Journal of Computational and Applied Mathematics 233 (2009) 793–801

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Rational interpolation to solutions of Riccati difference equations on elliptic lattices

Alphonse P. Magnus

Université catholique de Louvain, Institut mathématique, 2 Chemin du Cyclotron, B-1348 Louvain-La-Neuve, Belgium

ARTICLE INFO

Article history:
Received 7 November 2007

MSC:
42C05
41A20
39A70

Keywords:
Elliptic difference equations
Riccati difference equations
Orthogonal functions
Rational approximation

ABSTRACT

It is shown how to define difference equations on particular lattices $\{x_n\}$, $n \in \mathbb{Z}$, where the x_ns are values of an elliptic function at a sequence of arguments in arithmetic progression (*elliptic lattice*). Solutions to special difference equations (elliptic Riccati equations) have remarkable simple (!) interpolatory continued fraction expansions.

…opresivo y lento y plural.
J.L. Borges

1. Difference equations and lattices

Simplest difference equations relate two values of the unknown function f: say, $f(\varphi(x))$ and $f(\psi(x))$.

Most instances [1] are $(\varphi(x), \psi(x)) = (x, x+h)$, or the more symmetric $(x-h/2, x+h/2)$, or also (x, qx) in q-difference equations [2–4]. Recently, more complicated forms $(r(x)-\sqrt{s(x)}, r(x)+\sqrt{s(x)})$ have appeared [5,6,3,4,7–10], where r and s are rational functions.

This latter trend will be examined here: we need, for each x, two values $f(\varphi(x))$ and $f(\psi(x))$ for f.

A first-order difference equation is $\mathcal{F}(x, f(\varphi(x)), f(\psi(x))) = 0$, or $f(\varphi(x)) - f(\psi(x)) = \mathcal{G}(x, f(\varphi(x)), f(\psi(x)))$ if we want to emphasize the difference of f. There is of course some freedom in this latter writing. Only *symmetric* forms in φ and ψ will be considered here:

$$(\mathcal{D}f)(x) = \mathcal{F}(x, f(\varphi(x)), f(\psi(x))), \tag{1}$$

where $\mathcal{D}$ is the divided difference operator

$$(\mathcal{D}f)(x) = \frac{f(\psi(x)) - f(\varphi(x))}{\psi(x) - \varphi(x)}, \tag{2}$$

and where $\mathcal{F}$ is a symmetric function of its two last arguments.

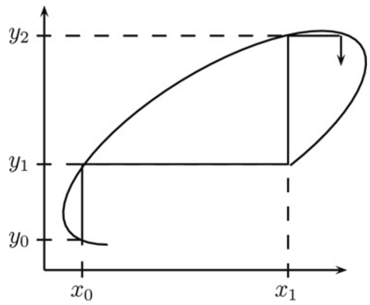
For instance, a linear difference equation of first order may be written as $a(x)f(\varphi(x)) + b(x)f(\psi(x)) + c(x) = 0$, as well as $\alpha(x)(\mathcal{D}f)(x) = \beta(x)[f(\varphi(x)) + f(\psi(x))] + \gamma(x)$, with $\alpha(x) = [b(x)-a(x)][\psi(x)-\varphi(x)]/2$, $\beta(x) = -[a(x)+b(x)]/2$, and $\gamma(x) = -c(x)$.

E-mail address: alphonse.magnus@uclouvain.be.

0377-0427/$ – see front matter
doi:10.1016/j.cam.2009.02.047

2. Elliptic grid, or lattice

2.1. Definition of elliptic grid



The simplest choice for φ and ψ is to take the two determinations of an algebraic function of degree 2, i.e., the two y-roots of

$$F(x,y) = X_0(x) + X_1(x)y + X_2(x)y^2 = 0, \tag{3a}$$

where X_0, X_1, and X_2 are rational functions.

Remark that the sum and the product of φ and ψ are the rational functions

$$\varphi + \psi = -X_1/X_2, \qquad \varphi\psi = X_0/X_2. \tag{3b}$$

When the divided difference operator $\mathcal{D}$ of (2) is applied to a rational function, the result is still a rational function. However, polynomials are normally not sent to polynomials, for instance,

$$\mathcal{D}x^2 = -X_1(x)/X_2(x),\ \mathcal{D}x^3 = (X_1^2(x) - X_0(x)X_2(x))/X_2^2(x).$$

But difference equations must allow the recovery of f on a whole set of points! An initial-value problem for a first-order difference equation starts with a value for $f(y_0)$ at $x = x_0$, where y_0 is one root of (3a) at $x = x_0$. The difference equation at $x = x_0$ relates then $f(y_0)$ to $f(y_1)$, where y_1 is the second root of (3a) at x_0. We need x_1 such that y_1 is one of the two roots of (3a) at x_1, so for one of the roots of $F(x, y_1) = 0$ which is not x_0. Here again, the simplest case is when F is of degree 2 in x:

$$F(x,y) = Y_0(y) + Y_1(y)x + Y_2(y)x^2 = 0. \tag{3c}$$

Both forms (3a) and (3c) hold simultaneously when F is *biquadratic*:

$$F(x,y) = \sum_{i=0}^{2}\sum_{j=0}^{2} c_{i,j}x^i y^j. \tag{4}$$

Definition. A sequence $\{\dots, x_{-1}, x_0, x_1, \dots\}$ is an elliptic lattice if there exists a sequence $\{\dots, y_{-1}, y_0, y_1, \dots\}$ and a biquadratic polynomial (4) such that $F(x_n, y_n) = 0$ and $F(x_n, y_{n+1}) = 0$, for $n \in \mathbb{Z}$.

As y_n and y_{n+1} are the two roots in t of $F(x_n, t) = X_0(x_n) + X_1(x_n)t + X_2(x_n)t^2 = 0$, useful identities are

$$y_n + y_{n+1} = -\frac{X_1(x_n)}{X_2(x_n)}, \qquad y_n y_{n+1} = \frac{X_0(x_n)}{X_2(x_n)}, \tag{5}$$

and the direct formula

$$y_n \quad \text{and} \quad y_{n+1} = \frac{-X_1(x_n) \pm \sqrt{P(x_n)}}{2X_2(x_n)}, \tag{6}$$

where

$$P = X_1^2 - 4X_0X_2 \tag{7}$$

is a polynomial of degree 4.

Also, as x_{n+1} and x_n are the two roots in t of $F(t, y_{n+1}) = 0$,

$$x_n + x_{n+1} = -\frac{Y_1(y_{n+1})}{Y_2(y_{n+1})}, \qquad x_n x_{n+1} = \frac{Y_0(y_{n+1})}{Y_2(y_{n+1})}. \tag{8}$$

Of course, the sequence $\{y_n\}$ is elliptic too.

Note that the names of the x- and y-lattices are sometimes inverted, as in [11, eq. (1.2)]

The construction above is called "T-algorithm" in [11, Theorem 6].

As the operators considered here are symmetric in $\varphi(x)$ and $\psi(x)$, we do not need to define precisely what φ and ψ are. However, once a starting point (x_0, y_0) is chosen, it will be convenient to define $\varphi(x_n) = y_n$ and $\psi(x_n) = y_{n+1}$, $n \in \mathbb{Z}$.

