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Irregular and multi-channel sampling of operators

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ABSTRACT

The classical sampling theorem for bandlimited functions has recently been generalized to apply to so-called bandlimited operators, that is, to operators with band-limited Kohn–Nirenberg symbols. Here, we discuss operator sampling versions of two of the most central extensions to the classical sampling theorem. In irregular operator sampling, the sampling set is not periodic with uniform distance. In multi-channel operator sampling, we obtain complete information on an operator by multiple operator sampling outputs.

1. Introduction

The so-called classical sampling theory addresses the problem of determining and reconstructing functions from countably many values that are attained on a discrete subset of the real line. The fundamental result in this theory is attributed to Whittaker, Kotel'nikov and Shannon. It asserts that a function bandlimited to an interval of length Ω can be recovered from the values of the function sampled regularly at Ω values per unit interval.

During the last few years the herein considered sampling theory for operators has been developed. It is motivated by the operator identification problem in communications engineering. There, the objective is to identify a channel operator from knowledge of the channel's action on a chosen input signal. A well known identification result states, for example, that any time-invariant channel operator is fully determined by its action on the Dirac impulse. Already in the 1960s, Kailath [14] and Bello [2] proclaimed that this simple identifiability result on time-invariant operators could be generalized to slowly time-varying operators, that is, to operators whose spreading functions are supported on sets of measure less than or equal to one. The spreading function is the symplectic Fourier transform of the operator's Kohn–Nirenberg symbol. The assertions of Kailath and Bello were confirmed in [19,28].

The identifiability results in [19,28] use weighted sums of regularly spaced delta impulses as identifiers for Hilbert–Schmidt operators with bandlimited Kohn–Nirenberg symbols. The discrete support of such a tempered distribution we shall refer to in the following as sampling set for operator sampling. Together with the fact that the classical sampling theorem can be seen as a special case of the identifiability results for bandlimited operators [19,28,30]—consider a bandlimited

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function as a multiplication operator which can be determined from its action on a regularly spaced sum of Dirac impulses—this has led to the development of the herein considered sampling theory for operators.

In this paper, we state and prove operator sampling versions of key generalizations of the classical sampling theorem for functions. Additional operator sampling results for classes of pseudodifferential operators are given in [30].

One central extension of the classical sampling theorem considers irregular or nonuniform sampling sets, see [23] and references therein. In practice, sampling with uniform distance is hardly realizable because of imperfections in data acquisition devices or perturbations during the collecting of data. As it is similarly challenging to generate regularly spaced sums of impulses for operator sampling, the consideration of irregularly placed impulses as identifiers for operators is natural. Here, we will give separation and Beurling density results for operator sampling which resemble corresponding results in the classical sampling theory for functions.

The feasibility of the classical sampling theorem in case of large bandwidth signals is limited by the sampling rates achievable in state of the art hardware. This problem is addressed through multi-channel sampling as pioneered by Papoulis [26]. Multi-channel sampling employs a number of samplers in parallel, thereby allowing the acquisition of samples to be carried out in each channel at a fraction of the sampling rate foreseen by the classical sampling theorem. Multi-channel sampling in the theory of operator sampling employs similarly the combination of multiple outputs from sampling procedures in order to reduce the rate at which impulses have to be sent. In addition, multi-channel sampling for operators, that is, multiple output sampling for operators, allows to identify operators whose Kohn–Nirenberg symbols are only bandlimited to an area of measure larger than one, as shown below. Note that in single output operator sampling, only operators with bandlimitations given by sets of measure less than one can be identified. Larger bandwidth of the Kohn–Nirenberg symbol cannot be compensated by an increase of the so-called sampling rate.[3]

We formulate and prove our results for Hilbert–Schmidt operators. Means for generalizing such results to non compact operators are outlined in [27,30].

This paper is organized as follows. In Section 2, we give some background for operator sampling and introduce operator Paley–Wiener spaces. In Section 3 we state the uniform operator sampling result for operator Paley–Wiener spaces as given in [29]. We include a new proof of this result, a proof that allows for generalizations to the setting of irregular and multi-channel operator sampling. We give a generalization to operator classes which have not necessarily bandlimited Kohn–Nirenberg symbols. In Section 4, we develop irregular operator sampling for operator Paley–Wiener spaces. Also, we consider irregular sampling of operators whose Kohn–Nirenberg symbols are not bandlimited in view of Kramer's Lemma setting. Multi-channel operator sampling is discussed in Section 5.

2. Preliminaries

The Fourier transform on $L^2(\mathbb{R}^d)$ is densely defined by

$$\mathcal{F}(f)(\xi)=\hat{f}(\xi)=\int_{\mathbb{R}^d} f(t)e^{-2\pi i t\cdot\xi}\,dt, \quad f\in L^1\big(\mathbb{R}^d\big)\cap L^2\big(\mathbb{R}^d\big).$$

Similarly, the symplectic Fourier transform $\mathcal{F}_s: L^2(\mathbb{R}^{2d})\to L^2(\mathbb{R}^{2d})$ is densely defined by

$$\mathcal{F}_s f(t,\nu)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x,\xi)\,e^{-2\pi i(\nu\cdot x-\xi\cdot t)}\,dx\,d\xi, \quad f\in L^1\big(\mathbb{R}^{2d}\big)\cap L^2\big(\mathbb{R}^{2d}\big).$$

For brevity of notation, we shall refrain from marking equalities and inequalities that hold in the L^2-sense with the customary a.e. whenever the context is unambiguous.

Let $S_\Omega=\prod_{k=1}^d[-\frac{\Omega_k}{2},\frac{\Omega_k}{2}]$ with $\Omega=(\Omega_1,\Omega_2,\dots,\Omega_d)$.

Definition 1. The Paley–Wiener space with bandwidth Ω is defined by

$$PW(S_\Omega)=\big\{f\in L^2\big(\mathbb{R}^d\big)\colon \operatorname{supp}\hat{f}\subseteq S_\Omega\big\}.$$

It is known that if an expansion for $f\in PW(S_\Omega)$ converges in the norm of $PW(S_\Omega)$, that is, in the L^2-norm, then it converges pointwise and uniformly over $\mathbb{R}^d$ (see p. 57 in [10]). Note that $PW(S_\Omega)$ is isometrically isomorphic to $L^2(S_\Omega)$ due to Plancherel's theorem. For simplicity of notation, we shall denote by $L^2(S_\Omega)$ the subspace of $L^2(\mathbb{R}^d)$ which consists of $L^2(\mathbb{R}^d)$ functions supported on S_Ω.

The classical sampling theorem shown independently by Whittaker, Kotel'nikov and Shannon is generalized by the following *oversampling* theorem [3]. It is based on possibly collecting samples more often than the sampling rate prescribes.

[3] Operator sampling is not simply an higher dimensional analogue of the 1-d Shannon sampling theorem. In the case of an operator acting on $L^2(\mathbb{R})$, the operator's 2-dimensional Kohn–Nirenberg symbol is to be determined from a signal defined on $\mathbb{R}$. No access to sample values of the Kohn–Nirenberg symbol is given, as is the case in 2-dimensional Shannon sampling theory.

The sampling rate, usually called Nyquist–Landau rate, for a function in $PW(S_\Omega)$ is defined to be the Lebesgue measure of the set S_Ω.

Here and in the following we use the notation

$$A(F) \asymp B(F), \quad F \in \mathcal{F},$$

if there exist positive constants c and C such that $cA(F) \leqslant B(F) \leqslant CA(F)$ for all objects F in the set $\mathcal{F}$.

Theorem 2. *Let $\Omega = (\Omega_1, \ldots, \Omega_d)$, $T = (T_1, \ldots, T_d)$ with $T_k, \Omega_k > 0$ and $T_k \cdot \Omega_k < 1$ for all $k \in \mathbb{Z}$. For $\frac{2}{T} = (\frac{2}{T_1}, \ldots, \frac{2}{T_d})$ and $\varphi \in PW(S_{\frac{2}{T}-\Omega})$ with $\hat{\varphi} = 1$ on S_Ω, we have the sampling expansion*

$$f(t) = T \sum_{n \in \mathbb{Z}^d} f(nT)\varphi(t - nT), \quad f \in PW(S_\Omega), \tag{1}$$

with uniform and $L^2(\mathbb{R}^d)$-convergence. Moreover, (1) is stable in the sense that

$$\|f\|_{L^2}^2 \asymp \sum_{n \in \mathbb{Z}^d} \left|f(nT)\right|^2, \quad f \in PW(S_\Omega).$$

In practice, a stable sampling expansion guarantees that perturbations in the sampling output and in the reconstruction procedure are controlled by error bounds on the input function and vice versa.

The development of operator sampling necessitates the use of some rudimentary distribution theory. The space of distributions chosen here is the dual of the Feichtinger algebra $S_0(\mathbb{R}^d)$. The dual $S_0'(\mathbb{R}^d)$ is a Banach space with $\mathcal{S}(\mathbb{R}^d) \subsetneq S_0(\mathbb{R}^d) \subsetneq L^2(\mathbb{R}^d) \subsetneq S_0'(\mathbb{R}^d) \subsetneq \mathcal{S}'(\mathbb{R}^d)$, where $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz class of rapidly decaying functions and $\mathcal{S}'$ its dual, the space of tempered distribution. There are several equivalent definitions of the Feichtinger algebra $S_0(\mathbb{R}^d)$ [7]. We choose the characterization of $S_0(\mathbb{R}^d)$ via the short time Fourier transform.

Definition 3. The Feichtinger algebra is defined by

$$S_0(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d)\colon V_{\mathfrak{g}} f(t, \nu) \in L^1(\mathbb{R}^{2d})\},$$

where $V_{\mathfrak{g}} f(t,\nu) = \langle f, M_\nu T_t \mathfrak{g}\rangle$ is the short-time Fourier transform of f with respect to the Gaussian $\mathfrak{g}(x) = e^{-\pi\|x\|^2}$. The norm on $S_0(\mathbb{R}^d)$ is given by $\|f\|_{S_0} = \|V_{\mathfrak{g}} f\|_{L^1}$.

In this paper we consider the sampling problem for Hilbert–Schmidt operators only.

Definition 4. The class of Hilbert–Schmidt operators $HS(L^2(\mathbb{R}^d))$ consists of bounded linear operators on $L^2(\mathbb{R}^d)$ which can be represented as integral operators of the form

$$Hf(x) = \int \kappa_H(x,t) f(t)\, dt, \quad f \in L^2(\mathbb{R}^d),$$

with kernel $\kappa_H \in L^2(\mathbb{R}^{2d})$.

The linear space of Hilbert–Schmidt operators $HS(L^2(\mathbb{R}^d))$ becomes a Hilbert space if it is endowed with the Hilbert space structure of $L^2(\mathbb{R}^d)$, that is, by

$$\langle H_1, H_2\rangle_{HS} = \langle \kappa_{H_1}, \kappa_{H_2}\rangle_{L^2}.$$

In view of pseudodifferential operators, the Kohn–Nirenberg symbol σ_H [8,17] of a Hilbert–Schmidt operator H is given by

$$\sigma_H(x,\xi) = \int \kappa_H(x, x-t) e^{-2\pi i t\cdot\xi}\, dt.$$

It leads to the operator representation

$$Hf(x) = \int \sigma_H(x,\xi) \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi, \quad f \in L^2(\mathbb{R}^d).$$

In time–frequency analysis and communication engineering, the spreading function η_H of a Hilbert–Schmidt operator H is commonly considered. It is given by

$$\eta_H(t,\nu) = \int \kappa_H(x, x-t) e^{-2\pi i x\cdot\nu}\, dx$$

and leads to

$$Hf(x) = \iint \eta_H(t,\nu) M_\nu T_t f(x)\, dt\, d\nu, \quad f \in L^2(\mathbb{R}^d), \tag{2}$$

where the time-shift (translation) and frequency-shift (modulation) operators T_t and M_ν are defined by $T_t f(x) = f(x-t)$ and $\widehat{M_\nu f}(\gamma) = \hat{f}(\gamma - \nu)$, respectively. That is, a Hilbert–Schmidt operator H is a continuous superposition of translation and modulation operators with coefficient function η_H. The identity (2) is understood weakly, namely

$$\langle Hf, g\rangle = \iint \eta_H(t,\nu) \langle M_\nu T_t f, g\rangle\, dt\, d\nu, \quad g \in L^2(\mathbb{R}^d).$$

As $\eta_H = \mathcal{F}_s \sigma_H$, operators with band-limited Kohn–Nirenberg symbols are operators whose spreading functions are compactly supported.

