Existence Theorems for a Class of Problems in Nonlinear Elasticity

J. Tinsley Oden

The University of Texas at Austin, TICOM Office WRW 305, Austin, Texas 78712

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1. INTRODUCTION

In this article, we develop some general theorems governing the existence of solutions of a class of nonlinear problems of elastostatics of continuous media. We arrive at these theorems in a deductive way; i.e., we develop some general existence theorems applicable to abstract equations defined on reflexive Banach spaces, and we then describe sufficient conditions under which certain problems in nonlinear elasticity fall within the framework of the general theorems. To simplify the analysis, we restrict ourselves to the problem of place in nonlinear elastostatics; i.e., we consider the class of nonlinear boundary-value problems in which the displacements are prescribed on the boundary. Our theory does not require the existence of a stored energy function, or is it limited to bodies subjected to conservative external forces. Moreover, it is applicable to one-, two-, and three-dimensional (indeed, to n-dimensional) problems. As such, it generalizes theories that have been put forth recently. However, in the present study we do not consider complications due to the constraint of local invertibility which is commonly assumed to hold in problems of finite elasticity. We hope to address this problem in later work.

A number of strong existence theorems for linear elliptic boundary value theorems of elasticity are known, and a comprehensive article on this subject was written by Fichera [13]. By comparison, however, the nonlinear theory is virtually untouched. An excellent summary account of the state of existence theory in nonlinear elasticity as it stood in 1973 can be found in the book of Wang and Truesdell [27]. For one-dimensional problems, the theory is more fully developed primarily because of the fact that, if strong ellipticity is assumed, the operators of one-dimensional elasticity are semi-monotone; i.e., they are monotone in the highest derivatives. As such, they fall into the class of operators studied extensively by Brézis [9], Lions [18], Browder [11], and others. Antman has exploited this fact repeatedly in his studies of elastic rods, plates, and shells (see, for example, [2–6] and the references therein). An exhaustive bibliography on monotone and pseudo-monotone operator theory, with an emphasis on
questions of existence of solutions to abstract problems, can be found in the monograph of Browder [11].

One important factor that has hampered progress in nonlinear elasticity theory is lack of agreement as to what constitutes a physically reasonable boundary-value problem in elastostatics. Indeed, in 1956 Truesdell [23] referred to the problem of determining restrictions on the form of the stored-energy function of a hyperelastic material so as to ensure physically reasonable results as "the main unsolved problem of the theory of finite strain." While it is universally agreed that response functionals should be frame-indifferent functionals of some appropriate measure of the deformation, that the rule of material symmetry be obeyed, and the deformations which are not locally invertible and orientation-preserving be disallowed, additional requirements that are needed have been the subject of debate for many years. A comprehensive survey and critique of various inequalities proposed in nonlinear elasticity can be found in Wang and Truesdell [27]. In recent years, several investigators have suggested that the strong ellipticity condition may prove to be an essential ingredient in the construction of physically sound theories. It has certainly proved to be useful in one-dimensional theories. Originally proposed for study by Truesdell and Toupin [24], the strong ellipticity condition provides an necessary and sufficient condition for the acoustic tensor for second- or higher-order waves to be positive definite; statical interpretations of the condition given by Hayes [15] and others (see also Wang and Truesdell [27, pp. 220-225]) show that it leads to physically reasonable conditions on elastic deformations. Antman (e.g. [2, 4]) and Ball [7] make use of versions of the strong ellipticity condition in their existence theories. Very recently, however, Konwles and Sternberg [16, 17] have presented examples in two-dimensional finite elasticity in which the strong ellipticity is violated.

Recently, Ball [7] presented an existence theory for two- and three-dimensional elasticity problems. Ball's theory is limited to classical solutions of problems involving hyperelastic materials and conservative forces and is based on classical minimization arguments of variational calculus. Ball's theory is based on the assumption that strong ellipticity is a constitutive requirement rather than a constraint on the motion. As a result, many common material laws cannot be accommodated by this theory. Moreover, he is unable to show that minimizers of the energy functional are weak solutions of the equilibrium equations.

The strong ellipticity condition is not assumed to hold in the theory described here. Rather, what is shown to be an important factor in our theory is the existence of a nonlinear generalization of the Gårding inequality, which arises from growth conditions on the stress operator.

In the present study, we consider a general class of variational boundary-value problems of nonlinear elasticity, posed in the context of operators on reflexive Banach spaces. After some mechanical and mathematical preliminaries in Sections 2 and 3, we describe the variational framework for the problem of place in elastostatics in Section 4, and we discuss briefly the relationship between the
formal operator intrinsic to the variational method and the operators of classical elasticity in Section 5. In Section 6, we develop an abstract existence theorem for a class of operators on reflexive Banach spaces, which we refer to as Gårding operators. The theory combines and generalizes some features of the theory of operators of the type of the calculus of variations introduced by Lions [18] and the theories of quasilinear elliptic operators of Viiik [26] and the semi-bounded operators of Dubinskii [12]. Our abstract Gårding operators are neither monotone nor semi-monotone, but under mild restrictions, they reduce to the pseudo-monotone operators studied by Brézis [9] and Lions [18]. Importantly, they do cover cases in which non-unique solutions are possible under dead loading. In Section 7 we prove several additional theorems in which sufficient conditions are laid down for the requirements of our abstract theorems to be met by specific elasticity operators on Sobolev spaces.

2. NOTATIONS AND CONVENTIONS

We adopt the usual index notations, when convenient. Latin and Greek indices have range 1, 2, 3 and are summed when they are repeated.

Vectors and tensors are denoted by bold face letters, and \( \mathbf{X} = (X^1, X^2, X^3) \) denoted a triple of real material coordinates. The usual set theoretic notations are used throughout this work.

Various vector and tensor fields appearing in this study are functions of the triple \( \mathbf{X} \) of material particle labels, although we shall frequently suppress the dependence on \( \mathbf{X} \) for simplicity in notations. For example, the value of a tensor field \( \mathbf{A}(\mathbf{w}(\mathbf{X}), \mathbf{X}) \) depending on the vector field \( \mathbf{w} \) at \( \mathbf{X} \) may occasionally be written simply \( \mathbf{A}(\mathbf{w}) \).

If \( \mathbf{A} \) and \( \mathbf{B} \) are second-order tensors and \( \mathbf{w} \) is a vector, we shall use the notations

\[
\mathbf{A} : \mathbf{B} = \text{tr}(\mathbf{A}\mathbf{B}^T) \quad (\text{tr} \mathbf{A} = \text{trace of } \mathbf{A})
\]

\[
| \mathbf{w} | = \| \mathbf{w} \|_{\mathbb{R}^3} = \left[ \sum_{i=1}^{3} w_i^2 \right]^{1/2}.
\]

We also make use of the properties of Sobolev spaces. If \( \Omega \) is an open bounded domain in \( \mathbb{R}^3 \), the class of functions \( u \) given by

\[
W^{m,p}(\Omega) = \{ u : D^\alpha u \in L^p(\Omega), |\alpha| \leq m, m \geq 0 \}
\]

is a linear space called the Sobolev space of order \( m, p \). Here \( L^p(\Omega) \) is the usual Lebesgue space of equivalence classes of measurable functions with \( p \)th powers.
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Lebesgue integrable on \( \Omega \), and \( D^{\alpha}u \) denotes the generalized derivative of \( u \) of order \( \alpha \); i.e., we employ the multi-index notation,

\[
D^{\alpha}u = \frac{\partial^{\alpha_1 + \alpha_2 + \cdots + \alpha_d} u(X)}{\partial X_1^{\alpha_1} \partial X_2^{\alpha_2} \cdots \partial X_d^{\alpha_d}}, \quad \alpha_i = \text{integer} \geq 0.
\]  

Throughout our analysis, we assume that the boundary \( \partial \Omega \) of \( \Omega \) is smooth (e.g. \( \partial \Omega \in C^\infty \)). For \( 1 \leq p \leq \infty \), \( W^{m, p}(\Omega) \) is a Banach space when equipped with the norm

\[
\| u \|_{W^{m, p}(\Omega)} = \left[ \int_{\Omega} \left\{ \sum_{|\alpha| \leq m} \left| D^\alpha u \right|^p dX_1 X_2 X_3 \right\}^{1/p} \right]^{1/p}
\]

and \( W^{m, p}(\Omega) \) is reflexive whenever \( p \) satisfies \( 1 < p < \infty \). Clearly, \( W^{0, p}(\Omega) = L^p(\Omega) \).

We denote by \( W^{m, p}_0(\Omega) \) the closure of the space \( C_0^{\infty}(\Omega) \) of infinitely differentiable functions with compact support in \( \Omega \) with respect to the norm in (2.4). Likewise, the topological dual of the spaces \( W^{m, p}_0(\Omega) \) are the so-called negative Sobolev spaces,

\[
W^{-m, p'}(\Omega) = (W^{m, p}_0(\Omega))', \quad \frac{1}{p} + \frac{1}{p'} = 1.
\]

For vector-valued functions \( X \rightarrow w(X) \in \mathbb{R}^3 \) whose components are in Sobolev spaces, we use the notations

\[
W^{m, p}(\Omega) = (W^{m, p}(\Omega))^3 = W^{m, p}(\Omega) \times W^{m, p}(\Omega) \times W^{m, p}(\Omega)
\]

\[
= \{ w = (w_1, w_2, w_3) : w_i \in W^{m, p}(\Omega), i = 1, 2, 3 \}
\]

\[
\| w \|_{W^{m, p}(\Omega)} = \left[ \sum_{i=1}^3 \| w_i \|_{W^{m, p}(\Omega)}^p \right]^{1/p}
\]

\[
W^{-m, p'}(\Omega) = (W^{m, p}_0(\Omega))', \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad m \geq 0.
\]

We also make use of the Sobolev imbedding theorem (see Sobolev [22], or, for a modern account, Adams [1]) which asserts that for a domain \( \Omega \subset \mathbb{R}^n \) which has the cone property,\(^1\)

\[
W^{m+s, p}(\Omega) \subset W^{s, q}(\Omega) \quad \text{if} \quad mp < n \quad \text{and} \quad p \leq q \leq np/(n - mp)
\]

\[
W^{m+s, p}(\Omega) \subset W^{s, q}(\Omega) \quad \text{if} \quad mp = n \quad \text{and} \quad p < q < \infty
\]

etc. In particular, the embedding of \( W^{m, p}(\Omega) \) into \( L^q(\Omega) \) is compact for \( mp < n \), \( p < q \leq np/(n - mp) \) and \( mp \leq n \), \( p \leq q < \infty \).