Special cases. We already encountered the usual difference operators $(\varphi(x), \psi(x)) = (x, x+h)$ or $(x-h, x)$ or $(x-h/2, x+h/2)$ corresponding to $X_2(x) \equiv 1$, X_1 of degree 1, X_0 of degree 2 with $P = X_1^2 - 4X_0X_2$ of degree 0. For the geometric difference operator, P is the square of a first degree polynomial. For the Askey–Wilson operator [5,6,12,3,7,8], P is an arbitrary second degree polynomial.

2.2. Equivalent definitions

The companion sequence $\{y_n\}$ is not needed in the definition of an elliptic lattice, but the definition above is best suited to the description of difference equations.

A relation involving only x_n and x_{n+1} is obtained by the elimination of y_{n+1} through the resultant of the two polynomials in y_{n+1} from (8) $P_1(y_{n+1}) = (x_n + x_{n+1})Y_2(y_{n+1}) + Y_1(y_{n+1})$ and $P_2(y_{n+1}) = x_nx_{n+1}Y_2(y_{n+1}) - Y_0(y_{n+1})$.

The form of this resultant is most easily found through interpolation at the two zeros u and v of Y_2: let $Y_2(y) = \alpha(y-u)(y-v)$, $Y_0(y) = \beta(y-u)(y-v) + \beta'(y-u) + \beta''(y-v)$, and $Y_1(y) = \gamma(y-u)(y-v) + \gamma'(y-u) + \gamma''(y-v)$, then, with $y = y_{n+1}$, $S = x_n + x_{n+1}$ and $\Pi = x_nx_{n+1}$,

$$S = -\frac{Y_1(y)}{Y_2(y)} = -\frac{\gamma(y-u)(y-v) + \gamma'(y-u) + \gamma''(y-v)}{\alpha(y-u)(y-v)},$$

$$\Pi = \frac{Y_0(y)}{Y_2(y)} = \frac{\beta(y-u)(y-v) + \beta'(y-u) + \beta''(y-v)}{\alpha(y-u)(y-v)},$$

$$\alpha S + \gamma = -\frac{\gamma'(y-u) + \gamma''(y-v)}{(y-u)(y-v)}, \qquad \alpha\Pi - \beta = \frac{\beta'(y-u) + \beta''(y-v)}{(y-u)(y-v)},$$

$$\beta'(\alpha S + \gamma) + \gamma'(\alpha\Pi - \beta) = \frac{\beta''\gamma' - \beta'\gamma''}{y-u},$$

$$\beta''(\alpha S + \gamma) + \gamma''(\alpha\Pi - \beta) = \frac{\beta'\gamma'' - \beta''\gamma'}{y-v},$$

and, eliminating y,

$$\frac{v-u}{\beta''\gamma' - \beta'\gamma''} = \frac{1}{\beta'(\alpha S + \gamma) + \gamma'(\alpha\Pi - \beta)} + \frac{1}{\beta''(\alpha S + \gamma) + \gamma''(\alpha\Pi - \beta)} \tag{9}$$

which leads clearly to a polynomial of degree 2 in $x_n + x_{n+1}$ and x_nx_{n+1}, so

2.2.1. Definition 2

An elliptic lattice, or grid, is a sequence satisfying a symmetric biquadratic relation [11, Theorem 5]

$$E(x_n, x_{n+1}) = d_{0,0} + d_{0,1}(x_n + x_{n+1}) + d_{0,2}(x_n + x_{n+1})^2 + d_{1,1}x_nx_{n+1} + d_{1,2}x_nx_{n+1}(x_n + x_{n+1}) + d_{2,2}x_n^2x_{n+1}^2 = 0. \tag{10}$$

Conversely, let us show that Definition 2 implies the main definition: from a sequence $\{x_n\}$ satisfying (10), let us build a valid sequence $\{y_n\}$. We must construct $F(x, y) = \alpha(y-u)(y-v)x^2 + [\gamma(y-u)(y-v) + \gamma'(y-u) + \gamma''(y-v)]x + \beta(y-u)(y-v) + \beta'(y-u) + \beta''(y-v)$ such that the resultant (9) is deduced from (10) when $S = x_n + x_{n+1}$ and $\Pi = x_nx_{n+1}$, i.e.,

$$d_{0,0} + d_{0,1}S + d_{0,2}S^2 + d_{1,1}\Pi + d_{1,2}\Pi S + d_{2,2}\Pi^2 = 0. \tag{11}$$

Let us decide that $\alpha = 1$. Then, one chooses $(-\gamma, \beta) = (S, \Pi)$ as a point[1] on the conic (11). We now have

$$d_{0,2}(S+\gamma)^2 + d_{1,2}(\Pi - \beta)(S+\gamma) + d_{2,2}(\Pi - \beta)^2 + d'_{0,1}(S+\gamma) + d'_{1,1}(\Pi - \beta) = 0,$$

with $d'_{0,1} = -2d_{0,2}\gamma + d_{1,2}\beta + d_{0,1}$ and $d'_{1,1} = -d_{1,2}\gamma + 2d_{2,2}\beta + d_{1,1}$. Let ρ_1 and ρ_2 be the two roots of $d_{2,2}\rho^2 + d_{1,2}\rho + d_{0,2} = 0$, then, $d_{2,2}[\Pi - \beta - \rho_1(S+\gamma)][\Pi - \beta - \rho_2(S+\gamma)] + d'_{0,1}(S+\gamma) + d'_{1,1}(\Pi - \beta) = 0$, and we divide by $[\Pi - \beta - \rho_1(S+\gamma)][\Pi - \beta - \rho_2(S+\gamma)]$:

$$d_{2,2} + \frac{\mu}{\Pi - \beta - \rho_1(S+\gamma)} + \frac{\eta}{\Pi - \beta - \rho_2(S+\gamma)} = 0,$$

with $\mu = (d'_{1,1}\rho_1 + d'_{0,1})/(\rho_1 - \rho_2)$ and $\eta = (d'_{1,1}\rho_2 + d'_{0,1})/(\rho_2 - \rho_1)$. We now compare with (9):

$$\gamma' = \frac{\eta(v-u)}{(\rho_2 - \rho_1)d_{2,2}}, \qquad \gamma'' = \frac{\mu(v-u)}{(\rho_2 - \rho_1)d_{2,2}}, \qquad \beta' = -\rho_1\gamma', \qquad \beta'' = -\rho_2\gamma''.$$

The degrees of freedom are therefore u, v, and $(-\gamma, \beta) = (S, \Pi)$ on the conic (11).

---

1 The importance of this conic has been stressed in [13].

2.2.2. Definition 3

An elliptic lattice is a sequence $x_n = \mathcal{E}(nh + u_0)$*, where* $\mathcal{E}$ *is any elliptic function of order* 2 *(i.e., with* 2 *zeros and* 2 *poles in a fundamental parallelogram of periods).*

From the main definition, one may establish that the biquadratic curve $F(x, y) = 0$ in (4) has genus 1 and a parametric representation

$$x = \mathcal{E}_1(s), \qquad y = \mathcal{E}_2(s),$$

with $\mathcal{E}_1$ and $\mathcal{E}_2$ elliptic functions of order 2.