In communications, the time-varying operator H is also commonly represented by its time-varying impulse response $h_H(t,x)$ with

$$Hf(x) = \int h_H(t,x) f(x-t)\, dt,$$

where $h_H(t,x) = \kappa_H(x, x-t) = \int \eta_H(t,\nu) e^{2\pi i x\nu}\, d\nu$ a.e. Note that

$$\|H\|_{HS} = \|\kappa_H\|_{L^2} = \|h_H\|_{L^2} = \|\sigma_H\|_{L^2} = \|\eta_H\|_{L^2}.$$

Definition 5. The operator Paley–Wiener space of operators bandlimited to $S \subseteq \mathbb{R}^{2d}$ is

$$OPW(S) = \{H \in HS(L^2(\mathbb{R}^d)):\ \operatorname{supp} \mathcal{F}_s \sigma_H \subseteq S\}.$$

In the engineering literature, channels modeled by operators with $\operatorname{supp} \mathcal{F}_s\sigma_H = \operatorname{supp} \eta_H \subseteq [a_1, b_1] \times \cdots \times [a_{2d}, b_{2d}]$ are commonly referred to as underspread or slowly time-varying operators if volume $([a_1, b_1] \times \cdots \times [a_{2d}, b_{2d}]) = (b_1 - a_1) \times \cdots \times (b_{2d} - a_{2d}) \leqslant 1$ and as overspread operators else (see, for example, respective references in [19]).

Observe that $H \in OPW(S)$, S compact, implies $\eta_H = \mathcal{F}_s \sigma_H \in L^2(\mathbb{R}^{2d})$ with compact support. As then $\eta_H(t,\cdot) \in L^2(\mathbb{R}^d)$ for almost every $t \in \mathbb{R}^d$, we have, with Ω sufficiently large, $h_H(t,\cdot) \in PW(-\frac{\Omega}{2}, \frac{\Omega}{2})$ for almost every $t \in \mathbb{R}^d$, a fact that will be used frequently below.

We formulate the operator identification and sampling problems as follows.[4]

Definition 6. An operator class $\mathcal{H} \subseteq HS(L^2(\mathbb{R}^d))$ is identifiable if all $H \in \mathcal{H}$ extend to a domain containing a so-called identifier $f \in S_0'(\mathbb{R}^d)$ with

$$\|H\|_{HS} \asymp \|Hf\|_{L^2}, \quad H \in \mathcal{H}. \tag{3}$$

The operator class $\mathcal{H} \subseteq HS(L^2(\mathbb{R}^d))$ permits operator sampling if one can choose f in (3) with discrete support in $\mathbb{R}^d$ in the distributional sense. In that case, $\operatorname{supp} f$ is called sampling set for $\mathcal{H}$.

Note that any $H \in OPW(S)$ with S compact can be extended to a bounded linear operator $H : S_0'(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$.

Proposition 7. *For S compact there exists $C_S > 0$ with*

$$\|Hf\|_{L^2} \leqslant C_S \|f\|_{S_0'} \|H\|_{HS}, \quad H \in OPW(S),\ f \in S_0'(\mathbb{R}^d). \tag{4}$$

Proposition 7 can be deduced from more involved results in [6,33]. We include a proof of Proposition 7 in Appendix A as the case considered here is significantly simpler than the general case.

Proposition 7 implies that proving identifiability of a Hilbert–Schmidt operator by an element in $S_0'(\mathbb{R}^d)$ is equivalent to providing the lower bound A in (3), as an upper bound is given by $B = C_S \|f\|_{S_0'}$.

Given a separable Hilbert space X, a sequence of elements $\{f_k\}_{k\in\mathbb{Z}}$ in X is called a frame for X if

$$\sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \asymp \|f\|_X^2, \quad f \in X.$$

To each frame $\{f_k\}_{k\in\mathbb{Z}}$ for X exists a so-called dual frame $\{\tilde{f}_k\}_{k\in\mathbb{Z}}$ of $\{f_k\}_{k\in\mathbb{Z}}$ for X with

$$f = \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle \tilde{f}_k = \sum_{k\in\mathbb{Z}} \langle f, \tilde{f}_k\rangle f_k, \quad f \in X.$$

Moreover, a frame which does not form a frame if we remove any element from it is called a Riesz basis, or, also, exact frame. A sequence $\{f_k\}_{k\in\mathbb{Z}}$ is called a Riesz sequence if it is a Riesz basis for $\overline{\operatorname{span}}\{f_k\}_{k\in\mathbb{Z}}$ [5,9,18].

[4] See [19] for a more general concept of operator identification.

3. Uniform sampling of Hilbert–Schmidt operators

Theorem 2 states that a bandlimited, square integrable function can be reconstructed by its values sampled at a sufficiently dense sampling grid. In this section, we first consider the sampling problem for operators whose Kohn–Nirenberg symbols are bandlimited in the time–frequency plane. We shall state and prove all theorems for $d=1$ for convenience.

Operators with rectangular bandlimitation on their Kohn–Nirenberg symbols are the starting point of operator sampling [29,30].

Theorem 8. *For $\Omega, T, T'>0$ and $0<T'\Omega\leqslant T\Omega\leqslant 1$, choose $\varphi\in PW([-(\frac{1}{T}-\frac{\Omega}{2}),\frac{1}{T}-\frac{\Omega}{2}])$ with $\hat\varphi=1$ on $[-\frac{\Omega}{2},\frac{\Omega}{2}]$ and $r\in L^\infty(\mathbb{R})$ with $\operatorname{supp} r\subset[-T+T',T]$ and $r=1$ on $[0,T']$. Then $OPW([0,T']\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$ permits operator sampling as*

$$\|H\|_{HS}=\sqrt{T}\left\|H\sum_{k\in\mathbb{Z}}\delta_{kT}\right\|_{L^2},\quad H\in OPW\big(\big[0,T'\big]\times\big[-\tfrac{\Omega}{2},\tfrac{\Omega}{2}\big]\big),$$

and operator reconstruction is possible by means of the L^2-convergent series

$$h_H(t,x)=r(t)T\sum_{n\in\mathbb{Z}}\left(H\sum_{k\in\mathbb{Z}}\delta_{kT}\right)(t+nT)\varphi(x-t-nT).$$

In [19], the proof of the identifiability of $OPW([0,T']\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$ is based on the unitarity of the Zak transform. For clarity and to indicate directions for generalizations of this theorem, we prove Theorem 8 through elementary orthonormal basis expansions based on Fourier series.

Proof. For almost every $t\in\mathbb{R}$, we have $\eta_H(t,\cdot)\in L^2[-\frac{\Omega}{2},\frac{\Omega}{2}]\subseteq L^2[-\frac{1}{2T},\frac{1}{2T}]$ and, by expanding for such t the function $\eta_H(t,\nu)$ with respect to the orthonormal basis $\{\sqrt{T}\,e^{-2\pi i(t+nT)\nu}\}_{n\in\mathbb{Z}}$ of $L^2[-\frac{1}{2T},\frac{1}{2T}]$, we obtain the $L^2[-\frac{\Omega}{2},\frac{\Omega}{2}]$-convergent series

$$\eta_H(t,\nu)=\sum_{n\in\mathbb{Z}}\big\langle\eta_H(t,\nu),\sqrt{T}\,e^{-2\pi i(t+nT)\nu}\big\rangle\sqrt{T}\,e^{-2\pi i(t+nT)\nu}=T\sum_{n\in\mathbb{Z}}h_H(t,t+nT)\,e^{-2\pi i(t+nT)\nu}. \tag{5}$$

This series extends to $L^2(\mathbb{R})$ convergence via

$$\eta_H(t,\nu)=T\hat\varphi(\nu)\sum_{n\in\mathbb{Z}}h_H(t,t+nT)\,e^{-2\pi i(t+nT)\nu}$$

with φ chosen to satisfy $\varphi\in PW([-(\frac{1}{T}-\frac{\Omega}{2}),\frac{1}{T}-\frac{\Omega}{2}])$ and $\hat\varphi=1$ on $[-\frac{\Omega}{2},\frac{\Omega}{2}]$. Then, for almost every $t\in\mathbb{R}$, we have

$$h_H(t,x)=T\sum_{n\in\mathbb{Z}}h_H(t,t+nT)\int\hat\varphi(\nu)e^{-2\pi i(t+nT)\nu}e^{2\pi i\nu x}\,d\nu \tag{6}$$

$$=T\sum_{n\in\mathbb{Z}}h_H(t,t+nT)\,\varphi(x-t-nT),\quad x\in\mathbb{R}, \tag{7}$$

where we used the L^2-continuity of the Fourier transform to interchange sum and integration in (6). On the other hand, we have $(H\sum_k\delta_{kT})(x)=\sum_k h_H(x-kT,x)\in L^2(\mathbb{R})$ so that $(H\sum_k\delta_{kT})(t+nT)=\sum_k h_H(t+nT-kT,t+nT)$. Now,

$$r(t)\left(H\sum_{k\in\mathbb{Z}}\delta_{kT}\right)(t+nT)=h_H(t,t+nT),\quad t\in\big[0,T'\big], \tag{8}$$

since $\operatorname{supp} h_H(\cdot,x)\subseteq[0,T']\subseteq[0,T]$. With (7), this gives

$$h_H(t,x)=r(t)T\sum_{n\in\mathbb{Z}}\left(H\sum_{k\in\mathbb{Z}}\delta_{kT}\right)(t+nT)\varphi(x-t-nT), \tag{9}$$

a formula which contains the identifier $\sum_{k\in\mathbb{Z}}\delta_{kT}$ for $OPW([0,T']\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$. Note that the L^2-convergent series in (9) converges also pointwise and uniformly in x for a.e. t [10].

Moreover, since (5) was an orthonormal basis expansion, Parseval's identity gives

$$\big\|\eta_H(t,\cdot)\big\|_{L^2}^2=T\sum_{n\in\mathbb{Z}}\big|h_H(t,t+nT)\big|^2=T\sum_{n\in\mathbb{Z}}\left|r(t)\left(H\sum_{k\in\mathbb{Z}}\delta_{kT}\right)(t+nT)\right|^2,\quad\text{a.e. } t\in\mathbb{R},$$

so that

$$\|h_H(t,\cdot)\|_{L^2}^2 = T\sum_{n\in\mathbb{Z}}\left|r(t)\left(H\sum_{k\in\mathbb{Z}}\delta_{kT}\right)(t+nT)\right|^2, \quad \text{a.e. } t\in\mathbb{R},$$

and

$$\begin{aligned}\|h_H\|_{L^2}^2 &= \int_0^{T'}\|h_H(t,\cdot)\|_{L^2}^2\,dt = T\int_0^{T'}\sum_{n\in\mathbb{Z}}\left|h_H(t,t+nT)\right|^2dt = T\int_0^{T}\sum_{n\in\mathbb{Z}}\left|h_H(t,t+nT)\right|^2dt\\ &= T\int_0^T\sum_{n\in\mathbb{Z}}\left|\left(H\sum_{k\in\mathbb{Z}}\delta_{kT}\right)(t+nT)\right|^2dt = T\int_{\mathbb{R}}\left|\left(H\sum_{k\in\mathbb{Z}}\delta_{kT}\right)(t)\right|^2dt = T\left\|H\sum_{k\in\mathbb{Z}}\delta_{kT}\right\|_{L^2}^2\end{aligned}$$

as $h_H(t,t+nT)$ and $(H\sum_k\delta_{kT})(t+nT)$ vanish on the interval $[T',T]$. The operator class $OPW([0,T']\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$ is identifiable by $\sum_{k\in\mathbb{Z}}\delta_{kT}$ as we showed $\|H\|_{HS}=\|h_H\|_{L^2}=\sqrt{T}\|H\sum_{k\in\mathbb{Z}}\delta_{kT}\|_{L^2}$. $\square$

One crucial ingredient in the proof above is the fact that for each t, the set $\mathcal{E}_t=\{\sqrt{T}\,e^{-2\pi i(t+nT)\nu}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2[-\frac{1}{2T},\frac{1}{2T}]$. Note that the functionals corresponding to $\mathcal{E}_t$ depend on t which is necessary to associate $h_H(t,t+nT)$ with $Hg(t+nT)$ for some identifier g. Another important ingredient is the support condition on $h_H(\cdot,x)$. It guarantees that no aliasing in the infinite summation $H(\sum_{k\in\mathbb{Z}}\delta_{kT})(t+nT)$ takes place as for each t the sum is reduced to a single non zero summand $h_H(t,t+nT)$.