\(^1\) See, for example, Adams [1, p. 65], or Oden and Reddy [19].
3. SOME KINEMATICAL PRELIMINARIES

Our objective here is to give a brief account of certain mechanical ideas so as to establish notations and conventions to be used in the next section.

We regard a bounded material body $B$ as an oriented, three-dimensional, differentiable manifold whose underlying set is the closure of a measurable and nondenumerable set $\Omega$, the elements $P$ of which are called material particles. Since $\Omega$ is homeomorphic to some region $\mathcal{R}_0$ of $\mathbb{R}^3$, we can identify with each particle $P$ an intrinsic set of particle labels $X = (X^1, X^2, X^3)$ so that as $P$ runs over $\Omega$, $X$ runs over $\mathcal{R}_0$. As a consequence of this homeomorphism, we make no distinction between $X$ and $P$ or $P$ and $\mathcal{R}_0$ in what follows. To identify places in Euclidean space (points in $\mathbb{R}^3$), we establish a spatial frame of reference to which there corresponds a curvilinear coordinate system $x = (x^1, x^2, x^3)$. As is customary, we choose $\mathcal{R}_0$ to correspond to the region occupied by the body in its reference configuration, and we select the labels $(X^1, X^2, X^3)$ identifying each particle $X$ so as to coincide with the coordinates $(x^1, x^2, x^3)$ of $X$ in the reference configuration.

The position of particle $X$ in the reference configuration relative to the fixed spatial frame is defined by the position vector $r_0(X)$, which is assumed to be continuously differentiable and invertible on the closure $\Omega$ of $\Omega$. If $g(X)$ denotes the determinant of the metric tensor at $X$, i.e.,

$$g(X) = \det \left( \frac{\partial r_0}{\partial X^\alpha} \cdot \frac{\partial r_0}{\partial X^\beta} \right) = \left[ \det \left( \frac{\partial r_0}{\partial x} \right) \right]^2$$

then the volume measure of a differential volume of material at $X$ is denoted

$$dv(X) = g(X)^{1/2} \, dX^1 \, dX^2 \, dX^3.$$ \hspace{1cm} (3.1)

Likewise, a differential element of material surface area of the boundary $\partial \Omega = \overline{\Omega} - \Omega$ of $\Omega$ at a particle $X \in \partial \Omega$ is denoted $ds(X)$. Thus, in the reference configuration,

$$\text{meas}(\Omega) = \int_{\Omega} dv \quad \text{and} \quad \text{meas}(\partial \Omega) = \int_{\partial \Omega} ds,$$

where here and in all the developments to follow, Lebesgue integration is implied.

If the position of $X$ in an arbitrary (deformed) configuration relative to the frame $x$ is given by the position vector $x(X)$, then

$$x(X) = r_0(X) + w(X),$$ \hspace{1cm} (3.2)

where $w(X)$ is the displacement of particle $X$. This relation defines a continuous and invertible map $x$ of particles (or, equivalently, places in $\mathcal{R}_0$ corresponding
to particles) into places \( x \in \mathbb{R}^3 \). We often require that this map be absolutely continuous; i.e., \( x \) is continuous and its partial derivatives with respect to \( X^\alpha \), \( \alpha = 1, 2, 3 \), are measurable on \( \Omega \). It is customary to require that the map \( x \) be locally invertible and orientation-preserving at each \( X \), except possibly on sets of zero measure. This requirement manifests itself in the invertibility condition,

\[
g(X)^{-1/2} [G_1(X) \cdot (G_2(X) \times G_3(X))] > 0 \quad \text{for almost every } X \in \Omega
\]  

(3.3)

where

\[
G_\alpha(X) = x,\alpha(X)
\]  

(3.4)

and commas denote differentiation with respect to the \( X^\alpha \). It is important to note that the differentiations \( x,\alpha = \partial x/\partial X^\alpha \) need be interpreted, at this point, only in a generalized sense. We elaborate on this point in the next section.

We denote by \( C_{ab}(X) \) the components of the Green-Saint-Venant strain tensor \( C(X) \) at \( X \):

\[
C_{ab}(X) = G_a(X) \cdot G_b(X) \in (L^2(\Omega))^9.
\]  

(3.5)

Then

\[
\det(C_{ab}(X)) = g(X)^{-1} [\det(\partial x(X)/\partial X)]^2
\]  

(3.6)

so that \( \partial x/\partial X \in (L^2(\Omega))^9 \) and the invertibility condition (3.3) implies that \( C(X) \) is positive definite almost everywhere in \( \Omega \). As indicated in the Introduction, we do not impose the local invertibility constraint in this work. A discussion of the difficulties inherent in imposing this condition is to be given in future work.

4. SOME HYPOTHESES AND INEQUALITIES

In classical theories of elasticity, it is convenient to describe the particular constitution of the material of which the body is composed by a material law giving the first Piola-Kirchhoff stress \( S(X) \) as a function of, for example, the Green-Saint Venant strain. Indeed, if \( T = T^{a\beta}(X) G_a(X) \otimes G_\beta(X) \) is the Cauchy stress at \( X \), and if

\[
S^{\kappa\lambda}(X) = G(X)^{1/2} T^{a\beta}(X) x,_{\beta}(X)
\]  

(4.1)

where \( G(X) = \det(C_{\alpha\beta}(X)) \), then the constitutive equation is, for example, of the form

\[
S(X) = Q(C(X), X) = Q(\nabla w(X), X)
\]  

(4.2)

\footnote{The invertibility condition (in a slightly stronger form) is well known in continuum mechanics; see, for example, Truesdell and Toupin [24] or Truesdell and Noll [25]. However, its importance in existence theory was first emphasized by Antman [2]; see also Antman [3–6].}
where $\nabla w(X)$ is the displacement gradient tensor at $X$ and $Q$ and $Q$ are frame-
indifferent response functionals satisfying appropriate material symmetry rules
for the material at hand. Also, in the classical theory, the function $Q$ is assumed
to be a differentiable function of (for example) the displacement gradients, and
we denote
\[
\frac{\partial Q(C(X), X)}{\partial C} \bigg|_{C = \nabla w} = A(\nabla w(X), X) \quad (4.3)
\]
where $A$ is a fourth order tensor.

Let us assume that the $x^i$ are cartesian and that $\nabla w = (\partial w/\partial x^a) = (w_{,a})$.
Then the components of $A$ of (4.3) are given by
\[
A^{k\alpha\beta}(\nabla w(X), X) = \left. \frac{\partial Q^{k\alpha\beta}(\nabla w, X)}{\partial \omega_{ij}} \right|_{C = \nabla w}. \quad (4.4)
\]
The condition
\[
A^{k\alpha\beta}(\nabla w(X), X) m^a n_a m^b n_b > 0 \quad (4.5)
\]
for a given displacement field $w$ and for arbitrary nonzero vectors $m$ and $n$ is
called the strong ellipticity condition at $X$ (see, e.g., Truesdell and Toupin [24],
Hayes [15], or Antman [2, 3]).

We remark that if $A$ is any constant fourth order tensor with the properties

(i) $A^{k\alpha\beta}m^a n_a m^b n_b > 0, \forall m, n \neq 0$, and

(ii) $A$ is continuous 4-linear form mapping of $S_1 \times S_1$ into $\mathbb{R}$, where $S_1$
is the unit sphere in $\mathbb{R}^3$,
\[
S_1 = \{ a \in \mathbb{R}^3 : \| a \|_{\mathbb{R}^3} = 1 \},
\]
then there exists a constant $\gamma_0 > 0$ such that
\[
A^{k\alpha\beta} m^a m^b n_a n_b \geq \gamma_0 \| m \|_{\mathbb{R}^3} \| n \|_{\mathbb{R}^3}. \quad (4.6)
\]

We next list a number of hypothesis that play an important role in subse-
quent developments:

I. Henceforth, we regard the displacement field $w$ as an element of a
separable reflexive Banach space $\mathcal{U}(\Omega)$ called the space of admissible displace-
ments. In this setting, stress is identified with a continuous, tensor-valued, linear
functional $S$ on the space $\mathcal{V}(\Omega)$, and this functional depends upon $w$. Moreover,
the dependence of the stress on $w$ is determined by a generalized response

\footnote{This follows immediately from the fact that $(A^{k\alpha\beta} m^a m^b n_a n_b) \| m \|_{\mathbb{R}^3} \| n \|_{\mathbb{R}^3}$ is continuous on the compact set $S_1 \times S_1$ and, therefore, achieves its infimum $\gamma_0$ there. Condition (i) guarantees that this infimum is positive.}

\footnote{The assumption of separability is convenient but not essential, since we can make use of filters in lieu of countably dense subspaces in subsequent compactness arguments.}
functional $Q$ defined on a subspace $U_d(\Omega)$ of $U(\Omega)'$, $U(\Omega)'$ being the topological dual of $U(\Omega)$, such that

I.1. For almost every $X \in \Omega$, the function $C \rightarrow Q(C, X) \in U(\Omega)'$, $C \in L^2(\mathbb{R}^3, \mathbb{R}^3)$, is a Gâteaux differentiable function of $C$, its derivative being the fourth-order tensor $A$ such that, for real $\theta$,

$$
\int_0^1 \frac{dQ(\nabla u + \theta \nabla v, X)}{d\theta} d\theta = \int_0^1 A(\nabla u + \theta \nabla v, X) d\theta \cdot \nabla v
$$

(4.6)

$\forall u, v \in U(\Omega)$ and almost every $X \in \Omega$;

I.2. For every $C \in L^2(\mathbb{R}^3, \mathbb{R}^3)$, the functions $X \rightarrow Q(C, X)$ and $X \rightarrow A(C, X)$ are measurable.

