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On the constants in a basic inequality for the Euler and Navier–Stokes equations

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ABSTRACT

We consider the incompressible Euler or Navier–Stokes (NS) equations on a d-dimensional torus $\mathbf{T}^d$; the quadratic term in these equations arises from the bilinear map sending two velocity fields $v, w : \mathbf{T}^d \to \mathbf{R}^d$ into $v \cdot \partial w$, and also involves the Leray projection $\mathfrak{L}$ onto the space of divergence free vector fields. We derive upper and lower bounds for the constants in two inequalities related to the above quadratic term; these bounds hold, in particular, for the sharp constants $K_{nd} \equiv K_n$ in the basic inequality $\|\mathfrak{L}(v \cdot \partial w)\|_n \leqslant K_n \|v\|_n \|w\|_{n+1}$, where $n \in (d/2, +\infty)$ and v, w are in the Sobolev spaces $\mathbb{H}^n_{\Sigma 0}, \mathbb{H}^{n+1}_{\Sigma 0}$ of zero mean, divergence free vector fields of orders n and $n+1$, respectively. As examples, the numerical values of our upper and lower bounds are reported for $d = 3$ and some values of n. Some practical motivations are indicated for an accurate analysis of the constants K_n, making reference to other works on the approximate solutions of Euler or NS equations.

1. Introduction

The incompressible Euler or Navier–Stokes (Euler/NS) equations in d space dimensions can be written as

$$\frac{\partial u}{\partial t} = -\mathfrak{L}(u \bullet \partial u) + \nu \Delta u + f, \tag{1.1}$$

where: $u = u(x, t)$ is the divergence free velocity field; $x = (x_s)_{s=1,\ldots,d}$ are the space coordinates (yielding the derivatives $\partial_s := \partial/\partial x_s$); $\Delta := \sum_{s=1}^d \partial_{ss}$ is the Laplacian; $(u \bullet \partial u)_r := \sum_{s=1}^d u_s \partial_s u_r$ $(r = 1, \ldots, d)$; $\mathfrak{L}$ is the Leray projection onto the space of divergence free vector fields; $\nu = 0$ for the Euler equations; $\nu \in (0, +\infty)$ for the NS equations (a case that can always be reduced to $\nu = 1$, by rescaling); $f = f(x, t)$ is the Leray projected density of external forces. In this paper we stick to the case of space periodic boundary conditions; so, x ranges in the d-dimensional torus $\mathbf{T}^d := (\mathbf{R}/2\pi\mathbf{Z})^d$. As well known (see, e.g., [7]), the analysis of (1.1) in the space periodic case can always be reduced, by suitable transformations, to the case where the (spatial) means $\langle u \rangle := (2\pi)^{-d} \int_{\mathbf{T}^d} u\, dx$ and $\langle f \rangle$ (defined similarly) are zero.

Our functional setting for the incompressible Euler/NS equations relies on Sobolev spaces of the H^n type. More precisely we consider, for suitable (integer or noninteger) values of n, the spaces

$$\mathbb{H}^n_0(\mathbf{T}^d) \equiv \mathbb{H}^n_0 := \{v : \mathbf{T}^d \to \mathbf{R}^d \mid \sqrt{-\Delta}^{\,n} v \in \mathbb{L}^2(\mathbf{T}^d),\ \langle v \rangle = 0\}, \qquad \mathbb{H}^n_{\Sigma 0}(\mathbf{T}^d) \equiv \mathbb{H}^n_{\Sigma 0} := \{v \in \mathbb{H}^n_0 \mid \operatorname{div} v = 0\} \tag{1.2}$$

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(where 0 and Σ indicate, respectively, the vanishing of the mean and of the divergence). We equip $\mathbb{H}^n_0$ with the standard inner product and norm $\langle v|w\rangle_n := \langle \sqrt{-\Delta}^n v|\sqrt{-\Delta}^n w\rangle_{L^2}$, $\|v\|_n := \sqrt{\langle v|v\rangle_n}$, which can be restricted to the subspace $\mathbb{H}^n_{\Sigma 0}$.

A fully quantitative treatment of several problems related to the above functional setting (see, e.g., the sequel of this Introduction) relies on the constants in some inequalities about the bilinear map sending two vector fields v, w on $\mathbf{T}^d$ into $v \bullet \partial w$, or about the composition of this map with $\mathfrak{L}$. Here, we wish to analyze some of these constants.

To describe precisely the contents of this paper, let us mention that the assumptions $n > d/2$, $v \in \mathbb{H}^n_{\Sigma 0}$ and $w \in \mathbb{H}^{n+1}_0$ imply $v \bullet \partial w \in \mathbb{H}^n_0$, whence $\mathfrak{L}(v \bullet \partial w) \in \mathbb{H}^n_{\Sigma 0}$. In this paper we consider the basic inequality

$$\|\mathfrak{L}(v \bullet \partial w)\|_n \leqslant K_n \|v\|_n \|w\|_{n+1} \quad \text{for } n \in (d/2, +\infty),\ v \in \mathbb{H}^n_{\Sigma 0},\ w \in \mathbb{H}^{n+1}_{\Sigma 0}; \tag{1.3}$$

our aim is to give quantitative upper and lower bounds on the sharp constant $K_n \equiv K_{nd}$ appearing therein. We use the fact that $K_n \leqslant K'_n$, where K'_n is the sharp constant in the (auxiliary) inequality

$$\|v \bullet \partial w\|_n \leqslant K'_n \|v\|_n \|w\|_{n+1} \quad \text{for } n \in (d/2, +\infty),\ v \in \mathbb{H}^n_{\Sigma 0},\ w \in \mathbb{H}^{n+1}_0. \tag{1.4}$$

Even though Eqs. (1.3) and (1.4) are well known, little information can be found in the literature about the numerical values of the constants therein. Our approach produces fully computable upper and lower bounds $K^{\pm}_n \equiv K^{\pm}_{nd}$ such that

$$K^-_n \leqslant K_n \leqslant K'_n \leqslant K^+_n \tag{1.5}$$

for all $n > d/2$. As examples, the numerical values of $K^{\pm}_n$ are given in dimension $d = 3$, for some values of n; all details on these numerical computations can be found in an extended version of this paper [9] posted in arXiv.