We assume in Theorem 8 that the area of the rectangle $[0,T']\times[-\frac{\Omega}{2},\frac{\Omega}{2}]$ is less than or equal to 1. This assumption coincides with the one given in Kailath's conjecture for identifiability of such operator classes [14]. For $1<T'\Omega\leqslant T\Omega$, perfect reconstruction of $h_H(t,x)$ from its samples is not possible since the sampling rate $\frac{1}{T}$ is strictly less than the Nyquist–Landau rate Ω for $h_H(t,\cdot)$. In this case, not only operator sampling, but also operator identification by any tempered distribution as single input signal is not possible as shown in [31].

Now we extend Theorem 8 to the case where $h_H(t,\cdot)$ lies in a shift-invariant space other than the Paley–Wiener space. Given a Riesz sequence $\{\varphi(\cdot-nT)\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$, let

$$V_T(\varphi)=\overline{\operatorname{span}}\{\varphi(\cdot-nT)\}=\left\{\sum_{n\in\mathbb{Z}}c_n\varphi(\cdot-nT)\colon\ \{c_n\}_{n\in\mathbb{Z}}\in l^2\right\}.$$

Let $\mathcal{H}_{T,\varphi}\subseteq HS(L^2(\mathbb{R}))$ consist of integral operators H with $h_H\in L^2[0,T]\otimes V_T(\varphi)$. Note that the proof of Proposition 7 relied on the compactness of the support of $\mathcal{F}_s\sigma_H$. As this is not stipulated in this more general setup, not every operator considered here maps boundedly $S_0'(\mathbb{R})$ to $L^2(\mathbb{R})$.

We require the shift-invariant space $V_T(\varphi)$ to be a reproducing kernel Hilbert space [10].

Definition 9. A Hilbert space X of complex-valued functions on a given domain $D\neq\emptyset$ is a reproducing kernel Hilbert space if there exists a kernel $k(s,t)$ defined on $D\times D$ satisfying $k(\cdot,t)\in X$ for all $t\in D$ and $f(t)=\langle f(\cdot),k(\cdot,t)\rangle_X$ for all $f\in X$ and $t\in D$. Such a function $k(s,t)$ is called a reproducing kernel.

For example, if φ is a complex-valued integrable function well-defined everywhere on $\mathbb{R}$, and if φ satisfies

$$\sum_{n\in\mathbb{Z}}\left|\varphi(t+n)\right|^2<\infty,\quad t\in[0,1],$$

then $V_1(\varphi)$ is a reproducing kernel Hilbert space [16]. Alternatively, if φ is continuous and belongs to the Wiener amalgam space $W(L^\infty,l^1)$, that is, to the subspace of $L^2(\mathbb{R})$ defined by the norm

$$\|\varphi\|_{W(L^\infty,l^1)}=\sum_{n\in\mathbb{Z}}\sup_{t\in[0,1]}\left|\varphi(t+n)\right|<\infty,$$

then $V_1(\varphi)$ is a reproducing kernel Hilbert space as well [1].

Theorem 10. *Assume that $V_T(\varphi)$ is a reproducing kernel Hilbert space and its reproducing kernel $k(s,t)$ satisfies the condition that $\{k(\cdot,t+nT)\}_{n\in\mathbb{Z}}$ is a frame for $V_T(\varphi)$ for each $t\in[0,T]$. Then $\sum_{k\in\mathbb{Z}}\delta_{kT}$ identifies $\mathcal{H}_{T,\varphi}$, that is,*

$$\|H\|_{HS}\asymp\left\|H\sum_{k\in\mathbb{Z}}\delta_{kT}\right\|_{L^2},\quad H\in\mathcal{H}_{T,\varphi}.$$

The reconstruction of operators is possible by

$$h_H(t,x)=\chi_{[0,T]}(t)\sum_{n\in\mathbb{Z}}\left(H\sum_{k\in\mathbb{Z}}\delta_{kT}\right)(t+nT)k_n^*(x,t),$$

where $\{k_n^*(\cdot,t)\}_{n\in\mathbb{Z}}$ *is a dual frame of* $\{k(\cdot,t+nT)\}_{n\in\mathbb{Z}}$ *for a.e.* $t\in[0,T]$.

Proof. Since $h_H(t,\cdot)\in V_T(\varphi)$ and $\{k(\cdot,t+nT)\}_{n\in\mathbb{Z}}$ is a frame for $V_T(\varphi)$ for each $t\in[0,T]$,

$$h_H(t,x)=\sum_{n\in\mathbb{Z}}\langle h_H(t,\cdot),k(\cdot,t+nT)\rangle k_n^*(x,t),$$

where $\{k_n^*(\cdot,t)\}_{n\in\mathbb{Z}}$ is a dual frame of $\{k(\cdot,t+nT)\}_{n\in\mathbb{Z}}$. Since $k(s,t)$ is a reproducing kernel of $V_T(\varphi)$,

$$h_H(t,x)=\sum_{n\in\mathbb{Z}}h_H(t,t+nT)k_n^*(x,t)$$

and

$$\|h_H(t,\cdot)\|_{L^2}^2\asymp\sum_{n\in\mathbb{Z}}\left|h_H(t,t+nT)\right|^2. \tag{10}$$

Formally, we have $(H\sum_{k\in\mathbb{Z}}\delta_{kT})(t+nT)=\sum_{k\in\mathbb{Z}}h_H(t+nT-kT,t+nT)$ so that

$$r(t)\left(H\sum_{k\in\mathbb{Z}}\delta_{kT}\right)(t+nT)=h_H(t,t+nT),$$

where $r(t)=\chi_{[0,T]}(t)$. Together with (10), we note $H\sum_{k\in\mathbb{Z}}\delta_{kT}\in L^2(\mathbb{R})$ and we conclude

$$\|h_H\|_{L^2}^2=\|H\|_{HS}^2\asymp\int_0^T\sum_{n\in\mathbb{Z}}\left|\left(H\sum_{k\in\mathbb{Z}}\delta_{kT}\right)(t+nT)\right|^2dt=\left\|H\sum_{k\in\mathbb{Z}}\delta_{kT}\right\|_{L^2}^2. \qquad \square$$

Example 11. Let $T=1$ and take $\varphi(t)=\chi_{[0,1)}(t)$. Then $V_1(\varphi)$ is a reproducing kernel Hilbert space and its reproducing kernel $k(s,t)$ allows $\{k(\cdot,t+n)\}_{n\in\mathbb{Z}}$ to be a frame, in fact, an orthonormal basis for $V_1(\varphi)$ since $k(s,t)=\sum_{n\in\mathbb{Z}}\chi_{[n,n+1)}(s)\chi_{[n,n+1)}(t)$. Hence, for $t\in[0,1]$,

$$k(s,t+n)=\sum_{m\in\mathbb{Z}}\chi_{[m,m+1)}(s)\chi_{[m,m+1)}(t+n)=\sum_{m\in\mathbb{Z}}\chi_{[m,m+1)}(s)\delta_{n,m}=\chi_{[n,n+1)}(s).$$

We conclude that $\sum_k\delta_k$ identifies $\mathcal{H}=\{H\in HS(L^2(\mathbb{R})):\ h_H\in L^2[0,1]\otimes V_1(\varphi)\}$. For $H\in\mathcal{H}$, we observe that $h_H(t,\cdot)\in V_1(\varphi)$ implies $\kappa_H(\cdot,\cdot-t)\in V_1(\varphi)$ and, hence, $\kappa_H(x,t)$ is a step function along the lines $x=t+c$, $c\in\mathbb{R}$.

4. Irregular sampling of Hilbert–Schmidt operators

First, we provide the background on irregular sampling of functions that is needed to develop operator sampling results based on irregular sampling sets.

4.1. Irregular sampling in Paley–Wiener spaces

Definition 12. Let $\Lambda=\{\lambda_k\}_{k\in\mathbb{Z}}\subseteq\mathbb{R}$, with $\lambda_k<\lambda_{k+1}$, $k\in\mathbb{Z}$.

(1) Λ is a set of sampling, also referred to as stable sampling set, for $PW([-\frac{\Omega}{2},\frac{\Omega}{2}])$ if

$$\|f\|_{L^2}^2\asymp\sum_{k\in\mathbb{Z}}\left|f(\lambda_k)\right|^2,\quad f\in PW\left(\left[-\tfrac{\Omega}{2},\tfrac{\Omega}{2}\right]\right).$$

(2) Λ is a set of interpolation for $PW([-\frac{\Omega}{2},\frac{\Omega}{2}])$ if the interpolation or moment problem

$$f(\lambda_k)=c_k,\quad k\in\mathbb{Z},$$

has a solution in $PW([-\frac{\Omega}{2},\frac{\Omega}{2}])$ for every $\{c_k\}\in l^2(\mathbb{Z})$.

(3) Λ is uniformly discrete if,

$$\lambda_{k+1}-\lambda_k\geqslant\delta>0,\quad k\in\mathbb{Z}.$$

In this case δ is called a separation constant.

(4) Λ is relatively uniformly discrete if Λ is a finite union of uniformly discrete sets.
(5) The upper and lower Beurling densities are defined, respectively, by

$$D^+(\Lambda) = \limsup_{h\to\infty} \frac{n^+(h)}{h} \quad \text{and} \quad D^-(\Lambda) = \liminf_{h\to\infty} \frac{n^-(h)}{h},$$

where for $h > 0$, $n^+(h)$ and $n^-(h)$ are the supremum and infimum of the number of points from Λ in $[x - \frac{h}{2}, x + \frac{h}{2})$, $x \in \mathbb{R}$, respectively. If $D^+(\Lambda) = D^-(\Lambda)$, then we say that Λ has uniform Beurling density $D(\Lambda) = D^+(\Lambda) = D^-(\Lambda)$.

We recall necessary and sufficient conditions on the Beurling density of a set $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ for its nonharmonic sequence to be a frame or a Riesz sequence for $L^2[-\frac{\Omega}{2}, \frac{\Omega}{2}]$ [32].

Theorem 13.

(1) *Λ is a set of sampling for $PW([-\frac{\Omega}{2}, \frac{\Omega}{2}])$ if and only if $\{e^{-2\pi i\lambda_k\xi}\}_{k\in\mathbb{Z}}$ is a frame for $L^2[-\frac{\Omega}{2}, \frac{\Omega}{2}]$. Moreover, for Λ being relatively uniformly discrete, a necessary condition for $\{e^{-2\pi i\lambda_k\xi}\}_{k\in\mathbb{Z}}$ to be a frame for $L^2[-\frac{\Omega}{2}, \frac{\Omega}{2}]$ is $D^-(\Lambda) \geqslant \Omega$, and a sufficient condition is $D^-(\Lambda) > \Omega$.*
(2) *Λ is a set of interpolation for $PW([-\frac{\Omega}{2}, \frac{\Omega}{2}])$ if and only if $\{e^{-2\pi i\lambda_k\xi}\}_{k\in\mathbb{Z}}$ is a Riesz sequence in $L^2[-\frac{\Omega}{2}, \frac{\Omega}{2}]$. Moreover, for Λ being uniformly discrete, a necessary condition for $\{e^{-2\pi i\lambda_k\xi}\}_{k\in\mathbb{Z}}$ to be a Riesz sequence in $L^2[-\frac{\Omega}{2}, \frac{\Omega}{2}]$ is $D^+(\Lambda) \leqslant \Omega$, and a sufficient condition is $D^+(\Lambda) < \Omega$.*

In general, it is highly non-trivial to determine whether a set $\{e^{-2\pi i\lambda_k\xi}\}_{k\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\frac{\Omega}{2}, \frac{\Omega}{2}]$. A famous affirmative result was given by Kadec [13].

