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Nonlinear Second-Order Boundary Value Problems: Intervals of Existence, Uniqueness and Continuous Dependence*

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1. INTRODUCTION

Consider the boundary value problem

$$y'' + f(x, y, y') = 0, \qquad a \leqslant x \leqslant b, \tag{1.1}$$

$$g(y(a), y'(a)) = A, \tag{1.2}$$

$$h(y(b), y'(b)) = B. \tag{1.3}$$

We shall assume throughout that $f(x, y, z)$ satisfies the following conditions:

(a) $f(x, y, z)$ is continuous on $R = \{(x, y, z): a \leqslant x \leqslant b, |y| + |z| < \infty\}$,

(b) $f(x, y, z)$ satisfies a Lipschitz condition in z on each compact subset of R: for each $\rho > 0$, there exist $M_i(\rho)$, $i = 1, 2$, so that

$$M_1(\rho)(z_2 - z_1) \leqslant f(x, y, z_2) - f(x, y, z_1) \leqslant M_2(\rho)(z_2 - z_1)$$

whenever $z_1 \leqslant z_2$ and the points (x, y, z_i) lie in the compact set

$$R_\rho = \{(x, y, z): a \leqslant x \leqslant b, |y| + |z| \leqslant \rho\},$$

(c) $f(x, y, z)$ satisfies a one-sided Lipschitz condition in y on R: there exists K so that

$$f(x, y_2, z) - f(x, y_1, z) \leqslant K(y_2 - y_1) \qquad \text{for } y_1 \leqslant y_2.$$

We are first of all interested in the question of existence and uniqueness of a solution $\phi(x, A, B)$ of (1.1), (1.2), (1.3) and the continuous dependence of

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$\phi(x, A, B)$ and its derivative $\phi'(x, A, B)$ on the triple (x, A, B). We have dealt already in [5, 6] with this problem in the case $K \leqslant 0$. In this paper we deal with the case $K > 0$.

The results we obtain contain the earlier results of Bailey, Shampine, and Waltman [1–4, 12] as special cases. These earlier results are closely related to the theorem obtained in the case $K \leqslant 0$ by Keller [10] and Bebernes and Gaines [8, 9]. A detailed discussion of that relationship has been given in [7], where in fact it is shown that the results of Bailey, Shampine, and Waltman in the case $K > 0$ are a consequence of the Keller–Bebernes–Gaines theorem for $K \leqslant 0$. The strategy used in [7] is actually a method of transplanting results valid for $K \leqslant 0$ to the situation $K > 0$ and is motivated in part by Protter and Weinberger [11, Chap. 1] and in part by the Bailey–Shampine–Waltman techniques. The underlying ideas are discussed in Section 2.

In Section 3, we shall obtain our results for $K > 0$ concerning existence, uniqueness, and continuous dependence of the solution of (1.1), (1.2), (1.3) by transplanting the corresponding results for $K \leqslant 0$ in [5, 6]. One of the main advantages of assuming $K \leqslant 0$ is that the maximum principle can be invoked, as in [11, Chap. 1], to obtain comparison theorems. In Section 4, we transplant such comparison theorems to the case $K > 0$. These comparison theorems are related to, and in some ways extend, the ones in [4, Chap. 5].

In Section 5, we use the results of Sections 3 and 4 to obtain existence and uniqueness theorems for the differential equation (1.1) and the generalized periodic boundary conditions

$$y(a) = y(b) + G(y'(a), y'(b)), \tag{1.4}$$

$$H(y'(a), y'(b)) = 0. \tag{1.5}$$

Our results on continuous dependence and the generalized periodic boundary value problem extend corresponding results in [12].

In comparing the various papers, the reader should note that in several instances, including [5, 6], Eq. (1.1) is written in the form $y'' = f(x, y, y')$, causing various sign changes in the statements of hypotheses.

During the last decade, some rather general existence results for nonlinear boundary value problems have appeared. Schmidt [22] extended the existence portion of the Bebernes–Gaines theorem [9] to situations of nonuniqueness. Gaines [16] also obtained an existence theorem that gave a slight improvement of the result in [9] and then [17] used the Leray–Schauder theorem to prove existence of solutions for certain nonlinear problems with nonlinear boundary conditions. Gaines and Mawhin [19] have used the techniques of coincidence degree theory to prove a general existence theorem which contains the earlier result of Erbe [14]. The trend in these papers is generally away from the explicit conditions on f which characterized the earlier papers and the results offered here; implicit hypotheses on f involving the Nagumo condition and the existence of upper and lower solutions are often invoked. A good summary of these developments is given in [20]. Gaines' results [15] on continuous dependence in the case of fixed endpoint conditions are closely related to our results here and in [6].

General existence theorems for the periodic boundary value problem with hypotheses not so explicit as ours and based on Leray–Schauder degree theory have been given by Bebernes [13] and Gaines [18]. Muldowney and Willett [21] have dealt with existence for two point boundary value problems and for the periodic boundary conditions; their approach is relatively elementary although again their hypotheses are not so explicit as ours.

2. Preliminary Results

Our basic strategy is a simple change of dependent variable. Suppose $w \in C^2[a,b]$, $w(x) > 0$ on $[a,b]$. It is easy to verify that $\phi(x) = w(x)\,\psi(x)$ is a solution of (1.1) if and only if $\psi(x)$ is a solution of

$$y'' + F_w(x, y, y') = 0, \tag{2.1}$$

where

$$F_w(x,y,z) = [f(x, w(x)\,y, w'(x)\,y + w(x)\,z) + 2w'(x)\,z + w''(x)\,y]/w(x). \tag{2.2}$$

It is clear that if $f(x,y,z)$ satisfies (a) and (b), so will $F_w(x,y,z)$. We would like for $F_w(x,y,z)$ to satisfy (c) with $K=0$. Since

$$\begin{aligned} F_w(x, y_2, z) - F_w(x, y_1, z) &= [f(x, wy_2, w'y_2 + wz) - f(x, wy_2, w'y_1 + wz) + f(x, wy_2, w'y_1 + wz) \\ &\quad - f(x, wy_1, w'y_1 + wz) + w''(y_2 - y_1)]/w, \end{aligned}$$

and $f(x,y,z)$ satisfies (b) and (c), we see that

$$F_w(x, y_2, z) - F_w(x, y_1, z) \leqslant (y_2 - y_1)[w'' + M_2(\rho)\,w' + Kw]/w \tag{2.3}$$

whenever $w'(x) \geqslant 0$ and all relevant points lie in R_ρ. Further,

$$F_w(x, y_2, z) - F_w(x, y_1, z) \leqslant (y_2 - y_1)[w'' + M_1(\rho)\,w' + Kw]/w \tag{2.4}$$

whenever $w'(x) \leqslant 0$ and all relevant points lie in R_ρ.

The following lemma is an immediate consequence of (2.3) and (2.4).