II. $U(\Omega)$ is densely and continuously embedded in a Hilbert space $U(\Omega)$ and $U(\Omega)$ is a pivot space; i.e., $U(\Omega)$ is identified with its topological dual $U(\Omega)'$. We then have the inclusions,

$$
U(\Omega) \subset H(\Omega) = H(\Omega)' \subset U(\Omega)'.
$$

(4.7)

Moreover, $U(\Omega)$ is contained in a reflexive Banach space $V(\Omega)$ with compact injection $i: U(\Omega) \rightarrow V(\Omega)$, and $V(\Omega) \subset H(\Omega)$. 6

III. Let $B_\mu(0)$ be the ball of radius $\mu$ in $U(\Omega)$. Then, for every $u, \eta \in B_\mu(0)$, 7

5 With rare exceptions, we generally have in mind here the space $L^2(\Omega) = (L^2(\Omega))^3$ in applications.

6 For example, if $U(\Omega) = W^{1,p}(\Omega)$, $p > 2$, where $W^{1,p}(\Omega)$ is the Sobolev space introduced in Section 2, then $U(\Omega)$ is compact in $L^p(\Omega)$ and $L^p(\Omega) \subset L^2(\Omega)$. Other examples, of course, could be cited.

7 In the classical linear theory of elasticity, the condition $A^{\alpha \beta}(X) m_{\alpha \beta} \geq y_0 \| m \|_{L^2(\Omega)} \| n \|_{L^2(\Omega)}$ implies the Gårding-inequality,

$$
\int_{\Omega} A^{\alpha \beta}(X) v_{\alpha \beta}(X) v_{\alpha \beta}(X) \, dx \geq c_0 \| v \|_{W^{1,2}(\Omega)^3}^2 + c_1 \| v \|_{L^2(\Omega)}^2
$$

where $c_0, c_1 > 0$; see, for example, Yosida [28] or Gurtin [14].

A nonlinear version of this type of inequality was described by Visik [24] for a certain class of quasilinear boundary-value problems. In certain nonlinear theories, it may be reasonable to assume a strong ellipticity condition of the form

$$
\int_0^1 A^{\alpha \beta}(\nabla u + \theta \nabla v, X) d\theta \cdot m_{\alpha \beta} n_{\alpha \beta} \geq \gamma(\nabla v) \| m \|_{L^2(\Omega)} \| n \|_{L^2(\Omega)}
$$

$\forall u, v \in U(\Omega)$, $\forall m, n \in \mathbb{R}^3$, and almost every $X \in \Omega$.

where function $\gamma(\nabla v(X), X)$ is strictly positive for every $v \in U(\Omega)$ and almost every $X \in \Omega$. Ball [7] has apparently used a strong ellipticity condition of this type in his work on hyperelasticity. Knowles and Sternberg [16, 17] have constructed examples of finite deformations of elastic bodies for which strong ellipticity is violated. We emphasize that we do not assume strong ellipticity at any point in this work.
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\[ \int_{\Omega} \int_{\tilde{\Omega}} A^{\alpha \beta}(\nabla \mathbf{u} + \theta \nabla \eta, \mathbf{X}) \, d \theta \eta_{\alpha \beta}(\mathbf{X}) \eta_{\beta \gamma}(\mathbf{X}) \, dv \geq \mathcal{F}(\| \eta \|_{\mathcal{W}^1(\Omega)}) - \mathcal{G}(\mu, \| \eta \|_{\mathcal{W}^1(\Omega)}) \]

where \( \mathcal{F} \) and \( \mathcal{G} \) are continuous non-negative, real-valued functions and \( \mathcal{G} \) has the property that

\[ \lim_{\theta \to 0^+} \frac{1}{\theta} \mathcal{G}(x, \theta y) = 0 \quad \forall x, y \in \mathbb{R}^+ \]

where \( \theta \) is a real parameter greater than zero.

IV. The operators entering the boundary-value problems of elasticity are coercive, i.e.

\[ \lim_{\| \eta \|_{\mathcal{W}^1(\Omega)} \to \infty} \| \eta \|_{\mathcal{W}^2(\Omega)}^2 \left\{ \int_{\Omega} (Q(\nabla \eta, \mathbf{X}) : \nabla \eta - \rho_0 F(\eta, \mathbf{X}) \cdot \eta) \, dv \right\} = +\infty \]

where \( F(\eta, \mathbf{X}) \) is a body force possibly depending on \( \eta \).

We emphasize that \( \mathcal{F}(x) - \mathcal{G}(\mu, x) \) may be negative for certain \( \mu, x \). We give more specific forms of \( \mathcal{F} \) and \( \mathcal{G} \) in Section 7.

5. A Variational Boundary-Value Problem in Elastostatics

We now consider a class of nonlinear Dirichlet problems of elastostatics (i.e., we consider the problem of place with homogeneous boundary data). In addition to the assumptions and conventions laid down in Sections 3 and 4, we assume that \( \mathcal{D}(\Omega) \) is dense in the space of admissible displacements \( \mathcal{U}(\Omega) \), where

\[ \mathcal{D}(\Omega) = (\mathcal{D}(\Omega))^3 \]

and \( \mathcal{D}(\Omega) \) is the space of test functions defined on \( \Omega \); i.e., \( \mathcal{D}(\Omega) \) is the linear space \( C_0^\infty(\Omega) \) of infinitely differentiable functions with compact support in \( \Omega \) equipped with the usual locally convex topology of test functions. The topological dual of \( \mathcal{D}(\Omega) \) is then the locally convex linear topological space \( \mathcal{D}(\Omega)' \) of vector-valued Schwartz distributions. \(^8\) We then have the inclusions,

\[ \mathcal{D}(\Omega) \subset \mathcal{U}(\Omega) \subset \mathcal{U}(\Omega)' \subset \mathcal{D}(\Omega)' \]  

We consider the following problem in nonlinear elasticity: Find the displacement field \( \mathbf{w} \in \mathcal{U}(\Omega) \) such that

\[ \int_{\Omega} (Q(\nabla \mathbf{w}) : \nabla \mathbf{v} - \rho_0 F(\mathbf{w}) \cdot \mathbf{v}) \, dv = 0 \quad \forall \mathbf{v} \in \mathcal{U}(\Omega) \]

\(^8\) See, for example, Schwartz [21] or, for a readable summary account, Yosida [28].
where \( \rho_0 = \rho_0(X) \) is the mass density in the reference configuration and \( F(w) \equiv F(w(X), X) \) is the body force, possibly dependent on \( w \), but not necessarily derivable from a potential.\(^9\) Here

\[
X \to \rho_0(X) F(w(X), X) \quad \text{is measurable} \quad \forall w \in \mathcal{U}(\Omega)
\]

\[
\xi \to \rho_0(X) F(\xi, X), \quad \xi \in \mathbb{R}^3, \quad \text{is continuous from } \mathbb{R}^3 \text{ into itself for almost every } X \in \Omega.
\]

Let us denote

\[
P(u, v) = Q(\nabla u) : \nabla v - \rho_0 F(u) \cdot v.
\]

Then in this case,

\[
\mathcal{H}(\Omega) = \{ w \colon P(w, w) \in L^1(\Omega); w(X) = 0, \text{ a.e. } X \in \partial \Omega \}. \tag{5.4}
\]

Remark. We remark that we do not necessarily assume that the material is hyperelastic, nor do we assume that the external forces \( F \) are conservative.\(^1\)

The Formal Operator \( \Lambda \colon \mathcal{U}(\Omega) \to \mathcal{U}(\Omega)' \). We shall now investigate the relationship between the variational boundary-value problem (5.2) and the "classical" boundary-value problem of place in nonlinear elasticity.

We observe that for each choice of a displacement field \( u \in \mathcal{U}(\Omega) \), the integral in (5.2) defines a continuous linear functional \( \Lambda(u) \in \mathcal{U}(\Omega)' \). Indeed, if \( \langle \cdot, \cdot \rangle \) denotes duality pairing on \( \mathcal{U}(\Omega)' \times \mathcal{U}(\Omega) \), then we may write

\[
\Lambda(u)(v) = \langle \Lambda(u), v \rangle = \int_\Omega (Q(\nabla u) : \nabla v - \rho_0 F(u) \cdot v) \, dv. \tag{5.5}
\]

Thus, (5.2) is equivalent to the following abstract problem on reflexive Banach spaces: find \( w \in \mathcal{U}(\Omega) \) such that

\[
\Lambda(w) = \theta \tag{5.5}
\]

where \( \Lambda \) is a nonlinear map from \( \mathcal{U}(\Omega) \) into its topological dual \( \mathcal{U}(\Omega)' \) and \( \theta \) is the zero elements of \( \mathcal{U}(\Omega)' \).

