

# Regularity criteria for suitable weak solutions of the Navier–Stokes equations near the boundary <sup>☆</sup>

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Received 27 April 2005; revised 11 December 2005

Available online 18 January 2006

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## Abstract

We present some new regularity criteria for “suitable weak solutions” of the Navier–Stokes equations near the boundary in dimension three. We prove that suitable weak solutions are Hölder continuous up to the boundary provided that the scaled mixed norm  $L_{x,t}^{p,q}$  with  $3/p + 2/q \leq 2$ ,  $2 < q \leq \infty$ ,  $(p, q) \neq (3/2, \infty)$  is small near the boundary. Our methods yield new results in the interior case as well. Partial regularity of weak solutions is also analyzed under some additional integral conditions.

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*Keywords:* Boundary regularity; Suitable weak solution; Navier–Stokes equations

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## 1. Introduction

In this note, we study the boundary regularity problem for *suitable weak solutions*  $(u, p)$ :  $Q_T \rightarrow \mathbb{R}^3 \times \mathbb{R}$  to the Navier–Stokes equations

$$\begin{cases} u_t - \Delta u + (u \cdot \nabla)u + \nabla p = f, \\ \operatorname{div} u = 0 \end{cases} \quad \text{in } Q_T = \Omega \times (0, T), \quad (1)$$

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<sup>☆</sup> The research of the first and third authors is partly supported by NSERC grants nos. 22R80976 and 22R81253. The second author is partly supported by a PIMS PDF.

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where  $\Omega$  is a domain in  $\mathbb{R}^3$ ,  $u(x, t)$  is the velocity field and  $p(x, t)$  is the pressure. By suitable weak solutions we mean functions which solve the Navier–Stokes equations in the sense of distribution, satisfy some integrability conditions, and satisfy the local energy inequality (for details, see Definition 4 in Section 2). We assume that  $\Omega$  and  $f$  are sufficiently regular and will give the specifics later. For a point  $z = (x, t)$  in  $\overline{\Omega} \times (0, T]$ , denote  $B_{x,r} = \{y \in \mathbb{R}^3: |y - x| < r\}$ ,

$$Q_{z,r} := B_{x,r} \times (t - r^2, t), \quad Q_{z,r}^+ := Q_{z,r} \cap Q_T.$$

A solution  $u$  is said to be *regular* at  $z$  if  $u$  is uniformly Hölder continuous (for some exponent) in both  $x$  and  $t$  in  $Q_{z,r}^+$  for some  $r > 0$ .

After the seminal work of Leray [12] and Hopf [8] on the existence of weak solutions, the problems of uniqueness and regularity of weak solutions remain unsolved. A set of criteria which guarantee uniqueness and regularity is the *zero-dimensional integral conditions*: any weak solution  $u$  of the Navier–Stokes equations is unique and regular in  $Q_T$  if it satisfies, for some  $p, q \geq 1$ ,

$$\|u\|_{L^{p,q}(Q_T)} < \infty, \quad \frac{3}{p} + \frac{2}{q} \leq 1. \tag{2}$$

Here

$$\|u\|_{L^{p,q}(Q_T)} := \left\| \|u(\cdot, t)\|_{L_x^p(\Omega)} \right\|_{L_t^q(0,T)}.$$

Note that a weak solution satisfying (2) is automatically a suitable weak solution: interpolating with  $u \in L^{2,\infty}(Q) \cap L^{6,2}(Q)$ , a weak solution  $u$  satisfying (2) belongs to  $L^{4,4}(Q)$  (see, e.g., [1, pp. 778–780] for definitions of weak solutions and suitable weak solutions). Hence one can use  $u$  multiplied by a cut-off function as a test function and derive the local energy inequality from the weak formulation of (1). We now briefly review regularity results for the zero-dimensional integral class. See [4,27] for more references. Assuming (2), Serrin [24,25] proved regularity when  $3/p + 2/q < 1$  (see also [10,15]). The cases  $3/p + 2/q = 1$ ,  $3 < p \leq \infty$  were proved by Fabes, Jones and Rivière [5] for  $\Omega = \mathbb{R}^3$ , by Sohr [26] and Giga [6] for  $\Omega$  a domain, and by Struwe [30] for the interior case. See [2,27] for results in the setting of Lorentz and Morrey spaces. These results were recently extended up to a flat boundary by the second author [9] and to a curved boundary by Solonnikov [29]. A flat boundary is a portion of the boundary which lies on a plane. The endpoint case  $(p, q) = (3, \infty)$  was recently resolved by Escauriaza, Seregin and Šverák [4] for the  $\mathbb{R}^3$  and interior cases, and by Seregin [22] for domains.

Recently, there have been many works on regularity criteria with conditions involving only  $p$ . We will not try to give a list here.

After the partial regularity theory of Scheffer in a series of papers [16–19], Caffarelli, Kohn and Nirenberg [1] proved that the one-dimensional parabolic Hausdorff measure of the set  $\mathcal{S}$  of possible interior singular points of suitable weak solutions is zero, denoted  $\mathcal{P}^1(\mathcal{S}) = 0$ . This implies that the one-dimensional Hausdorff measure of  $\mathcal{S}$  is also zero. See Section 2 for the definition of parabolic Hausdorff measures. The key to the analysis in [1] is the following regularity criterion. There is an absolute constant  $\epsilon > 0$  such that, if  $u$  is a suitable weak solution of the Navier–Stokes equations in  $Q_T$  and for an interior point  $z = (x, t) \in Q_T$ ,

$$\limsup_{r \rightarrow 0_+} \frac{1}{r} \int_{Q_{z,r}} |\nabla u(y, s)|^2 dy ds \leq \epsilon, \tag{3}$$

then  $u$  is regular at  $z$ . See [13] for a simplified proof and [11] for more details.

Recently, Seregin [21] extended the interior partial regularity result up to a flat boundary. More precisely, there exists an absolute constant  $\epsilon > 0$  such that, if a suitable weak solution  $u$  satisfies

$$\limsup_{r \rightarrow 0_+} \frac{1}{r} \int_{Q_{z,r}^+} |\nabla u(y, s)|^2 dy ds \leq \epsilon, \tag{4}$$

where  $z \in \Gamma \times (0, T)$  and  $\Gamma$  is a flat boundary of  $\Omega$ , then  $u$  is regular at  $z$ . Combining the results in [1,21], one can conclude that suitable weak solutions are Hölder continuous up to the flat boundary away from a closed set  $\mathcal{S} \subset \bar{Q}_T$  with  $\mathcal{P}^1(\mathcal{S}) = 0$ . The same assertion for a curved boundary is believed to be true, but there seems no written proof yet.

The objective of this paper is to present new sufficient conditions for the regularity of suitable weak solutions to the Navier–Stokes equations near the flat boundary (and in the interior). Our main result is that, in place of condition (4), Hölder continuity of  $u$  near the boundary can be ensured by the smallness of the scaled mixed  $L^{p,q}$ -norm of the velocity field  $u$ . We assume that  $f$  belongs to  $M_{2,\gamma}$  for some  $\gamma > 0$  (this is a parabolic Morrey space, to be defined in Section 2). We have the following theorem.

**Theorem 1 (Regularity criteria).** *Suppose  $f \in M_{2,\gamma}(Q)$  for some  $\gamma > 0$ , a parabolic Morrey space. For every pair  $p, q$  satisfying*

$$1 \leq 3/p + 2/q \leq 2, \quad 2 < q \leq \infty, \quad (p, q) \neq (3/2, \infty), \tag{5}$$

*there exists a constant  $\epsilon > 0$  depending only on  $p, q, \gamma$  and  $\|f\|_{M_{2,\gamma}}$  such that, if the pair  $u, p$  is a suitable weak solution of the Navier–Stokes equations (1) vanishing on a flat boundary  $\Gamma$  according to Definition 4, and for some point  $z = (x, t) \in \Gamma \times (0, T)$ ,  $u$  is locally in  $L^{p,q}$  near  $z$  and*

$$\limsup_{r \rightarrow 0_+} r^{-\left(\frac{3}{p} + \frac{2}{q} - 1\right)} \| \|u(y, s)\|_{L^p(B_{x,r}^+)} \|_{L^q(I_{t-r^2,t})} \leq \epsilon, \tag{6}$$

*then  $z$  is a regular point.*

*Comments for Theorem 1.*

1. The same statement for an interior point  $z$  remains true, see Appendix A.
2. The quantities in (6) are invariant under the scaling  $u(x, t) \rightarrow su(sx, s^2t)$ . Scaling invariant quantities have been important in the study of (1), see, e.g., [1].
3. The exponents  $(p, q)$  in Theorem 1 correspond to region II in Fig. 1, which is a solid parallelogram excluding its top borderline and the corner point  $(2/3, 0)$ . By Hölder’s inequality, it suffices to prove the cases  $\frac{3}{p} + \frac{2}{q} = 2, 2 < q < \infty$ , the right borderline of region II. Our method fails for the end points  $(p, q) = (3/2, \infty)$  and  $(3, 2)$  for the lack of  $L^{3/2,1}$  and  $L^{1,2}$  estimates for the Stokes system.
4. The usual zero-dimensional integral conditions correspond to region I and imply (6) pointwise. Thus, also by Hölder’s inequality, regularity under the zero-dimensional integral conditions is a corollary of Theorem 1, except in the endpoint cases  $(p, q) = (3, \infty)$  or  $(p, q) = (\infty, 2)$  (see Corollary 12 for the details). Regularity up to the boundary under the

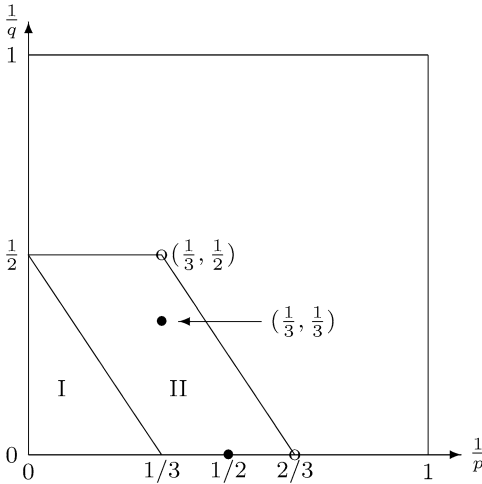


Fig. 1. Regularity criteria.

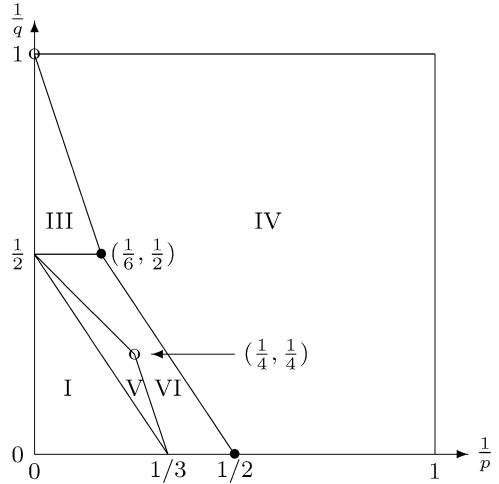


Fig. 2. Partial regularity.

zero-dimensional integral conditions is proved in [9,29] but the proof of Theorem 1 seems easier.