LEMMA 2.1. *Suppose* $w \in C^2[a,b]$, $w(x) > 0$ *on* $[a,b]$, *and suppose* $f(x,y,z)$ *satisfies* (a), (b), *and* (c) *with* $K > 0$. *Then* $F_w(x,y,z)$ *satisfies* (a) *and* (b). *Further, if either*

(i) $M_2(\rho) \leqslant M_2 < \infty$ *for all* $\rho > 0$ *and* w *is a solution of*

$$w'' + M_2 w' + Kw = 0, \qquad a \leqslant x \leqslant b,$$

with $w'(x) \geqslant 0$ *on* $[a,b]$, *or*

(ii) $M_1(\rho) \geqslant M_1 > -\infty$ *for all* $\rho > 0$ *and* w *is a solution of*

$$w'' + M_1 w' + Kw = 0, \qquad a \leqslant x \leqslant b,$$

with $w'(x) \leqslant 0$ *on* $[a,b]$, *or*

(iii) $M_2(\rho) \leqslant M_2 < \infty$, $M_1(\rho) \geqslant M_1 > -\infty$ *for all* $\rho > 0$ *and* w *satisfies*

$$w'' + M_2 w' + Kw = 0, \qquad w' \geqslant 0, \qquad \text{for } a \leqslant x \leqslant c,$$

$$w'' + M_1 w' + Kw = 0, \qquad w' \leqslant 0, \qquad \text{for } c \leqslant x \leqslant b,$$

with $a < c < b$.
Then $F_w(x, y, z)$ *satisfies* (c) *with* $K = 0$.

Our strategy consists of taking problems concerning the differential equation (1.1), which satisfies (a), (b), and (c) with $K > 0$ and transforming them into equivalent problems concerning the differential equation (2.1) where $F_w(x,y,z)$ given by (2.2) satisfies (a), (b), and (*c*) with $K = 0$. According to Lemma 2.1, we will need to use for our change of variable a function w which is constructed from solutions of one or both of the linear equations

$$w'' + M_2 w' + Kw = 0,$$

$$w'' + M_1 w' + Kw = 0.$$

Since we shall be interested in positive solutions with slopes of one sign, we shall analyze carefully the solution $w_+ = w_+(x, M, K, r)$ of the initial value problem

$$w'' + Mw' + Kw = 0, \qquad K > 0, \tag{2.5}$$

$$w(0) = r, \qquad w'(0) = 1 \tag{2.6}$$

and the solution $w_- = w_-(x, M, K, r)$ of the initial value problem (2.5) and

$$w(0) = r, \qquad w'(0) = -1 \tag{2.7}$$

where in both cases $r \geqslant 0$ and M is an arbitrary constant. Of course, $w_-(x, M, K, r) = w_+(-x, -M, K, r)$, so it suffices to study only w_+.

Let $\alpha(M, K, r)$ be the distance from 0 to the nearest positive zero of w'_+; if no such zero exists, put $\alpha(M, K, r) = +\infty$. Let $\beta(M, K, r)$ be the distance from 0 to the nearest negative zero of w'_-; if no such zero exists, put $\beta(M, K, r) = +\infty$. Of course, $\beta(M, K, r) = \alpha(-M, K, r)$. Further $\alpha(M, K, 0)$, $\beta(M, K, 0)$ are the same as the $\alpha(M, K)$, $\beta(M, K)$ of Bailey, Shampine, and Waltman. The value of $\alpha(M, K, r)$ can be computed explicitly by elementary methods to yield for $M > -2\sqrt{K}$

$$\begin{aligned} \alpha(M, K, r) &= 2(M^2 - 4K)^{-1/2} \cosh^{-1} \mu(M, K, r) && \text{if } M > 2\sqrt{K} \\ &= (rK + \sqrt{K})^{-1} && \text{if } M = 2\sqrt{K} \\ &= 2(4K - M^2)^{-1/2} \cos^{-1} \mu(M, K, r) && \text{if } -2\sqrt{K} < M < 2\sqrt{K}, \end{aligned}$$

where

$$\mu(M, K, r) = \frac{M + 2rK}{2\sqrt{K}(1 + rM + r^2K)^{1/2}}.$$

If $M \leqslant -2\sqrt{K}$, the value of $\alpha(M, K, r)$ depends on r. If $0 \leqslant r \leqslant K^{-1/2}$, then $\alpha(M, K, r) = +\infty$ for $M \leqslant -2\sqrt{K}$. However, if $r > K^{-1/2}$, we have the more complicated behavior

$$\begin{aligned} \alpha(M, K, r) &= +\infty && \text{if } \quad M \leqslant -r^{-1} - rK \\ &= 2(M^2 - 4K)^{-1/2} \cosh^{-1} \mu(M, K, r) \\ & && \text{if } \quad -r^{-1} - rK < M < -2\sqrt{K} \\ &= (rK - \sqrt{K})^{-1} && \text{if } \quad M = -2\sqrt{K}. \end{aligned}$$

For fixed values of $K > 0$, $r \geqslant 0$, one may verify that $\alpha(M, K, r)$ is continuous in M and decreases monotonically from the value $+\infty$ to 0 as $M \to +\infty$. We find it convenient to set

$$\alpha(+\infty, K, r) = 0 \qquad \text{for } K > 0,\ r \geqslant 0.$$

For $0 \leqslant r \leqslant K^{-1/2}$, the graph of $\alpha(M, K, r)$, as a function of M, has a vertical asymptote at $M = -2\sqrt{K}$; as r increases from 0 to $K^{-1/2}$, the ordinates of points on the graph to the right of this vertical asymptote decrease toward 0. As r increases from $K^{-1/2}$ to $+\infty$, the vertical asymptote moves left from $M = -2\sqrt{K}$ to $-\infty$, while the points on the graph to the right of the vertical asymptote continue to move down toward 0. Thus, increasing M or r decreases $\alpha(M, K, r)$. Since

$$\lim_{r \to +\infty} \alpha(M, K, r) = 0 \qquad \text{for } K > 0,\ M > -\infty,$$

we find it convenient to set

$$\alpha(M, K, +\infty) = 0 \qquad \text{for } 0 < K < +\infty, \ -\infty < M \leqslant +\infty.$$

It is clear that for fixed $M, K, M > -2\sqrt{K}$, then $\alpha(M, K, r)$ is continuous in r. The following lemma is straightforward to verify.

LEMMA 2.2. *For* $0 < K < +\infty$, $-\infty < M \leqslant +\infty$,

$$\alpha(M, K, p) \leqslant \alpha(M, K, r) \qquad \text{if} \quad 0 \leqslant r \leqslant p,$$

$$\lim_{p \to r^+} \alpha(M, K, p) = \alpha(M, K, r) \qquad \text{for } r \geqslant 0.$$

We also define

$$\beta(M, K, r) = 0 \qquad \text{for } r = +\infty \text{ or } M = -\infty,$$

and thus peserve the identity $\beta(M, K, r) = \alpha(-M, K, r)$.