Indeed, a birational transformation $(x, y) \leftrightarrow (\xi, \eta)$ sending the biquadratic curve (4) $F(x, y) = 0$ to the canonical elliptic curve $\eta^2 = Q(\xi)$, where Q is a polynomial of third degree, see [14, p.292]: from (6), choose w = a square root of $P(x)$, so that $y = (-X_1(x) + w)/(2X_2(x)) \leftrightarrow w = X_1(x) + 2yX_2(x)$, and $x = z_1 + 1/\xi$, where z_1 is one of the four roots of $P(x) = 0$. Then, with $P(z_1 + 1/\xi) = Q(\xi)/\xi^4$ and $\eta = w\xi^2$, $\eta^2 = Q(\xi)$.

Then, the Weierstrass representation holds $\xi = \wp(hu+u_0)$, $\eta = \wp'(hu+u_0)$. So, $x = z_1+1/\wp$, $y = (-X_1+\wp'/\wp^2)/(2X_2)$.

However, the authors of [15] recommend the biquadratic setting instead of the more familiar cubic one, see [15, pp. 300–301].

Now, let s_n and s'_n correspond to the two points (x_n, y_n) and (x_n, y_{n+1}). As $\mathcal{E}_1(s_n) = \mathcal{E}_1(s'_n)$ with y_{n+1} normally different from y_n, $s_n + s'_n$ = a constant, say γ_1 (as s_n and s'_n are integrals involving the square root of a polynomial on two paths with the same endpoints [the second endpoint being x_n], the square roots are opposite on a part of the paths). Similarly, $s'_n + s_{n+1}$ = another constant, say γ_2. Therefore, $s_{n+1} = s_n + h$, with $h = \gamma_2 - \gamma_1$, and this establishes Definition 3 with $\mathcal{E} = \mathcal{E}_1$.

Conversely, from Definition 3, one recovers Definition 2 by recognizing (10) as an addition formula for elliptic functions [11].

The essential parameters in the description of an elliptic sequence are the modulus k and the step h. The modulus is also related to the ratio ω_1/ω_2 of periods. Finally, in a multiplicative setting, the main parameters are the nome p and the multiplier q, which are basically (i.e., up to multiplication by constants) the exponentials of the periods' ratio and the step.

The modulus and the step depend only on F in (4) (or E in (10)). For each starting point (x_0, y_0), or $s_0 = h_0$, there is a different elliptic lattice with the same k and h.

It is always possible to relate $\mathcal{E}_2$ to $\mathcal{E}_1$ through a rational transformation of first degree $\mathcal{E}_2(s) = \frac{\alpha\mathcal{E}_1(s+h/2)+\beta}{\gamma\mathcal{E}_1(s+h/2)+\delta}$ [11, p. 298].

2.3. A brief history [16,17,15,11]

Elliptic lattices were developed by Baxter in the solution of special problems of statistical physics, they appear in works by Fritz John, in many treatments of a Poncelet problem [16,17] [11, Section 6], and go back to pioneering work by Euler[2] on the addition formulas of elliptic functions, that is why the symmetric biquadratic polynomial (10) has been called the Euler polynomial in [11, p. 294].

Even the name of our subject is not easy to choose: "elliptic sequences" seems perfect, but this name is used by other sequences related in another way to elliptic functions (sequences $\{A_n\}$ where $A_{n-1}A_{n+1}/A_n^2$ is our x_n, [18]), "elliptic lattice" may be used for the repetitions of the periods' parallelogram of an elliptic function, "elliptic grid" means a convenient mesh for discretizing over ellipses, and "elliptic difference operator" is a partial difference operator extending partial differential operator of elliptic type.

3. Elliptic Pearson equation

A famous theorem by Pearson [19, (2.25) p.152]; [20] relates the classical orthogonal polynomials to the differential equation $w' = rw$ satisfied by the weight function, where r is a rational function of degree $\leqslant 2$.

Even without this constraint on the degree, the Stieltjes transform $f(z) = \int_a^b \frac{w(t)\mathrm{d}t}{z-t}$ of w satisfies the differential equation $f'(z) = r(z)f(z)+s(z)$, where s is a rational function[3] too. A suitable continued fraction expansion of f leaves then important information on the relevant orthogonal polynomials (theory of Laguerre [21]).

The Pearson equation has of course been extended to various difference calculus settings [5,12,22,9,10], here is the elliptic version:

---

2 and perhaps even to Fermat [communicated by R. Askey]!

3 $s(z) = D(z)/A(z)$, where the polynomials A and D are related to boundary conditions for the weight function w at a and b: the product $Ar = C$ must be a polynomial and $A(t)w(t)$ must vanish at $t = a$ and $t = b$. Then $D(z) = -\int_a^b \left[\frac{A(z)-A(t)-(z-t)A'(t)}{(z-t)^2} + \frac{C(z)-C(t)}{z-t}\right] w(t)\mathrm{d}t$. So, even in the Legendre case, where $w(t) \equiv 1$, $r = C = 0$, one must take $A(z) = 1 - z^2$, and $D(z) = 2$ follows.

3.1. Theorem

Let (x'_0, y'_0) be a point on the biquadratic curve $F = 0$ of (4), $\{(x'_k, y'_k)\}$ the elliptic lattice starting from this point. If there are polynomials a and c, with

$$a(x'_0) - (y'_1 - y'_0)c(x'_0) = a(x'_N) + (y'_{N+1} - y'_N)c(x'_N) = 0, \tag{12}$$

and a sequence $\{w_0, \ldots, w_{N+1}\}$ such that

$$a(x'_k)\frac{\frac{w_{k+1}}{Y_2(y'_{k+1})(x'_{k+1}-x'_k)} - \frac{w_k}{Y_2(y'_k)(x'_k-x'_{k-1})}}{y'_{k+1} - y'_k} = c(x'_k)\left[\frac{w_{k+1}}{Y_2(y'_{k+1})(x'_{k+1} - x'_k)} + \frac{w_k}{Y_2(y'_k)(x'_k - x'_{k-1})}\right], \tag{13}$$

$k = 0, 1, \ldots, N$, and $w_0 = w_{N+1} = 0$, then,

$$f(x) = \sum_{k=1}^{N} \frac{w_k}{x - y'_k} \tag{14}$$

satisfies

$$a(x)\mathcal{D}f(x) = a(x)\frac{f(\psi(x)) - f(\varphi(x))}{\psi(x) - \varphi(x)} = c(x)[f(\varphi(x)) + f(\psi(x))] + d(x), \tag{15}$$

where d is a polynomial too.

We already used the writing $(x_n, y_n) = (\mathcal{E}_1(s_0 + nh), \mathcal{E}_2(s_0 + nh))$ for a generic elliptic lattice, normally to be used as interpolation points. We here need a function with poles on another lattice with same modulus and step, but with another starting point, and it is written here $(x'_n, y'_n) = (\mathcal{E}_1(s'_0 + nh), \mathcal{E}_2(s'_0 + nh))$.