In a companion paper [10], we have proposed upper and lower bounds for the constants $G_{nd} \equiv G_n$ in the inequality

$$|\langle v \bullet \partial w|w\rangle_n| \leqslant G_n \|v\|_n \|w\|^2_n \quad \text{for } n \in (d/2 + 1, +\infty),\ v \in \mathbb{H}^n_{\Sigma 0},\ w \in \mathbb{H}^{n+1}_{\Sigma 0}, \tag{1.6}$$

dating back to a seminal paper by Kato [3].

Let us illustrate some applications of the inequalities (1.3) and (1.6), depending on quantitative information on the constants K_n, G_n. To this purpose, following [11] we consider the Euler/NS equations (1.1) with a specified initial condition $u(x, 0) = u_0(x)$; let $u_a : \mathbf{T}^d \times [0, T_a) \to \mathbf{R}^d$ be an approximate solution of this Cauchy problem. Given $n \in (d/2 + 1, +\infty)$ (and assuming suitable regularity for u_0, f, u_a), let u_a possess the differential error estimator $\epsilon_n : [0, T_a) \to [0, +\infty)$, the datum error estimator $\delta_n \in [0, +\infty)$ and the growth estimators $\mathcal{D}_n, \mathcal{D}_{n+1} : [0, T_a) \to [0, +\infty)$; this means that, for $t \in [0, T_a)$,

$$\left\| \left(\frac{\partial u_a}{\partial t} + \mathfrak{L}(u_a \bullet \partial u_a) - \nu \Delta u_a - f \right)(t) \right\|_n \leqslant \epsilon_n(t),$$
$$\|u_a(0) - u_0\|_n \leqslant \delta_n, \qquad \|u_a(t)\|_n \leqslant \mathcal{D}_n(t), \qquad \|u_a(t)\|_{n+1} \leqslant \mathcal{D}_{n+1}(t) \tag{1.7}$$

(with $u_a(t) := u_a(\cdot, t)$, etc.). Furthermore, let us assume the existence of a function $\mathscr{R}_n \in C([0, T_c), [0, +\infty))$ with $T_c \in (0, T_a]$, fulfilling the *control inequalities*

$$\frac{d^+\mathscr{R}_n}{dt} \geqslant -\nu \mathscr{R}_n + (G_n \mathcal{D}_n + K_n \mathcal{D}_{n+1})\mathscr{R}_n + G_n \mathscr{R}^2_n + \epsilon_n \quad \text{on } [0, T_c),\ \mathscr{R}_n(0) \geqslant \delta_n \tag{1.8}$$

(with d^+/dt the right upper Dini derivative). Then, as shown in [11], the solution u of Eq. (1.1) with initial datum u_0 exists (in a classical sense) on the time interval $[0, T_c)$, and its distance from the approximate solution admits the bound

$$\|u(t) - u_a(t)\|_n \leqslant \mathscr{R}_n(t) \quad \text{for } t \in [0, T_c). \tag{1.9}$$

This somehow refines a previous result of [2], where the time of existence of u was estimated via an integral inequality involving $\delta_n, \epsilon_n, \mathcal{D}_n, \mathcal{D}_{n+1}$ and the constants K_n, G_n (but with no quantitative information on these constants). For a given datum u_0, the practical implementation of the setting of [11] is performed choosing a suitable u_a (say, a Galerkin approximate solution), computing the estimators $\epsilon_n, \mathcal{D}_n, \mathcal{D}_{n+1}$ and then using the inequalities ((1.8)–(1.9)).

To conclude, let us mention other papers [1,4–8,12–15] where a fully quantitative approach was considered for the NS equations, other nonlinear PDEs and/or some related inequalities, with the aim to derive conditions of existence or error bounds on approximation methods. In particular, in [7] we derived (fairly rough) upper bounds on the constants in a variant of the inequality (1.4) using an approach similar to the present one, but much less refined.

2. Sobolev spaces on $\mathbf{T}^d$ and the Euler/NS quadratic nonlinearity

Throughout the paper, we consider any space dimension $d \geqslant 2$; r, s are indices running from 1 to d. For $a = (a_r)$, $b = (b_r) \in \mathbf{C}^d$ we put $a \bullet b := \sum_{r=1}^d a_r b_r$ and $|a| := \sqrt{\overline{a} \bullet a}$, where $\overline{a} := (\overline{a_r})$ is the complex conjugate of a. The d-dimensional torus $\mathbf{T}^d$ is the Cartesian product of d copies of $\mathbf{T} := \mathbf{R}/(2\pi\mathbf{Z})$, and its elements are typically written $x = (x_r)_{r=1,\ldots,d}$. The expression "a vector field $\mathbf{T}^d \to \mathbf{R}^d$" can be understood, with very wide generality, as "an $\mathbf{R}^d$-valued distribution on $\mathbf{T}^d$"