We shall also need

LEMMA 2.3. *Suppose* $K > 0$, $r \geqslant 0$ *and* $\alpha(M, K, r) < \infty$. *Then*

$$W(x) \equiv \frac{w_+(x, M, K, r)}{w'_+(x, M, K, r)}$$

is strictly increasing for $0 < x < \alpha(M, K, r)$.

Proof. We may assume that M, r are both finite since otherwise the result is vacuously true. Clearly $W(x) > 0$ for $0 < x < \alpha(M, K, r)$. Since $w''_+ + Mw'_+ + Kw_+ = 0$, then $W' = 1 + MW + KW^2$. Thus $W'(x) > 0$ for $0 < x < \alpha(M, K, r)$ if and only if $M > -KW(x) - 1/W(x)$. The function $-Kt - 1/t$, for $t > 0$, attains the maximum value $-2\sqrt{K}$ at $t = K^{-1/2}$. For $0 \leqslant r \leqslant K^{-1/2}$, $\alpha(M, K, r) < \infty$ is equivalent to $M > -2\sqrt{K}$ so the desired conclusion is immediate. For $r > K^{-1/2}$, $\alpha(M, K, r) < \infty$ is equivalent to $M > -r^{-1} - Kr = -KW(0) - 1/W(0)$; hence $W'(0) > 0$. Suppose the desired conclusion is false and let $\bar{x} > 0$ be the first point at which $W'(x) = 0$. Then $W(\bar{x}) > W(0) = r$ and $M = -KW(\bar{x}) - 1/W(\bar{x}) < -Kr - 1/r$, a contradiction.

Henceforth, we put for $i = 1, 2$,

$$M_i = \sup\{M_i(\rho): \rho > 0\} \leqslant +\infty,$$

where $M_i(\rho)$ are the functions in condition (b). The next lemma gives the basic facts regarding the existence of appropriate functions $w(x)$.

LEMMA 2.4. *Suppose* $f(x, y, z)$ *satisfies* (a), (b), *and* (c) *with* $K > 0$.

(i) *If* $r \geqslant 0$ *and* $b - a < \alpha(M_2, K, r)$, *then there exists* $w \in C^2[a, b]$ *with* $w(x) > 0$, $w'(x) > 0$ *on* $[a, b]$ *and* $w(a) - rw'(a) > 0$ *such that* $F_w(x, y, z)$ *satisfies* (a), (b), *and* (c) *with* $K = 0$.

(ii) *If* $r \geqslant 0$ *and* $b - a < \beta(M_1, K, r)$, *then there exists* $w \in C^2[a, b]$ *with* $w(x) > 0$, $w'(x) < 0$ *on* $[a, b]$ *and* $w(b) + rw'(b) > 0$ *such that* $F_w(x, y, z)$ *satisfies* (a), (b), *and* (c) *with* $K = 0$.

(iii) *If* $r_1 \geqslant 0$, $r_2 \geqslant 0$, $0 < \alpha(M_2, K, r_2) < \infty$, $0 < \beta(M_1, K, r_1) < \infty$, *and* $b - a < \alpha(M_2, K, r_2) + \beta(M_1, K, r_1)$, *then there exist* $w \in C^2[a, b]$ *and* c, $a < c < b$, *with* $w(x) > 0$ *on* $[a, b]$, $w'(x) > 0$ *on* $[a, c)$, $w'(x) < 0$ *on* $(c, b]$, $w(a) - r_2 w'(a) > 0$, $w(b) + r_1 w'(b) > 0$ *such that* $F_w(x, y, z)$ *satisfies* (a), (b), *and* (c) *with* $K = 0$.

Proof. To prove (i), we can certainly assume M_2, r are both finite. If $b - a < \alpha(M_2, K, r)$, by Lemma 2.2, there exists $p > r$ such that $b - a < \alpha(M_2, K, p) \leqslant \alpha(M_2, K, r)$. Let $w(x) = w_+(x, M_2, K, p)$. Then $w(a) - rw'(a) = p - r > 0$ and $w \in C^2[a, b]$, $w(x) > 0$, $w'(x) > 0$ on $[a, b]$. Since w is a solution of $w'' + M_2 w' + Kw = 0$, then (i) follows from Lemma 2.1(i). A similar proof of (ii) may be given, but it is quicker to get (ii) from (i) by a change of variable (just replace x by $a + b - x$).

To prove (iii), choose c, $a < c < b$, so that $c - a < \alpha(M_2, K, r_2)$, $b - c < \beta(M_1, K, r_1)$. Put

$$\begin{aligned} w(x) &= w_+(x + \alpha(M_2, K, r_2) - c) && a \leqslant x \leqslant c \\ &= \frac{w_+(\alpha(M_2, K, r_2))}{w_-(-\beta(M_1, K, r_1))} w_-(x - \beta(M_1, K, r_1) - c) && c < x \leqslant b, \end{aligned}$$

where $w_+(x) = w_+(x, M_2, K, r_2)$, $w_-(x) = w_-(x, M_1, K, r_1)$. Clearly $w(x)$, $w'(x)$ are continuous on $[a, b]$; $w''(x)$ is also continuous on $[a, b]$ since

$$\begin{aligned} w''(x) &= -M_2 w'(x) - Kw(x) && a \leqslant x \leqslant c, \\ &= -M_1 w'(x) - Kw(x) && c < x \leqslant b, \end{aligned}$$

and $w'(c) = 0$. Thus $w \in C^2[a, b]$. Further, $w'(x) > 0$ on $[a, c)$, $w'(x) < 0$ on $(c, b]$. By Lemma 2.3,

$$\frac{w(a)}{w'(a)} = \frac{w_+(a + \alpha(M_2, K, r_2) - c)}{w'_+(a + \alpha(M_2, K, r_2) - c)} > \frac{w_+(0)}{w'_+(0)} = r_2;$$

thus $w(a) - r_2 w'(a) > 0$. Similarly $w(b) + r_1 w'(b) > 0$. Thus (iii) follows from Lemma 2.1(iii).

3. Existence, Uniqueness, and Continuous Dependence

We shall say that existence, uniqueness, and continuous dependence hold for the problem (1.1), (1.2), (1.3) if that problem has a unique solution $\phi(x, A, B)$ and both $\phi(x, A, B)$, $\phi'(x, A, B)$ are continuous functions of the triple (x, A, B).

We use the standard subscript notation for partial derivatives, e.g., $h_2(s, t) = \partial h/\partial t$. Our basic assumptions about the boundary conditions (1.2), (1.3) are that g, h are continuously differentiable everywhere and satisfy

(d) $g_1, h_1, h_2 \geqslant 0$, $g_2 \leqslant 0$, $g_1 - g_2 \geqslant \delta > 0$, $h_1 + h_2 \geqslant \delta$.