Remark that (13) is a recurrence relation for the $\tilde{w}_k = \frac{w_k}{Y_2(y'_k)(x'_k-x'_{k-1})}$, which is

$$[a(x'_k) - (y'_{k+1} - y'_k)c(x'_k)]\tilde{w}_{k+1} = [a(x'_k) + (y'_{k+1} - y'_k)c(x'_k)]\tilde{w}_k, \tag{16}$$

$k = 0, \ldots, N$, so that (12) ensures the boundary conditions $w_0 = w_{N+1} = 0$.

Proof.

$$\begin{aligned}\frac{f(\psi(x)) - f(\varphi(x))}{\psi(x) - \varphi(x)} &= -\sum_1^N \frac{w_k}{(\varphi(x) - y'_k)(\psi(x) - y'_k)} = -\sum_1^N \frac{w_k X_2(x)}{F(x, y'_k)} \\ &= -\sum_1^N \frac{w_k X_2(x)}{Y_2(y'_k)(x - x'_{k-1})(x - x'_k)} = X_2(x)\sum_0^N \frac{\tilde{w}_{k+1} - \tilde{w}_k}{x - x'_k}\end{aligned}$$

where $\tilde{w}_k = \frac{w_k}{Y_2(y'_k)(x'_k-x'_{k-1})}$, and with $w_0 = w_{N+1} = 0$,

$$\begin{aligned}f(\psi(x)) + f(\varphi(x)) &= -\sum_1^N \frac{w_k[X_1(x) + 2y'_kX_2(x)]}{X_2(x)(\varphi(x) - y'_k)(\psi(x) - y'_k)} = -\sum_1^N \frac{w_k[X_1(x) + 2y'_kX_2(x)]}{F(x, y'_k)} \\ &= -\sum_1^N \frac{w_k[X_1(x) + 2y'_kX_2(x)]}{Y_2(y'_k)(x - x'_{k-1})(x - x'_k)} = \sum_0^N \frac{\tilde{w}_{k+1}[X_1(x) + 2y'_{k+1}X_2(x)] - \tilde{w}_k[X_1(x) + 2y'_kX_2(x)]}{x - x'_k},\end{aligned}$$

therefore the rational functions $a\mathcal{D}f$ and $c(f(\varphi) + f(\psi))$ differ by a polynomial if all the residues are equal: $a(x'_k)X_2(x'_k)(\tilde{w}_{k+1} - \tilde{w}_k) = c(x'_k)(\tilde{w}_{k+1}[X_1(x'_k) + 2y'_{k+1}X_2(x'_k)] - \tilde{w}_k[X_1(x'_k) + 2y'_kX_2(x'_k)])$ for $k = 0, 1, \ldots, N$. Or, as $X_1(x'_k) = -(y'_k + y'_{k+1})X_2(x'_k)$, $a(x'_k)(\tilde{w}_{k+1} - \tilde{w}_k) = c(x'_k)(y'_{k+1} - y'_k)(\tilde{w}_{k+1} + \tilde{w}_k)$, which is exactly (16). □

4. Interpolatory continued fraction expansion and Riccati equations

Let $f_0(x) = f(x) - f(y_0)$ be expanded in an interpolatory continued fraction (R_{II}-fraction [12,23–25], or contracted Thiele's continued fraction [1, Chap. 5])

$$f_0(x) = \cfrac{x - y_0}{\alpha_0 x + \beta_0 - \cfrac{(x-y_1)(x-y_2)}{\ddots \over \alpha_{n-2}x+\beta_{n-2}+\cfrac{(x-y_{2n-3})(x-y_{2n-2})}{\alpha_{n-1}x+\beta_{n-1}+\cdots}}}$$

making clear that the nth approximant (stopped at, and including the $\alpha_{n-1}x + \beta_{n-1}$ term) is the rational function of degree n interpolating f_0 at $x = y_0, y_1, \ldots, y_{2n}$.

Let p_n be the denominator of this nth approximant. If f is a Stieltjes transform, it is known [23,24] [25,26, Section 5] that $\{p_n(x)/(x-y_0)(x-y_2)\cdots(x-y_{2n})\}$ and $\{p_m(x)/(x-y_1)(x-y_3)\cdots(x-y_{2m+1})\}$ are biorthogonal sequences of rational functions.

From $f_n(x)=\frac{x-y_{2n}}{\alpha_n x+\beta_n-(x-y_{2n+1})f_{n+1}(x)}$, $\alpha_n x+\beta_n$ is the polynomial interpolant of degree 1 to $(x-y_{2n})/f_n(x)$ at y_{2n+1} and y_{2n+2}, so we need $f_n(y_{2n+1})$ and $f_n(y_{2n+2})$ in order to find α_n and β_n.

If f_n satisfies a difference equation of first order $\mathcal{F}_n(x, f_n(\varphi(x)), f_n(\psi(x)))=0$, we find $f_n(y_{2n+1})$ from the equation at x_{2n}, as $\varphi(x_{2n})=y_{2n}$, $\psi(x_{2n})=y_{2n+1}$, and $f_n(y_{2n})=0$. Next, the equation at $x=x_{2n+1}$ yields $f_n(y_{2n+2})$.

As seen in Section 3 on the Pearson equation, a linear difference equation of first order easily involves a weight function useful in orthogonality considerations. However, Riccati equations are better suited to continued fraction constructions [2,27]. Of course, linear difference equations of first order are special cases of Riccati equations, that is why the coefficients in (15) are written $a(x)$, $c(x)$, and $d(x)$, whereas $b(x)$ is the coefficient of the nonlinear part of a Riccati equation.

So, if f_n satisfies the Riccati equation

$$a_n(x)\frac{f_n(\psi(x))-f_n(\varphi(x))}{\psi(x)-\varphi(x)}=b_n(x)f_n(\varphi(x))f_n(\psi(x))+c_n(x)(f_n(\varphi(x))+f_n(\psi(x)))+d_n(x), \tag{17}$$

one finds at $x=x_{2n}$, $\varphi(x)=y_{2n}$, $\psi(x)=y_{2n+1}$, and knowing that $f_n(y_{2n})=0$,

$$a_n(x_{2n})\frac{f_n(y_{2n+1})}{y_{2n+1}-y_{2n}}=c_n(x_{2n})f_n(y_{2n+1})+d_n(x_{2n})$$

yields

$$f_n(y_{2n+1})=\frac{d_n(x_{2n})}{\frac{a_n(x_{2n})}{y_{2n+1}-y_{2n}}-c_n(x_{2n})}$$

and

$$\alpha_n y_{2n+1}+\beta_n=\frac{y_{2n+1}-y_{2n}}{f_n(y_{2n+1})}=\frac{a_n(x_{2n})-(y_{2n+1}-y_{2n})c_n(x_{2n})}{d_n(x_{2n})}, \tag{18}$$

and at

$$\begin{aligned}&x=x_{2n+1},\\ &a_n(x_{2n+1})\frac{\frac{y_{2n+2}-y_{2n}}{\alpha_n y_{2n+2}+\beta_n}-\frac{y_{2n+1}-y_{2n}}{\alpha_n y_{2n+1}+\beta_n}}{y_{2n+2}-y_{2n+1}}=b_n(x_{2n+1})\frac{y_{2n+2}-y_{2n}}{\alpha_n y_{2n+2}+\beta_n}\frac{y_{2n+1}-y_{2n}}{\alpha_n y_{2n+1}+\beta_n}\\ &\qquad+c_n(x_{2n+1})\left[\frac{y_{2n+2}-y_{2n}}{\alpha_n y_{2n+2}+\beta_n}+\frac{y_{2n+1}-y_{2n}}{\alpha_n y_{2n+1}+\beta_n}\right]+d_n(x_{2n+1}),\end{aligned}$$