(see, e.g., [8]). We write $\mathbb{D}'(\mathbf{T}^d) \equiv \mathbb{D}'$ for the space of such distributions; this contains, for any $p \in [1, +\infty)$, the space $\mathbb{L}^p(\mathbf{T}^d) \equiv \mathbb{L}^p$ of p-summable vector fields $\mathbf{T}^d \to \mathbf{R}^d$. We are mainly interested in $\mathbb{L}^2$, with its inner product $\langle v|w\rangle_{L^2} := \int_{\mathbf{T}^3} v(x) \bullet w(x)dx$ and the induced norm $\|v\|_{L^2} = \sqrt{\int_{\mathbf{T}^3} |v(x)|^2 dx}$. Any $v \in \mathbb{D}'$ has a mean $\langle v\rangle := (2\pi)^{-d} \int_{\mathbf{T}^d} dx\, v(x) \in \mathbf{R}^d$ and a family of Fourier coefficients $v_k := (2\pi)^{-d/2} \int_{\mathbf{T}^d} dx\, e^{-ik\bullet x} v(x) \in \mathbf{C}^d$, labeled by $k \in \mathbf{Z}^d$ (where the last two integrals are understood as the action of v on the test functions 1 and $x \mapsto e^{-ik\bullet x}$); we note that $v_{-k} = \overline{v_k}$ and $\langle v\rangle = (2\pi)^{-d/2} v_0$. From now on, we are mainly interested in the zero mean vector fields, whose relevant Fourier coefficients are labeled by the nonzero wave vectors: throughout the paper we will put

$$\mathbb{D}'_0 := \{v \in \mathbb{D}' \mid \langle v\rangle = 0 \text{ (i.e., } v_0 = 0)\}; \qquad \mathbf{Z}^d_0 := \mathbf{Z}^d \setminus \{0\}. \tag{2.1}$$

The Sobolev space of zero mean vector fields of any order $n \in \mathbf{R}$, its inner product and the induced norm are

$$\mathbb{H}^n_0(\mathbf{T}^d) \equiv \mathbb{H}^n_0 := \left\{v \in \mathbb{D}'_0 \mid \sqrt{-\Delta}^{\,n} v \in \mathbb{L}^2, \right\} = \left\{v \in \mathbb{D}'_0 \mid \sum_{k\in\mathbf{Z}^d_0} |k|^{2n}|v_k|^2 < +\infty\right\}; \tag{2.2}$$

$$\langle v|w\rangle_n := \langle \sqrt{-\Delta}^{\,n} v|\sqrt{-\Delta}^{\,n} w\rangle_{L^2} = \sum_{k\in\mathbf{Z}^d_0} |k|^{2n}\overline{v_k} \bullet w_k, \qquad \|v\|_n = \|\sqrt{-\Delta}^{\,n} v\|_{L^2} = \sqrt{\sum_{k\in\mathbf{Z}^d_0} |k|^{2n}|v_k|^2}. \tag{2.3}$$

Defining div in a distributional sense, we can introduce the spaces of divergence free vector fields

$$\begin{aligned}&\mathbb{D}'_\Sigma := \{v \in \mathbb{D}' \mid \operatorname{div} v = 0\} = \{v \in \mathbb{D}' \mid k \bullet v_k = 0\ \forall k \in \mathbf{Z}^d\}; \qquad \mathbb{D}'_{\Sigma 0} := \mathbb{D}'_\Sigma \cap \mathbb{D}'_0;\\ &\mathbb{H}^n_{\Sigma 0} := \mathbb{D}'_\Sigma \cap \mathbb{H}^n_0 \quad (n \in \mathbf{R}).\end{aligned} \tag{2.4}$$

$\mathbb{H}^n_{\Sigma 0}$ is a closed subspace of the Hilbert space $\mathbb{H}^n_0$, that we equip with the restrictions of $\langle|\rangle_n$, $\|\ \|_n$. The *Leray projection* is

$$\mathfrak{L} : \mathbb{D}' \to \mathbb{D}'_\Sigma, \qquad v \mapsto \mathfrak{L}v := \sum_{k\in\mathbf{Z}^d} (\mathfrak{L}_k v_k)e_k, \tag{2.5}$$

where, for each k, $\mathfrak{L}_k$ is the orthogonal projection of $\mathbf{C}^d$ onto the orthogonal complement of k (if $c \in \mathbf{C}^d$, $\mathfrak{L}_0 c = c$ and $\mathfrak{L}_k c = c - (k \bullet c)k/|k|^2$ for $k \in \mathbf{Z}^d_0$). One has

$$\langle \mathfrak{L}v\rangle = \langle v\rangle \quad \text{for } v \in \mathbb{D}'; \qquad \mathfrak{L}\mathbb{D}'_0 = \mathbb{D}'_{\Sigma 0}; \qquad \mathfrak{L}\mathbb{H}^n_0 = \mathbb{H}^n_{\Sigma 0}; \qquad \|\mathfrak{L}v\|_n \leqslant \|v\|_n \quad \text{for } n \in \mathbf{R},\ v \in \mathbb{H}^n_0. \tag{2.6}$$

To discuss the quadratic Euler/NS nonlinearity we start from some known facts, reviewed in this lemma for completeness.

Lemma 2.1. *Let us consider two vector fields v, w on $\mathbf{T}^d$, such that $v \in \mathbb{L}^2$ and $\partial_s w \in \mathbb{L}^2$ for $s = 1, \ldots, d$ (a fact implying $w \in \mathbb{L}^2$); let us introduce the vector field $v \bullet \partial w$ on $\mathbf{T}^d$, of components $(v \bullet \partial w)_r := \sum_{s=1}^d v_s \partial_s w_r$. Then, we have the following.*

(i) *$v \bullet \partial w$ is well defined and belongs to $\mathbb{L}^1$. If $\operatorname{div} v = 0$, one has $\langle v \bullet \partial w\rangle = 0$ (whence $\langle \mathfrak{L}(v \bullet \partial w)\rangle = 0$, see (2.6)).*

(ii) *The Fourier coefficients of this vector field and of its Leray projection are*

$$\begin{aligned}&(v \bullet \partial w)_k = \frac{i}{(2\pi)^{d/2}} \sum_{h\in\mathbf{Z}^d} [v_h \bullet (k-h)]w_{k-h},\\ &[\mathfrak{L}(v \bullet \partial w)]_k = \frac{i}{(2\pi)^{d/2}} \sum_{h\in\mathbf{Z}^d} [v_h \bullet (k-h)]\mathfrak{L}_k w_{k-h} \quad \textit{for all } k \in \mathbf{Z}^d.\end{aligned} \tag{2.7}$$

Proof (*Sketch*). (i) Each component $(v \bullet \partial w)_r$, being a sum of products of L^2 functions, is evidently in L^1. The identity $\int_{\mathbf{T}^d} (v \bullet \partial w)\, dx = -\int_{\mathbf{T}^d} (\operatorname{div} v)\, w\, dx$ is easily proved via integration by parts, and implies $\langle v \bullet \partial w\rangle = 0$ if $\operatorname{div} v = 0$.