5. One key feature of Theorem 1 is that *condition (6) does not involve any scaled norm of the pressure p*. A previous such result is by Tian and Xin [31] for the special case of (6) with  $(p, q) = (3, 3)$ . Another such result is by Seregin and Šverák [23] for  $(p, q) = (2, \infty)$ . Both results are for interior points, and are included in region II.
6. Another regularity criterion in [31] is the uniform boundedness  $\sup_{r < R_0} (r^{-1/2} \|u\|_{L^{2,\infty}(Q_{z,r})}) \leq M$  for some  $M < \infty$ , and the condition (6) with  $(p, q) = (2, 2)$  and a small constant  $\epsilon$  depending on  $M$ . Although  $(p, q) = (2, 2)$  lies outside of region II, using  $\|u\|_{L^{2,4}} \leq \|u\|_{L^{2,\infty}}^{1/2} \|u\|_{L^{2,2}}^{1/2}$ , one obtains (6) with  $(p, q) = (2, 4)$ , which falls in region II. Thus this result is also implied by Theorem 1.
7. Equation (6) is a uniform estimate for  $r$  sufficiently small. There are conditions which only require one  $r$ . For example, there is an  $\epsilon > 0$  such that the condition

$$r^{-2} \int_{Q_{z,r}} (|u|^3 + |p|^{3/2}) dx dt \leq \epsilon \quad \text{for some } r > 0$$

implies regularity at  $z$ . This is essentially [1, Proposition 1] and is stated as above in [13,14]. Also see [21] and our Lemma 7 when  $z$  is on boundary.

The main tools of our analysis are a standard “blow up” method and the decomposition of the pressure as introduced in [21], which enable us to prove a decay property of the scaled Lebesgue norms of velocity and pressure in both the interior and boundary cases (see Lemma 8 and Appendix A). Combining this with the local estimate of the Stokes system for the pressure, we can estimate the pressure for the Navier–Stokes equations near the boundary (see Lemma 11).

As mentioned earlier, the best available estimate for the singular set is that  $\mathcal{P}^1(\mathcal{S}) = 0$  (in [3] the estimate of the Hausdorff measure of the singular set for suitable weak solutions was improved by a logarithmic factor for the interior case). In the following theorem we improve the estimate using Theorem 1, assuming some additional integral conditions of velocity fields.

**Theorem 2 (Partial regularity).** *Suppose  $f \in M_{2,\gamma}(Q)$  for some  $\gamma > 0$ , a parabolic Morrey space. Suppose  $(u, p)$  is a weak solution of the Navier–Stokes equations (1) according to Definition 2.2 and assume that*

$$u \in L^m((0, T); L^l(\Omega)), \quad (l, m) \in \text{region } V,$$

where region  $V$  is the triangular region in Fig. 2 satisfying  $\frac{3}{l} + \frac{2}{m} > 1$ ,  $\frac{1}{l} + \frac{1}{m} < \frac{1}{2}$ , and  $\frac{3}{l} + \frac{1}{m} < 1$ . Let  $S$  denote the singular set of  $u$  up to the flat boundary where  $u$  vanishes, and

$$d(l, m) = \begin{cases} 3 - m + \frac{2m}{l} & \text{if } l > m, \\ 2 - m + \frac{3m}{l} & \text{if } l \leq m. \end{cases}$$

Then the  $d(l, m)$ -dimensional parabolic Hausdorff measure of  $S$  is zero, i.e.,  $\mathcal{P}^{d(l,m)}(S) = 0$ .

In Theorem 2 we only require a weak solution. As for the zero-dimensional integral class, these solutions are automatically suitable weak solutions, see Section 4. Note that weak solutions are known to lie in region IV of Fig. 2, including the solid line from  $(\frac{1}{2}, 0)$  to  $(\frac{1}{6}, \frac{1}{2})$ . Assuming  $u \in L^{p,q}$ , one can use Theorem 1 to estimate the dimension of the singular set for all  $(p, q)$  in region II with  $q < \infty$ , but only those in region V give us dimensions less than 1.

The plan of this paper is as follows: In Section 2 we introduce the notion of suitable weak solutions near the boundary, which is a slightly modified version of that used in [21]. We also show the decay property of the velocity field and pressure (see Lemmas 7 and 8). In Section 3 we present the proof of the main Theorem 1. In Section 4, as an application, we investigate the size of the possible singular set under our additional integrability assumption on  $u$  (see Theorem 2). In Appendix A we present a brief sketch of the proof that the regularity criteria (6) is valid in the interior.

## 2. Preliminaries

In this section we introduce notation, define suitable weak solutions, and give some lemmas on the decay properties of the velocity and pressure.

We start with notation. Denote by  $\Omega$  an open domain in  $\mathbb{R}^3$  and by  $\partial\Omega$  its boundary.  $\Gamma$  indicates an open subset of  $\partial\Omega$  which lies on a plane. In this article, for simplicity, we assume  $\Gamma$  lies on the plane  $\{x_3 = 0\}$ .

For  $1 \leq q \leq \infty$ ,  $W^{k,q}(\Omega)$  denotes the usual Sobolev space, i.e.,  $W^{k,q}(\Omega) = \{u \in L^q(\Omega) : D^\alpha u \in L^q(\Omega), 0 \leq |\alpha| \leq k\}$ . As usual,  $W_0^{k,q}(\Omega)$  is the completion of  $C_0^\infty(\Omega)$  in the  $W^{k,q}(\Omega)$  norm. We also denote by  $W^{-k,q'}(\Omega)$  the dual space of  $W_0^{k,q}(\Omega)$  where  $q$  and  $q'$  are Hölder conjugates.

For a domain  $Q \subset \mathbb{R}^3 \times I$ , we denote by  $C_{x,t}^{\alpha,\frac{q}{2}}(Q)$  the Banach space of functions that are Hölder continuous with exponent  $\alpha \in (0, 1)$ , with respect to the parabolic metric  $d(z, z') = |x - x'| + |t - t'|^{\frac{1}{2}}$  where  $z = (x, t)$  and  $z' = (x', t')$ .

We denote by  $\int_E f$  the average of  $f$  on  $E$ ; i.e.,  $\int_E f = \int_E f / |E|$ .

We denote by  $M_{2,\gamma}$  a parabolic version of Morrey’s spaces (see, e.g., [21, p. 3]). For  $\omega \subset \mathbb{R}^3 \times \mathbb{R}$  and a positive number  $\gamma \in (0, 2]$ , we define the space

$$M_{2,\gamma}(\omega; \mathbb{R}^3) := \{f \in L_{2,\text{loc}}(\omega; \mathbb{R}^3) : m_\gamma(f; \omega) < \infty\},$$

where

$$m_\gamma(f; \omega) := \sup \left\{ \frac{1}{r^{\gamma-2}} \left( \int_{Q(z,r) \cap \omega} |f|^2 dz' \right)^{\frac{1}{2}} : z \in \bar{\omega}, r > 0 \right\}. \tag{7}$$

Parabolic Hausdorff measures are defined in [1] using parabolic cylinders instead of usual balls. For any  $X \subset \mathbb{R}^3 \times \mathbb{R}$  and  $k \geq 0$  one defines

$$\mathcal{P}^k(X) = \inf_{\delta \rightarrow 0^+} \mathcal{P}_\delta^k(X), \quad \mathcal{P}_\delta^k(X) = \inf \left\{ \sum_{i=1}^\infty r_i^k : X \subset \bigcup_i Q_{z_i, r_i}, r_i < \delta \right\}.$$

Finally, by  $N = N(\alpha, \beta, \dots)$  we denote a constant depending on the prescribed quantities  $\alpha, \beta, \dots$ , which may change from line to line.

Next, we define several scaling-invariant functionals. Let  $z = (x, t) \in \Gamma \times I$ . As in [1,11,13, 21], let

$$A(r) := \sup_{t-r^2 \leq s < t} \frac{1}{r} \int_{B_{x,r}^+} |u(y, s)|^2 dy,$$

$$C(r) := \frac{1}{r^2} \int_{Q_{z,r}^+} |u(y, s)|^3 dy ds, \quad E(r) := \frac{1}{r} \int_{Q_{z,r}^+} |\nabla u(y, s)|^2 dy ds.$$

Let  $\kappa, \kappa^*$  and  $\lambda$  be numbers satisfying

$$\frac{3}{\kappa} + \frac{2}{\lambda} = 4, \quad \frac{1}{\kappa^*} = \frac{1}{\kappa} - \frac{1}{3}, \quad 1 < \lambda < 2. \tag{8}$$

We also introduce new functionals, which are useful for us:

$$\tilde{D}(r) := \frac{1}{r} \left( \int_{t-r^2}^t \left( \int_{B_{x,r}^+} |p(y, s) - (p)_{B_{x,r}^+}(s)|^{\kappa^*} dy \right)^{\frac{\lambda}{\kappa^*}} ds \right)^{\frac{1}{\lambda}}, \tag{9}$$

where  $(p)_{B_{x,r}^+}(s) = \int_{B_{x,r}^+} p(y, s) dy$ ,

$$\tilde{D}_1(r) := \frac{1}{r} \left( \int_{t-r^2}^t \left( \int_{B_{x,r}^+} |\nabla p(y, s)|^\kappa dy \right)^{\frac{\lambda}{\kappa}} ds \right)^{\frac{1}{\lambda}}, \tag{10}$$

and finally,

$$G(r) := \frac{1}{r} \left( \int_{t-r^2}^t \left( \int_{B_{x,r}^+} |u(y, s)|^p dy \right)^{\frac{q}{p}} ds \right)^{\frac{1}{q}},$$

where  $p$  and  $q$  are the Hölder conjugate exponents of  $\kappa^*$  and  $\lambda$  in (8), i.e.,

$$\frac{1}{p} + \frac{1}{\kappa^*} = 1, \quad \frac{1}{q} + \frac{1}{\lambda} = 1. \tag{11}$$

It is straightforward from (8) that  $p$  and  $q$  satisfy

$$\frac{3}{p} + \frac{2}{q} = 2, \quad 2 < q < \infty. \tag{12}$$

**Remark 3.** In [21] the following functionals are used, instead of  $\tilde{D}(r), \tilde{D}_1(r)$ ,

$$D(r) := \frac{1}{r^2} \int_{Q_{z,r}^+} |p - (p)_{B_{x,r}^+}|^{\frac{3}{2}} dz, \quad D_1(r) := \frac{1}{r^{\frac{3}{2}}} \int_{t-r^2}^t \left( \int_{B_{x,r}^+} |\nabla p|^{\frac{9}{8}} dy \right)^{\frac{4}{3}} ds.$$

We note that  $D_1(r)$  is a special case of  $\tilde{D}_1$  with  $\kappa = \frac{9}{8}, \lambda = \frac{3}{2}$ .

Let  $Q = \Omega \times I$  where  $\Omega \subset \mathbb{R}^3$  and  $I = [0, T]$ . A weak solution of the Navier–Stokes equations (1) is a vector field  $u \in L^\infty(I; L^2(\Omega)) \cap L^2(I; W_0^{1,2}(\Omega))$  satisfying

$$\int_Q (u \cdot \phi_t - \nabla u \cdot \nabla \phi - (u \cdot \nabla)u \cdot \phi + f\phi) dz = 0, \quad \int_\Omega u(x, t) \cdot \nabla \psi(x) dx = 0$$

for any divergence-free vector field  $\phi \in C_0^\infty(Q)$  and any smooth function  $\psi \in C_0^\infty(\Omega)$  for all  $t \in [0, T]$  (see, e.g., [8,12]).