or

$$\begin{aligned}a_n(x_{2n+1})(\alpha_n y_{2n}+\beta_n)&=b_n(x_{2n+1})(y_{2n+2}-y_{2n})(y_{2n+1}-y_{2n})+c_n(x_{2n+1})[(y_{2n+2}-y_{2n})(\alpha_n y_{2n+1}+\beta_n)\\ &\quad+(y_{2n+1}-y_{2n})(\alpha_n y_{2n+2}+\beta_n)]+d_n(x_{2n+1})(\alpha_n y_{2n+1}+\beta_n)(\alpha_n y_{2n+2}+\beta_n).\end{aligned} \tag{19}$$

which shows how to extract α_n and β_n from $a_n,\ldots$ at x_{2n} and x_{2n+1}.

Remark also that at $x=x_{2n-1}$, knowing that $f_n(\psi(x_{2n-1}))=f_n(y_{2n})=0$, (17) yields

$$\left[\frac{a_n(x_{2n-1})}{y_{2n}-y_{2n-1}}+c_n(x_{2n-1})\right]f_n(y_{2n-1})+d_n(x_{2n-1})=0. \tag{20}$$

And here is how the *Riccati* form is well suited to continued fraction progression:

4.1. Theorem

If f_n satisfies the Riccati equation (17) with rational coefficients a_n, b_n, c_n, and d_n, and if $f_n(x)=\frac{x-y_{2n}}{\alpha_n x+\beta_n-(x-y_{2n+1})f_{n+1}(x)}$, then f_{n+1} satisfies an equation of same complexity (degree of the rational functions) of its coefficients.

(Actually, the degrees of a_n, etc. will at most exceed the degrees at $n=0$ by 3 units).

Proof. Let us start with (17) at $n=0$ with polynomial coefficients a_0, b_0, c_0, and d_0. Suppose that, at the nth step, a_n, etc. are polynomials with b_n and d_n containing the factors X_2 and $x-x_{2n-1}$, and c_n containing the factor X_2 (from (3a) and (4), X_2 is a polynomial of degree $\leqslant 2$).

Of course, if the initial coefficients a_0, etc. do not contain such factors, we may have to multiply the four coefficients of (17) at $n=0$ by one or several factors of $(x-x_{-1})X_2(x)$, that is why the degrees are liable to have to be augmented by up to 3 units, but this operation has to be done only at $n=0$.

We suppose therefore that $b_n(x) = (x - x_{2n-1})X_2(x)\tilde{b}_n(x)$, $c_n(x) = X_2(x)\tilde{c}_n(x)$, and $d_n(x) = (x - x_{2n-1})X_2(x)\tilde{d}_n(x)$ in (17), where $\tilde{b}_n$, $\tilde{c}_n$, and $\tilde{d}_n$ are polynomials, so that (17) is now

$$a_n(x)\frac{f_n(\psi(x)) - f_n(\varphi(x))}{\psi(x) - \varphi(x)} = (x - x_{2n-1})X_2(x)\tilde{b}_n(x)f_n(\varphi(x))f_n(\psi(x)) + X_2(x)\tilde{c}_n(x)(f_n(\varphi(x)) + f_n(\psi(x))) + (x - x_{2n-1})X_2(x)\tilde{d}_n(x), \tag{21}$$

in which we enter $f_n(x) = \frac{x - y_{2n}}{\alpha_n x + \beta_n - (x - y_{2n+1})f_{n+1}(x)}$.

After multiplication by $(\alpha_n\varphi + \beta_n - (\varphi - y_{2n+1})f_{n+1}(\varphi))(\alpha_n\psi + \beta_n - (\psi - y_{2n+1})f_{n+1}(\psi))$:

$$\begin{aligned}
&a_n\frac{(\psi - y_{2n})[\alpha_n\varphi + \beta_n - (\varphi - y_{2n+1})f_{n+1}(\varphi)] - (\varphi - y_{2n})[\alpha_n\psi + \beta_n - (\psi - y_{2n+1})f_{n+1}(\psi)]}{\psi - \varphi}\\
&= a_n\left[\alpha_n y_{2n} + \beta_n + \left(\varphi\psi - \frac{(y_{2n} + y_{2n+1})(\varphi + \psi)}{2} + y_{2n}y_{2n+1}\right)\frac{f_{n+1}(\psi) - f_{n+1}(\varphi)}{\psi - \varphi}\right.\\
&\quad\left. + \frac{y_{2n+1} - y_{2n}}{2}(f_{n+1}(\varphi) + f_{n+1}(\psi))\right]\\
&= (x - x_{2n-1})X_2\tilde{b}_n(\varphi - y_{2n})(\psi - y_{2n}) + X_2\tilde{c}_n[(\psi - y_{2n})[\alpha_n\varphi + \beta_n - (\varphi - y_{2n+1})f_{n+1}(\varphi)]\\
&\quad + (\varphi - y_{2n})[\alpha_n\psi + \beta_n - (\psi - y_{2n+1})f_{n+1}(\psi)]]\\
&\quad + (x - x_{2n-1})X_2\tilde{d}_n[\alpha_n\varphi + \beta_n - (\varphi - y_{2n+1})f_{n+1}(\varphi)][\alpha_n\psi + \beta_n - (\psi - y_{2n+1})f_{n+1}(\psi)]
\end{aligned}$$

which is the Riccati equation for f_{n+1}, $\hat{a}_{n+1}\frac{f_{n+1}(\psi) - f_{n+1}(\varphi)}{\psi - \varphi} = \hat{b}_{n+1}f_{n+1}(\varphi)f_{n+1}(\psi) + \hat{c}_{n+1}(f_{n+1}(\varphi) + f_{n+1}(\psi)) + \hat{d}_{n+1}$, where $\hat{a}_{n+1}$ etc. are symmetric functions of φ and ψ, so are rational functions thanks to (3b):

$$\begin{aligned}
\hat{a}_{n+1} &= \left(\varphi\psi - \frac{(y_{2n} + y_{2n+1})(\varphi + \psi)}{2} + y_{2n}y_{2n+1}\right)a_n + \frac{y_{2n+1} - y_{2n}}{2}(\psi - \varphi)^2X_2\tilde{c}_n\\
&\quad + \frac{\alpha_n y_{2n+1} + \beta_n}{2}(\psi - \varphi)^2(x - x_{2n-1})X_2\tilde{d}_n = \frac{\left(X_0 + \frac{(y_{2n} + y_{2n+1})X_1}{2} + y_{2n}y_{2n+1}X_2\right)}{X_2}a_n\\
&\quad + \frac{[y_{2n+1} - y_{2n}]\tilde{c}_n + (\alpha_n y_{2n+1} + \beta_n)(x - x_{2n-1})\tilde{d}_n}{2}\frac{X_1^2 - 4X_0X_2}{X_2}
\end{aligned} \tag{22a}$$