To interpret \( \Lambda \), we make use of (5.1) and (4.7), which lead to

\[
\mathcal{D}(\Omega) \subset \mathcal{U}(\Omega) \subset \mathcal{H}(\Omega) = \mathcal{H}(\Omega)' \subset \mathcal{U}(\Omega)' \subset \mathcal{D}(\Omega)'. \tag{5.6}
\]

Let \( \varphi \in \mathcal{D}(\Omega) \) and consider the distribution \( \mathcal{A}(w) \in \mathcal{D}(\Omega)' \) defined by

\[
\mathcal{A}(w)(\varphi) = \int_\Omega (Q(\nabla w) : \nabla \varphi - \rho_0 F(w) \cdot \varphi) \, dv \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{5.7}
\]

\(^9\) Here, and elsewhere in this section, we refrain from displaying the dependence of various values on \( X \) for simplicity in notation, except in those instances where this dependence is of special importance. Thus, \( Q(\nabla w(X)), \nabla v(X), \rho_0(X) F(w(X), X) \) are written \( Q(\nabla w), \nabla v, \rho_0 F(w), \) etc.
Clearly, if $[\cdot, \cdot]$ denotes duality pairing on $\mathcal{D}(\Omega)' \times \mathcal{D}(\Omega)$, and if we identify $Q(\nabla w)$ and $-F(w)$ with distributions in $\mathcal{D}(\Omega)'$, we may write

$$\mathcal{A}(w)(\varphi) = [-\text{Div} Q(\nabla w) - \rho_0 F(w), \varphi] \quad \forall \varphi \in \mathcal{D}(\Omega). \quad (5.8)$$

Here we have followed a standard practice of using the same notations for the distributions as that for the functions identified with them. In (5.8), $-\text{Div}$ denotes the distributional divergence operator.

Now let $\mathcal{J}$ denote the injection of $\mathcal{D}(\Omega)$ in $\mathcal{U}(\Omega)$ and $\mathcal{J}'$ its adjoint:

$$\mathcal{J} : \mathcal{D}(\Omega) \rightarrow \mathcal{U}(\Omega); \quad \mathcal{J}' : \mathcal{U}'(\Omega) \rightarrow \mathcal{D}(\Omega)' \quad (5.9)$$

Then, it is clear that for any $\varphi \in \mathcal{D}(\Omega)$,

$$\langle \mathcal{J}(w), \varphi \rangle = [\mathcal{J}' \Lambda(w), \varphi] \quad (5.10)$$

and since, by virtue of (5.7), the values of the distributions $\mathcal{A}(w)$ and $\mathcal{J}' \Lambda(w)$ must coincide, we conclude that

$$\mathcal{J}' \Lambda(w) = \mathcal{A}(w); \quad (5.11)$$

that is, the linear functional $\Lambda(w)$ associated with the variational boundary-value problem (5.2) is the restriction of the distribution

$$\mathcal{A}(w) = [-\text{Div} Q(\nabla w) - \rho_0 F(w) \quad (5.12)$$

to the topological dual $\mathcal{U}'(\Omega)$ of the space $\mathcal{U}(\Omega)$ of admissible displacements.

We obtain the classical problem of elastostatics,

$$\text{Div} Q(\nabla w) + \rho_0 F(w(X)) = 0; \quad w = 0 \text{ on } \partial \Omega$$

whenever

(1) $w(X)$ and the functions $\text{Div} Q(w(X)), F(w(X))$ are sufficiently smooth that (5.12) makes sense for every $X \in \Omega$; e.g., $w(X) \in C^2(\Omega) \cap C^0(\Omega)$, $F(w(X))$, $Q \in C^1$.

(2) We identify the distributions in (5.12) with the functions in (5.1).

If $-\text{Div} Q(\nabla( )) - \rho_0 F( )$ is the classical differential operator on, for example, $C^2(\Omega) \cap C^0(\Omega)$, then $\Lambda$ represents its extension to $\mathcal{U}(\Omega)$.

The role of the pivot space $\mathcal{H}(\Omega)$ in these interpretations is critical: the pivot space is the space of transition from functions to functionals; it is here that we are permitted to identify functions in $\mathcal{H}(\Omega)$ with functionals in $\mathcal{H}'(\Omega)$.

An important question which arises at this point (and one which we shall not address in this paper) is that of regularity: given $\Omega$, $Q$, and the data $\rho_0F$, and assuming a solution exists, exactly what is the "smallest" space in which the solution lies? This fundamental question seems to be open for virtually all nonlinear boundary-value problems of much significance.
6. ABSTRACT EXISTENCE THEOREMS FOR ELASTICITY

Throughout this section, we shall use the following conventions:

(1) $\mathfrak{U}$ and $\mathfrak{V}$ are separable reflexive Banach spaces with norms $\| \cdot \|_{\mathfrak{U}}$ and $\| \cdot \|_{\mathfrak{V}}$, respectively;

(2) $\mathfrak{U}$ is continuously and densely embedded in $\mathfrak{V}$ and the injection $i: \mathfrak{U} \to \mathfrak{V}$ is compact.

We recall that an operator $A: \mathfrak{U} \to \mathfrak{W}$ is said to be hemicontinuous if, for every $u, v \in \mathfrak{U}$, the function\(^{10}\)

$$\varphi(t) = \langle A(u + tv), v \rangle, \quad t \in \mathbb{R},$$

where $\langle \cdot, \cdot \rangle$ denotes duality pairing on $\mathfrak{U} \times \mathfrak{W}$, is a continuous function from $\mathbb{R}$ into $\mathbb{R}$. We also recall that for nonlinear operators, an operator $A: \mathfrak{U} \to \mathfrak{W}$ is bounded whenever it maps bounded sets in $\mathfrak{U}$ into bounded sets in $\mathfrak{W}$.

We next introduce a broad class of operators which includes many of those encountered in nonlinear elasticity.

**DEFINITION.** An operator $A: \mathfrak{U} \to \mathfrak{W}'$, where $\mathfrak{W}'$ is the topological dual of $\mathfrak{U}$, will be called a Gårding operator if $A$ can be expressed in the form

$$A(u) = A(u, u)$$

where $u, v \mapsto A(u, v)$ is an operator on $\mathfrak{U} \times \mathfrak{U} \to \mathfrak{W}'$ having the following properties:

(i) $\forall v \in \mathfrak{U}$, $u \mapsto A(u, v)$ is hemicontinuous from $\mathfrak{U}$ into $\mathfrak{W}'$.

(ii) There exists a continuous, non-negative valued function $H: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, where $\mathbb{R}^+ = [0, \infty)$, with the properties

$$\lim_{\theta \to 0^+} \frac{1}{\theta} H(x, \theta y) = 0 \quad \forall x, y \in \mathbb{R}^+$$

such that for every $u$ and $v$ in the ball

$$B_u(0) = \{ u \in \mathfrak{U}; \| u \|_{\mathfrak{U}} < \mu, \mu > 0 \}$$

the following inequality holds:

$$\langle A(u, u) - A(v, u), u - v \rangle \geq -H(\mu, \| u - v \|_{\mathfrak{U}}).$$

\(^{10}\) This definition of hemicontinuity is slightly more general than that found elsewhere (e.g., Lions [18, p. 157]) wherein the function $\chi(t) = \langle A(u) + tv), w \rangle$ is continuous for every $u, v, w \in \mathfrak{U}$. Here we require continuity only in the direction of the first variable $u$.\)
(iii) If \( \{u_n\} \) is a sequence converging weakly to an element \( u \in \mathcal{U} \), then

\[
\liminf_{n \to \infty} \langle A(v, u_n) - A(v, u), u_n - u \rangle \geq 0 \quad \forall v \in \mathcal{U},
\]

\[
\liminf_{n \to \infty} \langle A(v, u_n) - A(v, u), w \rangle = 0 \quad \forall v, w \in \mathcal{U}.
\]  

We shall now demonstrate that if a Gårding operator \( A \) is bounded, then it falls into the class of so-called pseudomonotone operators \(^{11}\) which have been studied by Brézis [9], Lions [18], and others.

**Theorem 6.1.** Let \( A: \mathcal{U} \to \mathcal{U}' \) be a Gårding operator. Then \( A \) has the following property: If \( \{u_n\} \) is a sequence converging weakly to an element \( u \in \mathcal{U} \), and if

\[
\liminf_{n \to \infty} \langle A(u_n), u_n - u \rangle < 0,
\]

then

\[
\liminf_{n \to \infty} \langle A(u_n), u_n - v \rangle \geq \langle A(u), u - v \rangle \quad \forall v \in \mathcal{U}.
\]

**Proof.** Let \( \{u_n\} \) converge weakly to \( u \) in \( \mathcal{U} \) and let \( w \) be an arbitrary element of \( \mathcal{U} \). If \( \mu > \max\{|| w ||_{\mathcal{U}'}, || u_n ||_{\mathcal{U}'}, || u ||_{\mathcal{U}'}\} \), then according to (6.3),

\[
\langle A(u_n, u_n) - A(w, u_n), u_n - w \rangle \geq -H(\mu, || u_n - w ||_{\mathcal{U}'}).
\]

Thus,

\[
\langle A(u_n), u - w \rangle + \langle A(u_n), u_n - u \rangle
\]

\[
\geq \langle A(w, u_n) - A(w, u), u_n - u \rangle + \langle A(w, u_n) - A(w, u), u - w \rangle
\]

\[
+ \langle A(w, u), u_n - w \rangle - H(\mu, || u_n - w ||_{\mathcal{U}'}).
\]

Let (6.6) hold. Then, since

\[
\liminf_{n \to \infty} \langle A(u_n), u_n - u \rangle \leq \limsup_{n \to \infty} \langle A(u_n), u_n - u \rangle \leq 0,
\]

we have

\[
\liminf_{n \to \infty} \langle A(u_n), u - w \rangle
\]

\[
\geq \liminf_{n \to \infty} \{ \langle A(w, u_n) - A(w, u), u_n - u \rangle + \langle A(w, u_n) - A(w, u), u - w \rangle \}
\]

\[
+ \lim_{n \to \infty} \langle A(w, u), u_n - w \rangle - H(\mu, || u_n - w ||_{\mathcal{U}'}).
\]