Next we define suitable weak solutions for the Navier–Stokes equations.

**Definition 4.** Let  $Q = \Omega \times I$  where  $\Omega \subset \mathbb{R}^3$  and  $I = [0, T]$  and  $\Gamma$  be an open subset of the set  $\partial\Omega$ . Suppose that  $f$  belongs to the parabolic Morrey space  $M_{2,\gamma}(Q)$  for some  $\gamma \in (0, 2]$ . A pair of  $(u, p)$  is a *suitable weak solution* to the Navier–Stokes equation (1) in  $Q$  near the boundary  $\Gamma$  and vanishing on  $\Gamma$  if the following conditions are satisfied.

(a) The functions  $u : Q \rightarrow \mathbb{R}^3$  and  $p : Q \rightarrow \mathbb{R}$  satisfy

$$u \in L^\infty(I; L^2(\Omega)) \cap L^2(I; W^{1,2}(\Omega)), \quad p \in L^\lambda(I; L^{\kappa^*}(\Omega)), \tag{13}$$

$$\nabla^2 u \in L^\lambda(I; L^\kappa(\Omega)), \quad \nabla p \in L^\lambda(I; L^\kappa(\Omega)), \tag{14}$$

where  $\kappa, \kappa^*$  and  $\lambda$  are fixed numbers satisfying (8).

(b)  $u$  and  $p$  solve the Navier–Stokes equations (1) in  $Q$  in the sense of distributions and  $u$  satisfies the boundary condition  $u = 0$  on  $\Gamma \times I$ .

(c)  $u$  and  $p$  satisfy the local energy inequality

$$\int_{\Omega} |u(x, t)|^2 \phi(x, t) dx + 2 \int_Q |\nabla u(x, t')|^2 \phi(x, t') dx dt' \leq \int_Q (|u|^2 (\partial_t \phi + \Delta \phi) + (|u|^2 + 2p) u \cdot \nabla \phi + 2f \cdot u \phi) dx dt' \tag{15}$$

for almost all  $t \in (0, T)$  and for all nonnegative functions  $\phi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$ , vanishing in a neighbourhood of the set  $(\Omega \times \{t = 0\}) \cup ((\partial\Omega \setminus \Gamma) \times (0, T))$ .

Let us make several comments on the above definition.

**Remark 5.** Sohr and Von Wahl [28] showed that, under reasonable assumptions on  $f$  and  $u_0$ , the pressure  $p$  of a weak solution belongs to  $L^{\frac{5}{3}}(\Omega \times I)$ , which corresponds to  $\kappa^* = \lambda = \frac{5}{3}$  in (13). Here  $\Omega \subset \mathbb{R}^3$  can be either a bounded domain, an exterior domain, or a half-space. Giga and Sohr [7] later proved that  $u_t, \nabla^2 u, \nabla p \in L^{\kappa, \lambda}(Q)$  and  $p \in L^{\kappa^*, \lambda}(Q)$  where  $\kappa, \kappa^*$  and  $\lambda$  are any numbers satisfying (8). Therefore, it seems reasonable to make assumptions (13) and (14) for suitable weak solutions.

**Remark 6.** The main difference between suitable weak solutions and the original Leray–Hopf weak solutions is the additional condition of the local energy inequality (15). The existence of suitable weak solutions is proved in [1,17]. Slightly modified definitions are used in [11,13,21]. As indicated in [1, Remark 4, p. 823], it is an open question if all weak solutions are suitable weak solutions.

Next we show the local regularity criterion near the boundary, which is analogous to [21, Proposition 2.6]. Although our proof is based on a standard “blow up” method similar to that of [21], we present its details since different functionals are used for the pressure, and therefore modifications are needed.

**Lemma 7.** *There exists  $\epsilon > 0$  depending only on  $\lambda, \gamma$  and  $\|f\|_{M_{2,\gamma}}$ , such that if  $u$  is a suitable weak solution of the Navier–Stokes equations satisfying Definition 4,  $z = (x, t) \in \Gamma \times I$ , and*

$$\liminf_{r \rightarrow 0_+} (C^{\frac{1}{3}}(r) + \tilde{D}(r)) < \epsilon,$$

*then  $z$  is a regular point.*

Before we prove Lemma 7, we first prove the following lemma, which gives a decay property of  $u$  and  $p$  in some Lebesgue spaces.

**Lemma 8.** *Let  $0 < \theta < 1/2$  and  $0 < \beta < \gamma \leq 2$ . There exist  $\epsilon_1, r_1 > 0$  depending on  $\lambda, \theta, \gamma$  and  $\beta$  such that if  $u$  is a suitable weak solution of the Navier–Stokes equations satisfying Definition 4,  $z = (x, t) \in \Gamma \times I$ , and  $C^{\frac{1}{3}}(r) + \tilde{D}(r) + m_\gamma(f)r^{\beta+1} < \epsilon_1$  for some  $r \in (0, r_1)$ , then*

$$(C^{\frac{1}{3}}(\theta r) + \tilde{D}(\theta r)) < N\theta^{1+\alpha} (C^{\frac{1}{3}}(r) + \tilde{D}(r) + m_\gamma(f)r^{\beta+1}),$$

*where  $0 < \alpha < 1$  and  $N > 0$  are absolute constants.*



**Proof.** For simplicity we assume  $f = 0$ . The general case follows similarly. For convenience, we denote  $\varphi(r) := C^{\frac{1}{3}}(r) + \tilde{D}(r)$ . Suppose the statement is not true. Then for any  $\alpha \in (0, 1)$  and  $N > 0$ , there exist  $z_n = (x_n, t_n)$ ,  $r_n \searrow 0$ , and  $\epsilon_n \searrow 0$  such that

$$\varphi(r_n) = \epsilon_n, \quad \varphi(\theta r_n) > N\theta^{1+\alpha} \varphi(r_n) = N\theta^{1+\alpha} \epsilon_n. \tag{16}$$

Let  $w = (y, s)$  where  $y = r_n^{-1}(x - x_n)$ ,  $s = r_n^{-2}(t - t_n)$  and we define  $v_n$  and  $q_n$  as follows:

$$v_n(w) = \epsilon_n^{-1} r_n u(z), \quad q_n(w) = \epsilon_n^{-1} r_n^2 (p(z) - (p)_{B_{r_n}^+}(t)).$$

For convenience we also define  $C(v_n, \theta)$ ,  $\tilde{D}(q_n, \theta)$ , and  $\tilde{D}_1(q_n, \theta)$  by

$$C(v_n, \theta) := \frac{1}{\theta^2} \int_{Q_\theta^+} |v_n|^3 dw,$$

$$\tilde{D}(q_n, \theta) := \frac{1}{\theta} \left( \int_{-\theta^2}^0 \left( \int_{B_\theta^+} |q_n - (q_n)_{B_\theta^+}|^{\kappa^*} dy \right)^{\frac{\lambda}{\kappa^*}} ds \right)^{\frac{1}{\lambda}},$$

$$\tilde{D}_1(q_n, \theta) := \frac{1}{\theta} \left( \int_{-\theta^2}^0 \left( \int_{B_\theta^+} |\nabla q_n|^\kappa dy \right)^{\frac{\lambda}{\kappa}} ds \right)^{\frac{1}{\lambda}},$$

where  $\kappa^*$ ,  $\kappa$  and  $\lambda$  are numbers in (8). By the change of variables, we have

$$\frac{1}{\epsilon_n} \varphi(\theta r_n) = C^{\frac{1}{3}}(v_n, \theta) + \tilde{D}(q_n, \theta). \tag{17}$$

For convenience, we denote  $\psi_n(\theta) := C^{\frac{1}{3}}(v_n, \theta) + \tilde{D}(q_n, \theta)$ . Due to (16) and (17), we get

$$\psi_n(1) = \|v_n\|_{L^3(Q_1^+)} + \|q_n\|_{L^{\kappa^*, \lambda}(Q_1^+)} = 1, \tag{18}$$

$$\psi_n(\theta) = C^{\frac{1}{3}}(v_n, \theta) + \tilde{D}(q_n, \theta) \geq N\theta^{1+\alpha}. \tag{19}$$

On the other hand,  $v_n, q_n$  solve the following system in a weak sense

$$\partial_s v_n - \Delta v_n + \epsilon_n (v_n \cdot \nabla) v_n + \nabla q_n = 0, \quad \operatorname{div} v_n = 0 \quad \text{in } Q_1^+$$

with

$$v_n = 0 \quad \text{on } (B_1 \cap \{x_3 = 0\}) \times (-1, 0).$$

Because of (18), we have following weak convergence (possibly subsequences of  $v_n$  and  $q_n$  should be taken, however we use the same symbol for simplicity)

$$v_n \rightharpoonup v \quad \text{in } L^3(Q_1^+), \quad q_n \rightharpoonup q \quad \text{in } L^{\kappa^*, \lambda}(Q_1^+), \quad (q)_{B_1^+}(s) = 0.$$

In addition, we note that  $\partial_s v_n$  is uniformly bounded in  $L^\lambda((-1, 0); (W^{2,2}(B_1^+))')$ , and, therefore, we also have

$$\partial_s v_n \rightharpoonup \partial_s v \quad \text{in } L^\lambda((-1, 0); (W^{2,2}(B_1^+))'). \tag{20}$$

Next we show that  $\nabla v_n$  is uniformly bounded in  $L^2(Q_{3/4}^+)$ . Let  $\phi$  be a standard cut off function satisfying  $\phi$  is smooth,

$$\phi = 1 \quad \text{on } Q_{3/4}, \quad \phi = 0 \quad \text{on } (\mathbb{R}^3 \times (-\infty, 0)) \setminus Q_1, \quad 0 \leq \phi \leq 1.$$

From the local energy inequality, for every  $\tau \in (-1, 0)$ , we obtain

$$\begin{aligned} & \int_{B_1^+} |v_n(\cdot, \tau)|^2 \phi^2(x, \tau) dy + \int_{-1}^\tau \int_{B_1^+} |\nabla v_n|^2 \phi^2 dy ds \\ & \leq N \left( \int_{-1}^\tau \int_{B_1^+} |v_n|^2 (|\partial_s \phi| + |\Delta \phi| + |\nabla \phi|) dy ds \right. \\ & \quad \left. + \epsilon_n \int_{-1}^\tau \int_{B_1^+} |v_n|^3 |\nabla \phi| dy ds + \int_{-1}^\tau \int_{B_1^+} |q_n v_n \cdot \nabla \phi \phi| dy ds \right). \end{aligned}$$