(ii) For $r = 1, \ldots, d$, the k-th Fourier coefficient of the real function $(v \bullet \partial w)_r = \sum_{s=1}^d v_s \partial_s w_r$ is easily computed noting that $(\partial_s w_r)_k = ik_s w_{rk}$, and recalling that the pointwise product corresponds to $(2\pi)^{-d/2}$ times the convolution of the Fourier coefficients; this gives the relation $(v \bullet \partial w)_{rk} = i(2\pi)^{-d/2} \sum_{s=1}^d \sum_{h\in\mathbf{Z}^d} v_{sh}(k-h)_s w_{r,k-h}$, whose vector form is the first equality (2.7). The second relation (2.7) follows from the first one and from (2.5). □

3. The basic inequality for the Euler/NS quadratic nonlinearity

From here to the end of the paper (including the Appendix) we assume $d \in \{2, 3, \ldots\}$ (as before) and

$$n \in (d/2, +\infty). \tag{3.1}$$

The forthcoming Proposition 3.1 is well known; as a matter of fact, the quantitative analysis performed in Section 4 will also give, as a byproduct, a novel proof of this statement.

Proposition 3.1. *Let* $v \in \mathbb{H}^n_{\Sigma 0}$, $w \in \mathbb{H}^{n+1}_0$; *then*, $v \bullet \partial w \in \mathbb{H}^n_0$. *The map* $\mathbb{H}^n_{\Sigma 0} \times \mathbb{H}^{n+1}_0 \to \mathbb{H}^n_0$, $(v, w) \mapsto v \bullet \partial w$ *is bilinear and continuous.*

Of course, continuity of the above map is equivalent to the existence of a nonnegative constant K', such that $\|v \bullet \partial w\|_n \leqslant K'\|v\|_n\|w\|_{n+1}$ for v, w as in the previous proposition; a similar inequality holds as well for $\mathfrak{L}(v \bullet \partial w) \in \mathbb{H}^n_{\Sigma 0}$, since $\|\mathfrak{L}(v \bullet \partial w)\|_n \leqslant \|v \bullet \partial w\|_n$. So, we have the "auxiliary inequality" (1.4) and the "basic inequality" (1.3) of the Introduction; the sharp constants therein can be defined as follows.

Definition 3.2. We put

$$K'_{nd} \equiv K'_n := \min\{K' \in [0, +\infty) \mid \|v \bullet \partial w\|_n \leqslant K'\|v\|_n\|w\|_{n+1} \text{ for all } v \in \mathbb{H}^n_{\Sigma 0}, w \in \mathbb{H}^{n+1}_0\}; \tag{3.2}$$

$$K_{nd} \equiv K_n := \min\{K \in [0, +\infty) \mid \|\mathfrak{L}(v \bullet \partial w)\|_n \leqslant K\|v\|_n\|w\|_{n+1} \text{ for all } v \in \mathbb{H}^n_{\Sigma 0}, w \in \mathbb{H}^{n+1}_{\Sigma 0}\}. \tag{3.3}$$

We note that w is divergence free in (3.3), but not in (3.2). The considerations after Proposition 3.1 ensure that

$$K_n \leqslant K'_n; \tag{3.4}$$

in the rest of the section we present computable upper bounds on K'_n and lower bounds on K_n, which are the main result of the paper. The upper bound requires a more lengthy analysis, based on the forthcoming Definition 3.3 and, especially, on the function $\mathcal{K}_{nd} \equiv \mathcal{K}_n$ of the subsequent Definition 3.4.

Definition 3.3. Here and in the sequel, the exterior power $\bigwedge^2 \mathbf{R}^d$ is identified with the space of real, skew-symmetric $d \times d$ matrices $A = (A_{rs})_{r,s,=1,\ldots,d}$; this is equipped with the (bilinear, skew-symmetric) operation of exterior product $\wedge$ and with the norm $|\ |$ defined as follows:

$$\wedge : \mathbf{R}^d \times \mathbf{R}^d \to \bigwedge^2 \mathbf{R}^d,\ (p \wedge q)_{rs} := p_r q_s - q_r p_s; \qquad |\ | : \bigwedge^2 \mathbf{R}^d \to [0, +\infty), \qquad |A| := \sqrt{\frac{1}{2}\sum_{r,s=1}^{d} |A_{rs}|^2}. \tag{3.5}$$

It is well known that, for all p, q in $\mathbf{R}^d$, the norm of $p \wedge q$ is the area of the parallelogram with these sides. So,

$$|p \wedge q| = |p||q| \sin\vartheta \leqslant |p||q|, \tag{3.6}$$

where $\vartheta \equiv \vartheta(p, q) \in [0; \pi]$ is the convex angle between p and q (defined arbitrarily, if $p = 0$ or $q = 0$).