Now the terms in braces on the right side of this are non-negative by virtue of

\(^{11}\) An operator \( A: \mathcal{U} \to \mathcal{U}' \) is pseudomonotone if it is bounded and if (6.7) holds whenever \( u_n \to u \) weakly in \( \mathcal{U} \) then (6.6) is satisfied.
(6.4) and (6.5). Likewise, since $\mathcal{H}$ is compact in $\mathcal{V}$, $\{u_n\}$ converges strongly to $u$ in $\mathcal{V}$. Also, $H(x, y)$ is, by hypothesis, continuous in $y$. Therefore,

$$\lim_{n \to \infty} \inf \langle A(u_n), u - w \rangle \geq \langle A(w, u), u - w \rangle - H(\mu, \| u - w \|_{\mathcal{V}}) \quad \forall w \in \mathcal{H}.$$ 

Next, we set $w = u - \theta(u - v)$, where $\theta$ is an arbitrary real number and $v$ is an arbitrary element of $\mathcal{H}$ ($\mu$ now being also large enough to bound $\| v \|_{\mathcal{V}}$). Then

$$\lim_{n \to \infty} \inf \langle A(u_n), u - v \rangle \geq \langle A(u - \theta(u - v), u), u - v \rangle - \frac{1}{\theta} H(\mu, \theta \| u - v \|_{\mathcal{V}}).$$

Now using the hemicontinuity guaranteed by property (i) and using property (6.1), we take the limit as $\theta \to 0^+$ to obtain

$$\lim_{n \to \infty} \inf \langle A(u_n), u - v \rangle \geq \langle A(u), u - v \rangle \quad \forall v \in \mathcal{H}. \quad (6.8)$$

Next, we observe that

$$\langle A(u_n), u_n - u \rangle = - H(\mu, \| u_n - u \|_{\mathcal{V}}).$$

Hence

$$\langle A(u_n), u_n - u \rangle \geq \langle A(u), u_n - u \rangle + \langle A(u_n), u_n - u \rangle - H(\mu, \| u_n - u \|_{\mathcal{V}})$$

and, therefore, in view of (6.4),

$$\lim_{n \to \infty} \inf \langle A(u_n), u_n - u \rangle \geq 0.$$ 

We conclude from this last result that

$$\lim_{n \to \infty} \inf \langle A(u_n), u_n - v \rangle = \lim_{n \to \infty} \inf \{ \langle A(u_n), u - v \rangle + \langle A(u_n), u_n - u \rangle \}$$

$$\geq \lim_{n \to \infty} \inf \langle A(u_n), u - v \rangle.$$

Thus, in view of (6.8),

$$\lim_{n \to \infty} \inf \langle A(u_n), u_n - v \rangle \geq \langle A(u), u - v \rangle \quad \forall v \in \mathcal{H}$$

which was to be proved. 

**Remark.** Gårding operators $A: \mathcal{H} \to \mathcal{H}'$ can, by definition, be of the form

$$A(u) = B(u) C(u)$$

where $B$ is a hemicontinuous operator such that $\forall u, v \in B_u(0)$

$$\langle B(u) C(u) - B(v) C(u), u - v \rangle \geq -H(\mu, \| u - v \|_{\mathcal{V}})$$
and $C$ is an operator such that $u_n \rightharpoonup u$ weakly in $\mathcal{U}$ implies that

$$\liminf_{n \to \infty} \langle B(v) C(u_n) - B(v) C(u), u_n - u \rangle \geq 0 \quad \forall v \in \mathcal{U}$$

and

$$\lim_{n \to \infty} \langle B(v) C(u_n) - B(v) C(u), w \rangle \geq 0 \quad \forall v, w \in \mathcal{U}.$$

The conditions on $C$ are very weak, and below we show that they can be replaced by somewhat stronger conditions which are easier to implement in specific applications.

We note also that Gårding operators may also be expressible as a sum:

$$A(u) = A(u, u), \quad A(u, v) = B(u) + C(v).$$

In the theory of pseudomonotone operators, $B$ is often assumed to be hemi-continuous, bounded, and monotone, and it generally is introduced to represent the terms in a nonlinear partial differential operator involving derivatives of the the highest order. The term $C$ generally involves lower order derivatives and is weakly continuous. Obviously, for the class of Gårding operators of this form, $B$ need not be monotone.

We finally arrive at a general existence theorem.

**THEOREM 6.2.** Let $A: \mathcal{U} \to \mathcal{U}'$ be a bounded Gårding operator and let $A$ be coercive; i.e., let

$$\|u\|_{\mathcal{U}}^2 \to +\infty.$$ (6.9)

Then $A$ is surjective; i.e., for every $f \in \mathcal{U}'$ there exists at least one $u \in \mathcal{U}$ such that

$$A(u) = f.$$ (6.10)

**Remark.** This result actually follows immediately from Theorem 6.1 and a theorem of Brézis [9]: Coercive, pseudomonotone operators on reflexive separable Banach spaces into their duals are surjective. (See also Lions [18, p. 180].) For completeness, we shall outline a constructive proof.

**Proof of Theorem 6.2.** We adopt a method of proof similar to those common to monotone operator theory. We make use of three devices.

1. The construction of a countable family of finite-dimensional subspaces $\mathcal{U}_n$ of $\mathcal{U}$ such that $\bigcup_{n=1}^{\infty} \mathcal{U}_n$ is dense in $\mathcal{U}$, and the construction of approximations of (6.10) on these subspaces;

2. The fact that restrictions of bounded pseudo-monotone operators on reflexive Banach spaces to finite-dimensional subspaces are continuous, and
3. The classical Brouwer fixed point theorem,\(^{12}\) or a well-known corollary to it: if \( F: \mathbb{R}^n \to \mathbb{R}^n \) is continuous and coercive (i.e., \( \frac{\langle F(x) \cdot x \rangle}{\|x\|^n} \to +\infty \) as \( \|x\| \to \infty \)), then for every \( p \in \mathbb{R}^n \) there is at least one \( x \in \mathbb{R}^n \) such that \( F(x) = p \).

We denote by \([w_1, w_2, \ldots]\) a basis for \( \mathcal{W} \) and by \([w_1, w_2, \ldots, w_m]\) a basis for a finite-dimensional subspace \( \mathcal{W}_m \) of \( \mathcal{W} \). The Faedo–Galerkin approximation of (6.10) on \( \mathcal{W}_m \) involves the following finite-dimensional problem: find \( u_m \in \mathcal{W}_m \) such that

\[
\langle A(u_m), w_j \rangle = \langle f, w_j \rangle, \quad j = 1, 2, \ldots, m
\]  

(6.11)

where, again, \( \langle \cdot, \cdot \rangle \) denotes duality pairing on \( \mathcal{W} \).

Observations 2 and 3 listed above establish that (6.11) has a solution \( u_m \) for each \( m \). Let \( \{u_m\} \) denote the sequence of solutions to the approximate problems, and let \( \gamma: \mathbb{R}^+ \to \mathbb{R} \) be such that

\[
\gamma(n, u_m, \|u_m\|_\mathcal{W}) = \frac{\langle A(u_m), u_m \rangle}{\|u_m\|_\mathcal{W}} = \frac{\langle f, u_m \rangle}{\|u_m\|_\mathcal{W}} \leq \|f\|_\mathcal{W} < \infty.
\]

By hypothesis, \( \gamma(r) \to +\infty \) as \( r \to \infty \). Hence, the sequence of approximations \( \{u_m\} \) is bounded. Since \( \mathcal{W} \) is reflexive, we can extract a subsequence \( \{u_{m_k}\} \) which converges weakly to an element \( u \in \mathcal{W} \).

Now for each \( k \), \( u_{m_k} \) is a solution of the finite dimensional problem. Hence,

\[
\langle A(u_{m_k}), w_j \rangle = \langle f, w_j \rangle, \quad 1 \leq j \leq m_k
\]

and, as \( j \to \infty \), \( \langle A(u_{m_k}), w \rangle = \langle f, w \rangle \). Hence \( \{A(u_{m_k})\} \) is bounded and, therefore, we can choose a subsequence which converges weakly to some element \( \chi \in \mathcal{W}' \). We continue to denote this subsequence by \( \{u_{m_k}\} \).

Since \( \langle A(u_{m_k}), u_{m_k} \rangle = \langle f, u_{m_k} \rangle \), we have

\[
\lim_{k \to \infty} \langle A(u_{m_k}), u_{m_k} \rangle = \langle f, \chi \rangle = \langle \chi, u \rangle.
\]

Hence,

\[
\lim_{k \to \infty} \sup \langle A(u_{m_k}), u_{m_k} - u \rangle = 0.
\]

Thus (6.6) is satisfied. It follows, then, from (6.7) that

\[
\lim_{k \to \infty} \inf \langle A(u_{m_k}), u_{m_k} - v \rangle \geq \langle A(u), u - v \rangle.
\]

Thus, \( \forall v \in \mathcal{W} \),

\[
\langle A(u), u - v \rangle \leq \lim_{k \to \infty} \inf \langle A(u_{m_k}), u_{m_k} - v \rangle \leq \langle \chi, u - v \rangle
\]

\(^{12}\) Brouwer [10]. For a comprehensive account of the Brouwer-theory and its generalizations, see Browder [11].

\(^{13}\) See, for example, Lions [18, p. 53].
or
\[
\langle A(u) - \chi, u - v \rangle \leq 0, \quad \forall v \in \mathcal{H}.
\]
Choosing \( v = 2u - w \) shows that \( \langle A(u) - \chi, w - v \rangle \leq 0 \ \forall w \). Hence,

\[
A(u) = \chi
\]

and the theorem is proved.

The theory developed up to this point is sufficiently general to encompass most of the applications to elasticity that we have in mind. However, with some minor modifications, we can bring the theory of operators of the type of the calculus of variations studied by Lions [18] into the framework of our theory. For this purpose, we introduce a slightly modified class of operators.

**Definition.** An operator \( A: \mathcal{H} \to \mathcal{H}' \) will be called a variational Gårding operator if \( A \) can be expressed in the form

\[
A(u) = A(u, u)
\]

where \( u, v \to A(u, v) \) is an operator on \( \mathcal{H} \times \mathcal{H} \to \mathcal{H}' \) having the following properties:

(i) \( \forall v \in \mathcal{H}, u \to A(u, v) \) is hemicontinuous from \( \mathcal{H} \) into \( \mathcal{H}' \).