Consider the last term in the above inequality. Using the Hölder inequality, we have

$$\int_{-1}^\tau \int_{B_1^+} |q_n v_n \cdot \nabla \phi \phi| \leq \left( \int_{-1}^\tau \left( \int_{B_1^+} |q_n \nabla \phi|^{\kappa^*} \right)^{\frac{\lambda}{\kappa^*}} \right)^{\frac{1}{\lambda}} \left( \int_{-1}^\tau \left( \int_{B_1^+} |v_n \phi|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}},$$

where  $\kappa^*$ ,  $\lambda$ ,  $p$ , and  $q$  are numbers in (8), (11), and (12). We recall that  $p$  and  $q$  are in the ranges  $3/2 < p < 3$  and  $2 < q < \infty$ . In case  $q \leq 3$ , since  $p, q \leq 3$ , we have

$$\|v_n \phi\|_{L^{p,q}((-1,\tau); B_1^+)} \leq N \|v_n \phi\|_{L^3(Q_1^+)}.$$

Therefore, in this case,  $\nabla v_n$  is uniformly bounded in  $L^2(Q_{3/4}^+)$  because of (18). It remains to consider the case  $3 < q < \infty$  (equivalently  $3/2 < p < 9/4$ ). Suppose  $2 < p < 9/4$ , which is equivalent to  $4 < q < \infty$ . In this case, by interpolation, one can see the following estimate:

$$\|v_n \phi\|_{L^{p,q}(Q_1^+)} \leq N \sup_{-1 < s < \tau} \|v_n \phi(\cdot, s)\|_{L^2(B_1^+)}^{\frac{2\alpha}{p}} \|v_n \phi\|_{L^3(Q_1^+)}^{\frac{3(1-\alpha)}{p}},$$

where  $\alpha = 3 - p$ . In the above inequality, we used that  $(1 - \alpha)q/p < 1$ . Since  $\tau$  is arbitrary between  $-1$  and  $0$ , we obtain

$$\begin{aligned} & \sup_{-1 < \tau < 0} \|v_n \phi(\cdot, \tau)\|_{L^2(B_1^+)} + \|\nabla v_n \phi\|_{L^2(Q_1^+)} \\ & \leq N \left( \|v_n\|_{L^2(Q_1^+)} + \epsilon_n \|v_n\|_{L^3(Q_1^+)} + \sup_{-1 < s < 0} \|v_n \phi(\cdot, s)\|_{L^2(B_1^+)}^{\frac{2(3-p)}{p}} \|v_n \phi\|_{L^3(Q_1^+)}^{\frac{3(p-2)}{p}} \|q_n\|_{L^{\kappa^*, \lambda}(Q_1^+)} \right). \end{aligned}$$

Therefore, using Young’s inequality, we obtain

$$\begin{aligned} & \sup_{-1 < \tau < 0} \|v_n \phi(\cdot, \tau)\|_{L^2(B_1^+)} + \|\nabla v_n \phi\|_{L^2(Q_1^+)} \\ & \leq N \left( \|v_n\|_{L^2(Q_1^+)} + \epsilon_n \|v_n\|_{L^3(Q_1^+)} + \|v_n \phi\|_{L^3(Q_1^+)} \|q_n\|_{L^{\kappa^*, \lambda}(Q_1^+)}^{\frac{p}{3(p-2)}} \right). \end{aligned}$$

Therefore, we have

$$\|\nabla v_n\|_{L^2(Q_{3/4}^+)} \leq N \left( \|v_n\|_{L^2(Q_1^+)} + \epsilon_n \|v_n\|_{L^3(Q_1^+)} + \|v_n \phi\|_{L^3(Q_1^+)} \|q_n\|_{L^{\kappa^*, \lambda}(Q_1^+)}^{\frac{p}{3(p-2)}} \right).$$

Therefore,  $\nabla v_n$  is also uniformly bounded in  $L^2(Q_{3/4}^+)$  for the case  $2 < p < 9/4$ . For the case  $3/2 < p \leq 2$  (equivalently  $3 < q \leq 4$ ) we have

$$\|v_n \phi\|_{L^{p,q}(Q_1^+)} \leq N \sup_{-1 < s < \tau} \|v_n \phi(\cdot, s)\|_{L^2(B_1^+)}.$$

By proceeding as for the previous case, we can obtain the uniform bound of  $\nabla v_n$  in  $L^2(Q_{3/4}^+)$ . So together with (20), we get

$$\nabla v_n \rightharpoonup \nabla v \quad \text{in } L^2(Q_{3/4}^+), \quad v_n \rightarrow v \quad \text{in } L^3(Q_{3/4}^+).$$

Moreover,  $v$  and  $q$  solve the following linear Stokes system

$$\partial_s v - \Delta v + \nabla q = 0, \quad \text{div } v = 0 \quad \text{in } Q_1^+$$

with

$$v = 0 \quad \text{on } (B_1 \cap \{x_3 = 0\}) \times (-1, 0).$$

Next we show that

$$\partial_s v_n, \nabla^2 v_n, \nabla q_n \rightharpoonup \partial_s v, \nabla^2 v, \nabla q \quad \text{in } L^{\kappa, \lambda}(Q_{\frac{3}{8}}^+), \quad \text{respectively.} \tag{21}$$

Indeed, after direct calculations, we obtain

$$\|(v_n \cdot \nabla) v_n\|_{L^{\kappa, \lambda}(Q_{3/4}^+)} \leq N \|\nabla v_n\|_{L^2(Q_{3/4}^+)}^{\frac{2}{\lambda}} \|v_n\|_{L^{2, \infty}(Q_{3/4}^+)}^{\frac{3-2\kappa}{\kappa}}. \tag{22}$$

Due to the local boundary estimate for the Stokes system (see [20, Proposition 1]), we have the following estimate for  $v_n$  and  $q_n$ :

$$\begin{aligned} & \|\partial_s v_n\|_{L^{\kappa,\lambda}(Q_{5/8}^+)} + \|\nabla^2 v_n\|_{L^{\kappa,\lambda}(Q_{5/8}^+)} + \|\nabla q_n\|_{L^{\kappa,\lambda}(Q_{5/8}^+)} \\ & \leq N(\|v_n\|_{L^{\kappa,\lambda}(Q_{3/4}^+)} + \|\nabla v_n\|_{L^{\kappa,\lambda}(Q_{3/4}^+)} + \|q_n\|_{L^{\kappa,\lambda}(Q_{3/4}^+)} + \epsilon_n \|(v_n \cdot \nabla)v_n\|_{L^{\kappa,\lambda}(Q_{3/4}^+)}) \end{aligned}$$

Therefore, we obtain

$$\|\partial_s v_n\|_{L^{\kappa,\lambda}(Q_{5/8}^+)} + \|\nabla^2 v_n\|_{L^{\kappa,\lambda}(Q_{5/8}^+)} + \|\nabla q_n\|_{L^{\kappa,\lambda}(Q_{5/8}^+)} \leq N(1 + \epsilon_n), \tag{23}$$

where we used (22). The assertion (21) is established.

According to Hölder estimate of the Stokes system near boundary (see [20, Lemma 1]),  $v$  is Hölder continuous in  $Q_{1/2}^+$  with the exponent  $\alpha$  with  $0 < \alpha < 2(1 - 1/\lambda)$ . Here we fix  $\alpha = 1 - 1/\lambda$ , denoted by  $\alpha_0$  from now on. Then, by Hölder continuity of  $v$  and strong convergence of the  $L^3$ -norm of  $v_n$ , we obtain

$$C^{\frac{1}{3}}(v, \theta) \leq N_1 \theta^{1+\alpha_0}, \quad C(v_n, \theta) \rightarrow C(v, \theta). \tag{24}$$

Let  $\tilde{B}^+$  be a domain with smooth boundary such that  $B_{11/16}^+ \subset \tilde{B}^+ \subset B_{3/4}^+$ , and  $\tilde{Q}^+ := \tilde{B}^+ \times (-3/4)^2, 0$ . Now we consider the following initial and boundary problem:

$$\begin{aligned} \partial_s \hat{v}_n - \Delta \hat{v}_n + \nabla \hat{q}_n &= -\epsilon_n (v_n \cdot \nabla)v_n, \quad \operatorname{div} \hat{v}_n = 0 \quad \text{in } \tilde{Q}^+, \\ (\hat{q}_n)_{\tilde{B}^+}(s) &= 0, \quad s \in \left(-\left(\frac{3}{4}\right)^2, 0\right), \\ \hat{v}_n &= 0 \quad \text{on } \partial \tilde{B}^+ \times \left[-\left(\frac{3}{4}\right)^2, 0\right], \quad \hat{v}_n = 0 \quad \text{on } \tilde{B}^+ \times \left\{s = -\left(\frac{3}{4}\right)^2\right\}. \end{aligned}$$

Using the global estimate of the Stokes system (see [7, Theorem 3.1]), we get the following estimate

$$\begin{aligned} & \|\partial_s \hat{v}_n\|_{L^{\kappa,\lambda}(\tilde{Q}^+)} + \|\hat{v}_n\|_{L^{\kappa}((-\frac{3}{4})^2, 0); W_0^{2,\lambda}(\tilde{B}^+)} + \|\hat{q}_n\|_{L^{\kappa}((-\frac{3}{4})^2, 0); W^{1,\lambda}(\tilde{B}^+)} \\ & \leq \epsilon_n \|(v_n \cdot \nabla)v_n\|_{L^{\kappa,\lambda}(\tilde{Q}^+)} \leq N\epsilon_n. \end{aligned} \tag{25}$$

Next we define  $\tilde{v}_n$  and  $\tilde{q}_n$  as follows:

$$\tilde{v}_n = v_n - \hat{v}_n, \quad \tilde{q}_n = q_n - \hat{q}_n.$$

Then it is straightforward that  $\tilde{v}_n$  and  $\tilde{q}_n$  solve

$$\begin{aligned} \partial_s \tilde{v}_n - \Delta \tilde{v}_n + \nabla \tilde{q}_n &= 0, \quad \operatorname{div} \tilde{v}_n = 0 \quad \text{in } \tilde{Q}^+, \\ \tilde{v}_n &= 0 \quad \text{on } (\tilde{B}^+ \cap \{x_3 = 0\}) \times \left[-\left(\frac{3}{4}\right)^2, 0\right] \end{aligned}$$

and  $\tilde{v}_n, \tilde{q}_n$  satisfy the following estimate

$$\|\nabla^2 \tilde{v}_n\|_{L^{\kappa,\lambda}(Q_{5/8}^+)} + \|\nabla \tilde{q}_n\|_{L^{\kappa,\lambda}(Q_{5/8}^+)} \leq N(1 + \epsilon_n),$$

and furthermore, for  $\tilde{\kappa}$  with  $3/\tilde{\kappa} + 2/\lambda = 1$  we obtain

$$\|\nabla^2 \tilde{v}_n\|_{L^{\tilde{\kappa}, \lambda}(Q_{9/16}^+)} + \|\nabla \tilde{q}_n\|_{L^{\tilde{\kappa}, \lambda}(Q_{9/16}^+)} \leq N(1 + \epsilon_n). \tag{26}$$

Those estimates are again due to the local boundary estimate for the Stokes system (see [20, Proposition 1-2]). By the Poincaré inequality, we have

$$\tilde{D}(q_n, \theta) \leq N(\tilde{D}_1(\hat{q}_n, \theta) + \tilde{D}_1(\tilde{q}_n, \theta)).$$

We note that  $\tilde{D}_1(\hat{q}_n, \theta)$  goes to zero as  $n \rightarrow \infty$  because of the estimate (25). On the other hand, using the Hölder inequality and the estimate (26), we can show

$$\begin{aligned} \tilde{D}_1(\tilde{q}_n, \theta) &= \frac{1}{\theta} \left( \int_{-\theta^2}^0 \left( \int_{B^+(\theta)} |\nabla \tilde{q}_n|^{\tilde{\kappa}} dy \right)^{\frac{\lambda}{\tilde{\kappa}}} ds \right)^{\frac{1}{\lambda}} \leq \theta^2 \left( \int_{-\theta^2}^0 \left( \int_{B^+(\theta)} |\nabla \tilde{q}_n|^{\tilde{\kappa}} dy \right)^{\frac{\lambda}{\tilde{\kappa}}} ds \right)^{\frac{1}{\lambda}} \\ &\leq N\theta^2(1 + \epsilon_n). \end{aligned}$$

So summing up, we obtain

$$\liminf_{n \rightarrow \infty} \tilde{D}(q_n, \theta) \leq \lim_{n \rightarrow \infty} N_2\theta^2(1 + \epsilon_n) \leq N_2\theta^{1+\alpha_0}, \tag{27}$$

where  $N_2$  is an absolute constant. At the beginning in (19) we can take an absolute constant  $N$  bigger than  $2(N_1 + N_2)$ , where  $N_1$  and  $N_2$  are absolute constants in (24) and (27), respectively. Then this leads to a contradiction since

$$2(N_1 + N_2)\theta^{1+\alpha_0} \leq N\theta^{1+\alpha_0} \leq \liminf_{n \rightarrow \infty} \psi_n(\theta) \leq (N_1 + N_2)\theta^{1+\alpha_0}.$$

This completes the proof.  $\square$

The lemma above is the main part of the Lemma 7. Since the rest of the proof of Lemma 7 can be achieved by following similar procedures in [21], we present only a brief sketch of the main idea of Lemma 7.