We begin with the theorem, already known from [5, 6], which holds for $K \leqslant 0$.

THEOREM 3.1. *Suppose $f(x, y, z)$ satisfies* (a), (b), *and* (c) *with* $K \leqslant 0$. *Suppose that $M_i(\rho) = o(\log \rho)$ as $\rho \to \infty$ for $i = 1, 2$. In addition to* (d), *suppose that either $g_1 \geqslant \delta$ or $h_1 \geqslant \delta$. Then existence, uniqueness, and continuous dependence hold for the boundary value problem* (1.1), (1.2), (1.3).

The existence and uniqueness statement was proved in [5]; the continuous dependence in [6]. We may transplant this theorem to the situation $K > 0$ via the variable change discussed in Section 2. Assuming (d) holds, we put

$$r_g \equiv \sup\{-g_2/g_1\}, \qquad r_h \equiv \sup\{h_2/h_1\}.$$

Since $g_1 - g_2 \geqslant \delta > 0$, then $-g_2 \geqslant \delta$ whenever $g_1 = 0$. Thus we interpret $r_g = +\infty$ if g_1 is ever 0. Of course, $r_g = +\infty$ is possible even if g_1 never vanishes. Similar comments apply to r_h.

THEOREM 3.2. *Suppose $f(x, y, z)$ satisfies* (a), (b), *and* (c) *with* $K > 0$. *Suppose that $M_i(\rho) = o(\log \rho)$ as $\rho \to \infty$ for $i = 1, 2$. If the boundary conditions* (1.2), (1.3) *satisfy* (d) *and* $b - a < \alpha(M_2, K, r_g) + \beta(M_1, K, r_h)$, *then existence, uniqueness, and continuous dependence hold for the boundary value problem* (1.1), (1.2), (1.3).

Proof. If $\alpha(M_2, K, r_g)$, $\beta(M_1, K, r_h)$ are both 0, the theorem is vacuously true. We deal with the proof in three cases.

Case (i). Either $\alpha(M_2, k, r_g) = +\infty$ or $\beta(M_1, K, r_h) = 0$. Then $b - a < \alpha(M_2, K, r_g)$ and M_2, r_g are both finite. Let $w(x)$ be the function whose existence is asserted in Lemma 2.4(i), with $r = r_g$. Then the boundary value problem (1.1), (1.2), (1.3) is transformed after replacing y by $w(x)\,y$ into the problem

$$y'' + F_w(x, y, y') = 0, \qquad a \leqslant x \leqslant b, \tag{2.1}$$

$$\tilde{g}((a), y'(a)) \equiv g(w(a)\, y(a), w'(a)\, y(a) + w(a)\, y'(a)) = A, \tag{3.1}$$

$$\tilde{h}(y(b), y'(b)) \equiv h(w(b)\, y(b), w'(b)\, y(b) + w(b)\, y'(b)) = B, \tag{3.2}$$

where $F_w(x, y, z)$ is given by (2.2). By Lemma 2.4(i), $F_w(x, y, z)$ satisfies (a), (b), and (c) with $K = 0$. It is easy to verify that the assumptions on $M_i(\rho)$ regarding $f(x, y, z)$ are also satisfied by $F_w(x, y, z)$. In order to apply Theorem 3.1, it remains to check that the boundary conditions (3.1), (3.2) are suitable. Now by Lemma 2.4(i),

$$\begin{aligned}
\tilde{g}_1 &= w(a)\, g_1 + w'(a)\, g_2 \\
&= g_1 \left[w(a) + w'(a) \frac{g_2}{g_1} \right] \geqslant g_1 [w(a) - r_g w'(a)], \\
\tilde{g}_2 &= w(a)\, g_2 \leqslant 0, \\
\tilde{h}_1 &= w(b)\, h_1 + w'(b)\, h_2 \geqslant \delta \min\{w(b), w'(b)\} > 0, \\
\tilde{h}_2 &= w(b)\, h_2 \geqslant 0.
\end{aligned}$$

Since $-g_2/g_1 \leqslant r_g$, then $g_1 \geqslant \delta/(1 + r_g)$ and thus $\tilde{g}_1 \geqslant \delta[w(a) - r_g w'(a)]/(1 + r_g)$. Thus Theorem 3.1 applies and the problem (2.1), (3.1), (3.2) has a unique solution $\psi(x, A, B)$ and both $\psi(x, A, B)$, $\psi'(x, A, B)$ are continuous functions of (x, A, B). Therefore, (1.1), (1.2), (1.3) has the unique solution $\phi(x, A, B) = w(x)\, \psi(x, A, B)$ and clearly $\phi(x, A, B)$, $\phi'(x, A, B)$ are continuous functions of (x, A, B).

Case (ii). Either $\alpha(M_2, K, r_g) = 0$ or $\beta(M_1, K, r_h) = +\infty$. Then $b - a < \beta(M_1, K, r_h)$ and M_1, r_h are both finite. A similar proof to case (i) can now be given using Lemma 2.4(ii) or the theorem in this case may be obtained directly from case (i) by a variable change (replace x by $a + b - x$).

Case (iii). Both $0 < \alpha(M_2, K, r_g) < +\infty$, $0 < \beta(M_1, K, r_h) < +\infty$. Then M_1, M_2, r_h, r_g are all finite. Let $w(x)$ be the function whose existence is asserted in Lemma 2.4(iii), with $r_2 = r_g$, $r_1 = r_h$. Then (1.1), (1.2), (1.3) is again transformed into (2.1), (3.1), (3.2), and the function $F_w(x, y, z)$ and the boundary conditions (3.1), (3.2) satisfy the requirements of Theorem 3.1, as before.

Theorem 3.2 generalizes and unifies the results of Bailey, Shampine, and Waltman which dealt separately with the boundary conditions

$$y(a) = A, \qquad y(b) = B, \tag{I}$$

$$y(a) = A, \qquad y'(b) = B, \tag{II}$$

$$y'(a) = A, \qquad y(b) = B. \tag{III}$$

In the case of (I), $r_g = r_h = 0$ so the interval restriction becomes $b - a < \alpha(M_2, K, 0) + \beta(M_1, K, 0)$. In the case of (II), $r_g = 0$, $r_h = +\infty$, so the interval restriction is $b - a < \alpha(M_2, K, 0) + \beta(M_1, K, +\infty) = \alpha(M_2, K, 0)$. Finally in the case of (III), $r_g = +\infty$, $r_h = 0$, and the interval restriction is $b - a < \alpha(M_2, K, +\infty) + \beta(M_1, K, 0) = \beta(M_1, K, 0)$.