Definition 3.4. We put

$$\mathbf{Z}^d_{0k} := \mathbf{Z}^d \setminus \{0, k\} \quad \text{for each } k \in \mathbf{Z}^d_0;$$
$$\mathcal{K}_{nd} \equiv \mathcal{K}_n : \mathbf{Z}^d_0 \to (0, +\infty), \qquad \mathcal{K}_n(k) := |k|^{2n} \sum_{h \in \mathbf{Z}^d_{0k}} \frac{|h \wedge k|^2}{|h|^{2n+2}|k-h|^{2n+2}}. \tag{3.7}$$

The sum in (3.7) is finite for each k since, for $h \to \infty$, $|h \wedge k|^2|h|^{-2n-2}|k-h|^{-2n-2} = O(|h|^{-4n-2})$ and $4n+2 > 2(d+1) > d$. Some information about the function $\mathcal{K}_n$ and its practical computation is given in Appendix and, with more details, in the extended version of this paper in arXiv [9]. One of the facts established via this analysis is that $\sup_{k \in \mathbf{Z}^d_0} \mathcal{K}_n(k) < +\infty$.

Proposition 3.5. *The constant* K'_n *defined by* (3.2) *has the upper bound*

$$K'_n \leqslant K^+_n, \quad K^+_n := \frac{1}{(2\pi)^{d/2}} \sqrt{\sup_{k \in \mathbf{Z}^d_0} \mathcal{K}_n(k)}. \tag{3.8}$$

Proof. See Section 4. □

Proposition 3.6. *The constant* K_n *defined by* (3.3) *has the lower bound*

$$K_n \geqslant K^-_n, \quad K^-_n := \frac{2^{n/2}}{(2\pi)^{d/2}} U_d, \qquad U_d := \begin{cases} (2-\sqrt{2})^{1/2} = 0.76536\ldots & \text{if } d = 2, \\ 1 & \text{if } d \geqslant 3. \end{cases} \tag{3.9}$$

Proof. See Section 4. □

Putting together Eqs. (3.4), (3.8) and (3.9) we obtain the chain of inequalities, anticipated in Eq. (1.5) of the Introduction,

$$K_n^- \leqslant K_n \leqslant K_n' \leqslant K_n^+;$$

here and in the sequel K_n^+ is the upper bound in (3.8) (or some upper approximant for it), while K_n^- is the lower bound in (3.9) (or some lower approximant for it).

Examples 3.7. For $d = 3$ and $n = 2, 3, 4, 5, 10$, we can take

$$\begin{aligned} &K_2^- = 0.126, \quad K_2^+ = 0.335; \quad K_3^- = 0.179, \quad K_3^+ = 0.323; \quad K_4^- = 0.253, \quad K_4^+ = 0.441; \\ &K_5^- = 0.359, \quad K_5^+ = 0.657; \quad K_{10}^- = 2.03, \quad K_{10}^+ = 6.21. \end{aligned} \tag{3.10}$$

In the above, the K_n^- are obtained by rounding down to three digits the bound $2^{n/2}(2\pi)^{-3/2}$ from (3.9); the K_n^+ are obtained from upper approximation of the bound in (3.8), estimating the sup therein with the method sketched in Appendix, and described in full detail in [9]. The ratios K_n^-/K_n^+ are $0.376\ldots, 0.554\ldots, 0.573\ldots, 0.546\ldots, 0.326\ldots$ for $n = 2, 3, 4, 5, 10$, respectively. One can see that K_n^-/K_n^+ is smaller (i.e., that we have a larger uncertainty on the sharp constant K_n) in the extreme cases $n = 2$, $n = 10$; we presume this to happen, in any space dimension d, when n approaches the limit values $d/2$ and $+\infty$.

4. Proof of Propositions 3.1, 3.5 and 3.6

Lemma 4.1. *Let $p, q \in \mathbf{R}^d \setminus \{0\}$, $z \in \mathbf{C}^d$, $p \bullet z = 0$ and let $\vartheta(p, q) \equiv \vartheta \in [0, \pi]$ be the convex angle between q and p. Then*

$$|q \bullet z| \leqslant \sin\vartheta\, |q||z| = \frac{|p \wedge q|}{|p|}\, |z|. \tag{4.1}$$

Proof. We choose an orthonormal basis $(\eta_r)_{r=1,\ldots,d}$ of $\mathbf{R}^d$ so that q be a positive multiple of η_1, p be in the span of η_1, η_2 and $p \bullet \eta_2 \geqslant 0$; then $q = |q|\eta_1$ and $p = |p|(\cos\vartheta\ \eta_1 + \sin\vartheta\ \eta_2)$.

The $(d-1)$ vectors $(-\sin\vartheta\ \eta_1 + \cos\vartheta\ \eta_2, \eta_3, \ldots, \eta_d)$ clearly form an orthonormal basis for $\{p\}^\perp := \{z \in \mathbf{C}^d \mid p \bullet z = 0\}$; so, any $z \in \{p\}^\perp$ has a unique expansion $z = z^{(2)}(-\sin\vartheta\ \eta_1 + \cos\vartheta\ \eta_2) + z^{(3)}\eta_3 + \cdots + z^{(d)}\eta_d$, with $z^{(t)} \in \mathbf{C}$ for $t = 2, \ldots, d$.

From these representations for q and z we get $q \bullet z = -\sin\vartheta\ |q| z^{(2)}$, which implies $|q \bullet z| = \sin\vartheta\ |q||z^{(2)}| \leqslant \sin\vartheta\, |q||z|$. So, the inequality in (4.1) is proved; the subsequent equality in (4.1) follows from (3.6). □

Proof of Propositions 3.1 and 3.5. We choose $v \in \mathbb{H}^n_{\Sigma 0}$, $w \in \mathbb{H}^{n+1}_0$ and proceed in two steps; let us recall that $v \bullet \partial w$ has zero mean, see Lemma 2.1.