(ii) Property (ii) of Gårding operators holds; i.e. (6.3) holds for a function \( H \) satisfying (6.1).

(iii) \( \forall u \in \mathcal{H}, v \to A(u, v) \) is bounded from \( \mathcal{H} \) into \( \mathcal{H}' \).

(iv) If \( \{u_n\} \) is a sequence converging weakly in \( \mathcal{H} \) to \( u \) and if

\[
\lim_{n \to \infty} \langle A(u_n , u_n) - A(u, u), u_n - u \rangle = 0
\]

then \( \forall v \in \mathcal{H}, A(v, u_n) \) converges weakly to \( A(v, u) \) in \( \mathcal{H}' \).

(v) If \( \{u_n\} \) is a sequence converging weakly to \( u \) in \( \mathcal{H} \) and if, \( \forall v \in \mathcal{H}, A(v, u_n) \) converges weakly to some element \( \chi \) in \( \mathcal{H}' \), then

\[
\lim_{n \to \infty} \langle A(v, u_n), u_n \rangle = \langle \chi, u \rangle.
\]

**Remark.** Obviously, we obtain variational Gårding operators from Gårding operators by replacing condition (iii) (particularly (6.4) and (6.5)) by conditions (iii)'--(v) above. These new conditions are similar to those found in Lion's definition of "operators of the type of the calculus of variations" in [18]. However, a variational Gårding operator, as we have defined it here, is considerably more general than the operators of the type of the calculus of variations. To obtain this latter class of operators, we must (1) add to property (i) the require-
moment that \( u \to A(u, v) \) be bounded, (2) take \( H := 0 \) in (i), and (3) demand that \( u \to A(v, u) \) be hemicontinuous.

In addition, the class of Gårding operators includes the quasilinear operators studied by Višik [25] and the semi-bounded quasi-linear operators studied by Dubinskii [12], the latter being obtained by replacing our assumed hemi-continuity in (i) by demicontinuity of the total operator, by requiring that (ii) hold for all of \( A \) rather than only a component part of \( A \), by replacing conditions (iii)–(v) with the gross hypothesis that \( A \) be completely continuous, and by choosing \( \mathcal{H} \) to be specifically the Sobolev space \( W^{1, \infty}_0(Q) \). It is also observed that \( A \) is not monotone, nor is it necessarily monotone in any of its components \( A(u) = A(u, u) \). However, we show below that if a variational Gårding operator \( A \) is bounded, then, again, it is pseudomonotone.

**THEOREM 6.3.** Let \( A: \mathcal{H} \to \mathcal{H} \) be a variational Gårding operator. Then \( A \) has the following property: If \( \{u_n\} \) is a sequence converging weakly to an element \( u \in \mathcal{H} \), and if (6.6) is satisfied, then (6.7) holds.

**Proof.** The plan of the proof involves the following four steps:

1. Suppose that \( \{u_n\} \) is a sequence in \( \mathcal{H} \) converging weakly to \( u \in \mathcal{H} \) such that (6.6) holds. We will show that a subsequence \( \{u_{n_k}\} \) can be chosen so that

\[
\lim_{n_k \to \infty} \inf \langle A(u_{n_k}), u_{n_k} - u \rangle = 0. \tag{6.14}
\]

2. Suppose that \( \{u_{n_k}\} \) is the subsequence in (6.14), \( v \) is an arbitrary element in \( \mathcal{H} \), and \( \mu \geq \max \{\|u_{n_k}\|_\mathcal{H}, \|v\|_\mathcal{H}, \|u\|_\mathcal{H}\} \) for all \( n_k \). Let

\[
w = \theta v + (1 - \theta) u, \quad \theta \in (0, 1).
\]

Then we shall show that

\[
\lim_{n_k \to \infty} \inf \langle A(u_{n_k}), u - v \rangle \geq \langle A(w, u), u - v \rangle - \frac{1}{\theta} H(\mu, \theta \|u - v\|_\mathcal{H}). \tag{6.15}
\]

3. Next we observe that (6.14) and (6.15) combine to give

\[
\lim_{n_k \to \infty} \inf \langle A(u_{n_k}), u - u_{n_k} + u_{n_k} - v \rangle
\]

\[
= \lim_{n_k \to \infty} \inf \langle A(u_{n_k}), u_{n_k} - v \rangle \geq \langle A(w, u), u - v \rangle - \frac{1}{\theta} H(\mu, \theta \|u - v\|_\mathcal{H}).
\]

Thus, using the hemicontinuity of \( A(w, u) \) in \( w \) and property (6.1) of \( H \), we take the limit as \( \theta \to 0 \) to obtain

\[
\lim_{n_k \to \infty} \inf \langle A(u_{n_k}), u_{n_k} - v \rangle \geq \langle A(u), u - v \rangle \quad \forall v \in \mathcal{H}. \tag{6.16}
\]
4. Finally, we argue that (6.16) must hold for arbitrary subsequences \( \{u_{n_k}\} \) and that, therefore, it also holds for the original sequence \( \{u_n\} \).

We proceed to prove (6.14). Since \( A(u, u_n) \) is bounded (property (iii)'), we can identify a subsequence \( \{u_{n_k}\} \), depending on \( u \), such that \( A(u, u_{n_k}) \) converges weakly to an element \( \chi_u \in \mathcal{W}' \). By property (v),

\[
\lim_{n_k \to \infty} \langle A(u, u_{n_k}), u_{n_k} - u \rangle = 0.
\]

Thus,

\[
\lim_{n_k \to \infty} \langle A(u_{n_k}), u_{n_k} - u \rangle = \lim_{n_k \to \infty} \langle A(u, u_{n_k}), u_{n_k} - u \rangle = \lim_{n_k \to \infty} \langle A(u_{n_k}), u_{n_k} - u \rangle \geq -\lim_{n_k \to \infty} H(\mu, \| u_{n_k} - u \|_\gamma) = 0.
\]

We must, therefore, conclude that \( \lim_{n_k \to \infty} \langle A(u_{n_k}), u_{n_k} - u \rangle = 0 \), which is precisely (6.14). The first step in our proof is, thus, easily concluded.

Next, let \( M_n \) be defined by

\[
M_n = \langle A(u_n, u_n) - A(u, u_n), u_n - u \rangle.
\]

If \( \{u_n\} \) is any sequence converging weakly to \( u \) and satisfying (6.6), then

\[
\limsup_{n \to \infty} M_n \leq -\liminf_{n \to \infty} \langle A(u, u_n), u_n - u \rangle.
\]

Hence, according to (6.14),

\[
\limsup_{n_k \to \infty} M_{n_k} \leq 0.
\]

But

\[
\lim_{n_k \to \infty} M_{n_k} \geq -\lim_{n_k \to \infty} H(\mu, \| u_{n_k} - u \|_\gamma) = 0
\]

because \( u_{n_k} \) must converge strongly to \( u \) in \( \gamma' \) and \( \lim_{r \to 0^+} H(\mu, r) = 0 \). Hence

\[
\lim_{n_k \to \infty} M_{n_k} = 0.
\]

This means, by virtue of property (iv), that

\[
A(v, u_{n_k}) \to A(v, u) \quad \text{weakly in} \quad \mathcal{W}', \quad \forall v \in \mathcal{W}. \quad (6.17)
\]

Next, we notice that (6.3) implies that

\[
\langle A(u_{n_k}) - A(v, u_{n_k}), u_{n_k} - w \rangle \geq -H(\mu, \| u_{n_k} - w \|_{\gamma'})
\]
where \( w = \theta w + (1 - \theta) u, \theta \in (0, 1) \). Thus

\[
\theta \langle A(u_{n_k}), u - v \rangle \geq \theta \langle A(w, u_{n_k}), u - v \rangle - \langle A(u_{n_k}), u_{n_k} - u \rangle \\
+ \langle A(w, u_{n_k}), u_{n_k} - u \rangle - H(\mu, \| u_{n_k} - w \|_Y).
\]

(6.18)

We shall examine the limit inferior of each of the terms on the right side of this inequality.

First, it is clear that the term \( \langle A(u_{n_k}), u - v \rangle \to \langle A(w, u), u - v \rangle \) by virtue of (6.17). The second term, \( \langle A(u_{n_k}), u_{n_k} - u \rangle \to 0 \) by (6.14). Likewise, (6.11) shows that \( A(w, u_{n_k}) \to A(w, u) \) weakly in \( Y' \) so that, by property (v) (or (6.13)), \( \langle A(w, u_{n_k}), u_{n_k} - u \rangle \to 0 \).

It now remains for us to examine the term \( H(\mu, \| u_{n_k} - w \|_Y) \) in (6.18). Since \( H \) is continuous in each of its arguments, and since \( \lim_{n_k \to \infty} \| u_{n_k} - w \|_Y = \lim_{n_k \to \infty} \| u_{n_k} - u + \theta(u - v) \|_Y \), we have

\[
\lim_{n_k \to \infty} H(\mu, \| u_{n_k} - w \|_Y) = H(\mu, \| u - v \|_Y).
\]

Thus, we arrive at the inequality

\[
\theta \liminf_{n_k \to \infty} \langle A(u_{n_k}), u - v \rangle \geq \theta \langle A(w, u), u - v \rangle - H(\mu, \| u - v \|_Y)
\]

which, upon dividing through by \( \theta \), gives (6.15).

At this point, we have established (6.14) and (6.15). Inequality (6.16) now follows immediately from the arguments given in step 3 of the plan described at the beginning of the proof. It remains only to complete step 4: to show that (6.16) holds for arbitrary subsequences. This, however, is not difficult.