4. Comparison Theorems

We begin with a statement of comparison theorems which hold if $K \leqslant 0$. These theorems are consequences of the maximum principle and are essentially the theorems of [11, Chap. 1]; however we assume less smoothness and, consequently, initial value problems may have more than one solution. For example, $f(x, y, z) = -6y^{1/3}$ satisfies (a), (b), and (c) with $K = 0$, but the initial value problem $y'' + f(x, y, y') = 0$, $y(0) = 0$, $y'(0) = 0$, has, in addition to the trivial solution, $y = x^3$ and infinitely many other solutions. Since the usual comparison theorems for initial value problems imply uniqueness, our statement must be modified.

Theorem 4.1. *Suppose $f(x, y, z)$ satisfies* (a), (b), *and* (c) *with $K \leqslant 0$. Suppose that $u(x)$, $v(x)$ satisfy*

$$u'' + f(x, u, u') \geqslant v'' + f(x, v, v'), \qquad a \leqslant x \leqslant b.$$

(i) *If $v(a) \leqslant u(a)$, $v'(a) < u'(a)$, then $v'(x) < u'(x)$, $v(x) < u(x)$, for $a < x \leqslant b$.*

(ii) *If $v(a) < u(a)$, $v'(a) \leqslant u'(a)$, then $v'(x) \leqslant u'(x)$, $v(x) < u(x)$, for $a \leqslant x \leqslant b$.*

Proof. Let $y = u - v$. Then on any interval where $u(x) > v(x)$, y satisfies

$$y'' + p(x)y' + q(x)y \geqslant 0, \tag{4.1}$$

where

$$p(x) = \left[\frac{f(x, u(x), u'(x)) - f(x, u(x), v'(x))}{u'(x) - v'(x)}\right] \quad \text{if} \quad u'(x) \neq v'(x),$$

$$= 0 \quad \text{if} \quad u'(x) = v'(x), \tag{4.2}$$

$$q(x) = \frac{f(x, u(x), v'(x)) - f(x, v(x), v'(x))}{u(x) - v(x)} \quad \text{if} \quad u(x) > v(x). \tag{4.3}$$

In case (i), we have $y(a) \geqslant 0$, $y'(a) > 0$. If $y'(x) > 0$ for $a \leqslant x \leqslant b$ is false, there exists c, $a < c \leqslant b$, at which $y'(c) = 0$, $y'(x) > 0$ for $a \leqslant x < c$. Thus for

$0 < \varepsilon < c - a$, $y(x) > 0$ on $[a + \varepsilon, c]$. Our hypotheses (a), (b), and (c) with $K \leqslant 0$ imply that $p(x)$ is bounded and $q(x) \leqslant 0$ is continuous on $[a + \varepsilon, c]$. The maximum principle for (4.1) [11, pp. 6–7] then implies, since the maximum of $y(x)$ on $[a + \varepsilon, c]$ must occur at c, that $y'(c) > 0$, a contradiction. Thus $y'(x) > 0$ on $(a, b]$ and consequently, $y(x) > y(a) \geqslant 0$ on $(a, b]$ also.

In case (ii), if $y(x)$ is constant on $[a, b]$, the result is obvious. Otherwise, let us assume $y'(x) \geqslant 0$ on $[a, b]$ is false. Then there exists c, $a < c \leqslant b$, with $y'(c) < 0$ and $y(x) > 0$ on $[a, c]$. Since the maximum of $y(x)$ on $[a, c]$ clearly does not occur at c, the maximum principle for (4.1) implies that the maximum occurs at a and $y'(a) < 0$, a contradiction since $y'(a) = u'(a) - v'(a) \geqslant 0$. Thus $y'(x) \geqslant 0$ on $[a, b]$ and hence $y(x) \geqslant y(a) > 0$ on $[a, b]$.

Our hypotheses (a), (b), and (c) with $K \leqslant 0$ do, however, give uniqueness in boundary value problems, so the usual comparison theorem holds.

THEOREM 4.2. *Suppose $f(x, y, z)$ satisfies* (a), (b), *and* (c) *with $K \leqslant 0$. Suppose that $u(x)$, $v(x)$ satisfy*

$$u'' + f(x, u, u') \geqslant v'' + f(x, v, v'), \qquad a \leqslant x \leqslant b,$$

$$g(u(a), u'(a)) \leqslant g(v(a), v'(a)), \tag{4.4}$$

$$h(u(b), u'(b)) \leqslant h(v(b), v'(b)), \tag{4.5}$$

where g_1, $h_1, h_2 \geqslant 0$, $g_2 \leqslant 0$, $g_1 - g_2 > 0$, $h_1 + h_2 > 0$, and at least one of these three: (i) $g_1 > 0$, (ii) $h_1 > 0$, (iii) *there exists $\bar{x} \in [a, b]$ for which $f(\bar{x}, y, z)$ is strictly increasing in y for each fixed z. Then $u(x) \leqslant v(x)$, for $a \leqslant x \leqslant b$.*

Proof. Supposing the contrary, then $y(x) = u(x) - v(x)$ has a positive maximum in $[a, b]$ and satisfies (4.1) at all points where $u(x) > v(x)$. If $y(x)$ is a positive constant, then $q(x) \equiv 0$ and (4.3) gives $f(x, u(x), v'(x)) \equiv f(x, v(x), v'(x))$ on $[a, b]$ and (iii) is violated. From (4.4) and the mean value theorem, $g_1(\bar{s}, v'(a))\, y(a) \leqslant 0$, where $v(a) < \bar{s} < u(a)$; hence $g_1(\bar{s}, v'(a)) \leqslant 0$, violating (i). Similarly, (4.5) shows that (ii) is violated. Thus $y(x)$ is not constant on $[a, b]$. Suppose the positive maximum of $y(x)$ occurs at $c \in (a, b)$. Then there exists an interval $[c_1, c_2] \subset [a, b]$ with $c_1 < c < c_2$, $y(x) > 0$ on $[c_1, c_2]$, but $y(x)$ not constant on $[c_1, c_2]$. But this gives an interior maximum on $[c_1, c_2]$, contradicting the maximum principle for (4.1) on $[c_1, c_2]$. Thus the maximum must occur at a or b. If the maximum occurs at a, we may choose $\varepsilon > 0$ so that $y(x) > 0$ on $[a, a + \varepsilon]$, but not constant on $[a, a + \varepsilon]$. It follows from the maximum principle that $y'(a) < 0$. Then (4.4) and the mean theorem give

$$g_1(\bar{s}, \bar{t})\, y(a) + g_2(\bar{s}, \bar{t})\, y'(a) \leqslant 0,$$

where $(\bar{s}, \bar{t})$ lies on the line segment joining $(u(a), u'(a))$ and $(v(a), v'(a))$; but this inequality is inconsistent with $y(a) > 0$, $y'(a) < 0$ and $g_1 \geqslant 0$, $g_2 \leqslant 0$, $g_1 - g_2 > 0$. If the maximum occurs at b, a similar contradiction follows from (4.5) and the proof is complete.

As a corollary, we obtain uniqueness with less hypotheses than Theorem 3.1.

