Note

On Multiply Critically $h$-Connected Graphs

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A conjecture of Slater states that $K_{h+1}$ is the unique $k$-critically $h$-connected noncomplete graph for $2k > h$. We prove here that there is no $k$-critically $h$-connected graph with order $\geq h + k - 2$ for $2k > h + 1$. We prove also that there is no $k$-critically $h$-connected line graph for $2k > h$. The last result was conjectured by Maurer and Slater. We apply in our proofs a method introduced by Mader.

1. Introduction

We use the terminology of Berge [1]. A graph $G = (V, E)$ is said to be $k$-critically $h$-connected (or simply an $(h, k)$-graph) if for every $S \subseteq V$ with $0 \leq |S| \leq k$, $\kappa(G_{V-S}) = h - |S|$. This notion is introduced by Maurer and Slater in [7]. The following conjecture is proposed [2, 7].

CONJECTURE (Slater). The complete graph $K_{h+1}$ is the unique $k$-critically $h$-connected graph for $2k > h$.

The above conjecture is verified for $h \leq 6$ by Maurer and Slater [7] and for $h \leq 10$ by Mader [5, 6].

We prove in this paper that there is no $k$-critically $h$-connected graph with order $\geq h + k - 2$ for $2k > h + 1$. This implies that a counterexample to Slater's conjecture, if it exists, must of relatively small order for even $h$. Our methods can be used to prove Slater's conjecture for even $h$ not exceeding 18.

We prove also that for $2k > h$, there is no $(h, k)$-noncomplete line graph, answering a conjecture of Maurer and Slater [8].
2. Atomic Sequence of a Graph

Let \( G = (V, E) \) be a connected noncomplete graph and \( F \) be a subset of \( V \). The subset \( F \) is said to be a fragment of \( G \) if \(|N(F)| = k(G)\) and \( \bar{F} \neq \emptyset \), where \( \bar{F} = V \setminus (F \cup \Gamma(F)) \) and \( N(F) = \Gamma(F) \setminus F \). We see easily that \( \bar{F} \) is also a fragment of \( G \). A fragment of minimal cardinality is called an atom. A fundamental property of atoms is the following.

**Theorem A (Mader [4]).** Let \( G \) be a connected noncomplete graph, \( A \) be an atom of \( G \) and \( T \) be a minimum cutset of \( G \). If \( A \cap T \neq \emptyset \), then \( A \subseteq T \) and \(|A| \leq \frac{1}{2}k(G)\).

Let \( G = (V, E) \) be a graph and \( \{A_i; 1 \leq i \leq r\} \) be a family of subsets of \( V \). We say that \( \{A_i; 1 \leq i \leq r\} \) is an atomic sequence of \( G \) if \( A_i \) is an atom of \( G_{V - A_{i+1}} \); \( 1 \leq i \leq r \). We can verify easily using Mader's Theorem A that every \( k \) critically \( h \)-connected noncomplete graph has an atomic sequence of length \( k + 1 \) (cf. [3]). We verify also that the elements of this atomic sequence are fragments of \( G \). Therefore \( |A_1| \leq |A_2| \leq \cdots \leq |A_r| \), for any atomic sequence \( \{A_i; 1 \leq i \leq r\} \). The proof of these two results is a direct application of the definitions and of Theorem A.

**Proposition 2.1.** Let \( F \) and \( F' \) be two fragments of a noncomplete connected graph \( G = (V, E) \). If \( G_{N(F) - N(F')} \) is connected, then \( F \cap N(F') = \emptyset \) or \( \bar{F} \cap N(F') = \emptyset \).

**Proof:** Put \( N(F) = T \) and \( N(F') = T' \). Let \( C \) be a connected component of \( G_{V - T} \), such that \( C \cap (T - T') = \emptyset \) (such a component exists since \( G_{T - T'} \) is a connected subgraph of \( G_{V - T} \)). Clearly \( G_C \) is a connected subgraph of \( G_{V - T} \). Hence \( C \subseteq F \) or \( C \subseteq \bar{F} \). In the case \( C \subseteq F \), we have \( T' = N(C) \subseteq F \cup N(F) \). Therefore \( T' \cap \bar{F} = \emptyset \). Similarly \( T' \cap F = \emptyset \) in the case \( C \subseteq \bar{F} \).