Let

\[
P_n = \langle A(u_n), u_n - v \rangle - \langle A(u), u - v \rangle.
\]

We have shown that if \( \{u_n\} \) is a sequence converging weakly to \( u \) which satisfies (6.6), then a subsequence \( \{u_{n_k}\} \) can be chosen such that

\[
\lim_{n_k \to \infty} \inf P_{n_k} \geq 0.
\]

Suppose this were not the case; i.e. suppose it is possible to pick a subsequence \( \{u_{n_k}\} \) such that \( P_{n_k} \leq - \mu n \). Since \( u_n \) satisfies the original hypotheses, we can extract a subsequence \( \{u_{n_{k_m}}\} \) such that

\[
\lim \inf P_{n_{k_m}} \geq 0
\]

a contradiction. Hence (6.7) holds for \( \{u_n\} \), and the proof is complete.

We next have the obvious corollary:
THEOREM 6.4. Let \( A : \mathcal{U} \to \mathcal{U}' \) be a bounded, coercive, variational Gårding operator. Then \( A \) is surjective.

It is known that pseudomonotone operators belong to a larger and very general class of operators introduced by Brézis [9] called operators of type \( M \). In particular, an operator \( A : \mathcal{U} \to \mathcal{U}' \) is of type \( M \) if the condition

\[
\{\{u_n\}\} \text{ converges weakly to } u \text{ in } \mathcal{U}, \ A(u_n) \text{ converges weakly to } \chi \text{ in } \mathcal{U}', \text{ and } \limsup_{n \to \infty} \langle A(u_n), u_n \rangle \leq \langle \chi, u \rangle \tag{6.19}
\]

implies that

\[ \chi = A(u). \]

A very general existence theorem involving operators of type \( M \) can be constructed as a straightforward generalization of the theory of monotone operators. We shall record it as a lemma:\(^{14}\)

**LEMMA 6.1.** Let \( A : \mathcal{U} \to \mathcal{U}' \) be a bounded hemicontinuous, coercive operator of type \( M \). Then \( A \) is surjective.

If we now examine carefully the properties of Gårding operators, we observe that property (iii) serves to ensure that \( A(u, v) \) is completely continuous in \( v \) and that (i) and (ii) produce the required behavior in \( u \). Once this observation is made, the following straightforward generalization of the proofs of Theorems 6.1 and 6.2 is possible.\(^{15}\)

**THEOREM 6.3.** Let \( A : \mathcal{U} \to \mathcal{U}' \) be a Gårding operator. Then \( A \) is an operator of type \( M \); i.e., \( A \) satisfies (6.19).

**Proof.** Let \( \{u_n\} \) be weakly convergent to \( u \) and let \( v \) be arbitrary. If

\[ \mu \geq \max\{\|u_n\|_{\mathcal{U}'}, \|u\|_{\mathcal{U}'}, \|v\|_{\mathcal{U}'}\}, \]

property (ii) reveals that

\[ \langle A(u_n, u_n) - A(v, u_n), u_n - v \rangle \geq -H(\mu, \|u_n - v\|_{\mathcal{U}'}). \]

Further, suppose that \( A(u_n) \to \chi \) weakly in \( \mathcal{U}' \) and that

\[ \langle \chi, u \rangle \geq \limsup_{n \to \infty} \langle A(u_n, u_n), u_n \rangle. \]

Taking lim sup of the above inequality,

\[ \langle \chi, u - v \rangle - \liminf_{n \to \infty} \langle A(v, u_n), u_n - v \rangle \geq -\liminf_{n \to \infty} H(\mu, \|u_n - v\|_{\mathcal{U}'}). \]

\(^{14}\) For a proof, see, for example, Brézis [9] or Lions [18].

\(^{15}\) This generalization was suggested by N. Kikuchi after he had read an earlier version of the manuscript of this paper.
However, from (6.21) and (6.27),
\[
\lim_{n \to \infty} \inf \langle A(v, u_n), u_n - v \rangle = \lim_{n \to \infty} \inf \{ \langle A(v, u_n) - A(v, u), u_n - u \rangle \\
+ \langle A(v, u), u_n - u \rangle + \langle A(v, u_n), u - v \rangle \} \\
\geq \langle A(v, u), u - v \rangle.
\]
Therefore,
\[
\langle \chi, u - v \rangle - \langle A(v, u), u - v \rangle \geq -\lim_{n \to \infty} H(\mu, \| u_n - v \|_{\mathcal{V}}).
\]

Let \( w \) be arbitrary and choose \( \mu \) again large enough so that \( B_w(0) \) also contains \( w \). If we set \( v = \theta w + (1 - \theta) u \), then
\[
\langle \chi - A(u + \theta(w - u)), w \rangle \geq -\frac{1}{\theta} H(\mu, \theta \| w - u \|_{\mathcal{V}})
\]
so that, upon taking the limit as \( \theta \to 0^+ \), we have
\[
\langle \chi - A(u), w \rangle \geq 0, \quad \forall w \in \mathcal{V}.
\]
Hence,
\[
\chi = A(u)
\]
i.e., \( A \) is type \( M \). 

We may summarize the relationship between various classes of nonlinear operators as follows:

1. \( A \) is a bounded, hemicontinuous, monotone operator
   \( \Rightarrow \) \( A \) is a bounded Gårding operator
   \( \Rightarrow \) \( A \) is pseudo-monotone \( \Rightarrow \) \( A \) is of type \( M \).

2. \( A \) is an operator of the type of the calculus of variations
   \( \Rightarrow \) \( A \) is a bounded variational Gårding operator
   \( \Rightarrow \) \( A \) is a bounded Gårding operator
   \( \Rightarrow \) \( A \) is pseudomonotone \( \Rightarrow \) \( A \) is of type \( M \)

3. \( A \) is a Gårding operator
   \( \Rightarrow \) \( A \) is of type \( M \).

It appears that the theory of operators of the type of the calculus of variations are sufficiently general to cover most one-dimensional theories of nonlinear elasticity. However, for two- and three-dimensional problems, a more general framework than Gårding operators may be necessary.
EXISTENCE THEOREMS

7. APPLICATIONS TO NONLINEAR BOUNDARY-VALUE PROBLEMS

We shall now consider several applications of the theory of Section 6 to some specific boundary-value problems. We are able to state sufficient conditions for existence of solutions for some concrete examples of operators on Sobolev spaces.

APPLICATION 1 (A Problem of Place in Elastotatics). We first consider problem (5.2), where \( F(w, X) = f(X) \); i.e., \( F \) depends on \( X \) but not on \( w \), and \( \mathcal{U}(\Omega) \) is defined by (5.4). We immediately have:

**Theorem 7.1.** Let conditions I, II, III, and IV given in Section 4 hold. Then there exists a solution to (5.2).

**Proof.** Since \( F \) is independent of \( w \), we can take \( A(u, v) = A_1(u) + A_2(v) \) where \( A_1(u) = -\text{Div} Q(Vu, X) \) and \( A_2 = 0 \). Hence, \( A \) possesses property (iii) of Gårding operators. Property (i) follows from hypothesis I and (ii) follows from (4.8):

\[
\langle A(u, v) - A(v, u), u - v \rangle = \int_{\Omega} (Q(Vu, X) - Q(Vv, X)) : \nabla(u - v) \, dv
\]

for \( u, v \in B_m(0) \).

APPLICATION 2 (Non-conservative Forces Dependent on the Displacement Field). We return to the variational boundary-value problem (5.2) of place. The hypotheses made in Sections 4 and 5 are still in force, except that now we must be more specific about the structure of the spaces \( \mathcal{W}(\Omega), \mathcal{V}(\Omega), \) and \( \mathcal{K}(\Omega) \) and the functions \( \mathcal{F} \) and \( \mathcal{G} \) of (4.8).

We shall consider here the following example problem for a homogeneous, isotropic, elastic body: Find \( w \in W_0^{1,p}(\Omega) \) such that

\[
\int_{\Omega} (Q(Vw) : \nabla v - \rho_0 F(w) \cdot v) \, dv = 0 \quad \forall v \in W_0^{1,p}(\Omega)
\]

where \( p \geq 2 \) and \( W_0^{1,p}(\Omega) \) is the Sobolev space described in Section 2. In the present case, we have

\[
\mathcal{W}(\Omega) = W_0^{1,p}(\Omega), \quad \mathcal{V}(\Omega) = L^p(\Omega), \quad \mathcal{K}(\Omega) = L^2(\Omega)
\]

and the following hypotheses are in force:
Hypotheses I and II of Section 4 for the choices of spaces indicated in (7.2); in particular, we assume that constants \( \alpha_k, \beta_k > 0 \) such that

\[
\| \eta \|^k_{W_0^{1,k}(\Omega)} - \| \nabla \eta \|^p_{L^p(\Omega)}.
\]

\( \Pi' \). \( Q \) is bounded and hemicontinuous from \( W_0^{1,p}(\Omega) \) into \( W^{-1,p'}(\Omega) \), \( p' = \frac{p}{p-1} \), and \( Q(0) = 0 \).

\( \Pi'' \). \( \rho_0 F(w) \) satisfies (5.3) and, moreover, for almost every \( x \in \Omega \), \( w \rightarrow \rho_0(X) F(w, X) \) is bounded from \( W_0^{1,p}(\Omega) \) into \( W^{-1,p'}(\Omega) \) and satisfies the Lipschitz condition

\[
\| \rho_0 F(u, \cdot) - \rho_0 F(v, \cdot) \|_{L^{p'}(\Omega)} \leq c_0 \| u - v \|_{L^p(\Omega)}.
\]

wherein \( c_0 \) is a positive constant and \( 0 < \alpha < p \).

We recall that property (iii) of Gårding operators trivially holds in the cases in which \( \rho_0 F \) is a function of only \( X \). Thus, the problem of place with dead loading is a special case of (7.1).

**Theorem 7.1.** Let conditions I', II', and III' stated above hold. Then there exists at least one solution to the variational boundary-value problem (7.1).