Theorem 14 *(Kadec's 1/4-theorem). Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$. If there is $L \geqslant 0$ such that*

$$\left|\lambda_k - \frac{k}{\Omega}\right| \leqslant L < \frac{1}{4\Omega}, \quad k \in \mathbb{Z}, \tag{11}$$

then $\{e^{-2\pi i\lambda_k\xi}\}_{k\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\frac{\Omega}{2}, \frac{\Omega}{2}]$, and $\frac{1}{4\Omega}$ is the best possible constant for (11) *to hold.*

4.2. Irregular sampling in operator Paley–Wiener spaces

Definition 15. A sequence $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ in $\mathbb{R}$ is a set of sampling for an operator class $\mathcal{H}$, if for some $\{c_k\}_{k\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$, we have $\sum_{k\in\mathbb{Z}} c_k\delta_{\lambda_k} \in S_0'(\mathbb{R})$ and $\sum_{k\in\mathbb{Z}} c_k\delta_{\lambda_k}$ identifies $\mathcal{H}$.

We start our discussion of irregular sampling in $OPW([0,T]\times[-\frac{\Omega}{2}, \frac{\Omega}{2}])$ with a simple observation.

Proposition 16. *If $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is a uniformly discrete set of sampling for $OPW([0,T]\times[-\frac{\Omega}{2}, \frac{\Omega}{2}])$, then Λ is a set of sampling for $PW([-\frac{\Omega}{2}, \frac{\Omega}{2}])$.*

Proof. Note that Λ uniformly discrete implies that there exists $C > 0$ with $\sum |f(\lambda_k)|^2 \leqslant C\|f\|_{L^2}^2$ for all $f \in PW([-\frac{\Omega}{2}, \frac{\Omega}{2}])$ [32]. To obtain the reverse inequality for some $\widetilde{C} > 0$, we choose a separation constant δ of Λ with $0 < \delta \leqslant T$. For each $f \in PW([-\frac{\Omega}{2}, \frac{\Omega}{2}])$ define H_f by $\kappa_{H_f}(x,y) = f(y)$ for $0 \leqslant x - y < \delta$, $\kappa_{H_f}(x,y) = 0$ else. As $\sum_{k\in\mathbb{Z}} c_k\delta_{\lambda_k}$ identifies $OPW([0,T]\times[-\frac{\Omega}{2}, \frac{\Omega}{2}])$, we conclude

$$\begin{aligned}
\delta\|f\|_{L^2}^2 &= \|\chi_{[0,\delta]}\|_{L^2}^2\|f\|_{L^2}^2 = \|\kappa_{H_f}\|_{L^2}^2 = \|H_f\|_{HS}^2 \asymp \left\|H_f \sum_{k\in\mathbb{Z}} c_k\delta_{\lambda_k}\right\|_{L^2}^2 \\
&= \left\|\sum c_k\kappa_{H_f}(x,\lambda_k)\right\|_{L^2}^2 = \left\|\sum c_k\chi_{[\lambda_k,\lambda_k+\delta)} f(\lambda_k)\right\|_{L^2}^2 \\
&= \sum \left|c_k f(\lambda_k)\right|^2 \int \left|\chi_{[\lambda_k,\lambda_k+\delta)}(x)\right|^2 \leqslant \delta\|c\|_{l^\infty}^2 \sum \left|f(\lambda_k)\right|^2, \quad f \in PW\big(\big[-\tfrac{\Omega}{2}, \tfrac{\Omega}{2}\big]\big).
\end{aligned} \tag{12}$$

To obtain (12), we utilized the fact that the functions $\chi_{[\lambda_k,\lambda_k+\delta)}$, $k \in \mathbb{Z}$, have disjoint support. □

Proposition 16 shows that restrictions on sampling sets for functions generally apply to the operator sampling setup as well. For example, we conclude that sampling sets for $OPW([0,T]\times[-\frac{\Omega}{2}, \frac{\Omega}{2}])$ cannot be finite.

Operator sampling is operator identification with discretely supported identifiers. Consequently, irregular operator sampling for $OPW([0,T]\times[-\frac{\Omega}{2}, \frac{\Omega}{2}])$ is *a-priori* possible only if $T\Omega \leqslant 1$ (see, for example, Theorem 3.6 in [19] and Theorem 4.1 in [28]). Sufficiency of the condition $T\Omega \leqslant 1$ for operator sampling follows from Theorem 8.

Theorem 17. *$T\Omega \leqslant 1$ is a necessary and sufficient condition for the existence of a sampling set for $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$.*

In the following, we shall address the question whether a given set Λ is a set of sampling for $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$, $T\Omega \leqslant 1$.

Analogous to Theorem 13, we have the following result.

Theorem 18. *If $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$, $\lambda_{k+1} > \lambda_k$, is uniformly discrete, then a necessary condition for Λ being a set of sampling for $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$ is $D^-(\Lambda) \geqslant \Omega$, and a sufficient condition is $D^-(\Lambda) > \Omega$ and $\lambda_{k+1} - \lambda_k \geqslant T$.*

Proof. Necessity of $D^-(\Lambda) \geqslant \Omega$ follows from Theorem 13(1) combined with Proposition 16.

Now we shall establish the sufficient condition for Λ to be a set of sampling. As Λ is uniformly discrete and $D^-(\Lambda) > \Omega$, $\{e^{-2\pi i\lambda_k\xi}\}_{k\in\mathbb{Z}}$ is a frame for $L^2[-\frac{\Omega}{2},\frac{\Omega}{2}]$ by Theorem 13(1). Theorem 13(1) implies also that then Λ is a set of sampling for $PW([-\frac{\Omega}{2},\frac{\Omega}{2}])$ and for a.e. $t\in\mathbb{R}$,

$$\left\|h_H(t,\cdot)\right\|_{L^2}^2 \asymp \sum_{n\in\mathbb{Z}}\left|h_H(t,t+\lambda_n)\right|^2, \quad H\in OPW\left([0,T]\times\left[-\tfrac{\Omega}{2},\tfrac{\Omega}{2}\right]\right).$$

As $\lambda_{k+1} - \lambda_k \geqslant T$, $k\in\mathbb{Z}$, we conclude that Λ is a set of sampling for $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$ since for $H\in OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$,

$$\begin{aligned}\|H\|_{HS}^2 = \|h_H\|_{L^2}^2 &= \int_0^T \left\|h_H(t,\cdot)\right\|^2 dt \asymp \int_0^T \sum_{n\in\mathbb{Z}}\left|h_H(t,t+\lambda_n)\right|^2 dt \\ &= \int_0^T \sum_{n\in\mathbb{Z}}\left|\left(H\sum_{k\in\mathbb{Z}}\delta_{\lambda_k}\right)(t+\lambda_n)\right|^2 dt \leqslant \left\|H\sum_{k\in\mathbb{Z}}\delta_{\lambda_k}\right\|_{L^2}^2. \qquad \square\end{aligned}$$

In the remainder of this section, we shall discuss the separation condition $(\lambda_{k+1} - \lambda_k) \geqslant T$, $k\in\mathbb{Z}$, on the sampling sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$ for $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$. This condition rules out aliasing in the sense that the summands in $\sum_{n\in\mathbb{Z}} h_H(\cdot-\lambda_n,\cdot) = \sum_{n\in\mathbb{Z}}\kappa_H(\cdot,\lambda_n)$ have non-overlapping support. Consequently, the sufficient condition on $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ for being a set of sampling for $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$ given in Theorem 18 implies

$$\Omega < D^-(\Lambda) \leqslant D^+(\Lambda) \leqslant \frac{1}{T}$$

so that $T\Omega < 1$. Theorem 17 shows that this is only a mild additional restriction on operator Paley–Wiener spaces to allow for irregular operator sampling.

Corollary 19. *If $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$, $\lambda_{k+1} > \lambda_k$, satisfies $|\lambda_k - kT| \leqslant L < \frac{T}{4}$ for some $L \geqslant 0$ and $\lambda_{k+1} - \lambda_k \geqslant T$, $k\in\mathbb{Z}$, then Λ is a set of sampling for $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$, $T\Omega < 1$.*

Proof. Since $|\lambda_k - kT| \leqslant L < \frac{T}{4}$, $\{e^{-2\pi i\lambda_k\xi}\}$ is a Riesz basis for $L^2[-\frac{1}{2T},\frac{1}{2T}]$ by Theorem 14. If $T\Omega < 1$, the result follows directly from Theorem 18. $\square$

Note that the hypothesis on Λ in Corollary 19 is satisfied if and only if $\lambda_k = kT + \epsilon_k$ with

$$-\frac{T}{4} < -L \leqslant \cdots \leqslant \epsilon_{-2} \leqslant \epsilon_{-1} \leqslant \epsilon_0 \leqslant \epsilon_1 \leqslant \epsilon_2 \leqslant \cdots \leqslant L < \frac{T}{4}.$$

Theorem 20. *Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$, $\lambda_{k+1} > \lambda_k$, be a set of sampling for $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$ and let $\{e^{-2\pi i\lambda_k\xi}\}_{k\in\mathbb{Z}}$ be a Riesz basis for $L^2[-\frac{\Omega}{2},\frac{\Omega}{2}]$, then $\lambda_{k+1} - \lambda_k \geqslant T$, $k\in\mathbb{Z}$.*

Proof. It is easy to see that if $\{e^{-2\pi i\lambda_k\xi}\}_{k\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\frac{\Omega}{2},\frac{\Omega}{2}]$ and $\sum_{k\in\mathbb{Z}} c_k\delta_{\lambda_k}$ is an identifier of $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$, then $c_k \neq 0$ for all $k\in\mathbb{Z}$. In fact, if $c_{k_0} = 0$ for some $k_0\in\mathbb{Z}$, then $\sum_{k\neq k_0} c_k\delta_{\lambda_k}$ also identifies $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$ so that $\{e^{-2\pi i\lambda_k\xi}\}_{k\neq k_0}$ remains a frame for $L^2[-\frac{\Omega}{2},\frac{\Omega}{2}]$. But this is a contradiction as a Riesz basis ceases to form a frame when we remove any element [5].

Assume $\lambda_{l+1} - \lambda_l < T$ for some l. For such l, set

$$\kappa_H(x,\lambda_k) = 0 \quad \text{if } k \neq l, l+1$$

and

$$\kappa_H(x,\lambda_l)=\begin{cases} c_{l+1} & \text{if } \lambda_{l+1}\leqslant x\leqslant \lambda_l+T,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\kappa_H(x,\lambda_{l+1})=\begin{cases} -c_l & \text{if } \lambda_{l+1}\leqslant x\leqslant \lambda_l+T,\\ 0 & \text{otherwise.}\end{cases}$$

The freedom of choice of values for $\kappa_H(x,\lambda_k)$ is justified by Theorem 13(2). Then $H\sum_{k\in\mathbb{Z}}c_k\delta_{\lambda_k}(x)=\sum_{k\in\mathbb{Z}}c_k\kappa_H(x,\lambda_k)=0$ for all $x\in\mathbb{R}$, but as $c_l,c_{l+1}\neq 0$, we have $\kappa_H\neq 0$ and therefore $H\neq 0$. $\square$

Example 21. Theorem 20 implies that $\{\lambda_n\}_{n\in\mathbb{Z}}=\{2n\}_{n\in\mathbb{Z}}\cup\{2n+\alpha\}_{n\in\mathbb{Z}}$ with $0<\alpha<1$ is not a set of sampling for $OPW([0,1]\times[-\frac{1}{2},\frac{1}{2}])$.

The condition $(\lambda_{k+1}-\lambda_k)\geqslant T$, $k\in\mathbb{Z}$ is not necessary for operator sampling of $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$ if $\{e^{-2\pi i\lambda_k\xi}\}_{k\in\mathbb{Z}}$ is not a Riesz basis but a frame for $L^2[-\frac{\Omega}{2},\frac{\Omega}{2}]$.

Example 22. The tempered distribution $\sum_{k\in\mathbb{Z}}(-1)^k\delta_{\frac{k}{2}}$ identifies $OPW([0,1]\times[-\frac{1}{2},\frac{1}{2}])$ [28].

To illustrate the rigidity of operator sampling in comparison to function sampling, we add the following simple example.