**The sketch of the proof of Lemma 7.** We first note that the lemma above allows iterations (compare [21, Lemma 4.2]), and therefore, there exists a positive constant  $\alpha_1 < 1$  such that (compare [21, Lemmas 4.3 and 4.4])

$$(C^{\frac{1}{3}}(r) + \tilde{D}(r)) \leq N \left( \frac{r}{\rho} \right)^{1+\alpha_1} (C^{\frac{1}{3}}(\rho) + \tilde{D}(\rho) + m_\gamma(f)\rho^{\beta+1}), \quad r \leq \rho.$$

We consider for any  $w \in \overline{Q_{z, r_1/2}^+}$

$$\tilde{C}(w, r) := \frac{1}{r^2} \int_{Q_{w, r} \cap Q_{z, r_1/2}^+} |u - (u)_a|^3 dz, \quad (u)_a = \int_{Q_{w, r} \cap Q_{z, r_1/2}^+} u(z) dz,$$

and we can show that for any  $r < r_1/4$

$$\tilde{C}^{\frac{1}{3}}(w, r) \leq Nr^{1+\alpha_1},$$

where  $N = N(\lambda, \gamma, \|f\|_{M^{2,\gamma}})$  is an absolute constant. This argument can be proved using the same method as in [21, Lemma 5.2], and therefore we omit the details. The regularity of  $u$  at  $z$  is a standard consequence of this estimate. This completes the sketch of the proof.  $\square$

### 3. Local boundary regularity

In this section, we will present the proof of our main theorem (see Theorem 1). We first begin with an estimate for the scaled  $L^3$ -norm of suitable weak solutions.

**Lemma 9.** *Suppose (without loss of generality)  $z = (0, 0)$ . Let  $p, q$  be the numbers in (12) and  $Q_r^+ = B_r^+ \times (-r^2, 0)$ . Suppose  $u$  is a suitable weak solution of the Navier–Stokes equations satisfying Definition 4. If  $u \in L^{p,q}(Q_r^+)$  and  $u = 0$  on  $(B_r \cap \{x_3 = 0\}) \times (-r^2, 0)$ , then*

$$C(r) \leq NA^{\frac{1}{q}}(r)E^{1-\frac{1}{q}}(r)G(r). \tag{28}$$

**Proof.** We take  $\alpha, \beta$ , and  $\delta$  such that  $\alpha = 1/q, \beta = (1/3)(1 - 1/q)$ , and  $\delta = 1/p$ . Thus  $2\alpha + 6\beta + p\delta = 3$ , and, therefore, using the Hölder inequality, we obtain

$$\begin{aligned} \int_{B_r^+} |u|^3 dx &\leq \left( \int_{B_r^+} |u|^2 dx \right)^\alpha \left( \int_{B_r^+} |u|^6 dx \right)^\beta \left( \int_{B_r^+} |u|^p dx \right)^\delta \\ &\leq N \left( \int_{B_r^+} |u|^2 dx \right)^\alpha \left( \int_{B_r^+} |\nabla u|^2 dx \right)^{3\beta} \left( \int_{B_r^+} |u|^p dx \right)^\delta, \end{aligned}$$

where Sobolev embedding is used. Integrating in time, we obtain

$$\begin{aligned} \int_{Q_r^+} |u|^3 dz &\leq N \left( \sup_{-r^2 \leq t \leq 0} \int_{B_r^+} |u|^2 dx \right)^\alpha \int_{-r^2}^0 \left( \int_{B_r^+} |\nabla u|^2 dx \right)^{3\beta} \left( \int_{B_r^+} |u|^p dx \right)^\delta dt \\ &\leq N \left( \sup_{-r^2 \leq t \leq 0} \int_{B_r^+} |u|^2 dx \right)^\alpha \left( \int_{Q_r^+} |\nabla u|^2 dz \right)^{3\beta} \left( \int_{-r^2}^0 \left( \int_{B_r^+} |u|^p dx \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}, \end{aligned}$$

where we used  $3\beta + \alpha = 1$  and Hölder inequality. Dividing both sides by  $r^2$ , we obtain (28). This completes the proof.  $\square$

**Remark 10.** The above estimate (28) is also true in the case  $1 < q \leq \infty$  and  $1 < p < \infty$ , although we restrict to numbers  $p, q$  satisfying (12).

An immediate consequence of the local energy inequality is

$$\begin{aligned}
 A\left(\frac{r}{2}\right) + E\left(\frac{r}{2}\right) &\leq N\left(C^{\frac{2}{3}}(r) + C(r) + G(r)\tilde{D}(r) + r \int_{Q_r^+} |f|^2 dz\right), \\
 &\leq N\left(C^{\frac{2}{3}}(r) + C(r) + G(r)\tilde{D}(r) + r^{2(\gamma+1)}m_\gamma^2\right). \tag{29}
 \end{aligned}$$

For those exponents  $\kappa$  and  $\lambda$  in (8) and (11), we can show (compare (22))

$$\|(u \cdot \nabla)u\|_{L^{\kappa,\lambda}(Q_\rho^+)} \leq N\rho E^{\frac{1}{\lambda}}(\rho)A^{\frac{3-2\kappa}{2\kappa}}(\rho). \tag{30}$$

Its verification is straightforward, and we omit the details.

In next lemma we prove an estimate for the pressure.

**Lemma 11.** *Suppose  $z = (x, t), x \in \Gamma, t - \rho^2 > 0$ , and  $t < T$ . Then for  $0 \leq r \leq \rho/4$ ,*

$$\tilde{D}_1(r) \leq N\left(\left(\frac{\rho}{r}\right)\left(E^{\frac{1}{\lambda}}(\rho)A^{\frac{3-2\kappa}{2\kappa}}(\rho) + \rho^{\gamma+1}m_\gamma\right) + \left(\frac{r}{\rho}\right)\left(E^{\frac{1}{2}}(\rho) + \tilde{D}_1(\rho)\right)\right), \tag{31}$$

where  $\kappa$  and  $\lambda$  are numbers in (8) and (11).

**Proof.** Without loss of generality, we assume  $x = 0$ . We choose a domain  $\tilde{B}^+$  with a smooth boundary such that  $B_{\rho/2}^+ \subset \tilde{B}^+ \subset B_\rho^+$ , and we denote  $\tilde{Q}^+ := \tilde{B}^+ \times (t - \rho^2, t)$ . We note first that, by the definition of  $m_\gamma$  and the Hölder inequality, we have

$$\|f\|_{L^{\kappa,\lambda}(Q_\rho^+)} \leq N\rho^{\gamma+2}m_\gamma, \quad \|\nabla u\|_{L^{\kappa,\lambda}(Q_\rho^+)} \leq N\rho^2 E^{\frac{1}{2}}(\rho). \tag{32}$$

Let  $v$  and  $p_1$  be the unique solution to the following initial boundary value problem for the Stokes system

$$\begin{aligned}
 \partial_t v - \Delta v + \nabla p_1 &= (u \cdot \nabla)u + f, \quad \operatorname{div} v = 0 \quad \text{in } \tilde{Q}^+, \\
 (p_1)_{\tilde{B}^+}(t) &= \int_{\tilde{B}^+} p_1(y, t) dy = 0, \quad t \in (t - \rho^2, t), \\
 v &= 0 \quad \text{on } \partial\tilde{B}^+ \times [t - \rho^2, t], \quad v = 0 \quad \text{on } \tilde{B}^+ \times \{t = t - \rho^2\}.
 \end{aligned}$$

Then  $v$  and  $p_1$  satisfy the following estimate (see [7, Theorem 3.1])

$$\begin{aligned}
 \frac{1}{\rho^2}\|v\|_{L^{\kappa,\lambda}(\tilde{Q}^+)} + \frac{1}{\rho}\|\nabla v\|_{L^{\kappa,\lambda}(\tilde{Q}^+)} + \|\partial_t v\|_{L^{\kappa,\lambda}(\tilde{Q}^+)} + \|\nabla^2 v\|_{L^{\kappa,\lambda}(\tilde{Q}^+)} \\
 + \frac{1}{\rho}\|p_1\|_{L^{\kappa,\lambda}(\tilde{Q}^+)} + \|\nabla p_1\|_{L^{\kappa,\lambda}(\tilde{Q}^+)}
 \end{aligned}$$

$$\begin{aligned} &\leq N(\|u\nabla u\|_{L^{\kappa,\lambda}(\tilde{Q}^+)} + \|f\|_{L^{\kappa,\lambda}(\tilde{Q}^+)}) \leq N(\|u\nabla u\|_{L^{\kappa,\lambda}(Q_{\rho}^+)} + \|f\|_{L^{\kappa,\lambda}(Q_{\rho}^+)}) \\ &\leq N(\rho E^{\frac{1}{\lambda}}(\rho)A^{\frac{3-2\kappa}{2\kappa}}(\rho) + \rho^{\gamma+2}m_{\gamma}), \end{aligned}$$

where we used (30) and (32).