COROLLARY (Uniqueness). *If $f(x, y, z)$ and the boundary conditions* (1.2), (1.3) *satisfy the hypotheses of Theorem* 4.2, *then the boundary value problem* (1.1), (1.2), (1.3) *has at most one solution.*

We now transplant these comparison theorems to the situation $K > 0$. As might be expected, the transplanted results are more delicate; not only must the length of $[a, b]$ be restricted, but, depending on which of the three functions $w(x)$ of Lemma 2.4 we use, further changes are necessary. We deal first with the initial value Theorem 4.1.

THEOREM 4.3. *Suppose $f(x, y, z)$ satisfies* (a), (b), *and* (c) *with $K > 0$. Suppose that $u(x)$, $v(x)$ satisfy*

$$u'' + f(x, u, u') \geqslant v'' + f(x, v, v'), \qquad a \leqslant x \leqslant b.$$

(i) *If $b - a < \alpha(M_2, K, 0)$ and if $v(a) = u(a)$, $v'(a) < u'(a)$, then $v'(x) < u'(x)$, $v(x) < u(x)$ for $a < x \leqslant b$.*

(ii) *If $b - a < \beta(M_1, K, 0)$ and if $v(b) = u(b)$, $v'(b) > u'(b)$, then $v'(x) > u'(x)$, $v(x) < u(x)$ for $a \leqslant x < b$.*

(iii) *If $b - a < \alpha(M_2, K, 0) + \beta(M_1, K, 0)$ and if $v(a) = u(a)$, $v'(a) < u'(a)$, then $v(x) < u(x)$ for $a < x \leqslant b$.*

Proof. Suppose $w \in C^2[a, b]$, $w(x) > 0$ on $[a, b]$. Let $u(x) = w(x)\,\phi(x)$, $v(x) = w(x)\,\psi(x)$. Then (cf. (2.1))

$$\phi'' + F_w(x, \phi, \phi') \geqslant \psi'' + F_w(x, \psi, \psi'), \qquad a \leqslant x \leqslant b.$$

If $v(a) = u(a)$, $v'(a) < u'(a)$, it follows that $\psi(a) = \phi(a)$, $\psi'(a) < \phi'(a)$. In case (iii), we may choose the $w(x)$ of Lemma 2.4(iii), after which Theorem 4.1(i) applies to give $\psi(x) < \phi(x)$, $\psi'(x) < \phi'(x)$ for $a < x \leqslant b$, and (iii) follows. In case (i), we may choose the $w(x)$ of Lemma 2.4(i), after which Theorem 4.1(i) applies to give $\psi(x) < \phi(x)$, $\psi'(x) < \phi'(x)$ for $a < x \leqslant b$. Thus $v(x) < u(x)$ on $(a, b]$ and since $w'(x) > 0$ on $[a, b]$, $v'(x) < u'(x)$ on $(a, b]$, which proves (i). Statement (ii) follows from (i) by the familiar change of variable; of course, one may also change variables first in Theorem 4.1(i) and then use the $w(x)$ of Lemma 2.4(ii) to prove (ii).

Several remarks are in order regarding this last theorem.

(1) One may replace x by $a+b-x$ in part (iii) to get a statement involving terminal values.

(2) If one tries, for example, to replace the hypothesis $v(a)=u(a)$ in (i) by $v(a) \leqslant u(a)$, not only does the proof fail, but the result is false. One may construct simple counterexamples with solutions of $y''+y=0$.

(3) If one assumes in Theorem 4.1 that $f(x, y, z)$ satisfies a (two-sided) Lipschitz condition on each compact set with respect to y, the problem of nonuniqueness disappears, and in the proof of Theorem 4.1, $q(x)$ may be defined as in (4.3) if $u(x) \neq v(x)$, and extended by $q(x)=0$ if $u(x)=v(x)$. Then $q(x) \leqslant 0$ is bounded and the proof, less fussy, extends to give Theorem 4.1(ii) with both strict inequalities replaced by weak inequalities. Continuing with these more restrictive hypotheses on $f(x, y, z)$, comparable changes can be made in Theorem 4.3.

(4) The comparison Theorem 4.3 and its modified version just indicated should be compared with the first part of each of Theorem 5.1–5.3 of [4, pp. 73–74] and their modified forms described in [4, pp. 80–81].

We now pass to a comparison theorem for boundary value problems in the case $K>0$. The next theorem should be compared to the second part of each of Theorems 5.1–5.3 in [4, pp. 73–74], particularly the modifications discussed in [4, pp. 80–81].

THEOREM 4.4. *Suppose* $f(x, y, z)$ *satisfies* (a), (b), *and* (c) *with* $K>0$. *Suppose that* $u(x)$, $v(x)$ *satisfy*

$$u''+f(x, u, u') \geqslant v''+f(x, v, v'), \qquad a \leqslant x \leqslant b,$$

$$g(u(a), u'(a)) \leqslant g(v(a), v'(a)),$$

$$h(u(b), u'(b)) \leqslant h(v(b), v'(b)),$$

where g_1, h_1, $h_2 \geqslant 0$, $g_2 \leqslant 0$, $g_1-g_2>0$, $h_1+h_2>0$. *Let* r_g, r_h *have the same meaning as in Theorem* 3.2. *If* $b-a<\alpha(M_2, K, r_g)+\beta(M_1, K, r_h)$, *then* $u(x) \leqslant v(x)$, $a \leqslant x \leqslant b$.

Proof. This is the transplanted version of Theorem 4.2. The details of the proof are close to those of the proof of Theorem 3.2.

COROLLARY (Uniqueness). *Suppose* $f(x, y, z)$ *and the boundary conditions* (1.2), (1.3) *satisfy the hypotheses of Theorem* 4.4. *If* $b-a<\alpha(M_2, K, r_g)+\beta(M_1, K, r_h)$, *then the boundary value problem* (1.1), (1.2), (1.3) *has at most one solution.*

5. Generalized Periodic Boundary Value Problems

We now turn to the questions of existence and uniqueness of a solution of (1.1), (1.4), (1.5). Our basic assumptions regarding the boundary conditions (1.4) and (1.5) are that G, H are continuously differentiable everywhere and

(e) $H_1 \geqslant 0,\ H_2 \leqslant 0,\ H_1 - H_2 \geqslant \delta > 0,\ G_2 H_1 - G_1 H_2 \geqslant 0.$

The special case $G(s,t) \equiv 0$, $H(s,t) = s - t$ gives the periodic boundary conditions. Since the case $K \leqslant 0$ has been already considered in [6], we deal here with the case $K > 0$. As observed by Shampine [12], we cannot expect uniqueness or existence without further restrictions. For example, if $f(x,c,0) \equiv 0$ for two different constants c, then the periodic boundary value problem will have these two constants as distinct solutions. We shall rule out such behavior by strengthening (c) to read

(c′) $f(x,y,z)$ satisfies a two-sided Lipschitz condition in y on R: there exists $K > 0$ so that

$$0 < f(x,y_2,z) - f(x,y_1,z) \leqslant K(y_2 - y_1) \qquad \text{for } y_1 < y_2.$$

Of course, the hypotheses (a), (b), and (c′) are sufficient for uniqueness in initial value problems, and we shall use this fact later.