Step 1. *The Fourier coefficients of $v \bullet \partial w$, and some estimates for them*. First of all $(v \bullet \partial w)_0 = 0$. Moreover,

$$(v \bullet \partial w)_k = \frac{i}{(2\pi)^{d/2}} \sum_{h \in \mathbf{Z}^d_{0k}} [v_h \bullet\ (k-h)] w_{k-h} \quad \text{for } k \in \mathbf{Z}^d_0; \tag{4.2}$$

this follows from (2.7) taking into account that, in the sum therein, the term with $h = 0$ vanishes due to $v_0 = 0$, and the term with $h = k$ is zero for evident reasons. Let us consider the above term $v_h \bullet\ (k-h)$; we have $h \bullet v_h = 0$ due to the assumption $\operatorname{div} v = 0$, so we can apply Eq. (4.1) with $p = h$, $q = k - h$ and $z = v_h$ to infer

$$|v_h \bullet\ (k-h)| \leqslant \frac{|h \wedge (k-h)|}{|h|}|v_h| = \frac{|h \wedge k|}{|h|}|v_h| \tag{4.3}$$

(concerning the last passage, note that $h \wedge h = 0$). Eqs. (4.2) and (4.3) imply the following, for each $k \in \mathbf{Z}^d_0$:

$$|(v \bullet \partial w)_k| \leqslant \frac{1}{(2\pi)^{d/2}} \sum_{h \in \mathbf{Z}^d_{0k}} \frac{|h \wedge k|}{|h|}|v_h||w_{k-h}| = \frac{1}{(2\pi)^{d/2}} \sum_{h \in \mathbf{Z}^d_{0k}} \frac{|h \wedge k|}{|h|^{n+1}|k-h|^{n+1}} \Big(|h|^n|v_h||k-h|^{n+1}|w_{k-h}|\Big); \tag{4.4}$$

now, the Cauchy–Schwarz inequality $(\sum_h a_h b_h)^2 \leqslant (\sum_h a_h^2)(\sum_h b_h^2)$ (for real a_h, b_h) gives the following, for all $k \in \mathbf{Z}^d_0$:

$$\begin{aligned} &|(v \bullet \partial w)_k|^2 \leqslant \frac{1}{(2\pi)^d}\, \mathcal{C}_n(k)\mathcal{D}_n(k), \\ &\mathcal{C}_n(k) := \sum_{h \in \mathbf{Z}^d_{0k}} \frac{|h \wedge k|^2}{|h|^{2n+2}|k-h|^{2n+2}}, \qquad \mathcal{D}_n(k) := \sum_{h \in \mathbf{Z}^d_{0k}} |h|^{2n}|v_h|^2|k-h|^{2n+2}|w_{k-h}|^2 \end{aligned} \tag{4.5}$$

(in the definition of $\mathcal{D}_n(k)$ one can write as well $\sum_{h\in\mathbf{Z}_0^d}$, since the general term of the sum vanishes for $h = k$). We now multiply both sides of (4.5) by $|k|^{2n}$; it appears that $|k|^{2n}\mathcal{C}_n(k) = \mathcal{K}_n(k)$ with $\mathcal{K}_n(k)$ as in (3.7), so

$$|k|^{2n}|(v \bullet \partial w)_k|^2 \leqslant \frac{1}{(2\pi)^d}\mathcal{K}_n(k)\mathcal{D}_n(k). \tag{4.6}$$

Step 2. *Completing the proofs of* Propositions 3.1 and 3.5. Due to (4.6),

$$\sum_{k\in\mathbf{Z}_0^d}|k|^{2n}|(v \bullet \partial w)_k|^2 \leqslant \frac{1}{(2\pi)^d}\sum_{k\in\mathbf{Z}_0^d}\mathcal{K}_n(k)\mathcal{D}_n(k) \leqslant \frac{1}{(2\pi)^d}\Big(\sup_{k\in\mathbf{Z}_0^d}\mathcal{K}_n(k)\Big)\Bigg(\sum_{k\in\mathbf{Z}_0^d}\mathcal{D}_n(k)\Bigg) = (K_n^+)^2\sum_{k\in\mathbf{Z}_0^d}\mathcal{D}_n(k), \tag{4.7}$$

(where the last passage uses the definition of K_n^+ in (3.8)). On the other hand, the definition of $\mathcal{D}_n$ gives

$$\sum_{k\in\mathbf{Z}_0^d}\mathcal{D}_n(k) = \sum_{h\in\mathbf{Z}_0^d}|h|^{2n}|v_h|^2\sum_{k\in\mathbf{Z}_{0h}^d}|k-h|^{2(n+1)}|w_{k-h}|^2 = \Bigg(\sum_{h\in\mathbf{Z}_0^d}|h|^{2n}|v_h|^2\Bigg)\Bigg(\sum_{\ell\in\mathbf{Z}_{0h}^d}|\ell|^{2(n+1)}|w_\ell|^2\Bigg)$$
$$\leqslant \|v\|_n^2\,\|w\|_{n+1}^2 \tag{4.8}$$

(where the last inequality follows from the inclusion $\mathbf{Z}_{0h}^d \subset \mathbf{Z}_0^d$). Returning to (4.7), we obtain

$$\sum_{k\in\mathbf{Z}_0^d}|k|^{2n}|(v \bullet \partial w)_k|^2 \leqslant (K_n^+)^2\|v\|_n^2\,\|w\|_{n+1}^2. \tag{4.9}$$