Let  $w = u - v$  and  $p_2 = p - (p)_{B_{\rho/2}^+} - p_1$ . Then  $w, p_2$  solve the following boundary value problem:

$$\partial_t w - \Delta w + \nabla p_2 = 0, \quad \operatorname{div} w = 0 \quad \text{in } \tilde{Q}^+, \quad w = 0 \quad \text{on } (\partial\tilde{B}^+ \cap \{x_3 = 0\}) \times [t - \rho^2, t].$$

Now we take  $\kappa'$  ( $\kappa'$  is different than  $\kappa^*$ ) such that  $3/\kappa' + 2/\lambda = 2$ . Then from the local estimate near the boundary for the Stokes system (see [20, Proposition 2]), we obtain

$$\begin{aligned} &\|\nabla^2 w\|_{L^{\kappa',\lambda}(Q_{\frac{\rho}{4}}^+)} + \|\nabla p_2\|_{L^{\kappa',\lambda}(Q_{\frac{\rho}{4}}^+)} \\ &\leq \frac{N}{\rho^2} \left( \frac{1}{\rho^2} \|w\|_{L^{\kappa,\lambda}(Q_{\frac{\rho}{2}}^+)} + \frac{1}{\rho} \|\nabla w\|_{L^{\kappa,\lambda}(Q_{\frac{\rho}{2}}^+)} + \frac{1}{\rho} \|p_2\|_{L^{\kappa,\lambda}(Q_{\frac{\rho}{2}}^+)} \right) \equiv \frac{N}{\rho^2} I. \end{aligned}$$

Using Sobolev imbedding, the right side can be estimated as follows:

$$I \leq \left( \frac{1}{\rho} \|\nabla u\|_{L^{\kappa,\lambda}(Q_{\frac{\rho}{2}}^+)} + \|\nabla p\|_{L^{\kappa,\lambda}(Q_{\frac{\rho}{2}}^+)} + \frac{1}{\rho} \|\nabla v\|_{L^{\kappa,\lambda}(Q_{\frac{\rho}{2}}^+)} + \frac{1}{\rho} \|p_1\|_{L^{\kappa,\lambda}(Q_{\frac{\rho}{2}}^+)} \right).$$

Due to the second inequality in (32), we obtain

$$\begin{aligned} \|\nabla p_2\|_{L^{\kappa',\lambda}(Q_{\frac{\rho}{4}}^+)} &\leq \frac{N}{\rho^2} (\rho E^{\frac{1}{2}}(\rho) + \rho \tilde{D}_1(\rho) + \rho E^{\frac{1}{\lambda}}(\rho)A^{\frac{3-2\kappa}{2\kappa}}(\rho) + \rho^{\gamma+2}m_{\gamma}) \\ &= \frac{N}{\rho} (E^{\frac{1}{2}}(\rho) + \tilde{D}_1(\rho) + E^{\frac{1}{\lambda}}(\rho)A^{\frac{3-2\kappa}{2\kappa}}(\rho) + \rho^{\gamma+1}m_{\gamma}). \end{aligned}$$

Now we assume  $0 \leq r \leq \rho/4$ . Noting that  $\|\nabla p_2\|_{L^{\kappa,\lambda}(Q_r^+)} \leq Nr^2 \|\nabla p_2\|_{L^{\kappa',\lambda}(Q_{\rho}^+)}$ , we have

$$\begin{aligned} \tilde{D}_1(r) &= \frac{1}{r} \|\nabla p\|_{L^{\kappa,\lambda}(Q_r^+)} \leq \frac{1}{r} (\|\nabla p_1\|_{L^{\kappa,\lambda}(Q_r^+)} + \|\nabla p_2\|_{L^{\kappa,\lambda}(Q_r^+)}) \\ &\leq \frac{1}{r} (\|\nabla p_1\|_{L^{\kappa,\lambda}(Q_{\rho}^+)} + r^2 \|\nabla p_2\|_{L^{\kappa',\lambda}(Q_{\rho}^+)}) \\ &\leq N \left( \frac{\rho}{r} \right) (E^{\frac{1}{\lambda}}(\rho)A^{\frac{3-2\kappa}{2\kappa}}(\rho) + \rho^{\gamma+1}m_{\gamma}) \\ &\quad + N \left( \frac{r}{\rho} \right) (E^{\frac{1}{2}}(\rho) + \tilde{D}_1(\rho) + E^{\frac{1}{\lambda}}(\rho)A^{\frac{3-2\kappa}{2\kappa}}(\rho) + \rho^{\gamma+1}m_{\gamma}) \\ &\leq N \left( \frac{\rho}{r} \right) (E^{\frac{1}{\lambda}}(\rho)A^{\frac{3-2\kappa}{2\kappa}}(\rho) + \rho^{\gamma+1}m_{\gamma}) + N \left( \frac{r}{\rho} \right) (E^{\frac{1}{2}}(\rho) + \tilde{D}_1(\rho)). \end{aligned}$$

This completes the proof.  $\square$



Now we are ready to present the proof of Theorem 1.

**Proof of Theorem 1.** We recall first, due to Lemma 9,

$$C(r) \leq NA^\alpha(r)E^{3\beta}(r)G(r), \quad \alpha = \frac{1}{q}, \quad \beta = \frac{q-1}{3q}, \tag{33}$$

and, due to Sobolev imbedding, we have

$$\tilde{D}(r) \leq N_d \tilde{D}_1(r). \tag{34}$$

Let  $4r < \rho$ . We consider  $C(r) + \tilde{D}_1(r)$ . Recalling the estimate (31) for the pressure, we obtain

$$\begin{aligned} \tilde{D}_1(r) + C(r) &\leq NA^\alpha(r)E^{3\beta}(r)G(r) + N\left(\frac{\rho}{r}\right)\left(E^{\frac{1}{\lambda}}\left(\frac{\rho}{4}\right)A^{\frac{3-2\kappa}{2\kappa}}\left(\frac{\rho}{4}\right) + m_\gamma\rho^{\gamma+1}\right) \\ &\quad + N\left(\frac{r}{\rho}\right)\left(E^{\frac{1}{2}}\left(\frac{\rho}{4}\right) + \tilde{D}_1\left(\frac{\rho}{4}\right)\right) \\ &\equiv \text{I} + \text{II} + \text{III}. \end{aligned}$$

We first consider the first term I. Since  $\alpha + 3\beta = 1$ , by using the local energy inequality (29), we have

$$\begin{aligned} \text{I} &\leq N(C^{\frac{2}{3}}(2r) + C(2r) + G(2r)\tilde{D}(2r) + r^{2(\gamma+1)}m_\gamma^2)G(r) \\ &\leq N\left(\left(\frac{\rho}{r}\right)^{\frac{7}{3}}C^{\frac{2}{3}}(\rho)G(\rho) + \left(\frac{\rho}{r}\right)^3C(\rho)G(\rho) + \left(\frac{\rho}{r}\right)^3G^2(\rho)\tilde{D}_1(\rho) + \left(\frac{\rho}{r}\right)r^{2(\gamma+1)}m_\gamma^2G(\rho)\right) \\ &\leq N\left(\left(\frac{\rho}{r}\right)^3C(\rho)G(\rho) + \left(\frac{\rho}{r}\right)^3G^2(\rho)\tilde{D}_1(\rho) + \left(\frac{\rho}{r}\right)G(\rho) + \left(\frac{\rho}{r}\right)\rho^{2(\gamma+1)}m_\gamma^2G(\rho)\right), \tag{35} \end{aligned}$$

where we used Young’s inequality and

$$C(2r) \leq N\left(\frac{\rho}{r}\right)^2C(\rho), \quad \tilde{D}_1(2r) \leq N\left(\frac{\rho}{r}\right)\tilde{D}_1(\rho), \quad G(2r) \leq N\left(\frac{\rho}{r}\right)G(\rho).$$

For the third term III, again using energy inequality (29), we have

$$\begin{aligned} \text{III} &\leq N\left(\frac{r}{\rho}\right)\left(\left(C^{\frac{1}{3}}\left(\frac{\rho}{2}\right) + C^{\frac{1}{2}}\left(\frac{\rho}{2}\right) + G^{\frac{1}{2}}\left(\frac{\rho}{2}\right)\tilde{D}_1^{\frac{1}{2}}\left(\frac{\rho}{2}\right) + m_\gamma\rho^{\gamma+1}\right) + \tilde{D}_1\left(\frac{\rho}{2}\right)\right) \\ &\leq N\left(\frac{r}{\rho}\right)\left(G\left(\frac{\rho}{2}\right) + C^{\frac{1}{3}}\left(\frac{\rho}{2}\right) + C\left(\frac{\rho}{2}\right) + \tilde{D}_1\left(\frac{\rho}{2}\right) + m_\gamma\rho^{\gamma+1}\right), \end{aligned}$$

where we used Young’s inequality, i.e.,  $ab \leq a^l/l + b^m/m$  where  $1/l + 1/m = 1$ ,  $1 < l, m < \infty$ . By (35), note that

$$\begin{aligned}
 C^{\frac{1}{3}}\left(\frac{\rho}{2}\right) &\leq NA^{\frac{\alpha}{3}}\left(\frac{\rho}{2}\right)E^{\beta}\left(\frac{\rho}{2}\right)G^{\frac{1}{3}}\left(\frac{\rho}{2}\right) \\
 &\leq N\left(C^{\frac{2}{3}}(\rho) + C(\rho) + G(\rho)\tilde{D}_1(\rho) + m_{\gamma}^2\rho^{2(\gamma+1)}\right)^{\frac{1}{3}}G^{\frac{1}{3}}(\rho) \\
 &\leq N\left(C^{\frac{2}{9}}(\rho) + C^{\frac{1}{3}}(\rho) + G^{\frac{1}{3}}(\rho)\tilde{D}_1^{\frac{1}{3}}(\rho) + m_{\gamma}^{\frac{2}{3}}\rho^{\frac{2}{3}(\gamma+1)}\right)G^{\frac{1}{3}}(\rho).
 \end{aligned}$$

Again applying Young’s inequality, we obtain

$$C^{\frac{1}{3}}\left(\frac{\rho}{2}\right) \leq N\left(C(\rho) + \tilde{D}_1(\rho) + G^{\frac{3}{7}}(\rho) + G(\rho) + m_{\gamma}^{\frac{2}{3}}\rho^{\frac{2}{3}(\gamma+1)}\right)G^{\frac{1}{3}}(\rho).$$

Summing up, we obtain

$$\text{III} \leq N\left(\frac{r}{\rho}\right)\left(C(\rho) + \tilde{D}_1(\rho) + G(\rho) + G^{\frac{3}{7}}(\rho) + m_{\gamma}\rho^{\gamma+1} + m_{\gamma}^{\frac{2}{3}}\rho^{\frac{2}{3}(\gamma+1)}\right)G^{\frac{1}{3}}(\rho). \tag{36}$$

It remains to consider the second term II. Since  $1/\lambda + (3 - 2\kappa)/(2\kappa) = 1$ , by (29), we obtain

$$\text{II} \leq N\left(\frac{\rho}{r}\right)\left(C^{\frac{2}{3}}\left(\frac{\rho}{2}\right) + C\left(\frac{\rho}{2}\right) + G\left(\frac{\rho}{2}\right)\tilde{D}_1\left(\frac{\rho}{2}\right) + \rho^{2(\gamma+1)}m_{\gamma}^2 + m_{\gamma}\rho^{\gamma+1}\right).$$

Using the same procedure as above, using (33) and Young’s inequality, we obtain

$$\begin{aligned}
 C^{\frac{2}{3}}\left(\frac{\rho}{2}\right) &\leq N\left(C^{\frac{4}{9}}(\rho) + C^{\frac{2}{3}}(\rho) + G^{\frac{2}{3}}(\rho)\tilde{D}_1^{\frac{2}{3}}(\rho) + m_{\gamma}^{\frac{4}{3}}\rho^{\frac{4}{3}(\gamma+1)}\right)G^{\frac{2}{3}}(\rho) \\
 &\leq N\left((G(\rho) + G^{\frac{1}{2}}(\rho))C(\rho) + G(\rho)\tilde{D}_1(\rho) + G^2(\rho) + G^{\frac{4}{5}}(\rho) + m_{\gamma}^{\frac{4}{3}}\rho^{\frac{4}{3}(\gamma+1)}\right)G^{\frac{2}{3}}(\rho),
 \end{aligned}$$

and

$$\begin{aligned}
 C\left(\frac{\rho}{2}\right) &\leq N\left(C^{\frac{2}{3}}(\rho) + C(\rho) + G(\rho)\tilde{D}_1(\rho) + m_{\gamma}^2\rho^{2(\gamma+1)}\right)G(\rho) \\
 &\leq N\left(C(\rho)G(\rho) + G^2(\rho)\tilde{D}_1(\rho) + G(\rho) + m_{\gamma}^2\rho^{2(\gamma+1)}\right)G(\rho).
 \end{aligned}$$