$$\hat{b}_{n+1} = (x - x_{2n-1})X_2\tilde{d}_n(\varphi - y_{2n+1})(\psi - y_{2n+1}) = (x - x_{2n-1})\tilde{d}_n(X_0 + y_{2n+1}X_1 + y_{2n+1}^2X_2) \tag{22b}$$

$$\begin{aligned}
\hat{c}_{n+1} &= -(y_{2n+1} - y_{2n})a_n - \left(\varphi\psi - \frac{(y_{2n} + y_{2n+1})(\varphi + \psi)}{2} + y_{2n}y_{2n+1}\right)X_2\tilde{c}_n\\
&\quad - \left(\alpha_n\left(\varphi\psi - y_{2n+1}\frac{\varphi + \psi}{2}\right) + \beta_n\left(\frac{\varphi + \psi}{2} - y_{2n+1}\right)\right)(x - x_{2n-1})X_2\tilde{d}_n\\
&= -(y_{2n+1} - y_{2n})a_n - \left(X_0 + \frac{(y_{2n} + y_{2n+1})X_1}{2} + y_{2n}y_{2n+1}X_2\right)\tilde{c}_n\\
&\quad - \left(\alpha_n\left(X_0 + y_{2n+1}\frac{X_1}{2}\right) - \beta_n\left(\frac{X_1}{2} + y_{2n+1}X_2\right)\right)(x - x_{2n-1})\tilde{d}_n
\end{aligned} \tag{22c}$$

$$\begin{aligned}
\hat{d}_{n+1} &= -(\alpha_n y_{2n} + \beta_n)a_n + (x - x_{2n-1})X_2\tilde{b}_n(\varphi - y_{2n})(\psi - y_{2n})\\
&\quad + X_2\tilde{c}_n[(\psi - y_{2n})(\alpha_n\varphi + \beta_n) + (\varphi - y_{2n})(\alpha_n\psi + \beta_n)] + (x - x_{2n-1})X_2\tilde{d}_n(\alpha_n\varphi + \beta_n)(\alpha_n\psi + \beta_n)\\
&= -(\alpha_n y_{2n} + \beta_n)a_n + (x - x_{2n-1})\tilde{b}_n(X_0 + y_{2n}X_1 + y_{2n}^2X_2)\\
&\quad + \tilde{c}_n[\alpha_n(2X_0 + y_{2n}X_1) - \beta_n(X_1 + 2y_{2n}X_2)] + (x - x_{2n-1})\tilde{d}_n(\alpha_n^2X_0 - \alpha_n\beta_nX_1 + \beta_n^2X_2).
\end{aligned} \tag{22d}$$

The first coefficient $\hat{a}_{n+1}$ is not a polynomial, but a rational function of denominator X_2. We recover polynomials by multiplying the four coefficients $\hat{a}_{n+1}$, $\hat{b}_{n+1}$, $\hat{c}_{n+1}$, and $\hat{d}_{n+1}$ by X_2. Moreover, this already restores the factor X_2 in $X_2\hat{b}_{n+1}$, $X_2\hat{c}_{n+1}$, and $X_2\hat{d}_{n+1}$!

However, the degrees of the new coefficients are higher than before. The problem is settled by seeing that the four polynomials $X_2\hat{a}_{n+1}$, $\hat{b}_{n+1}$, $\hat{c}_{n+1}$, and $\hat{d}_{n+1}$ vanish at $x = x_{2n-1}$ and at $x = x_{2n}$. Then, we will simply divide the four of them by $(x - x_{2n-1})(x - x_{2n})$.

(1) The most obvious case is (22b): $\hat{b}_{n+1}(x) = (x - x_{2n-1})\tilde{d}_n(x)(X_0(x) + y_{2n+1}X_1(x) + y_{2n+1}^2X_2(x)) = (x - x_{2n-1})\tilde{d}_n(x)F(x, y_{2n+1}) = (x - x_{2n-1})\tilde{d}_n(x)Y_2(y_{2n+1})(x - x_{2n})(x - x_{2n+1})$, shows indeed the factors $x - x_{2n-1}$ and $x - x_{2n}$,

as well as $x - x_{2n+1}$, so that

$$\tilde{b}_{n+1}(x) = \frac{\hat{b}_{n+1}(x)}{(x - x_{2n-1})(x - x_{2n})(x - x_{2n+1})} = Y_2(y_{2n+1})\tilde{d}_n(x).$$

(2) Next, from (22d),

$$\hat{d}_{n+1}(x_{2n-1}) = (\alpha_n y_{2n} + \beta_n)[-a_n(x_{2n-1}) - (y_{2n} - y_{2n-1})c_n(x_{2n-1})] = 0$$

from (20), knowing that

$$d_n(x_{2n-1}) = 0,$$

$$\begin{aligned}\hat{d}_{n+1}(x_{2n}) &= (\alpha_n y_{2n} + \beta_n)[-a_n(x_{2n}) + (y_{2n+1} - y_{2n})c_n(x_{2n}) + (\alpha_n y_{2n+1} + \beta_n)d_n(x_{2n})]\\ &= (\alpha_n y_{2n} + \beta_n)(y_{2n+1} - y_{2n})\left[-\frac{a_n(x_{2n})}{y_{2n+1} - y_{2n}} + c_n(x_{2n}) + \frac{d_n(x_{2n})}{f_n(y_{2n+1})}\right] = 0,\end{aligned}$$

as the last factor is the Riccati equation (17) at $x = x_{2n}$ divided by $f_n(y_{2n+1})$ (see also (18)).

$$\begin{aligned}\hat{d}_{n+1}(x_{2n+1}) = &-(\alpha_n y_{2n} + \beta_n)a_n(x_{2n+1}) + b_n(x_{2n+1})(y_{2n+1} - y_{2n})(y_{2n+2} - y_{2n})\\ &+ c_n(x_{2n+1})[(y_{2n+2} - y_{2n})(\alpha_n y_{2n+1} + \beta_n)\\ &+ (y_{2n+1} - y_{2n})(\alpha_n y_{2n+2} + \beta_n)] + d_n(x_{2n+1})(\alpha_n y_{2n+1} + \beta_n)(\alpha_n y_{2n+2} + \beta_n) = 0,\end{aligned}$$

from (19).

(3) In order to show that $\hat{a}_{n+1}$ and $\hat{c}_{n+1}$ both vanish at $x = x_{2n-1}$ and x_{2n}, we consider

$$\begin{aligned}&\frac{a_n}{\psi - \varphi} \pm c_n : \frac{\hat{a}_{n+1}(x)}{\psi(x) - \varphi(x)} + \hat{c}_{n+1}(x)\\ &= (\varphi(x) - y_{2n+1})\left[(\psi(x) - y_{2n})\left(\frac{a_n(x)}{\psi(x) - \varphi(x)} - c_n(x)\right) - (\alpha_n\psi(x) + \beta_n)d_n(x)\right]\end{aligned}$$

vanishes at $x = x_{2n}$, as the big factor is $a_n(x_{2n}) - (y_{2n+1} - y_{2n})c_n(x_{2n}) - (\alpha_n y_{2n+1} + \beta_n)d_n(x_{2n}) = 0$ from (18). At $x = x_{2n-1}$, we already encountered the condition $a_n(x_{2n-1}) + (y_{2n} - y_{2n-1})c_n(x_{2n-1}) = 0$ from (20) and $d_n(x_{2n-1}) = 0$.