Example 23. Let $\Lambda_r=\{\lambda_k\}_{k\in\mathbb{Z}}$ be given by $\lambda_k=k$ for $k\neq 0$ and $\lambda_0=r\in\mathbb{R}$. The set Λ_r is a set of sampling for $PW([-\frac{1}{2},\frac{1}{2}])$ if and only if $r\notin\mathbb{Z}\setminus\{0\}$. To see this, note that since $\{e^{2\pi ik\xi}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $L^2[-\frac{1}{2},\frac{1}{2}]$ and since for $r\notin\mathbb{Z}\setminus\{0\}$ we have $\langle e^{2\pi ir\xi},e^{2\pi i0\xi}\rangle\neq 0$, we conclude that $\{e^{2\pi ik\xi}\}_{k\neq 0}\cup\{e^{2\pi ir\xi}\}$, $r\notin\mathbb{Z}\setminus\{0\}$, is a Riesz basis.

On the other hand, applying Theorem 20 we obtain $\Lambda_r=\{\lambda_k\}_{k\in\mathbb{Z}}$ is a set of sampling for $OPW([0,1]\times[-\frac{1}{2},\frac{1}{2}])$ if and only if $r=0$.

4.3. An operator version of Kramer's Lemma

Kramer's Lemma plays a crucial role in the proofs of a number of important sampling theorems. For example, it allows for sampling series expansions for functions which are integral transforms of type other than the Fourier one. For example, Bessel–Hankel, Legendre, Jacobi, Laguerre, Gegenbauer, Chebyschev, prolate spheroidal, and Hermite transforms can be considered, where each transform is defined as an integral transform whose kernel is its special function [10,24,35]. In particular, if a function on $\mathbb{R}^2$ has a circular symmetry, then a multi-dimensional Fourier transform can be reduced to a one-dimensional Bessel–Hankel transform [25].

After recalling Kramer's Lemma below, we shall state and prove an operator sampling version of Kramer's Lemma, Theorem 25, and point out restrictions in the applicability of Kramer's method in operator sampling (Remark 26).

Theorem 24 *(Kramer's Lemma). Let $I\subseteq\mathbb{R}$ be a bounded interval and $k(\cdot,t)\in L^2(I)$ for each fixed t in $D\subseteq\mathbb{R}$. If there is a sampling sequence $\{t_n\}_{n\in\mathbb{Z}}$ in D such that $\{k(\xi,t_n)\}_{n\in\mathbb{Z}}$ forms a frame for $L^2(I)$, then for any $f(t)=\langle F,\overline{k(\cdot,t)}\rangle_{L^2(I)}$, $F\in L^2(I)$, we have*

$$f(t)=\sum_{n\in\mathbb{Z}}f(t_n)S_n(t),$$

where the reconstruction functions S_n are given by

$$S_n(t)=\int_I\tilde{k}_n(\xi)k(\xi,t)\,d\xi,$$

with $\{\tilde{k}_n(\xi)\}_{n\in\mathbb{Z}}$ being a dual frame of $\{\overline{k(\xi,t_n)}\}_{n\in\mathbb{Z}}$.

The original Kramer's Lemma assumed that $\{k(\xi,t_n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(I)$ [10,20]. However, a close look at the proof of Kramer's Lemma in [10] shows that the result extends to the case where $\{k(\xi,t_n)\}_{n\in\mathbb{Z}}$ forms a frame. We remark that the classical sampling theorem addressing f in $PW([-\frac{1}{2},\frac{1}{2}])$ is given by Kramer's Lemma if we take $k(\xi,t)=e^{2\pi i\xi t}$, $F=\hat{f}$ and use the fact that $\{e^{2\pi in\xi}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2[-\frac{1}{2},\frac{1}{2}]$.

Let $I\subseteq\mathbb{R}$ be bounded. For

$$\mathcal{K}:L^2(I)\longrightarrow L^2(\mathbb{R}),\qquad F\longmapsto\big\langle F(\cdot),k(x,\cdot)\big\rangle_{L^2(I)},\tag{13}$$

bounded, set

$$\mathcal{H}^k\big([0,d]\times I\big)=\big\{H\in HS\big(L^2(\mathbb{R})\big):\ h_H(t,x)=\big\langle\zeta_H(t,\cdot),k(x,\cdot)\big\rangle_{L^2},\ \zeta_H\in L^2\big([0,d]\times I\big)\big\}.$$

Clearly, for $k(x,\nu)=e^{-2\pi i\nu x}$ we have $\mathcal{H}^k([0,d]\times I)=OPW([0,d]\times I)$.

Theorem 25. *Let $H\in\mathcal{H}^k([0,d]\times I)$, $d>0$, I in $\mathbb{R}$ bounded. If there is a set $\{y_n\}_{n\in\mathbb{Z}}$ in $\mathbb{R}$, such that $\{k(t+y_n,\nu)\}_{n\in\mathbb{Z}}$ forms a frame for $L^2(I)$ for every $t\in[0,d]$, and $d'=\inf(y_{n+1}-y_n)\geqslant d$, then exists $c>0$ with*

$$\Big\|H\Big(\sum_{k\in\mathbb{Z}}\delta_{y_k}\Big)\Big\|_{L^2}\geqslant c\|H\|_{HS},\quad H\in\mathcal{H}^k\big([0,d]\times I\big).\tag{14}$$

If the map $\mathcal{K}$ in (13) is bounded below, then

$$\Big\|H\Big(\sum_{k\in\mathbb{Z}}\delta_{y_k}\Big)\Big\|_{L^2}\asymp\|H\|_{HS},\quad H\in\mathcal{H}^k\big([0,d]\times I\big),\tag{15}$$

and operator reconstruction is possible as

$$h_H(t,x)=r(t)\sum_{n\in\mathbb{Z}}\Big(H\sum_{k\in\mathbb{Z}}\delta_{y_k}\Big)(t+y_n)\phi_n(t,x),$$

where $\phi_n(t,x)$ is given by

$$\phi_n(t,x)=\int_I k_n^*(t,\nu)\overline{k(x,\nu)}\,d\nu$$

with $\{k_n^*(t,\nu)\}_{n\in\mathbb{Z}}$ being a dual frame of $\{k(t+y_n,\nu)\}_{n\in\mathbb{Z}}$ for each t and $r\equiv 1$ on $[0,d]$ with $\operatorname{supp} r\subseteq[d-d',d']$.

Proof. As $\inf_k(y_{k+1}-y_k)=d'\geqslant d$ we have

$$r(t)\Big(H\sum_{k\in\mathbb{Z}}\delta_{y_k}\Big)(t+y_n)=r(t)\sum_{k\in\mathbb{Z}}h_H(t+y_n-y_k,t+y_n)=h_H(t,t+y_n),\quad t\in\mathbb{R},$$

where $r\equiv 1$ on $[0,d]$ and $\operatorname{supp} r\subseteq[d-d',d']$. Consequently,

$$\begin{aligned}\|\zeta_H\|_{L^2}^2&=\int\big\|\zeta_H(t,\cdot)\big\|_{L^2}^2\,dt\asymp\int\big\|\big\{\big\langle\zeta_H(t,\cdot),k(t+y_n,\cdot)\big\rangle_{L^2}\big\}\big\|_{l^2}^2\,dt\\&=\int\big\|\big\{h_H(t,t+y_n)\big\}\big\|_{l^2}^2\,dt=\int\sum_{n\in\mathbb{Z}}\big|h_H(t,t+y_n)\big|^2\,dt\\&=\int\sum_{n\in\mathbb{Z}}\Big|r(t-y_n)\Big(H\sum_{k\in\mathbb{Z}}\delta_{y_k}\Big)(t)\Big|^2\,dt=\sum_{n\in\mathbb{Z}}\int_{y_n}^{y_{n+1}}\Big|H\sum_{k\in\mathbb{Z}}\delta_{y_k}(t)\Big|^2\,dt\\&=\int\Big|H\sum_{k\in\mathbb{Z}}\delta_{y_k}(t)\Big|^2\,dt=\Big\|H\sum_{k\in\mathbb{Z}}\delta_{y_k}\Big\|^2,\quad H\in\mathcal{H}^k\big([0,d]\times I\big).\end{aligned}$$

We used the fact that $r(t-y_n)(H\sum_{k\in\mathbb{Z}}\delta_{y_k})(t)=(H\sum_{k\in\mathbb{Z}}\delta_{y_k})(t)$ for $t\in[y_n,y_n+d]$ and $r(t-y_n)(H\sum_{k\in\mathbb{Z}}\delta_{y_k})(t)=0$ for $t\in[y_n+d,y_{n+1})$, $n\in\mathbb{Z}$.

As $\|H\|_{HS}=\|h_H\|_{L^2}$ and $h_H(t,\cdot)=\mathcal{K}\zeta_H(t,\cdot)$, Eqs. (14) and (15) follow from the fact that $\mathcal{K}$ is bounded and the hypothesis that $\mathcal{K}$ is bounded below, respectively.

Moreover, we have, for $\nu\in I$,

$$\zeta_H(t,\nu)=\sum_{n\in\mathbb{Z}}\big\langle\zeta_H(t,\cdot),k(t+y_n,\cdot)\big\rangle_{L^2}k_n^*(t,\nu)=\sum_{n\in\mathbb{Z}}h_H(t,t+y_n)k_n^*(t,\nu),$$

where $\{k_n^*(t,\nu)\}_{n\in\mathbb{Z}}$ is a dual frame of $\{k(t+y_n,\nu)\}_{n\in\mathbb{Z}}$ for each t. Multiplying $\overline{k(x,\nu)}$ and integrating with respect to ν on both sides, we have for fixed $t\in[0,d]$

$$h_H(t,x)=\sum_{n\in\mathbb{Z}}h_H(t,t+y_n)\phi_n(t,x),\quad x\in\mathbb{R},$$

where $\phi_n(t,x)=\int_I k_n^*(t,\nu)\overline{k(x,\nu)}\,d\nu$. $\square$

Note that a set of sampling $\{y_n\}_{n\in\mathbb{Z}}$ satisfying the hypotheses of Theorem 25 is uniformly discrete with separation constant d' greater than or equal to d. On the other hand, the condition that $\{k(t+y_n,\nu)\}_{n\in\mathbb{Z}}$ forms a frame for $L^2(I)$ indicates that the sampling points should not be too sparse.

Remark 26. Unlike Kramer's Lemma for functions, we do not have any explicit example for operator sampling other than the $OPW([0,d]\times I)$ case. Sampling theorems based on various kinds of orthogonal polynomials generally do not satisfy all hypotheses in Theorem 25. For instance, for the Bessel–Hankel transform, we have $k(x,\nu)=\sqrt{x\nu}\,J_n(x\nu)$ where J_n is the Bessel function of the first kind of order n. Taking λ_k as the k-th positive root of $J_n(x)$, one can, in fact, obtain a sampling expansion induced from the Bessel–Hankel transform [10]. However, the kernel $k(x,\nu)=\sqrt{x\nu}\,J_n(x\nu)$ does not allow for $\{k(t+\lambda_k,\nu)\}_{k\in\mathbb{Z}}$ being a frame for each t.

As we saw already in the proof of Theorem 8, the Fourier kernel $k(x,\nu)=e^{2\pi ix\nu}$ satisfies the hypotheses in Theorem 25 as, for example, $\{e^{2\pi it\nu}e^{-2\pi in\nu}\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2[-\frac{1}{2},\frac{1}{2}]$ for all $t\in\mathbb{R}$. In fact, whenever $k(x,\nu)$ satisfies $k(t+x,\nu)=m(t,\nu)k(x,\nu)$ with $0<c\leqslant|m(t,\nu)|\leqslant C<\infty$, $t\in[0,d]$, $\nu\in I$, then $\{k(y_n,\nu)\}_{n\in\mathbb{Z}}$ being a frame for $L^2(I)$ implies that $\{k(t+y_n,\nu)\}_{n\in\mathbb{Z}}$ is also a frame for $L^2(I)$ for all $t\in[0,d]$. This observation applies to, for example, the Hilbert transform kernel $k(x,\nu)=-i\,\mathrm{sgn}(\nu)\,e^{-2\pi ix\nu}$. But again, substantial examples seem rare.

5. Multi-channel sampling for Hilbert–Schmidt operators

In classical multi-channel sampling, a signal is reconstructed using discrete values from the outputs of N different time-invariant operators applied to a single input signal. Generally, each of the N outputs is sampled at $\frac{1}{N}$-th of the Nyquist–Landau rate of the input signal. For example, when the signal's bandwidth is Ω Hz, then we should collect at least Ω samples per second. But if we design N channel filters appropriately, then it suffices to obtain Ω/N samples per second from each channel and combine the samples in order to reconstruct the signal. This allows us to reduce sampling rate requirements on sampling hardware at the cost of employing multiple samplers.