Summing up all together, we have

$$\begin{aligned}
 \text{II} &\leq N\left(\frac{\rho}{r}\right)\left((G(\rho) + G^{\frac{1}{2}}(\rho))C(\rho) + (G(\rho) + G^2(\rho))\tilde{D}_1(\rho) \right. \\
 &\quad \left. + G^{\frac{4}{5}}(\rho) + G^2(\rho) + m_{\gamma}^{\frac{4}{3}}\rho^{\frac{4}{3}(\gamma+1)}\right)G^{\frac{2}{3}}(\rho) + \rho^{2(\gamma+1)}m_{\gamma}^2 + m_{\gamma}\rho^{\gamma+1}. \tag{37}
 \end{aligned}$$

Adding (35), (36), and (37), we obtain

$$\begin{aligned}
 C(r) + \tilde{D}_1(r) &\leq N \left( \left( \frac{\rho}{r} \right)^3 G(\rho) + \left( \frac{\rho}{r} \right) (G(\rho) + G^{\frac{1}{2}}(\rho)) + \left( \frac{r}{\rho} \right) \right) C(\rho) \\
 &\quad + N \left( \left( \frac{\rho}{r} \right)^3 G(\rho) + \left( \frac{\rho}{r} \right) (G(\rho) + G^2(\rho)) + \left( \frac{r}{\rho} \right) \right) \tilde{D}_1(\rho) \\
 &\quad + \left( \frac{\rho}{r} \right) (G^{\frac{3}{7}}(\rho) + G^2(\rho)) + \left( \frac{\rho}{r} \right) m_\gamma \rho^{\gamma+1} + \left( \frac{\rho}{r} \right) m_\gamma^2 \rho^{2(\gamma+1)} G(r) \\
 &\quad + \left( \frac{\rho}{r} \right) m_\gamma^2 \rho^{2(\gamma+1)} + m_\gamma^{\frac{2}{3}} \rho^{\frac{2}{3}(\gamma+1)} G^{\frac{1}{3}}(\rho) + m_\gamma^{\frac{4}{3}} \rho^{\frac{4}{3}(\gamma+1)} G^{\frac{2}{3}}(\rho).
 \end{aligned}$$

We first choose  $\theta \in [0, 1/2]$  such that  $N\theta < 1/4$  where  $N$  is an absolute constant in the above inequality. By replacing  $r, \rho$  by  $\theta r$  and  $r$ , we obtain

$$\begin{aligned}
 C(\theta r) + \tilde{D}_1(\theta r) &\leq N \left( \left( \frac{1}{\theta^3} G(r) + \frac{1}{\theta} G^{\frac{1}{2}}(r) + \theta \right) C(r) \right. \\
 &\quad \left. + \left( \frac{1}{\theta^3} G(r) + \frac{1}{\theta} G^2(r) + \theta \right) \tilde{D}_1(r) + \phi(r) \right), \tag{38}
 \end{aligned}$$

where

$$\begin{aligned}
 \phi(r) &= \frac{1}{\theta} (G^{\frac{3}{7}}(r) + G^2(r)) + \frac{1}{\theta} m_\gamma r^{\gamma+1} + \frac{1}{\theta} m_\gamma^2 r^{2(\gamma+1)} G(r) \\
 &\quad + \frac{1}{\theta} m_\gamma^2 r^{2(\gamma+1)} + m_\gamma^{\frac{2}{3}} r^{\frac{2}{3}(\gamma+1)} G^{\frac{1}{3}}(r) + m_\gamma^{\frac{4}{3}} r^{\frac{4}{3}(\gamma+1)} G^{\frac{2}{3}}(r).
 \end{aligned}$$

Now we fix  $r_0 < \min\{1, \theta\epsilon^3/(1 + m_\gamma)\}$  such that for all  $r \leq r_0$

$$G(r) < \min \left\{ \frac{\theta^3}{2^{10}N}, \frac{\theta^2}{2^{10}N}, \frac{\epsilon^3\theta}{2^{10}N(N_d + 1)m_\gamma^2}, \left( \frac{\epsilon^3\theta}{2^{10}N(N_d + 1)} \right)^{\frac{7}{3}}, \frac{\epsilon^9}{(2^{10}N(N_d + 1))^3 m_\gamma^2} \right\},$$

where  $N, N_d$  are absolute constants in (38) and (34), respectively, and  $\epsilon$  is the fixed positive number in Lemma 7. Then one can check that  $\phi(r) < \epsilon^3/64N(N_d + 1)$  and moreover, we can show that for any  $r < r_0$

$$C(\theta r) + \tilde{D}_1(\theta r) \leq \frac{1}{2} (C(r) + \tilde{D}_1(r)) + \phi(r).$$

By iterating, we have

$$\begin{aligned}
 C(\theta^k r) + \tilde{D}_1(\theta^k r) &\leq \left( \frac{1}{2} \right)^k (C(r) + \tilde{D}_1(r)) + \sum_{i=0}^{k-1} \frac{N}{2^{k-1-i}} \phi(\theta^i r). \\
 &\leq \left( \frac{1}{2} \right)^k (C(r) + \tilde{D}_1(r)) + \frac{\epsilon^3}{64(N_d + 1)}. \tag{39}
 \end{aligned}$$

If  $z$  is a singular point, then there exists  $r_1 > 0$  such that  $C^{\frac{1}{3}}(r) + D(r) \geq \epsilon$  for every  $r \leq r_1$  by Lemma 7. However, this leads to a contradiction since for a sufficiently small  $r_2 < r_1$

$$C(r_2) + \tilde{D}(r_2) \leq C(r_2) + N_d \tilde{D}_1(r_2) \leq \frac{\epsilon^3}{64},$$

which immediately implies that  $C^{\frac{1}{3}}(r) + D(r) \leq \epsilon/2$ . This completes the proof.  $\square$

The following is a direct consequence of Theorem 1.

**Corollary 12.** *Let  $u$  be a weak solution of the Navier–Stokes equations satisfying Definition 4. Assume further that  $z = (x, t) \in \Gamma \times I$  and for some  $r_0 > 0$*

$$u \in L^{r,s}(Q_{z,r_0}^+), \quad \frac{3}{r} + \frac{2}{s} = 1, \quad 3 < r < \infty. \tag{40}$$

Then  $z$  is a regular point.

**Proof.** We observe that, as mentioned in the introduction,  $u \in L^4(Q_{z,r_0}^+)$  for weak solutions satisfying (40). To be more precise, we can show by interpolation that

$$\|u\|_{L^4(Q_{z,r_0}^+)} \leq \|u\|_{L^{r,s}(Q_{z,r_0}^+)}^{\frac{1}{2}} \|u\|_{L^{2,\infty}(Q_{z,r_0}^+)}^{\frac{1}{q}} \|u\|_{L^{6,2}(Q_{z,r_0}^+)}^{\frac{3}{2p}}.$$

The above estimate is true even in case  $(r, s) = (3, \infty)$  or  $(r, s) = (\infty, 2)$ , although our analysis does not include such cases. We conclude by the above estimate that  $u$  is a suitable weak solution in  $Q_{z,r_0}^+$ , namely  $u$  satisfies the local energy inequality (15) in a neighborhood of  $z$ . We also note that there exists a number  $\tilde{r}$  such that  $3/\tilde{r} + 2/s = 2$  and, by the Hölder inequality, we have

$$\frac{1}{\rho} \|u\|_{L^{\tilde{r},s}(Q_{z,\rho}^+)} \leq C \|u\|_{L^{r,s}(Q_{z,\rho}^+)}, \quad \text{for any } \rho \leq r_0.$$

Since the right-hand side above is finite, and it can be arbitrary small for sufficiently small  $\rho$  by assumption (40), condition (6) in Theorem 1 is satisfied. This completes the proof.  $\square$

**Remark 13.** Condition (40), including the case  $(r, s) = (\infty, 2)$ , is called a zero-dimensional integral condition. It is known that weak solutions of the Navier–Stokes equations are locally regular at an interior point provided that a zero-dimensional integral condition is assumed near the point (see [24,30]). This result, recently, was extended up to the boundary (see [9,29]), and, therefore, Corollary 12 is already implied by [9,29]. Our regularity criterion (6), however, gives a simple proof of the regularity of weak solutions near the boundary under a zero-dimensional integral condition (although the case  $(r, s) = (\infty, 2)$  is not covered by our analysis).

#### 4. Partial regularity

In this section, as an application of Theorem 1, we investigate the size of the possible singular set under additional integrability assumptions on weak solutions (see Assumption 14 below). As we saw in Corollary 12, we have a simple proof for weak solutions that a zero-dimensional

integral condition implies regularity up to the boundary. It is, however, an open question whether or not weak solutions (or suitable weak solutions) satisfy the zero-dimensional integral conditions. It was proved that the size of a possible singular set for suitable weak solutions is of 1-dimensional Hausdorff measure zero (see [1,21] for the interior case and for the boundary case, respectively). We remark that in [3] the estimate of the Hausdorff measure of the singular set for suitable weak solutions was improved by a logarithmic factor for the interior case.

Our aim in this section is to present the proof of Theorem 2, which says that the size of singular set for weak solutions can be reduced under additional integrability assumptions, which are weaker than zero-dimensional integral conditions. We note, however, that our result is weaker than what one gets from the zero-dimensional integral conditions; in that case full regularity is implied, but in our case we have partial regularity.

We start with recalling the following condition that is assumed in Theorem 2 for weak solutions.

**Assumption 14.** Let  $u$  be a weak solution of the Navier–Stokes equations satisfying Definition 4 and

$$u \in L^{m,l}(Q) = L^m((0, T); L^l(\Omega)),$$

where either

$$1 < \frac{3}{l} + \frac{2}{m}, \quad \frac{2}{l} + \frac{2}{m} < 1, \quad l > m, \quad \text{or} \tag{41}$$

$$1 < \frac{3}{l} + \frac{2}{m}, \quad \frac{3}{l} + \frac{1}{m} < 1, \quad l \leq m. \tag{42}$$

We remark that although Assumption 14 is not justified by the formulation of weak solutions, it seems to be of independent interest to characterize the size of the singular set depending on the mixed norm  $L^{l,m}$  of  $u$ .