We already know that $v \bullet \partial w$ has zero mean; Eq. (4.9) indicates the finiteness of $\sum_{k\in\mathbf{Z}_0^d}|k|^{2n}|(v \bullet \partial w)_k|^2$, so $v \bullet \partial w \in \mathbb{H}_0^n$. Eq. (4.9) also gives

$$\|v \bullet \partial w\|_n \leqslant K_n^+\|v\|_n\|w\|_{n+1}. \tag{4.10}$$

Now, we let (v, w) vary. The map $\mathbb{H}_{\Sigma 0}^n \times \mathbb{H}_0^{n+1} \to \mathbb{H}_0^n$, $(v, w) \mapsto v \bullet \partial w$ is clearly bilinear, and (4.10) indicates its continuity; so, Proposition 3.1 is proved. Eq. (4.10) indicates as well that the sharp constant K_n' in the inequality $\|v \bullet \partial w\|_n \leqslant K_n'\|v\|_n\|w\|_{n+1}$ fulfills $K_n' \leqslant K_n^+$, thus proving Eq. (3.8) and Proposition 3.5. □

Proof of Proposition 3.6. Let us consider any $v \in \mathbb{H}_{\Sigma 0}^n \setminus \{0\}$, $w \in \mathbb{H}_{\Sigma 0}^{n+1} \setminus \{0\}$, and note that (1.3) gives

$$K_n \geqslant \frac{\|\mathfrak{L}(v \bullet \partial w)\|_n}{\|v\|_n\|w\|_{n+1}}. \tag{4.11}$$

Hereafter we choose v, w with Fourier coefficients as below (δ denoting the usual Kronecker symbol):

$$v_k = A\delta_{k,a} + \overline{A}\delta_{k,-a}, \qquad w_k = B\delta_{k,b} + \overline{B}\delta_{k,-b}, \tag{4.12}$$

$$a := (1, 0, \dots, 0), \qquad b := (0, 1, 0, \dots, 0), \qquad A := (0, \alpha, \mathfrak{a}),$$
$$B := (\beta, 0, \mathfrak{b}), \quad \alpha, \beta \in \mathbf{C},\ \mathfrak{a}, \mathfrak{b} \in \mathbf{C}^{d-2},\ (\alpha, \mathfrak{a}), (\beta, \mathfrak{b}) \neq (0, 0). \tag{4.13}$$

The above choices of A, B ensure the conditions of zero divergence $A \bullet a = 0$, $B \bullet b = 0$; if $d = 2$, one understands $\mathfrak{a}, \mathfrak{b}$ to be missing from Eq. (4.13) (i.e., $A := (0, \alpha)$ and $B := (\beta, 0)$). As shown in [9], one has

$$\|v\|_n^2 = 2|A|^2 = 2(|\alpha|^2 + |\mathfrak{a}|^2), \qquad \|w\|_{n+1}^2 = 2|B|^2 = 2(|\beta|^2 + |\mathfrak{b}|^2),$$
$$\|\mathfrak{L}(v \bullet \partial w)\|_n^2 = \frac{2^{n+2}}{(2\pi)^d}|\alpha|^2\left((2-\sqrt{2})|\beta|^2 + |\mathfrak{b}|^2\right) \tag{4.14}$$

(the last result depending on an elementary, but tedious computation of $[\mathfrak{L}(v \bullet \partial w)]_k$ via Eq. (2.7)); Eqs. (4.11) and (4.14) give

$$K_n^2 \geqslant \frac{2^n}{(2\pi)^d}\frac{|\alpha|^2\left((2-\sqrt{2})|\beta|^2 + |\mathfrak{b}|^2\right)}{(|\alpha|^2 + |\mathfrak{a}|^2)(|\beta|^2 + |\mathfrak{b}|^2)}. \tag{4.15}$$

If $d = 2$, one understands $\mathfrak{a}, \mathfrak{b}$ to be missing from the above formula; so, (4.15) gives $K_n^2 \geqslant 2^n(2\pi)^{-2}(2-\sqrt{2})$, an inequality that coincides with (3.9) in this case.

If $d \geqslant 3$, we choose $(\alpha, \mathfrak{a}), (\beta, \mathfrak{b}) \neq (0, 0)$ so as to maximize the right hand side of Eq. (4.15). The maximum is attained with $\mathfrak{a} = 0$, $\beta = 0$ and arbitrary $\alpha \in \mathbf{C} \setminus \{0\}$, $\mathfrak{b} \in \mathbf{C}^{d-2} \setminus \{0\}$; this choice gives $K_n^2 \geqslant 2^n(2\pi)^{-d}$, yielding Eq. (3.9) for this case. □

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Appendix. The function $\mathcal{K}_n$

Hereafter we present a result allowing the practical computation of the function $\mathcal{K}_n$ in Eq. (3.7) and of its sup. The basic idea is to introduce a "cutoff" ρ and to replace the infinite sums in Eq. (3.7) by finite sums over balls of radius ρ in $\mathbf{Z}_0^d$; this gives an approximant $\mathscr{K}_n(k)$ of $\mathcal{K}_n(k)$, up to a k-independent error $\delta\mathscr{K}_n$. The behavior of $\mathscr{K}_n(k)$ for large k can be described as well; all these facts emerge from the following statement.

Proposition A.1. *Let us fix a cutoff $\rho \in (2\sqrt{d}, +\infty)$; then the following holds (with the functions and quantities $\mathscr{K}_n, \delta\mathscr{K}_n, \ldots$ mentioned in the sequel depending parametrically on d and ρ: $\mathscr{K}_n(k) \equiv \mathscr{K}_{nd}(\rho, k)$, $\delta\mathscr{K}_n \equiv \delta\mathscr{K}_{nd}(\rho), \ldots$).*

(i) *The function $\mathcal{K}_n$ can be evaluated using the inequalities*

$$\mathscr{K}_n(k) < \mathcal{K}_n(k) \leqslant \mathscr{K}_n(k) + \delta\mathscr{K}_n \quad \text{for all } k \in \mathbf{Z}_0^d. \tag{A.1}$$

