be defined as a rcll process such that for any \( 0 = t_0 < t_1 < t_2 < \cdots < t_n \) and \( f \in C_b(K^{n+1}) \),

\[
E[f(k_{t_0}, k_{t_1}, k_{t_2} \ldots, k_{t_n})] = \int f(x_0, x_0x_1, x_0x_1x_2, \ldots, x_0x_1 \ldots x_n) \mu_0(dx_0)\mu_{t_0,t_1}(dx_1) \mu_{t_1,t_2}(dx_2) \cdots \mu_{t_{n-1},t_n}(dx_n)
\] (21)

for a probability measure \( \mu_0 \) (initial distribution) and a continuous two-parameter convolution semigroup \( \mu_{s,t} \) on \( K \).

On a homogeneous space \( Z = K/M \), the convolution of \( M \)-invariant probability measures is defined and a nonhomogeneous Lévy process \( z_t \) in \( Z \) may be defined as a rcll process satisfying (21) with a continuous two-parameter convolution semigroup \( \mu_{s,t} \) of \( M \)-invariant probability measures on \( Z \). We may formulate a product structure on \( Z = K/M \) so that these probability measures may be regarded as the distributions of the increments. Feinsilver [4] obtained a martingale representation for nonhomogeneous Lévy processes in Lie groups. This is extended to homogeneous spaces in [9].

When given a constant radial path \( y_t = y \), the conditioned angular process \( z_t \) is a (homogeneous) Lévy process in \( Z = K/M \). It is interesting to compare its generator with the generator of the induced Lévy process \( z_t^y \) given by (20).

As in [9], assume \( L^{0,Y} \) is nondegenerate (that is, the symmetric matrix \( a_{ij}(y) \) is positive definite). Then at least locally, \( L^{0,Y}_{1,0} = \sum_{i=1}^q \xi_i \xi_i + \xi_0 \) for some (smooth) vector fields \( \xi_0, \xi_1, \ldots, \xi_q \) on \( Y \) and \( L^{0,Y}_{1,1} = \sum_{i=1}^q \eta_i \xi_i \) for some \( K \)-invariant vector fields \( \eta_1, \ldots, \eta_q \) on \( Z \) which may depend on \( y \in Y \). Let \( \lambda^{0,Z} = (1/2) \sum_{i=1}^q \eta_i \xi_i \), which is independent of the choice of vector fields, and \( L^{0,Z}_{1,2} = L^{0,Z} - L^{0,Z}_{1,0} \). We may write \( L^{0,Z}_{2,2} \) to indicate its dependence on \( y \in Y \). For a measure \( \mu \), let \( \mu_H \) denote its restriction to a set \( H \). By [9, Theorem 5], the generator of the conditioned angular process \( z_t \) given a constant radial path \( y_t = y \) is

\[
L_Z^y \phi(z) = L^{0,Z}_2(y)\phi(z) + \int_Z [\phi(zz') - \phi(z)][J_2 \eta(y)K_y](dz') \quad \text{for } \phi \in C^\infty(Z).
\] (22)

Comparing (20) and (22), we see that the induced Lévy process \( z_t^y \) obtained by forcing and the conditioned angular process \( z_t \) given \( y_t = y \) have the same generator (and hence the same distribution) if and only if \( \eta(y) \) is concentrated on \( K_y \) and \( L_{0,Y}^{0,Z} = 0 \).
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References