The obvious vanishing of the first factor at $x = x_{2n+1}$ will allow the same condition at x_{2n+1}: $a_{n+1}(x_{2n+1}) + (y_{2n+2} - y_{2n+1})c_{n+1}(x_{2n+1}) = 0$, and this will imply $f_{n+1}(y_{2n+2}) = 0$ at the next step.

(4)

$$\frac{\hat{a}_{n+1}(x)}{\psi(x) - \varphi(x)} - \hat{c}_{n+1}(x) = (\psi(x) - y_{2n+1})\left[(\varphi(x) - y_{2n})\left(\frac{a_n(x)}{\psi(x) - \varphi(x)} + c_n(x)\right) + (\alpha_n\varphi(x) + \beta_n)d_n(x)\right]$$

obviously vanishes now at $x = x_{2n}$; at $x = x_{2n-1}$, the big factor is $-a_n(x_{2n-1}) - (y_{2n} - y_{2n-1})c_n(x_{2n-1}) = 0$ as already encountered (in (20), together with $d_n(x_{2n-1}) = 0$).

We proceed therefore with $a_{n+1}(x) = \frac{X_2(x)\hat{a}_{n+1}(x)}{(x-x_{2n-1})(x-x_{2n})}$, $b_{n+1}(x) = \frac{X_2(x)\hat{b}_{n+1}(x)}{(x-x_{2n-1})(x-x_{2n})}$, $c_{n+1}(x) = \frac{X_2(x)\hat{c}_{n+1}(x)}{(x-x_{2n-1})(x-x_{2n})}$, $d_{n+1}(x) = \frac{X_2(x)\hat{d}_{n+1}(x)}{(x-x_{2n-1})(x-x_{2n})}$. □

A very interesting identity is $\hat{a}_{n+1}^2 - (\psi - \varphi)^2\hat{c}_{n+1}^2 = (\varphi - y_{2n})(\psi - y_{2n})(\varphi - y_{2n+1})(\psi - y_{2n+1})(a_n^2 - (\psi - \varphi)^2 c_n^2) - (\psi - \varphi)^2\hat{b}_{n+1}[\hat{d}_{n+1} - (\varphi - y_{2n})(\psi - y_{2n})b_n]$, or $\hat{a}_{n+1}^2 - (\psi - \varphi)^2(\hat{c}_{n+1}^2 - \hat{b}_{n+1}\hat{d}_{n+1}) = (\varphi - y_{2n})(\psi - y_{2n})(\varphi - y_{2n+1})(\psi - y_{2n+1})(a_n^2 - (\psi - \varphi)^2(c_n^2 - b_n d_n))$

$$\frac{c_n^2(x) - b_n(x)d_n(x)}{X_2^2(x)}P(x) - a_n^2(x) = C_n\frac{x - x_{2n-1}}{x - x_{-1}}\left[\frac{c_0^2(x) - b_0(x)d_0(x)}{X_2^2(x)}P(x) - a_0^2(x)\right], \tag{23}$$

where $C_n = \frac{Y_2(y_{2n-1})Y_2(y_{2n-3})\cdots Y_2(y_1)}{Y_2(y_{2n-2})Y_2(y_{2n-4})\cdots Y_2(y_0)}$, and P from (7).

5. Classical elliptic biorthogonal rational functions

The smallest possible degree for a_n, b_n, c_n, and d_n according to the theory above, happens to be three. Then, $b_n(x) = \xi_n(x - x_{2n-1})X_2(x)$ and $d_n(x) = \zeta_n(x - x_{2n-1})X_2(x)$ are already known, up to a single constant each. The equations (22a) and (22c) should care for the evolution of a_n and c_n with n, but (23) yields directly the values of $a_n^2(x)$ at the four zeros of $P = X_1^2 - 4X_0X_2$. Equation (22a) also provides a simple relation between $\hat{a}_{n+1}(x)$ (therefore, $a_{n+1}(x)$), and $a_n(x)$ at each of these four zeros, which is enough for a full determination of the third degree polynomials a_n.

I hope to recover in this way the results of Spiridonov and Zhedanov [28,29], obtained through elliptic hypergeometric identities [30–34].

Note however that Theorem 4.1 also makes room for coefficients of degree > 3, therefore to new families of biorthogonal functions.

Acknowledgments

Many thanks to Jacek Gilewicz, Roland Triay, and Galliano Valent, who organized the Conference at Marseille-Luminy on July 2–6, 2007.

Many thanks too to R. Askey, L. Haine, M. Ismail, F. Nijhoff, A. Ronveaux, and, of course, V. Spiridonov and A. Zhedanov for their preprints, interest, remarks, and kind words.

Many thanks to the referee(s) for expert and careful reading.

"This paper presents research results of the Belgian Programme on Interuniversity Attraction Poles, initiated by the Belgian Federal Science Policy Office. The scientific responsibility rests with its author(s)."