Here

$$\mathscr{K}_n(k) := |k|^{2n} \sum_{h\in\mathbf{Z}^d_{0k},\,|h|<\rho \text{ or } |k-h|<\rho} \frac{|h\wedge k|^2}{|h|^{2n+2}|k-h|^{2n+2}}$$
$$= |k|^{2n} \sum_{h\in\mathbf{Z}^d_{0k},\,|h|<\rho} \big[1+\theta(|k-h|-\rho)\big] \frac{|h\wedge k|^2}{|h|^{2n+2}|k-h|^{2n+2}}, \tag{A.2}$$

where $\theta(z) := 1$ if $z \in [0, +\infty)$ and $\theta(z) := 0$ if $z \in (-\infty, 0)$; moreover,

$$\delta\mathscr{K}_n := \frac{2^{2n+3}\pi^{d/2}(n+1)^{n+1}}{\Gamma(d/2)(n+2)^{n+2}} \sum_{i=0}^{d-1} \binom{d-1}{i} \frac{d^{d/2-1/2-i/2}}{(2n-i-1)(\rho-2\sqrt{d})^{2n-i-1}}. \tag{A.3}$$

(ii) *One has*

$$\mathscr{K}_n(k) \to Z_n \quad \text{for } k \to \infty, \tag{A.4}$$

for a suitable constant $Z_n > 0$, defined hereafter. The approach of $\mathscr{K}_n(k)$ to this limit is described quantitatively as follows, for any $t \in \{2, 4, 6, \ldots\}$:

$$Z_n + \sum_{\ell=2,4,\ldots,t-2} \frac{\mathcal{Q}_{n\ell}(k/|k|)}{|k|^\ell} + \frac{v_{nt}}{|k|^t} \leqslant \mathscr{K}_n(k) \leqslant Z_n + \sum_{\ell=2,4,\ldots,t-2} \frac{\mathcal{Q}_{n\ell}(k/|k|)}{|k|^\ell} + \frac{V_{nt}}{|k|^t} \quad \text{for } k \in \mathbf{Z}_0^d,\ |k| \geqslant 2\rho. \tag{A.5}$$

Here: $\sum_{\ell=2,\ldots,t-2} \ldots := 0$ if $t = 2$; Z_n, v_{nt}, V_{nt} are nonnegative constants and $Q_{n\ell}$ are real polynomial functions on the $(d-1)$-dimensional unit spherical surface, defined as follows:

$$Z_n := 2\left(1-\frac{1}{d}\right) \sum_{h\in\mathbf{Z}_0^d,\,|h|<\rho} \frac{1}{|h|^{2n}}; \qquad v_{nt} := 2\mu_{nt} \sum_{h\in\mathbf{Z}_0^d,\,|h|<\rho} \frac{1}{|h|^{2n-t}}; \qquad V_{nt} := 2M_{nt} \sum_{h\in\mathbf{Z}_0^d,\,|h|<\rho} \frac{1}{|h|^{2n-t}}; \tag{A.6}$$

$$\mathcal{Q}_{n\ell} : \mathbf{S}^{d-1} \to \mathbf{R}, \qquad u \mapsto \mathcal{Q}_{n\ell}(u) := 2 \sum_{h\in\mathbf{Z}_0^d,\,|h|<\rho} \frac{\hat{E}_{n\ell}(u \bullet h/|h|)}{|h|^{2n-\ell}}. \tag{A.7}$$

To define μ_{nt}, M_{nt} and $\hat{E}_{n\ell}$ consider the Taylor expansion

$$\frac{1-c^2}{(1-2c\xi+\xi^2)^{n+1}} = \sum_{\ell=0}^{t-1} E_{n\ell}(c)\,\xi^\ell + R_{nt}(c,\xi)\,\xi^t \quad \text{for } c \in [-1, 1],\ \xi \in [0, 1/2], \tag{A.8}$$

where the remainder contains the continuous function $(c, \xi) \mapsto R_{nt}(c, \xi)$, and each coefficient $E_{n\ell}(c)$ is found to be a polynomial in c of degree $\ell + 2$ with the parity of ℓ. Then

$$\mu_{nt} := \min_{c\in[-1,1],\,\xi\in[0,1/2]} R_{nt}(c,\xi), \qquad M_{nt} := \max_{c\in[-1,1],\,\xi\in[0,1/2]} R_{nt}(c,\xi) \tag{A.9}$$

and, for any even ℓ,

$$\hat{E}_{n\ell}(c) := e_{n\ell 0} + \frac{e_{n\ell 2}}{d} + e_{n\ell 4}c^4 + \cdots + e_{n\ell,\ell+2}c^{\ell+2}$$
$$\textit{if } E_{n\ell}(c) = e_{n\ell 0} + e_{n\ell 2}c^2 + e_{n\ell 4}c^4 + \cdots + e_{n\ell,\ell+2}c^{\ell+2}. \tag{A.10}$$

(iii) *Items* (i)–(ii) *imply*

$$\sup_{k\in\mathbf{Z}_0^d} \mathscr{K}_n(k) < +\infty; \qquad \sup_{k\in\mathbf{Z}_0^d} \mathscr{K}_n(k) \leqslant \sup_{k\in\mathbf{Z}_0^d} \mathcal{K}_n(k) \leqslant \Big(\sup_{k\in\mathbf{Z}_0^d} \mathscr{K}_n(k)\Big) + \delta\mathscr{K}_n. \tag{A.11}$$

Proof. See [9], Appendices A and B. □

The upper bounds K_n^+ given in (3.10) for $d = 3$ and $n = 2, 3, 4, 5, 10$ have been derived from Eq. (3.8), using Proposition A.1 to estimate the function $\mathcal{K}_n$ and its sup; as for the cutoff, we have chosen $\rho = 20$ for $n = 2$, and $\rho = 10$ for $n = 3, 4, 5, 10$. We refer again to [9] for the details of these computations.

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