Let \( \{A_i; 1 \leq i \leq k\} \) be an atomic sequence of a graph \( G = (V, E) \). We take \( B_j = \bigcup_{i \leq j} A_i; 1 \leq j \leq k \).

**Lemma 2.2 [3].** Let \( G = (V, E) \) be a \( k \)-critically \( h \)-connected graph and \( \{A_i; 1 \leq i \leq k + 1\} \) be an atomic sequence of \( G \). Then \( N(A_i) \cap B_{i-1} \); \( 1 \leq i \leq k + 1 \). Moreover we have \( 2 |A_k| \leq h - |B_{k-1}| \).

The proof of this lemma is an application of Mader's Theorem A.

**Proposition 2.3.** Let \( G = (V, E) \) be a noncomplete \((h, k)\)-graph and \( \{A_i; 1 \leq i \leq k\} \) be an atomic sequence of \( G \). Then the subgraph spanned by \( B_j \) is \((j - 1)\)-connected; \( 1 \leq j \leq k \).

**Proof:** Suppose the proposition false and let \( j \) be the smallest value for which this occurs. We have \( j \geq 2 \), since an atom is connected (cf. Mader
$|4|$. Clearly $|B_j| \geq j$, hence $G_{B_j}$ has a cutset $S$ such that $|S| \leq j - 2$. We prove the following.

(1) $A_j \cap S \neq \emptyset$.

Suppose the contrary. Then $S \subset B_{j-1}$. But $G_{A_j}$ is connected (observe that $A_j$ is an atom of the graph spanned by $V - B_{j-1}$, but an atom is connected (Mader $|4|)$). Using Lemma 2.2 we have $N(A_j) \supset B_{j-1} \supset B_{j-1} - S$. The above two relations imply that the subgraph induced by $B_j = A_j \cup (B_{j-1} - S)$ is connected, which is a contradiction.

(2) By (1) and the minimality of $j$, the subgraph induced by $B_{j-1} - S$ is connected. It follows that $A_j$ contains one component $P$ of the subgraph spanned by $B_j - S$. Therefore $N(P) \subset S \cup (N(A_j) - B_{j-1})$. By Lemma 2.2, we have $B_{j-1} \subset N(A_j)$. It follows that $h \leq |N(P)| \leq |S| + h - |B_{j-1}|$. Hence $|S| > |B_{j-1}| > j - 1$. This contradiction proves the proposition.

**Theorem 2.4.** Let $k$ and $h$ be two natural numbers such that $k > \lfloor h/2 \rfloor$. Then there is no $k$-critically $h$-connected graph with order $\geq h + k - 2$.

**Proof.** Suppose the contrary and let $G = (V, E)$ be an $(h, k)$-graph such that $|V| \geq h + k - 2$. Consider an atomic sequence $\{A_i; 1 \leq i \leq k + 1\}$ of $G$. Put $T_i = N(A_i)$; $1 \leq i \leq k + 1$ and $R = T_{k+1} - B_k$. By Lemma 2.2, we have $|R| \leq h - k \leq k - 2$. Let $x \in A_{k+1}$ and $y \in A_{k+1}$. Therefore $|R \cup \{x, y\}| \leq k$. We prove the following.

(1) $|R \cup (V - T_{k+1})| \geq k$.

This is true if $|R| \geq 2$, as $|V| \geq h + k - 2$. Suppose $|R| \leq 1$. The last part of Lemma 2.2 can be written $|A_k| \leq |R|$. Hence $|A_k| = 1$. It follows that $|B_k| = k$. Therefore $|R| = h - k = 1$, which is a contradiction since $K_{k+1}$ is the unique $k$-critically $(k + 1)$-connected graph (cf. Maurer and Slater [7]).