It does not seem possible to transplant the existence and uniqueness results for the problem (1.1), (1.4), (1.5) in the case $K \leqslant 0$ from [6]; the change of variable does too much damage in the boundary conditions. Instead we use the same strategy as in [6] and base our proofs on the theorems of Sections 3 and 4 which hold in the case $K > 0$.

For the moment, we shall replace (e) by the superfically more demanding

(e′) $H_1 \geqslant 0,\ H_2 \leqslant 0,\ H_1 - H_2 \geqslant \delta > 0,\ G_1 \geqslant 0,\ G_2 \geqslant 0.$

Note that the periodic boundary conditions satisfy (e′).

THEOREM 5.1 (Uniqueness). *Suppose* $f(x,y,z)$ *satisfies* (a), (b), *and* (c′) *and that the boundary conditions* (1.4), (1.5) *satisfy* (e′). *Put* $r_1 = \sup G_1$, $r_2 = \sup G_2$. *If* $b - a < \alpha(M_2, K, r_1) + \beta(M_1, K, r_2)$, *then the boundary value problem* (1.1), (1.4), (1.5) *has at most one solution.*

Proof. Suppose (1.1), (1.4), (1.5) has two solutions $u(x)$ and $v(x)$. Let $y = u - v$. Then (1.4) and the mean value theorem give

$$y(a) - G_1(s,t)\, y'(a) = y(b) + G_2(s,t)\, y'(b), \tag{5.1}$$

where (s,t) lies on the line segment joining $(u'(a), u'(b))$ to $(v'(a), v'(b))$.

Let k denote the common value of the two sides of (5.1). Interchanging the roles of u and v if necessary, we may assume that $k \geqslant 0$ Thus y satisfies

$$y'' + F(x, y, y') = 0, \qquad a \leqslant x \leqslant b, \tag{5.2}$$

$$y(a) - G_1(s, t)\, y'(a) = k, \tag{5.3}$$

$$y(b) + G_2(s, t)\, y'(b) = k, \tag{5.4}$$

where

$$F(x, y, z) = f(x, v(x) + y, v'(x) + z) - f(x, v(x), v'(x)). \tag{5.5}$$

It is easy to check that $F(x, y, z)$ satisfies (a), (b), and (c$'$) and that Theorem 4.4 and its corollary may be applied to the problem (5.2), (5.3), (5.4). If $k = 0$, the corollary to Theorem 4.4 implies that $y(x) \equiv 0$ so $u(x) \equiv v(x)$ and we are done. If $k > 0$, the comparison Theorem 4.4 gives $y(x) \geqslant 0$ for $a \leqslant x \leqslant b$, and clearly $y(x) \not\equiv 0$. Now $y(x)$ is not constant, for then (5.2), (5.5) give

$$f(x, v(x) + y(x), v'(x)) - f(x, v(x), v'(x)) \equiv 0, \qquad a \leqslant x \leqslant b,$$

contradicting (c$'$). The minimum of $y(x)$ occurs at some point c in $[a, b]$. If $a < c < b$, then $y'(c) = 0$ and $y(c) > 0$ since $y(c) = 0$ would contradict uniqueness for initial value problems. But then at the point $x = c$, (5.2) gives

$$y''(c) = f(c, v(c), v'(c)) - f(c, v(c) + y(c), v'(c)) < 0$$

by (c$'$), contradicting the minimum at c. Thus the minimum occurs at a or at b. Suppose the minimum occurs at a (the other alternative is handled similarly); then $y'(a) \geqslant 0$ and from (5.3), $y(a) \geqslant k > 0$. Also $y'(a) > 0$ since $y'(a) = 0$ implies from (5.2) that

$$y''(a) = f(a, v(a), v'(a)) - f(a, v(a) + y(a), v'(a)) < 0$$

by (c$'$), contradicting the minimum at a. Then using (1.5) and the mean value theorem, we get

$$H_1(\bar{s}, \bar{t})\, y'(a) + H_2(\bar{s}, \bar{t})\, y'(b) = 0,$$

where $(\bar{s}, \bar{t})$ lies on the line segment joining $(u'(a), u'(b))$ to $(v'(a), v'(b))$. If $H_1(\bar{s}, \bar{t}) = 0$, then by (e$'$), $H_2(\bar{s}, \bar{t}) \neq 0$ and hence $y'(b) = 0$. If $H_1(\bar{s}, \bar{t}) > 0$, then $y'(a) > 0$ implies $y'(b) > 0$. Thus in either case, $y'(b) \geqslant 0$. By (5.4), $y(b) \leqslant k$. Since $y(a) \geqslant k$, the minimum occurs also at b, with $y(b) = y(a) = k$ and $y'(b) = 0$. But now (5.2) gives

$$y''(b) = f(b, v(b), v'(b)) - f(b, v(b) + k, v'(b)) < 0$$

by (c′), contradicting the minimum at b. This final contradiction completes the proof.

It has been shown in [6, Lemma 5] that if G, H satisfy (e), it is possible to obtain an equivalent boundary value problem by replacing G with a new function $\tilde{G}$ after which H, $\tilde{G}$ satisfy (e′) with $\tilde{G}_1 = \tilde{G}_2 = G_2 H_1 - G_1 H_2 \geqslant 0$.

COROLLARY. *Suppose* $f(x, y, z)$ *satisfies* (a), (b), *and* (c′), *and that the boundary conditions* (1.4), (1.5) *satisfy* (e). *Put* $r = \sup\{G_2 H_1 - G_1 H_2\}$. *If* $b - a < \alpha(M_2, K, r) + \beta(M_1, K, r)$, *then* (1.1), (1.4), (1.5) *has at most one solution.*

We next turn to the question of existence of solutions of (1.1), (1.4), and (1.5). Since this work closely parallels that in [6], we shall only sketch the arguments, which use the following lemma [6, Lemma 7].

LEMMA 5.2. *Let* $c = (a + b)/2$. *Suppose* (e) *is satisfied and let* A *be given. Then there exist constants* B, C, *independent of* A, *so that*

$$u(x, A) = A + B(x - c) + C(x - c)^2$$

satisfies the boundary conditions (1.4), (1.5).