References

[1] L.M. Milne-Thomson, The Calculus of Finite Differences, Macmillan And Company, Limited, 1933, http://www.archive.org/details/calculusoffinite032017mbp.
[2] F.A. Grünbaum, L. Haine, On a q-analogue of Gauss equation and some q-Riccati equations, in: Mourad E.H. Ismail, et al. (Eds.), Special functions, q-series and related topics, in: Fields Inst. Commun, vol. 14, American Mathematical Society, Providence, RI, 1997, pp. 77–81.
[3] R. Koekoek, R.F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Delft University of Technology, Faculty of Information Technology and Systems, Department of Technical Mathematics and Informatics, Report no. 98-17, 1998. See also the online version in http://aw.twi.tudelft.nl/~koekoek/askey.html.
[4] T.H. Koornwinder, Compact quantum groups and q-special functions, in: V. Baldoni, M.A. Picardello (Eds.), Representations of Lie Groups and Quantum Groups, in: Pitman Research Notes in Mathematics Series, vol. 311, Longman Scientific & Technical, 1994, pp. 46–128. See also the part "q-Special functions, a tutorial", in ftp://unvie6.un.or.at/siam/opsf_new/koornwinder/koornwinder3/koornwinder3.tex and ftp://unvie6.un.or.at/siam/opsf_new/koornwinder/koornwinder3/wilsonqhahn.ps.
[5] G.E. Andrews, R. Askey, R. Roy Ranjan, Special Functions, in: Encyclopedia of Mathematics and its Applications, vol. 71, Cambridge University Press, Cambridge, 1999.
[6] R. Askey, J. Wilson, Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, Memoirs Amer. Math. Soc. 54 (319) (1985).
[7] A.P. Magnus, Associated Askey–Wilson polynomials as Laguerre–Hahn orthogonal polynomials, in: M. Alfaro, et al. (Eds.), Orthogonal Polynomials and their Applications, Proceedings, Segovia 1986, in: Lecture Notes in Mathematics, vol. 1329, Springer Verlag, Berlin, 1988, pp. 261–278.
[8] A.P. Magnus, Special non uniform lattice (*snul*) orthogonal polynomials on discrete dense sets of points, J. Comput. Appl. Math. 65 (1995) 253–265.
[9] A.F. Nikiforov, S.K. Suslov, Classical orthogonal polynomials of a discrete variable on nonuniform lattices, Lett. Math. Phys. 11 (1986) 27–34.
[10] A.F. Nikiforov, S.K. Suslov, V.B. Uvarov, Classical Orthogonal Polynomials of a Discrete Variable, Springer, Berlin, 1991.
[11] V.P. Spiridonov, A.S. Zhedanov, Elliptic grids, rational functions, and the Padé interpolation, Ramanujan J. 13 (1–3) (2007) 285–310.
[12] M.E.H. Ismail, Classical and Quantum Orthogonal Polynomials in One Variable, with Two Chapters by Walter Van Assche, in: Encyclopedia of Mathematics and its Applications., vol. 98, Cambridge University Press, Cambridge, 2005.
[13] A. Ronveaux, Private communication, October 2006.
[14] P. Appell, E. Goursat, Théorie des fonctions algébriques et de leurs intégrales: étude des fonctions analytiques sur une surface de Riemann, Gauthier-Villars, 1895. http://www.archive.org/details/theoriefonctions00apperich.
[15] M. Maillard, S. Boukraa, Modular invariance in lattice statistical mechanics, Annales de la Fondation Louis de Broglie volume 26 numéro spécial, 2001, pp. 287–328. http://www.ensmp.fr/aflb/AFLB-26j/aflb26jp287.pdf.
[16] V.P. Burskii, A.S. Zhedanov, The Dirichlet and the Poncelet problems, Reports of RIAM Symposium No.16ME-S1, in: Physics and Mathematical Structures of Nonlinear Waves, Proceedings of a symposium held at Chikushi Campus, Kyushu University, Kasuga, Fukuoka, Japan, November 15–17, 2004. http://www.riam.kyushu-u.ac.jp/fluid/meeting/16ME-S1/papers/Article_No_24.pdf.
[17] V.P. Burskii, A.S. Zhedanov, Dirichlet and Neumann Problems for String Equation, Poncelet Problem and Pell–Abel Equation, in: Symmetry, Integrability and Geometry: Methods and Applications, vol. 2, 2006, Paper 041, 5 pages. arXiv:math.AP/0604278v1 12 April 2006.
[18] A.J. Van Der Poorten, C.S. Swart, Recurrence relations for elliptic sequences: Every somos 4 is a somos k, Bull. London. Math. Soc. 38 (2006) 546–554.
[19] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.
[20] M. Hazewinkel (Ed.), Encyclopaedia of Mathematics, Springer-Verlag, 2002. http://eom.springer.de, entry "Classical orthogonal polynomials", http://eom.springer.de/C/c022420.htm.
[21] E. Laguerre, Sur la réduction en fractions continues d'une fraction qui satisfait à une équation différentielle linéaire du premier ordre dont les coefficients sont rationnels, J. Math. Pures Appl. (Série 4) 1 (1885) 135–165 in: Oeuvres, vol. II, Chelsea, New-York 1972, pp. 685–711. See http://math-tenere.ujf-grenoble.fr/JMPA/feuilleter.php?id=JMPA_1885_4_1http://portail.mathdoc.fr/cgi-bin/oetoc?id=OE_LAGUERRE__2.
[22] F. Marcellán, J.C. Medem, q—Classical orthogonal polynomials: A very classical approach, Electron. Trans. Numer. Anal. 9 (1999) 112–127. http://www.maths.tcd.ie/EMIS/journals/ETNA/vol.9.1999/pp112-127.dir/pp112-127.pdf.
[23] M.E.H. Ismail, D.R. Masson, Generalized orthogonality and continued fractions, J. Approx. Theory 83 (1995) 1–40.
[24] V.P. Spiridonov, S. Tsujimoto, A.S. Zhedanov, Integrable discrete time chains for the Frobenius–Stickelberger–Thiele polynomials, Comm. Math. Phys. 272 (2007) 139–165.
[25] A.S. Zhedanov, Biorthogonal rational functions and the generalized eigenvalue problem, J. Approx. Theory 101 (2) (1999) 303–329.
[26] A.S. Zhedanov, Padé interpolation table and biorthogonal rational functions, in: Proceedings of RIMS workshop on elliptic integrable systems, Kyoto, November 8–11, 2004 to be published http://www.math.kobe-u.ac.jp/publications/rlm18/20.pdf.
[27] A.P. Magnus, Riccati acceleration of Jacobi continued fractions and Laguerre–Hahn orthogonal polynomials, in: H. Werner, H.J. Bünger (Eds.), Padé Approximation and its Applications, Bad Honnef 1983, in: Lecture Notes in Mathematics, vol. 1071, Springer Verlag, Berlin, 1984, pp. 213–230.
[28] V.P Spiridonov, A.S. Zhedanov, Generalized eigenvalue problem and a new family of rational functions biorthogonal on elliptic grids, in: Joaquin Bustoz, et al. (Eds.), Special Functions 2000: Current Perspective and Future Directions, Proceedings of the NATO Advanced Study Institute, Tempe, AZ, USA, May 29–June 9, 2000, in: NATO Sci. Ser. II, Math. Phys. Chem., vol. 30, Kluwer Academic Publishers, Dordrecht, 2001, pp. 365–388.
[29] V.P. Spiridonov, A.S. Zhedanov, Classical biorthogonal rational functions on elliptic grids, C. R. Math. Acad. Sci. Soc. R. Can. 22 (2) (2000) 70–76.
[30] V.P. Spiridonov, Classical elliptic hypergeometric functions and their applications, Lectures delivered at the workshop Elliptic integrable systems (RIMS, Kyoto, November 8–11, 2004), 253–287 http://www.math.kobe-u.ac.jp/publications/rlm18/17.pdf.
[31] V.P. Spiridonov, Theta hypergeometric integrals, Algebra i Analiz 15 (2003) 6. Russian original version (English version: St. Petersburg Math. J. 15 (2004) 929–967).
[32] V.P. Spiridonov, Elliptic hypergeometric functions. This is a brief overview of the status of the theory of elliptic hypergeometric functions to the end of 2006 written as a complement to a Russian edition (to be published by the Independent University press, Moscow, 2008) of the book by G.E. Andrews, R. Askey, and R. Roy, Special Functions, Encycl. of Math. Appl. vol. 71, Cambridge Univ. Press, 1999. Report number: RIMS-1589 Cite as: math.CA.
[33] V.P. Spiridonov, Essays on the theory of elliptic hypergeometric functions, Uspekhi Mat. Nauk 63 (3(381)) (2008) 3–72 (in Russian); Russian Math. Surveys 63 (2008) 405–472 (in English).
[34] V.P. Spiridonov, Elliptic hypergeometric functions and models of Calogero–Sutherland type, Teoret. Mat. Fiz. 150 (2) (2007) 311–324 (in Russian); English translation in Theoret. Math. Phys. 150 (2007) (2) 266–277.