A number of important theorems, for example, on periodic nonuniform sampling, on derivative sampling and on samples of Hilbert transforms can be explained in the framework of multi-channel sampling [4,10]. In [12], multi-channel sampling has been developed for abstract Hilbert spaces, allowing each channel output to be sampled at different, irregular points.

In Sections 3 and 4, it has been shown that a slowly time-varying/underspread operator is identifiable by a single channel output while in [19,27] it is shown that an overspread operator is not identifiable in this sense. However, in this section we shall show that overspread operators may be recoverable from multiple channel outputs. In addition, we seek to reduce the rate at which delta impulses are produced for channel identification.

Throughout this section, we shall consider Hilbert–Schmidt operators whose Kohn–Nirenberg symbols are bandlimited to rectangular domains.

Theorem 27. *For $M,N\in\mathbb{N}$, $OPW([0,N]\times[-\frac{M}{2},\frac{M}{2}])$ permits multi-channel operator sampling as*

$$\|H\|_{HS}^2=\frac{1}{M^2N}\sum_{j=0}^{MN-1}\left\|H\left(\sum_{n\in\mathbb{Z}}e^{2\pi ijn/MN}\delta_{\frac{n}{M}}\right)\right\|^2,\quad H\in OPW\big([0,N]\times\big[-\tfrac{M}{2},\tfrac{M}{2}\big]\big).$$

Proof. Suppose that $\eta_H\in L^2([0,N]\times[-\frac{M}{2},\frac{M}{2}])$ for some $M,N\in\mathbb{N}$. Consider an orthonormal basis $\{\frac{1}{\sqrt{M}}\,e^{-2\pi i(t+\frac{n}{M})\nu}\}_{n\in\mathbb{Z}}$ for $L^2[-\frac{M}{2},\frac{M}{2}]$, $t\in[0,N]$. Then

$$\begin{aligned}\eta_H(t,\nu)&=\sum_{n\in\mathbb{Z}}\left\langle\eta_H(t,\nu),\frac{1}{\sqrt{M}}e^{-2\pi i(t+\frac{n}{M})\nu}\right\rangle\frac{1}{\sqrt{M}}e^{-2\pi i(t+\frac{n}{M})\nu}\\&=\sum_{n\in\mathbb{Z}}h_H\left(t,t+\frac{n}{M}\right)\frac{1}{M}e^{-2\pi i(t+\frac{n}{M})\nu},\quad\nu\in\big[-\tfrac{M}{2},\tfrac{M}{2}\big],\end{aligned}$$

and

$$h_H(t,x)=\sum_{n\in\mathbb{Z}}h_H\left(t,t+\frac{n}{M}\right)\mathrm{sinc}\,M\left(x-t-\frac{n}{M}\right),\quad x\in\mathbb{R},\ \text{a.e.}\ t\in[0,N].$$

Since $\{\sqrt{M}\,\mathrm{sinc}\,M(\cdot-t-\frac{n}{M})\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $PW([-\frac{M}{2},\frac{M}{2}])$, Parseval's identity gives

$$\big\|h_H(t,\cdot)\big\|_2^2=\frac{1}{M}\sum_{n\in\mathbb{Z}}\left|h_H\left(t,t+\frac{n}{M}\right)\right|^2,\quad t\in[0,N].$$

Hence, we have

$$\begin{aligned}
\|h_H\|^2_{L^2(\mathbb{R}^2)} &= \int_0^N \|h_H(t,\cdot)\|^2_{L^2(\mathbb{R})}\,dt = \frac{1}{M}\int_0^N \sum_{n\in\mathbb{Z}} \left|h_H\left(t, t+\frac{n}{M}\right)\right|^2 dt \\
&= \frac{1}{M}\sum_{r=0}^{MN-1} \int_{\frac{r}{M}}^{\frac{r+1}{M}} \sum_{n\in\mathbb{Z}} \left|h_H\left(t, t+\frac{n}{M}\right)\right|^2 dt \\
&= \frac{1}{M}\int_0^{\frac{1}{M}} \sum_{r=0}^{MN-1} \sum_{n\in\mathbb{Z}} \left|h_H\left(t+\frac{r}{M}, t+\frac{r}{M}+\frac{n}{M}\right)\right|^2 dt \\
&= \frac{1}{M}\int_0^{\frac{1}{M}} \sum_{r=0}^{MN-1} \sum_{n\in\mathbb{Z}} \left|h_H\left(t+\frac{r}{M}, t+\frac{n}{M}\right)\right|^2 dt.
\end{aligned}$$

Since $(H\sum_n \frac{1}{\sqrt{MN}}\delta_{\frac{n}{M}})(t+\frac{k}{M}) = \sum_n \frac{1}{\sqrt{MN}} h_H(t+\frac{k}{M}-\frac{n}{M}, t+\frac{k}{M})$, for fixed $t\in[0,\frac{1}{M}]$, we have

$$r(t)\sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{MN}} h_H\left(t+\frac{k}{M}-\frac{n}{M}, t+\frac{k}{M}\right) = \frac{1}{\sqrt{MN}} \sum_{r=0}^{MN-1} h_H\left(t+\frac{r}{M}, t+\frac{k}{M}\right),$$

where $r(t) = \chi_{[0,\frac{1}{M}]}(t)$. Similarly, consider $(H\sum_n \frac{1}{\sqrt{MN}}\omega_j^n \delta_{\frac{n}{M}})(x)$ where $\omega_j = e^{2\pi i j/MN}$, $j = 0, 1, \ldots, MN-1$. Then

$$\left(H\sum_n \frac{1}{\sqrt{MN}}\omega_j^n \delta_{\frac{n}{M}}\right)\left(t+\frac{k}{M}\right) = \sum_n \frac{\omega_j^n}{\sqrt{MN}} h_H\left(t+\frac{k}{M}-\frac{n}{M}, t+\frac{k}{M}\right).$$

For fixed $t\in[0,\frac{1}{M}]$,

$$r(t)\sum_{n\in\mathbb{Z}} \frac{\omega_j^n}{\sqrt{MN}} h_H\left(t+\frac{k}{M}-\frac{n}{M}, t+\frac{k}{M}\right) = \sum_{r=0}^{MN-1} \frac{\omega_j^{k-r}}{\sqrt{MN}} h_H\left(t+\frac{r}{M}, t+\frac{k}{M}\right).$$

Consider the system of linear equations

$$r(t)\begin{bmatrix} (H\sum_n \frac{1}{\sqrt{MN}}\delta_{\frac{n}{M}})(t+\frac{k}{M}) \\ (H\sum_n \frac{\omega_1^n}{\sqrt{MN}}\delta_{\frac{n}{M}})(t+\frac{k}{M}) \\ \vdots \\ (H\sum_n \frac{\omega_{MN-1}^n}{\sqrt{MN}}\delta_{\frac{n}{M}})(t+\frac{k}{M}) \end{bmatrix} = A_k \begin{bmatrix} h_H(t, t+\frac{k}{M}) \\ h_H(t+\frac{1}{M}, t+\frac{k}{M}) \\ \vdots \\ h_H(t+\frac{MN-1}{M}, t+\frac{k}{M}) \end{bmatrix},$$

where all matrices A_k are unitary $MN\times MN$ DFT matrices with entries given by $(A_k)_{j,l} = \frac{1}{\sqrt{MN}} e^{2\pi i(j-1)(k-(l-1))/MN}$. Since all A_k's are unitary, we have

$$\left\|\begin{bmatrix} \|\{(H\sum_n \frac{1}{\sqrt{MN}}\delta_{\frac{n}{M}})(t+\frac{k}{M})\}_k\|_{l^2} \\ \|\{(H\sum_n \frac{\omega_1^n}{\sqrt{MN}}\delta_{\frac{n}{M}})(t+\frac{k}{M})\}_k\|_{l^2} \\ \vdots \\ \|\{(H\sum_n \frac{\omega_{MN-1}^n}{\sqrt{MN}}\delta_{\frac{n}{M}})(t+\frac{k}{M})\}_k\|_{l^2} \end{bmatrix}\right\|^2 = \left\|\begin{pmatrix} \|\{h_H(t, t+\frac{k}{M})\}_k\|_{l^2} \\ \|\{h_H(t+\frac{1}{M}, t+\frac{k}{M})\}_k\|_{l^2} \\ \vdots \\ \|\{h_H(t+\frac{MN-1}{M}, t+\frac{k}{M})\}_k\|_{l^2} \end{pmatrix}\right\|^2,$$

where $\|\cdot\|$ denotes the $L^2[0,\frac{1}{M}]^{MN}$ norm. Therefore

$$\|h_H\|^2 = \frac{1}{M}\int_0^{\frac{1}{M}} \sum_{k\in\mathbb{Z}} \sum_{j=0}^{MN-1} \left| \left(H \sum_{n\in\mathbb{Z}} \frac{\omega_j^n}{\sqrt{MN}} \delta_{\frac{n}{M}} \right) \left(t + \frac{k}{M} \right) \right|^2 dt$$
$$= \frac{1}{M}\int_{\mathbb{R}} \sum_{j=0}^{MN-1} \left| \left(H \sum_{n\in\mathbb{Z}} \frac{\omega_j^n}{\sqrt{MN}} \delta_{\frac{n}{M}} \right)(t) \right|^2 dt$$
$$= \frac{1}{M^2 N} \sum_{j=0}^{MN-1} \left\| H \sum_{n\in\mathbb{Z}} \omega_j^n \delta_{\frac{n}{M}} \right\|_{L^2(\mathbb{R})}^2. \qquad \square$$

Clearly, the matrices A_k can be replaced by appropriate sequences of matrices whose norms are bounded above and away from zero.

Theorem 28. *Let $M, N \in \mathbb{N}$ and $\{c_{j,n}\}_{j=1,n\in\mathbb{Z}}^{MN}$ bounded. Let A_k, $k \in \mathbb{Z}$, be invertible with $\|A_k^{-1}\| \leqslant C < \infty$ where $(A_k)_{j,l=1,\dots,MN} = c_{j,k-l+1}$. For $f_j = \sum_{n\in\mathbb{Z}} c_{j,n}\delta_{\frac{n}{M}}$, $1 \leqslant j \leqslant MN$, we have*

$$\|H\|_{HS}^2 \asymp \sum_{j=1}^{MN} \|Hf_j\|_{L^2}^2, \quad H \in OPW\big([0,N]\times\big[-\tfrac{M}{2}, \tfrac{M}{2}\big]\big).$$

The assumption in Theorem 28 is satisfied if for all j, $\{c_{j,n}\}_{n\in\mathbb{Z}}$ is MN-periodic and $A_{j,0}$ is invertible with $\|A_{j,0}^{-1}\|$ also uniformly bounded. The proof of Theorem 28 closely resembles the proof of Theorem 27—norm equalities are replaced by norm equivalences—and is therefore omitted.

Now we consider periodic nonuniform sampling as first proposed by Yen [34]. We first recall a periodic nonuniform sampling theorem for functions in $PW([-\frac{\Omega}{2}, \frac{\Omega}{2}])$ which is stated and proven in [34].

Theorem 29. *There exists a Riesz basis $\{S_j(t - \frac{nN}{\Omega})\}_{j=1,n\in\mathbb{Z}}^N$ for $PW([-\frac{\Omega}{2}, \frac{\Omega}{2}])$ such that*

$$f(t) = \sum_{j=1}^N \sum_{n\in\mathbb{Z}} f\left(\frac{nN}{\Omega} + \alpha_j\right) S_j\left(t - \frac{nN}{\Omega}\right), \quad f \in PW\big(\big[-\tfrac{\Omega}{2}, \tfrac{\Omega}{2}\big]\big),$$

where $0 \leqslant \alpha_j < \frac{N}{\Omega}$, $1 \leqslant j \leqslant N$, and $\alpha_i \neq \alpha_j$ for $i \neq j$.

We show that $OPW([0,N]\times[-\frac{M}{2}, \frac{M}{2}])$ is identifiable by MN identifiers which are given by delta-trains whose supports are periodically nonuniformly distributed.