As in [6], we prove existence first in the case that G not only satisfies (e′), but can be separated:

$$G(y'(a), y'(b)) = G_a(y'(a)) + G_b(y'(b)), \tag{5.6}$$

$$G_1 = G_a' \geqslant 0, \qquad G_2 = G_b' \geqslant 0. \tag{5.7}$$

Letting $u = u(x, A)$, we have

$$u'' + f(x, u, u') = 2C + f(x, u(x, A), B + 2C(x - c)). \tag{5.8}$$

If we choose m, M so that

$$m \leqslant B(x - c) + C(x - c)^2 \leqslant M, \qquad a \leqslant x \leqslant b, \tag{5.9}$$

and assume that $f(x, y, z)$ is a nondecreasing function of y, then (5.8) gives

$$\begin{aligned} 2C &+ f(x, A + m, B + 2C(x - c) \\ &\leqslant u'' + f(x, u, u') \leqslant 2C + f(x, A + M, B + 2C(x - c)). \end{aligned} \tag{5.10}$$

THEOREM 5.3 (Existence). *Suppose that* $f(x, y, z)$ *satisfies* (a), (b), *and* (c) *with* $K > 0$, *that* $M_i(\rho) = o(\log \rho)$ *as* $\rho \to \infty$, *and that* $f(x, y, z)$ *is a nondecreasing function of* y. *Suppose that* G *satisfies* (5.6) *and* (5.7) *and that*

$H_1 \geqslant 0$, $H_2 \leqslant 0$, $H_1 - H_2 \geqslant \delta > 0$. Define B, C, $u(x, A)$ by Lemma 5.2 and m, M by (5.9). Suppose there exist A_1, A_2 so that

$$f(x, A_2 + m, B + 2C(x-c)) \geqslant -2C, \qquad a \leqslant x \leqslant b, \tag{5.11}$$

$$f(x, A_1 + M, B + 2C(x-c)) \leqslant -2C, \qquad a \leqslant x \leqslant b. \tag{5.12}$$

Put $r_1 = \sup G_1 = \sup G_a'$, $r_2 = \sup G_2 = \sup G_b'$. If $b - a < \alpha(M_2, K, r_1) + \beta(M_1, K, r_2)$, then (1.1), (1.4), (1.5) *has at least one solution.*

Proof. By Theorem 3.2, for each A, the boundary value problem

$$y'' + f(x, y, y') = 0, \qquad a \leqslant x \leqslant b, \tag{1.1}$$

$$y(a) - G_a(y'(a)) = u(a, A) - G_a(u'(a, A)) \tag{5.13}$$

$$y(b) + G_b(y'(b)) = u(b, A) + G_b(u'(b, A)) \tag{5.14}$$

has a unique solution $\phi(x, A)$ and both $\phi(x, A)$ and $\phi'(x, A)$ are continuous functions of A, because the right sides of (5.13) and (5.14) are continuous (actually linear) functions of A. Note that since $u(x, A)$ satisfies (1.4), so does $\phi(x, A)$. We need to chose A so that $\phi(x, A)$ satisfies (1.5). Our hypotheses (5.11), (5.12) together with (5.10) and the comparison Theorem 4.4 give

$$u(x, A_2) \leqslant \phi(x, A), \qquad a \leqslant x \leqslant b,\ A \geqslant A_2, \tag{5.15}$$

$$u(x, A_1) \geqslant \phi(x, A), \qquad a \leqslant x \leqslant b,\ A \leqslant A_1. \tag{5.16}$$

We then verify, exactly as in [6], that

$$H(\phi'(a, A_1), \phi'(b, A_1)) \leqslant 0, \tag{5.17}$$

$$H(\phi'(a, A_2), \phi'(b, A_2)) \geqslant 0. \tag{5.18}$$

Thus, by continuity, there exists A, $A_1 \leqslant A \leqslant A_2$, such that

$$H(\phi'(a, A), \phi'(b, A)) = 0, \tag{5.19}$$

and then $\phi(x, A)$ is the desired solution of (1.1), (1.4), (1.5).

One further interesting bit of information may be gleaned from the proof of Theorem 5.3. If in (5.6), it happens that $G_a \equiv 0$, we see from (5.13) that $\phi(a, A) = u(a, A)$ and since $A_1 \leqslant A \leqslant A_2$,

$$A_1 + m \leqslant u(a, A_1) \leqslant \phi(a, A) \leqslant u(a, A_2) \leqslant A_2 + M.$$

In particular, if $B = C = 0$,

$$A_1 \leqslant \phi(a, A) \leqslant A_2.$$

Similar comments apply if, in (5.6), $G_b \equiv 0$.

The following corollaries, similar to those of [6] in the case $K \leqslant 0$, are now rather immediate.

COROLLARY 5.4. *Suppose that $f(x, y, z)$ satisfies the same hypotheses as in Theorem* 5.3. *Suppose that G, H satisfy* (e)

(i) *If* $H_1 \geqslant \delta$, $r_2 = \sup\{(G_2H_1 - G_1H_2)/H_1\}$, *and* $b - a < \alpha(M_2, K, 0) + \beta(M_1, K, r_2)$, *then* (1.1), (1.4), (1.5) *has at least one solution* $\phi(x)$ *with* $A_1 + m \leqslant \phi(a) \leqslant A_2 + M$.

(ii) *If* $H_2 \leqslant -\delta$, $r_1 = \sup\{(G_1H_2 - G_2H_1)/H_2\}$, *and* $b - a < \alpha(M_2, K, r_1) + \beta(M_1, K, 0)$, *then* (1.1), (1.4), (1.5) *has at least one solution* $\phi(x)$ *with* $A_1 + m \leqslant \phi(b) \leqslant A_2 + M$.

Proof. In case (i), one solves $H(y'(a), y'(b)) = 0$ for $y'(a)$ in terms of $y'(b)$ and thus eliminates $y'(a)$ from (1.4), after which Theorem 5.3 may be applied.

COROLLARY 5.5. *The hypothesis in Theorem* 5.3 *and Corollary* 5.4 *that there exists A_1, A_2 satisfying* (5.11), (5.12) *may be replaced by the hypothesis that there exist $K > 0$, $y_0 > 0$ such that*

$$f(x, y_2, B + 2C(x - c)) - f(x, y_1, B + 2C(x - c)) \geqslant K(y_2 - y_1),$$

for $y_0 \leqslant y_1 \leqslant y_2$ and for $y_1 \leqslant y_2 \leqslant -y_0$.

COROLLARY 5.6. *Suppose that* (a) *and* (b) *are satisfied with $M_i(\rho) = o(\log \rho)$ as $\rho \to \infty$, and that there exists K_0, K with $K > K_0 > 0$ such that*

$$K_0(y_2 - y_1) \leqslant f(x, y_2, z) - f(x, y_1, z) \leqslant K(y_2 - y_1) \qquad \text{for } y_1 \leqslant y_2.$$

Suppose that G, H satisfy the hypotheses of Theorem 5.3. *If $b - a < \alpha(M_2, K, r_1) + \beta(M_1, K, r_2)$, where r_1, r_2 are defined as in Theorem* 5.3, *then* (1.1), (1.4), (1.5) *has a unique solution.*

Corollary 5.6 contains the theorem [12, Theorem 6] of Shampine as a special case.

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