We observe first that solutions satisfying Assumption 14 are in fact suitable weak solutions. This can be done by the interpolation argument of Corollary 12. More precisely, we can show

$$\|u\|_{L^4(Q)} \leq \|u\|_{L^{l,m}(Q)}^\sigma \|u\|_{L^{2,\infty}(Q)}^{\frac{(6-\alpha)(1-\sigma)}{2\alpha}} \|u\|_{L^{6,2}(Q)}^{\frac{2}{\beta}(1-\sigma)},$$

where

$$\alpha = \frac{\frac{3}{l} + \frac{2}{m} - \frac{5}{4}}{\frac{1}{l} + \frac{1}{2m} - \frac{3}{8}}, \quad \beta = \frac{4(\frac{3}{l} + \frac{2}{m} - \frac{5}{4})}{3(\frac{1}{l} + \frac{1}{m} - \frac{1}{2})}, \quad \sigma = \frac{l(4-l)}{4(l-\alpha)}.$$

Now we are ready to prove Theorem 2.

**Proof of Theorem 2.** We consider first the case that  $l, m$  satisfy (42). For convenience, we define  $\varsigma$  as  $\varsigma := 3/l + 2/m - 1$ . For given  $R > 0$  we set  $S_R := S \cap B_R(0)$ , where  $S$  is the singular set of  $u$ . Let  $z = (x, t) \in S_R$  and  $\tilde{Q}_r(z) = \tilde{Q}_r(x, t) := Q_r(x, t) \cap Q$ . Using Theorem 1 (boundary case) and Theorem 18 (interior case) in Appendix A, we see that there exists  $e_0$  such that

$$\limsup_{r \rightarrow 0_+} r^{-\varsigma} \|u\|_{L^m L^l_x(\tilde{Q}_r(z))} \geq e_0. \tag{43}$$

It is clear that the Lebesgue measure of  $S_R$  is zero, and so we can choose an open bounded set  $V \subset \mathbb{R}^3$  containing  $\overline{S_R}$  with the volume of  $V$  as small as we like. Moreover, due to (43), for any given  $\delta > 0$ , and for any  $z = (x, t) \in S_R$ , there exists  $r_z \in (0, \delta)$  such that  $\|u\|_{L_t^m L_x^l(\tilde{Q}_{r_z}(z))} \geq r_z^\zeta e_0$ . We denote by  $\mathcal{O} = \{\tilde{Q}_{r_z}(z) : z \in S_R\}$  the collection of such open neighbourhoods. We note that  $\mathcal{O}$  is an open covering of  $S_R$ , and, therefore, by a covering lemma (e.g., see [1, Lemma 6.1]), we can find a countable subfamily of disjoint cylinders  $\tilde{Q}_j = \tilde{Q}_{r_{z_j}}(z_j)$ ,  $j \in J$  such that  $S_R \subset \bigcup_{z \in S_R} \tilde{Q}_{r_z}(z) \subset \bigcup_{j \in J} \tilde{Q}_{5r_j}(z_j)$ . Denoting  $u_j(z) = u(z)$  if  $z \in Q_j^+$ ,  $u_j(z) = 0$  otherwise, we have

$$e_0^m \sum_{j \in J} r_j^{5m} \leq \sum_{j \in J} \int_I \left( \int_{\Omega} |u_j|^l dx \right)^{m/l} dt \leq \int_I \left( \sum_{j \in J} \int_{\Omega} |u_j|^l dx \right)^{m/l} dt \leq \|u\|_{L_t^m L_x^l(V)}^m,$$

where we used  $l \leq m$  in the second inequality. Since the volume of  $V$  can be taken arbitrarily small, as can  $\delta$ , we conclude that the  $\zeta m$ -dimensional Hausdorff measure of  $S_R$  is zero:  $\mathcal{P}^{\zeta m}(S_R) = 0$ . Since  $R$  is arbitrary, we conclude that the singular set  $S$  is of  $\zeta m = (2 - m + 3m/l)$ -dimensional parabolic Hausdorff measure zero. Observe that  $0 < \zeta m < 1$ .

Next we consider the case  $l, m$  satisfy (41). In this case we show, by interpolation, that

$$\|u\|_{L^k(Q)} \leq \|u\|_{L^{l,m}(Q)}^{1-\sigma} \|u\|_{L^{2,\infty}(Q)}^\sigma,$$

where

$$k = 2 \left( 1 - \frac{m}{l} \right) + l \frac{m}{l} = 2 + m - \frac{2m}{l}, \quad \sigma = \frac{m}{k}.$$

It is clear that  $4 < k < 5$  and  $m/k \leq 1$ , and therefore, due to the analysis of the case  $l \leq m$ , we conclude that the singular set is at most of  $(5 - k)$ -dimensional parabolic Hausdorff measure zero up to the boundary. This completes the proof.  $\square$

### Appendix A

In this appendix we show that the regularity criterion (6) holds also for the interior case, whose proof requires slightly different estimates. Since its verification for the interior case can be done by following a procedure similar to that of the boundary case, with no significant difficulty, we just present a sketch of the proof. From now on we replace  $Q_{z,r}^+$  and  $B_{x,r}^+$  by  $Q_{z,r}$  and  $B_{x,r}$  in the scaling invariant functionals below, because we are concerned with local regularity at an interior point. The interior case is in fact simpler than the boundary case, because the pressure is much easier to handle. We begin with the following lemma, which is analogous to Lemma 8 of the boundary case.

**Lemma 15.** *Let  $0 < \theta < 1/2$  and  $0 < \beta < \gamma \leq 2$ . There exist  $\epsilon_2, r_2 > 0$  depending on  $\lambda, \theta, \gamma$  and  $\beta$  such that if  $u$  is a suitable weak solution of the Navier–Stokes equations satisfying Definition 4,  $z = (x, t) \in Q = \Omega \times I$  is an interior point, and  $C^{\frac{1}{3}}(r) + \tilde{D}(r) + m_\gamma(f)r^{\beta+1} < \epsilon_2$  for some  $r \in (0, r_2)$ , then*

$$(C^{\frac{1}{3}}(\theta r) + \tilde{D}(\theta r)) < N\theta^{1+\alpha} (C^{\frac{1}{3}}(r) + \tilde{D}(r) + m_\gamma(f)r^{\beta+1}),$$

where  $0 < \alpha < 1$  and  $N > 0$  are absolute constants.

**The sketch of the proof of Lemma 15.** Again assume  $f = 0$  for simplicity. Suppose that the assertion is not true. Then there exist  $z_n = (x_n, t_n)$ ,  $r_n \searrow 0$ , and  $\epsilon_n \searrow 0$  such that  $\varphi(r_n) = \epsilon_n$  but  $\varphi(\theta r_n) > N\theta^{1+\alpha}\epsilon_n$ , where  $\varphi(r) = C^{\frac{1}{3}}(r) + \tilde{D}(r)$ . Using the change of variables  $y = r_n^{-1}(x - x_n)$  and  $s = r_n^{-2}(t - t_n)$ , we set  $v_n(w) := \epsilon_n^{-1}r_n u(z)$  and  $q_n(w) := \epsilon_n^{-1}r_n^2(p(z) - (p)_{B_{r_n}^+(t)})$ . By the “blow up” procedure and compactness arguments, the limit equations become the Stokes system. Since the pressure of the Stokes system is harmonic in the spatial variables for the interior case, our arguments are much simpler than in the boundary case. The other parts of the arguments are the same as in Lemma 8, and we omit the details.  $\square$

Due to the Lemma 15, we have the following lemma (compare to Lemma 7). Since the arguments are straightforward, we state it without proof.

**Lemma 16.** *There exists a constant  $\epsilon > 0$  depending on  $\lambda, \gamma$  and  $\|f\|_{M_{2,\gamma}}$  such that if  $u$  is a suitable weak solution of the Navier–Stokes equations satisfying Definition 4,  $z = (x, t) \in Q = \Omega \times I$  is an interior point, and*

$$\liminf_{r \rightarrow 0_+} (C^{\frac{1}{3}}(r) + \tilde{D}(r)) < \epsilon,$$

then  $z$  is a regular point.

Next we need the estimate for the pressure. To do that, we observe that at an interior point, instead of (28) for the boundary case, we can show

$$C(r) \leq N(A^{\frac{1}{q}}(r)E^{1-\frac{1}{q}}(r)G(r) + A^{\frac{1}{2}}(r)G^2(r)). \tag{A.1}$$

Indeed,

$$\int_{B_r} |u|^3 dy \leq N \left( \int_{B_r} |u - (u)_a|^3 dy + \int_{B_r} |(u)_a|^3 dy \right),$$

where  $(u)_a = \int_{B_r} u(y) dy$ . We note that the first of the above inequalities can be estimated as the same way as Lemma 9, and thus it is enough to consider the second one. We observe that

$$|(u)_a| \leq \frac{N}{r^{3/2}} \left( \int_{B_r} |u(y)|^2 dy \right)^{\frac{1}{2}} \leq \frac{N}{r} A^{\frac{1}{2}}(r), \quad |(u)_a| \leq \frac{N}{r^{3/p}} \left( \int_{B_r} |u(y)|^p dy \right)^{\frac{1}{p}}.$$

The second one is estimated as follows:

$$\int_{B_r} |(u)_a|^3 dy \leq \frac{N}{r^{6/p-2}} A^{\frac{1}{2}}(r) \left( \int_{B_r} |u(y)|^p dy \right)^{\frac{2}{p}}.$$

Integrating in time, and using  $q > 2$ , we obtain (A.1). Since the computations are straightforward, the details are skipped. In a similar manner, we also have the following estimate in the interior case (compare to (30) for the boundary case).

$$\|(u \cdot \nabla)u\|_{L^{\kappa,\lambda}(Q_\rho)} \leq N\rho \left( E^{\frac{1}{\lambda}}(\rho) A^{\frac{3-2\kappa}{2\kappa}}(\rho) + E^{\frac{1}{2}}(\rho) A^{\frac{2-\kappa}{2\kappa}}(\rho) G^{\frac{\kappa-1}{\kappa}}(\rho) \right). \quad (\text{A.2})$$

Since its verification is similar to (A.1), we omit the details.

Using the above estimate, we have the pressure estimate, equivalent to Lemma 11 of the boundary case. Since the estimates (A.2) and (A.1) are slightly different than (28) and (30) of the boundary case, the estimate of the pressure is slightly modified in the interior case. But it can be derived in the same manner, and so we skip its proof and just state it.

**Lemma 17.** *Suppose that  $z = (x, t) \in \Omega \times I$  is an interior point, and  $t - \rho^2 > 0, t < T$ . Then for  $0 \leq r \leq \rho/4$ ,*

$$\begin{aligned} \tilde{D}_1(r) \leq N \left( \frac{\rho}{r} \right) & \left( E^{\frac{1}{\lambda}}(\rho) A^{\frac{3-2\kappa}{2\kappa}}(\rho) + E^{\frac{1}{2}}(\rho) A^{\frac{2-\kappa}{2\kappa}}(\rho) G^{\frac{\kappa-1}{\kappa}}(\rho) + \rho^{\gamma+1} m_\gamma \right) \\ & + N \left( \frac{r}{\rho} \right) \left( E^{\frac{1}{2}}(\rho) + \tilde{D}_1(\rho) \right), \end{aligned}$$

where  $\kappa$  and  $\lambda$  are numbers satisfying (8) and (11).

Using the estimate of the pressure in Lemma 17, the same regularity criterion for interior points can be proved as in the boundary case. We have:

**Theorem 18.** *The same statement of Theorem 1 remains correct when  $z \in Q$  is an interior point, with  $B_{x,r}^+$  replaced by  $B_{x,r}$ .*

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