Theorem 30. *For $M, N \in \mathbb{N}$, and $0 \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_{MN} < N$, we have*

$$\|H\|_{HS}^2 \asymp \sum_{j=1}^{MN} \left\| H\left(\sum_{n\in\mathbb{Z}} \delta_{nN+\alpha_j} \right) \right\|^2, \quad H \in OPW\big([0,N]\times\big[-\tfrac{M}{2}, \tfrac{M}{2}\big]\big).$$

Proof. If we apply Theorem 29 to $h_H(t,\cdot) \in PW([-\frac{M}{2}, \frac{M}{2}])$ with MN channels, then we obtain

$$h_H(t,x) = \sum_{j=1}^{MN} \sum_{n\in\mathbb{Z}} h_H(t, t+nN+\alpha_j)\varphi_j(x-t-nN), \quad x \in \mathbb{R}, \text{ a.e. } t \in [0,N],$$

and

$$\big\|h_H(t,\cdot)\big\|^2 \asymp \sum_{j=1}^{MN} \sum_{n\in\mathbb{Z}} \big|h_H(t, t+nN+\alpha_j)\big|^2, \quad \text{a.e. } t \in [0,N],$$

where $\{\varphi_j(\cdot - t - nN)\}_{j=1,n\in\mathbb{Z}}^{MN}$ is a Riesz basis for $PW([-\frac{M}{2}, \frac{M}{2}])$ for each $t \in [0,N]$. Therefore, we have

$$\|H\|_{HS}^2 = \|h_H\|_{L^2}^2 \asymp \int_0^N \sum_{j=1}^{MN} \sum_{n\in\mathbb{Z}} \big|h_H(t, t+nN+\alpha_j)\big|^2 dt$$

$$= \sum_{j=1}^{MN} \int_0^N \sum_{n\in\mathbb{Z}} \left| H\left(\sum_{k\in\mathbb{Z}} \delta_{kN+\alpha_j} \right)(t+nN+\alpha_j) \right|^2 dt$$

$$= \sum_{j=1}^{MN} \left\| H\left(\sum_{k\in\mathbb{Z}} \delta_{kN+\alpha_j} \right) \right\|_{L^2}^2,$$

that is, $\{\sum_{k\in\mathbb{Z}} \delta_{kN+\alpha_j}\}_{j=1}^{MN}$ identifies $OPW([0,N]\times[-\frac{M}{2},\frac{M}{2}])$. $\square$

Remark 31. Note that as in the multichannel sampling theory for functions, Theorem 30 can be applied to reduce the rate of impulse transmission. For example, as $OPW([0,1]\times[-\frac{1}{2},\frac{1}{2}]) \subseteq OPW([0,N]\times[-\frac{1}{2},\frac{1}{2}])$, we can identify any operator in $OPW([0,1]\times[-\frac{1}{2},\frac{1}{2}])$ by its action on the tempered distributions $\sum_{n\in\mathbb{Z}} \delta_{nN+j\alpha}$, $0 \leqslant \alpha \leqslant 1$, $j = 0,\dots,N-1$, each of which has impulse rate $1/N$.

Another important multi-channel sampling concept can be applied to operator sampling, namely, derivative sampling (see Section 3.2 in [11]). We apply a multi-channel sampling formula consisting of samples of f and its $MN-1$ derivatives [21,22].

Theorem 32. *For $M, N \in \mathbb{N}$, we have*

$$\|H\|_{HS}^2 \asymp \sum_{j=0}^{MN-1} \left\| \sum_{r=0}^{j} \binom{j}{r} (-1)^r \left(H \sum_k \delta_{kN}^{(r)} \right)^{(j-r)} \right\|^2, \quad H \in OPW\big([0,N]\times\big[-\tfrac{M}{2},\tfrac{M}{2}\big]\big).$$

Here, $f^{(r)}$ denotes the r-th derivative of f in the distributional sense.

Proof. For almost every $t \in [0,N]$, we can apply the multi-channel derivative sampling theorem with MN-channels to $h_H(t,\cdot) \in PW([-\frac{M}{2},\frac{M}{2}])$. That is, we use the formula

$$f(t) = \sum_{n\in\mathbb{Z}} f(nNT)S_{0,n}(t) + \dots + f^{(N-1)}(nNT)S_{N-1,n}(t), \quad f \in PW\big(\big[-\tfrac{M}{2},\tfrac{M}{2}\big]\big),$$

where $T\Omega = 1$ [11]. This leads, for a.e. t, to the well-defined and L^2-convergent series

$$h_H(t,x) = \sum_{j=0}^{MN-1} \sum_{n\in\mathbb{Z}} \frac{\partial^j}{\partial x^j} h_H(t,x)\big|_{x=t+nN} \varphi_j(x-t-nN).$$

Moreover, we have

$$\big\|h_H(t,\cdot)\big\|^2 \asymp \sum_{j=0}^{MN-1} \sum_{n\in\mathbb{Z}} \left| \frac{\partial^j}{\partial x^j} h_H(t,x)\big|_{x=t+nN} \right|^2, \quad \text{a.e. } t \in [0,N],$$

as $\{\varphi_j(x-t-nN)\}_{j=0,n\in\mathbb{Z}}^{MN-1}$ is a Riesz basis for $PW([-\frac{M}{2},\frac{M}{2}])$ for each fixed $t \in [0,N]$ [11]. Note that an explicit form of the reconstruction functions φ_j is given in [11].

Let $H_j \in HS(L^2(\mathbb{R}))$ be the operator defined through the j-th derivative of $h_H(t,\cdot)$, that is, formally, we have

$$H_j f(x) = \int \frac{\partial^j}{\partial x^j} h_H(t,x) f(x-t)\, dt.$$

Since $\operatorname{supp} \frac{\partial^j}{\partial x^j} h_H(\cdot,x) \subseteq \operatorname{supp} h_H(\cdot,x)$, we have, similar to (8),

$$\chi_{[0,N]}(t) \left(H_j \sum_{k\in\mathbb{Z}} \delta_{kN} \right)(t+nN) = \frac{\partial^j}{\partial x^j} h_H(t,x)\big|_{x=t+nN},$$

so that

$$\|h_H\|^2 = \int_0^N \big\|h_H(t,\cdot)\big\|^2 dt \asymp \int_0^N \sum_{j=0}^{MN-1} \sum_{n\in\mathbb{Z}} \left| \left(H_j \sum_{k\in\mathbb{Z}} \delta_{kN} \right)(t+nN) \right|^2 dt = \sum_{j=0}^{MN-1} \left\| H_j \sum_{k\in\mathbb{Z}} \delta_{kN} \right\|^2. \tag{16}$$

By Leibniz rule, we have, formally,

$$\begin{aligned}(Hf)'(x) &= \frac{d}{dx}\int h_H(t,x)f(x-t)\,dt \\ &= \int \frac{\partial}{\partial x}h_H(t,x)f(x-t)\,dt + \int h_H(t,x)\frac{\partial}{\partial x}f(x-t)\,dt \\ &= \int \frac{\partial}{\partial x}h_H(t,x)f(x-t)\,dt + \int h_H(t,x)f'(x-t)\,dt \\ &= H_1 f(x) + H\big(f'\big)(x),\end{aligned}$$

so that $H_1 f = (Hf)' - Hf'$. Applying this procedure to $(Hf)''$ gives $H_2 f = (Hf)'' - 2(Hf')' + Hf''$, and, iterating this further, we obtain the Leibniz' formula

$$H_j f(x) = \sum_{r=0}^{j} \binom{j}{r} (-1)^r \big(Hf^{(r)}\big)^{(j-r)}. \tag{17}$$

Both sides of (17) are well defined for $f = \sum \delta_{kN}$ and equality follows from standard density arguments. Inserting the right-hand side of (17) into the right-hand side of (16) completes the proof. □

We conclude this section by presenting two explicit multi-channel reconstruction formulas for operators.

Example 33. Consider the Riesz bases $\{e^{-2\pi i(t+2n)\nu}\}_{n\in\mathbb{Z}} \cup \{-2\pi i\nu e^{-2\pi i(t+2n)\nu}\}_{n\in\mathbb{Z}}$, $t \in \mathbb{R}$, of $L^2[-1/2, 1/2]$, as well as their Riesz basis duals given by $\{2(1-2|\nu|)e^{-2\pi i(t+2n)\nu}\}_{n\in\mathbb{Z}} \cup \{\frac{2i}{\pi}\operatorname{sgn}(\nu)e^{-2\pi i(t+2n)\nu}\}_{n\in\mathbb{Z}}$ [10]. We obtain

$$\begin{aligned}\eta_H(t,\nu) = {}&\sum_n \big\langle \eta_H(t,\cdot), e^{-2\pi i(t+2n)\cdot}\big\rangle 2\big(1-2|\nu|\big)e^{-2\pi i(t+2n)\nu} \\ &+ \sum_n \big\langle \eta_H(t,\cdot), -2\pi i\cdot e^{-2\pi i(t+2n)\cdot}\big\rangle \frac{2i}{\pi}\operatorname{sgn}(\nu)e^{-2\pi i(t+2n)\nu}, \quad \nu \in \big[-\tfrac{1}{2}, \tfrac{1}{2}\big].\end{aligned}$$

Taking the inverse Fourier transform with respect to the variable ν, we have

$$h_H(t,x) = \sum_n h_H(t,t+2n)S_n(t,x) + \frac{\partial}{\partial x}h_H(t,x)\Big|_{x=t+2n} T_n(t,x), \ x \in \mathbb{R}, \quad t \in [0,2],$$

where

$$S_n(t,x) = \operatorname{sinc}^2 \frac{1}{2}(x-t-2n)$$

and

$$T_n(t,x) = \frac{2}{\pi}\operatorname{sinc}\frac{1}{2}(x-t-2n)\sin\frac{\pi}{2}(x-t-2n).$$

We give a second reconstruction formula for the operator class $OPW([0,2]\times[-\frac{1}{2},\frac{1}{2}])$.

Example 34. Let $H \in OPW([0,2]\times[-\frac{1}{2},\frac{1}{2}])$. Then for $0 < \alpha < 1$,

$$h_H(t,x) = \sum_{n\in\mathbb{Z}} \bigg(H\sum_{k\in\mathbb{Z}}\delta_{2k}\bigg)(t+2n)S_1(x-t-2n) + \bigg(H\sum_{k\in\mathbb{Z}}\delta_{2k+\alpha}\bigg)(t+2n+\alpha)S_2(x-t-2n-\alpha),$$

where the reconstruction functions are

$$S_1(x) = \frac{2}{e^{\pi i\alpha}-1}\mathcal{F}^{-1}\big(e^{\pi i\alpha}\chi_{[-\frac{1}{2},0)}(\nu) - \chi_{[0,\frac{1}{2}]}(\nu)\big)(x)$$

and

$$S_2(x) = \frac{2}{e^{\pi i\alpha}-1}\mathcal{F}^{-1}\big(-\chi_{[-\frac{1}{2},0)}(\nu) + e^{\pi i\alpha}\chi_{[0,\frac{1}{2}]}(\nu)\big)(x).$$

Appendix A

To prove Proposition 7, we use some advanced concepts from time–frequency analysis. See [9] for a detailed introduction to the underlying theory. In the following $\mathcal{S}(\mathbb{R}^d)$ denotes the class of Schwartz class functions and $\mathcal{S}'(\mathbb{R}^d)$ its dual of tempered distributions.

Definition 35. For $1 \leqslant p, q \leqslant \infty$, the modulation space $M^{p,q}(\mathbb{R}^d)$ is defined by

$$M^{p,q}(\mathbb{R}^d) = \{ f \in \mathcal{S}'(\mathbb{R}^d): \ V_{\mathfrak{g}} f(t,\nu) \in L^{pq}(\mathbb{R}^{2d}) \}, \tag{18}$$

where $V_{\mathfrak{g}} f(t,\nu)$ is the short-time Fourier transform of f with respect to the Gaussian $\mathfrak{g}(x) = e^{-\pi \|x\|^2}$. The norm on $M^{p,q}(\mathbb{R}^d)$ is given by

$$\|f\|_{M^{p,q}} = \|V_{\mathfrak{g}} f\|_{L^{p,q}} = \left(\int \left(\int |V_{\mathfrak{g}} f(t,\nu)|^p \, dt \right)^{\frac{q}{p}} d\nu \right)^{\frac{1}{q}}$$

with the usual adjustments if $p = \infty$ or $q = \infty$.