(2) Let $S$ be a subset of $R \cup (V - T_{k+1})$ such that $|S| = k$ and $S \supset R \cup \{x, y\}$ (such a subset exists by (1) and the relation $|R \cup \{x, y\}| \leq k$). Let $T$ be a minimum cutset of $G$ containing $S$. By Proposition 2.1 and since $T \cap A_{k+1} \neq \emptyset$ and $T \cap A_{k+1} \neq \emptyset$, the subgraph induced by $T_{k+1} - T$ is not connected. But $T_{k+1} - T \subset B_k$. Using Proposition 2.3, we have $|T_{k+1} - T| \leq |B_k| - k + 1$. Therefore

$$k - |R| \leq |T - T_{k+1}| = |T_{k+1} - T| \leq |B_k| - k + 1 \quad (|T| = |T_{k+1}|).$$

It results that $2k - 1 \leq |B_k| + |R| = h$. Thus $k \leq \lfloor h/2 \rfloor$, which is a contradiction. This contradiction proves the theorem.

**Conjecture 2.5.** There is no noncomplete $(h, k)$-graph for $k > \lfloor h/2 \rfloor$. 
This conjecture is weaker than Slater’s conjecture for odd $h$. The two conjectures coincide for even $h$. We can see easily using Theorem 2.4, that conjecture 2.5 is false if and only if there is a $(2p, p+1)$-graph $G = (V, E)$ such that $2p + 4 \leq |V| \leq 3p - 2$ (take a counterexample of minimal cardinality and observe that the deletion of a vertex from a $k$-critically $h$-connected graph gives an $(h-1, k-1)$-graph; observe also that this counterexample must be of order not less than $h + 4$, otherwise $|A_{k+1}| = 1$ and hence $|R| = h - k$, using the notations of Theorem 2.4). In particular to prove Slater’s conjecture for even $h$, it is sufficient to prove it for a graph $G = (V, E)$ such that $h + 4 \leq |V| \leq \lfloor \frac{3h}{2} \rfloor - 2$.

Remark. The above methods can be adapted to prove Slater’s conjecture for even $h$ not exceeding 18. Such a proof contains a tedious examination of cases.

3. $k$-Critically $h$-Connected Line Graphs

Maurer and Slater showed in [8] that the connectivity of the line graph of a graph is related to its separation into nontrivial components. They formulate Slater’s conjecture for the case of a line graph. We will prove this using Proposition 2.1 and the following result.

**Theorem B (Entringer and Slater [2]).** Let $k$ and $h$ be two nonnull natural numbers such that $k > \lfloor h/2 \rfloor$. Then every $k$-critically $h$-connected graph contains a vertex of degree $h$.

We note that this theorem is a consequence of Lemma 2.2. We proved in [3] that there are at least two such vertices, answering a conjecture of Entringer and Slater [2].

**Lemma 3.1.** Let $H$ be a line graph and $x$ be a vertex of $H$. Then $N(x)$ can be covered by two cliques of $H$.

The proof of this lemma is easy.

**Theorem 3.2.** There is no $k$-critically $h$-connected line graph for $k > \lfloor h/2 \rfloor$.

**Proof.** Suppose the contrary and let $H$ be a $k$-critically $h$-connected line graph. By Theorem B, $H$ contains a vertex $x$ of degree $h$. By Lemma 3.1, $N(x)$ contains a clique of cardinality not less than $h/2$. Let $C$ be such a clique. We have $|N(x) - C| < k$. Let $T$ be a minimum cutset of $H$ containing $(N(x) - C) \cup \{x\}$. As $N(x) - T$ is connected (it is contained in $C$), we have $T \cap \{x\} = \emptyset$ (observe that $\{x\}$ is a fragment), using Proposition 2.1.
Therefore $N(x) - T$ consists of a unique vertex $c$. But $c$ is connected to each component of $H_{V - N(x)}$, where $V$ is the vertex-set of $H$. This contradicts the fact that $T$ is a cutset. This contradiction proves the theorem.

Remark. Theorem 3.2 is equivalent to conjecture 3.5 [8].

REFERENCES