Observe that $M^{1,1}(\mathbb{R}^d) = S_0(\mathbb{R}^d)$, $M^{2,2}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$, and $M^{\infty,\infty}(\mathbb{R}^d) = S_0'(\mathbb{R}^d)$ with equivalent norms. Similarly, $M^{p,q}(\mathbb{R}^d)' = M^{p',q'}(\mathbb{R}^d)$ where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$, $1 \leqslant p, q < \infty$. Moreover, replacing $\mathfrak{g}(x) = e^{-\pi\|x\|^2}$ in (18) by any non-trivial function $\varphi \in S_0(\mathbb{R}^d)$ does not change the defined modulation space and leads to a norm equivalent to the norm induced by choosing $\mathfrak{g}(x) = e^{-\pi\|x\|^2}$ [9].

Lemma 36. *For $f \in S_0'(\mathbb{R}^d) = M^{\infty,\infty}(\mathbb{R}^d)$ and $g \in L^2(\mathbb{R}^d) = M^{2,2}(\mathbb{R}^d)$ we have $\mathcal{F}_s V_f g \in M^{2,\infty}(\mathbb{R}^{2d})$. Moreover, there exists $C > 0$ with*

$$\|\mathcal{F}_s V_f g\|_{M^{2,\infty}} \leqslant C \|f\|_{S_0'} \|g\|_{L^2}. \tag{19}$$

Proof. Assume first $g, f \in S_0(\mathbb{R}^d)$, the general case follows from the density of $S_0(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$, respectively the weak-$*$ density of $S_0(\mathbb{R}^d)$ in $S_0'(\mathbb{R}^d)$ [9].

The formal computation

$$\begin{aligned} G(x,\xi) = \mathcal{F}_s V_f g(x,\xi) &= \iint \int g(x') e^{-2\pi i x'\nu} \bar{f}(x'-t) \, dx' \, e^{-2\pi i(t\xi - x\nu)} \, d\nu \, dt \\ &= \int \iint g(x') e^{-2\pi i(x'-x)\nu} \bar{f}(x'-t) d\nu \, dx' \, e^{-2\pi i t\xi} \, dt \\ &= \int g(x) \bar{f}(x-t) e^{-2\pi i t\xi} \, dt = g(x) \overline{\hat{f}(\xi)} e^{-2\pi i x\xi} \end{aligned}$$

is easily justified by considering G as a distribution acting on $\mathcal{S}(\mathbb{R}^{2d})$.

Let now $\varphi \in C_c^\infty(\mathbb{R}^d)$ be real valued and set $\Phi(x,\xi) = e^{-2\pi i x\xi} \varphi(x)\varphi(\xi)$ for $(x,\xi) \in \mathbb{R}^{2d}$. Then $\Phi \in \mathcal{S}(\mathbb{R}^{2d}) \subseteq S_0(\mathbb{R}^{2d})$. We compute

$$\begin{aligned} V_\Phi G(x,\xi,t,\nu) &= \iint g(x') \bar{\hat{f}}(\xi') e^{-2\pi i x'\xi'} e^{-2\pi i(x'\nu - t\xi')} e^{2\pi i(x'-x)(\xi'-\xi)} \varphi(x'-x) \varphi(\xi'-\xi) \, d\xi' \, dx' \\ &= \iint g(x') \bar{\hat{f}}(\xi') e^{2\pi i(-x'\xi' - x'\nu + t\xi' + x'\xi' - x\xi' - x'\xi + x\xi)} \varphi(x'-x) \varphi(\xi'-\xi) \, d\xi' \, dx' \\ &= e^{2\pi i x\xi} \int \bar{\hat{f}}(\xi') e^{2\pi i(t-x)\xi'} \varphi(\xi'-\xi) \, d\xi' \int g(x') e^{-2\pi i x'(\nu+\xi)} \varphi(x'-x) \, dx' \\ &= e^{2\pi i x\xi} \langle M_{t-x} T_\xi \varphi, \hat{f} \rangle \, V_\varphi g(x, \nu+\xi) \\ &= e^{2\pi i x\xi} e^{2\pi i(t-x)\xi} \langle M_\xi T_{t-x} \check{\varphi}, f \rangle V_\varphi g(x, \nu+\xi) \\ &= e^{2\pi i t\xi} \overline{V_{\check{\varphi}} f}(t-x, \xi) V_\varphi g(x, \nu+\xi) \end{aligned}$$

and, hence,

$$\begin{aligned} \|V_\Phi G\|_{M^{2,\infty}}^2 &= \sup_{t,\nu} \iint |V_{\check{\varphi}} f(t-x,\xi) V_\varphi g(x,\nu+\xi)|^2 \, dx \, d\xi \\ &= \sup_{t,\nu} \iint |V_{\check{\varphi}} f(x,\xi)|^2 |V_\varphi g(t-x,\nu+\xi)|^2 \, dx \, d\xi \\ &\leqslant \|(V_{\check{\varphi}} f)^2\|_{L^\infty} \sup_{t,\nu} \iint |V_\varphi g(t-x,\nu+\xi)|^2 \, dx \, d\xi \\ &= \|(V_{\check{\varphi}} f)^2\|_{L^\infty} \|(V_\varphi g)^2\|_{L^1} = \|V_{\check{\varphi}} f\|_{L^\infty}^2 \|V_\varphi g\|_{L^2}^2 \leqslant C \|f\|_{S_0'}^2 \|g\|_{L^2}^2, \end{aligned}$$

where the constant C is obtained from $\|V_\varphi f\|_{L^2} \asymp \|V_{\mathfrak{g}} f\|_{L^2} \asymp \|f\|_{L^2}$ and $\|V_{\check{\varphi}} f\|_{L^\infty} \asymp \|V_{\mathfrak{g}} f\|_{L^\infty} \asymp \|f\|_{S_0'}$. $\square$

Proof of Proposition 7. A simple computation shows that for $f, g \in S_0(\mathbb{R}^d)$ and $H \in HS(L^2(\mathbb{R}^d))$ we have

$$\langle Hf, g\rangle = \langle \eta_H, V_f g\rangle = \langle \sigma_H, \mathcal{F}_s V_f g\rangle.$$

Note that S compact implies

$$\|\sigma_H\|_{L^2} \asymp \|\sigma_H\|_{M^{2,1}}, \quad H \in OPW(S),$$

[15], and we conclude that

$$\begin{aligned}\|Hf\|_{L^2} &= \sup_{\|g\|_{L^2}=1} \left|\langle Hf, g\rangle\right| = \sup_{\|g\|_{L^2}=1} \left|\langle \sigma_H, \mathcal{F}_s V_f g\rangle\right| \leqslant \sup_{\|g\|_{L^2}=1} \|\sigma_H\|_{M^{2,1}} \|\mathcal{F}_s V_f g\|_{M^{2,\infty}} \\ &\leqslant C \sup_{\|g\|_{L^2}=1} \|\sigma_H\|_{L^2} \|f\|_{S_0'} \|g\|_{L^2} \leqslant \widetilde{C} \|H\|_{HS} \|f\|_{S_0'}. \qquad \Box\end{aligned}$$

References

[1] A. Aldroubi, K. Gröchenig, Beurling–Landau-type theorems for non-uniform sampling in shift invariant spline spaces, J. Fourier Anal. Appl. 6 (1) (2000) 93–103.
[2] P.A. Bello, Measurement of random time-variant linear channels, IEEE Trans. Commun. 15 (1969) 469–475.
[3] J.J. Benedetto, Harmonic Analysis and Applications, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1997.
[4] J. Brown, Multi-channel sampling of low-pass signals, IEEE Trans. Circuits Syst. 28 (2) (1981) 101–106.
[5] O. Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, Boston, 2003.
[6] E. Cordero, K. Gröchenig, Time–frequency analysis of localization operators, J. Funct. Anal. 205 (1) (2003) 107–131.
[7] H.G. Feichtinger, G. Zimmermann, A Banach space of test functions for Gabor analysis, in: H.G. Feichtinger, T. Strohmer (Eds.), Gabor Analysis and Algorithms: Theory and Applications, Birkhäuser, Boston, MA, 1998, pp. 123–170.
[8] G.B. Folland, Harmonic Analysis in Phase Space, Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 1989.
[9] K. Gröchenig, Foundations of Time–Frequency Analysis, Book Series of Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, MA, 2001.
[10] J.R. Higgins, Sampling Theory in Fourier and Signal Analysis, Foundations, Clarendon Press, Oxford, 1996.
[11] J.R. Higgins, R.L. Stens, Sampling Theory in Fourier and Signal Analysis, Advanced Topics, Oxford University Press, USA, 2000.
[12] Y. Hong, J. Kim, K. Kwon, Sampling theory in abstract reproducing kernel Hilbert space, Sampl. Theory Signal Image Process. 6 (1) (2007) 109–121.
[13] M.Ĭ. Kadec′, The exact value of the Paley–Wiener constant, Dokl. Akad. Nauk SSSR 155 (1964) 1253–1254 (in Russian).
[14] T. Kailath, Measurements of time-variant communication channels, IEEE Trans. Inform. Theory 8 (1962) 229–236.
[15] K.A. Okoudjou, A Beurling–Helson type theorem for modulation spaces, J. Funct. Spaces Appl. 7 (1) (2009) 33–41.
[16] J. Kim, K. Kwon, Sampling expansion in shift invariant spaces, Int. J. Wavelets Multiresolut. Inf. Process. 6 (2) (2008) 223–248.
[17] J.J. Kohn, L. Nirenberg, An algebra of pseudo-differential operators, Comm. Pure Appl. Math. 18 (1965) 269–305.
[18] J. Kovačević, A. Chebira, Life beyond bases: The advent of frames (Parts I and II), IEEE Signal Proc. Mag. 24 (5) (2007) 115–125.
[19] W. Kozek, G.E. Pfander, Identification of operators with bandlimited symbols, SIAM J. Math. Anal. 37 (3) (2005) 867–888.
[20] H.P. Kramer, A generalized sampling theorem, J. Math. Phys. 38 (1959) 68–72.
[21] D.A. Linden, A discussion of sampling theorems, Proc. IRE 47 (1959) 1219–1226.
[22] D.A. Linden, N.M. Abramson, A generalization of the sampling theorem, Inf. Control 3 (1960) 26–31.
[23] F. Marvasti, Nonuniform Sampling: Theory and Practice, Information Technology: Transmission, Processing, and Storage Series, Kluwer Academic/Plenum Publishers, New York, 2001.
[24] F.C. Mehta, A general sampling expansion, Inform. Sci. 16 (1978) 41–46.
[25] A. Papoulis, Systems and Transforms with Applications in Optics, McGraw–Hill, New York, 1968.
[26] A. Papoulis, Generalized sampling expansion, IEEE Trans. Circuits Syst. CAS-24 (1977) 652–654.
[27] G.E. Pfander, D.F. Walnut, Operator identification and Feichtinger's algebra, Sampl. Theory Signal Image Process. 5 (2) (2006) 183–200.
[28] G.E. Pfander, D.F. Walnut, Measurement of time-variant linear channels, IEEE Trans. Inform. Theory 52 (11) (2006) 4808–4820.
[29] G.E. Pfander, D.F. Walnut, On the sampling of functions and operators with an application to multiple-input multiple-output channel identification, in: Dimitri Van De Ville, Vivek K. Goyal, Manos Papadakis (Eds.), Wavelets XII, in: Proc. SPIE, vol. 6701, 2007, pp. 67010T-1–67010T-14.
[30] G.E. Pfander, Sampling of operators, preprint, 2009.
[31] G.E. Pfander, Measurement of time-varying multiple-input multiple-output channels, Appl. Comput. Harmon. Anal. 24 (2008) 393–401.
[32] K. Seip, On the connection between exponential bases and certain related sequences in $L^2(-\pi, \pi)$, J. Funct. Anal. 130 (1) (1995) 131–160.
[33] J. Toft, Continuity and Schatten properties for pseudo-differential operators on modulation spaces, in: Modern Trends in Pseudo-Differential Operators, in: Oper. Theory Adv. Appl., vol. 172, Birkhäuser, Basel, 2007, pp. 173–206.
[34] J.L. Yen, On nonuniform sampling of bandwidth-limited signals, IRE Trans. Circuit Theory 3 (1956) 251–257.
[35] A.I. Zayed, Kramer's sampling theorem for multidimensional signals and its relationship with Lagrange-type interpolations, Multidimens. Syst. Signal Process. 3 (1992) 323–340.