**Proof.** We shall show that conditions I, II, and III are sufficient for the conditions of Theorem 6.2 to hold. Since \( \mathcal{D}(\Omega) \subseteq W_0^{1,p}(\Omega) \), we take \( v \in \mathcal{D}(\Omega) \) and easily conclude that the formal operator associated with (7.1) is

\[
A(w) = -\text{Div} Q(Vw) - \rho_0(X) F(w, X)
\]

and

\[
\langle A(w), v \rangle = \int_{\Omega} (Q(Vw) : Vv - \rho_0 F(w) \cdot v) \, dv.
\]

We easily show that \( A \) is coercive from \( W_0^{1,p}(\Omega) \) into \( W^{-1,p'}(\Omega) \). Let

\[
\Gamma(u) = \frac{\langle A(u), u \rangle}{\| u \|_{W_0^{1,p}(\Omega)}}.
\]
By virtue of (4.6), (4.8), and (7.3),

\[
\int_\Omega (Q(\nabla u + \nabla \eta) - Q(\nabla u)) \cdot \nabla \eta \, dv = \int_\Omega \int_0^1 \frac{dQ(\nabla u + \theta \nabla \eta)}{d\theta} \cdot \nabla \eta \, dv
\]

\[
= \int_\Omega \int_0^1 A(\nabla u + \theta \nabla \eta, X) \cdot \nabla \eta \cdot \nabla \eta \, d\theta \, dv
\]

\[
\geq \sum_{k=0}^p \alpha_k \| \eta \|_{W_0^k, p}^k - \sum_{k=0}^{p-1} \beta_k \| \eta \|_{L^p}^k.
\]

Since \(Q(0) = 0\), we have

\[
\| u \|_{W_0^{1,p}} \Gamma(u) = \int_\Omega (Q(\nabla u) \cdot \nabla u - \rho_0 F(u) \cdot u) \, dv
\]

\[
\geq \sum_{k=0}^p \alpha_k \| u \|_{W_0^k, p}^k - \sum_{k=0}^{p-1} \beta_k \| u \|_{L^p}^k - c_0 \| u \|_{L^p}^k,
\]

Recalling the Poincare inequality,

\[
\| u \|_{L^p} \leq c_1 \| u \|_{W_0^{1,p}}.
\]

c_1 being a positive constant, we observe that

\[
\Gamma(u) \geq \sum_{k=0}^p \alpha_k \| u \|_{W_0^k, p}^{k-1} - \sum_{k=0}^{p-1} \beta_k c_1^{k-1} \| u \|_{W_0^k, p}^{k-1} - c_0 \| u \|_{W_0^k, p}^{k-1}.
\]

Thus, \(\Gamma(u) \to +\infty\) as \(\| u \|_{W_0^k, p} \to \infty\); i.e., \(A\) is coercive, as asserted.

It remains to be shown that \(A\) is an abstract Garding operator. We first observe that \(A\) can be represented as a map from \(W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)\) into \(W^{-1,p}(\Omega)\) via

\[
A(u, v) = -\text{Div} Q(\nabla u) - \rho_0 F(v).
\]

Thus

\[
\langle A(u, u) - A(v, u), u - v \rangle = \int_\Omega (Q(\nabla u) - Q(\nabla v)) \cdot \nabla (u - v) \, dv
\]

\[
\geq -H(\| u - v \|_{L^p(\Omega)})
\]

where

\[
H(x) = \sum_{k=0}^{p-1} \beta_k x^k + c_0 x^\alpha, \quad \alpha < p.
\]

Clearly, \(H\) is a continuous function, satisfying property (ii) of Garding operators.

Property (i), of course, follows from assumption II'.
To verify properties (iii)-(v) of Gårding operators, first note that (iii) follows from inequality (7.3). Next, let \( \{u_n\} \) be a sequence converging weakly in \( W_0^{1,p}(\Omega) \). Since \( W_0^{1,p}(\Omega) \) is compact in \( L^p(\Omega) \), this sequence converges strongly to an element \( u \in L^p(\Omega) \). Clearly,

\[
M_n = \langle A(u_n, u_n) - A(u_n, u), u_n - u \rangle = \int_{\Omega} \rho_0(X) (F(u_n, X) - F(u, u)) \cdot (u_n - u) \, dv
\]

\[
\leq \| u_n - u \|_{L^p(\Omega)} \| \rho_0 F(u_n, X) - \rho_0 F(u, u) \|_{L^p(\Omega)} \leq c_0 \| u_n - u \|_{L^p(\Omega)}^{q+1}
\]

Thus \( M_n \to 0 \) as \( u_n \to u \), etc. We omit additional details.

We remark that for \( p > 2 \) the last term in (7.3) can be replaced by terms of the form \( -\beta(\mu) \| \eta \|_{L^p(\Omega)}^p \) without altering the basic results.

**Application 3 (A Quasi-linear Problem with Initial Stress and Nonhomogeneous Boundary Data).** We shall now consider a class of boundary-value problems in which "initial" stresses are present \( (Q(0) \neq 0) \) and the boundary conditions are nonhomogeneous \( (w \neq 0) \). The model problem we consider here is nonlinear and "non-unique" solutions exist. However, it falls outside the class of problems in nonlinear elasticity because the growth condition (7.13) we assume may not be met in physically reasonable constitutive laws for elastic materials. However, this example does illustrate how the conditions of Theorem 6.2 can be verified for a large class of nonlinear operators.

Again, denoting by \( \Omega \) an open bounded domain in \( \mathbb{R}^3 \) with a smooth boundary \( \partial \Omega \), we consider the problem of finding \( w \in W^{1,p}(\Omega) \), \( p \) an integer greater than or equal to 2, such that, formally,

\[
\Lambda(w) = \rho_0 f \quad \text{in} \quad \Omega
\]

\[
w = g_0 \quad \text{in} \quad \partial \Omega
\]

where

\[
\Lambda(w) \equiv -\text{Div} \, Q(\nabla w(X), w(X), X)
\]

\[
\rho_0(X) f(X) \in (W_0^{1,p}(\Omega))' = W^{-1,\prime}(\Omega), \quad \frac{1}{p} + \frac{1}{p'} = 1
\]

\[
g_0 \in W^{1-1/p, p}(\partial \Omega), \quad 2 \leq p < \infty.
\]

This last requirement on the boundary data guarantees that there exist functions \( g \in W^{1,p}(\Omega) \) continuously extendable to the boundary such that \( g |_{\partial \Omega} = g_0 \). Thus,

\[
\Lambda : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'
\]
and if a solution \( w \) to (7.2) exists,

\[
    w - g \in W^{1,p}_0(\Omega). \tag{7.7}
\]

Thus, (7.1) is equivalent to the following variational boundary-value problem: given \( g \in W^{1,p}(\Omega) \) such that \( g |_{\partial \Omega} = g_0 \), find \( w \in W^{1,p}(\Omega) \), with \( w - g \in W^{1,p}_0(\Omega) \), such that

\[
    \langle A(w), v \rangle = \langle p_0 f, v \rangle \quad \forall v \in W^{1,p}_0(\Omega) \tag{7.8}
\]

where \( \langle \cdot, \cdot \rangle \) denotes duality pairing on \( W^{1,p}(\Omega) \times (W^{1,p}(\Omega))^\prime \).

We shall make the following assumptions concerning this problem.

I". The function \( Q \) is of the form

\[
    Q(\nabla w, w, X) = Q_0(\nabla w, X) + Q_1(\nabla w, w, X) \tag{7.9}
\]

where \( Q_0(C, X) \), \( C \) being any second-order tensor, is differentiable in \( C \) and continuous in \( X \) and \( Q_1(C, w, X) \) is differentiable with respect to \( C \) and \( w \) and is continuous in \( X \).

II". There exist positive constants \( \gamma_0 \) and \( \gamma_1 \) such that for all \( u \) and \( v \) with the property that \( u - g, v - g \in W^{1,p}_0(\Omega) \),

\[
    \int_0^1 \frac{dQ_0(\nabla u + \theta \nabla v)}{d\theta} v_{i,\alpha} \, d\theta \geq \gamma(\nabla v) |\nabla v|^2 
\]

\[
    \gamma(\nabla v) \equiv \gamma_0 |\nabla v|^{p-2}; \quad p > 2 \tag{7.10}
\]

and

\[
    |\nabla v|^2 = \sum_{i,\alpha} |v_{i,\alpha}|^2
\]

and

\[
    |Q_0(C, X)| \leq \gamma_1(|C|^{p-1} + 1). \tag{7.11}
\]

III". There exist positive constants \( \beta_0 \) and \( \beta_1 \) such that for all \( C \), all \( w \in W^{1,p}(\Omega) \), and all \( X \),

\[
    |Q_1(C, w, X)| \leq \beta_0(1 + |w|^q + |\nabla w|^q) \tag{7.12}
\]

and

\[
    \left| \frac{\partial Q_1(C, v, X)}{\partial v} \right| \leq \beta_1(1 + |v|^{q-1} + |C|^{q-1}) \tag{7.13}
\]

\[
    \frac{\partial Q_1(C, v, X)}{\partial C} \geq 0.
\]

**Theorem 7.1.** Let conditions I", II", and III" above hold. Then there exists at least one solution to (7.8) (or (7.4)).
Proof. We return to Theorem 6.2 and the definition of a Gårding operator, and we shall show that the conditions of Theorem 6.2 are fulfilled.

In the present case, we can take $\Lambda(u, u) = \Lambda(u)$, but $\Lambda(u, v) = \Lambda(u) + 0$. Condition (iii) of Gårding operators is trivially satisfied and condition (i) is satisfied owing to the assumed continuity of $Q$ (condition I"). It remains to be shown that $\Lambda$ of (7.5) satisfies condition (ii) of Gårding operators and is coercive.

We begin by observing that

$$\langle \Lambda(u, u) - \Lambda(v, u), u - v \rangle = \langle \Lambda(u) - \Lambda(v), u - v \rangle$$

$$= J_0(u, v) + J_1(u, v) \quad \forall u, v \in W^{1,p}(\Omega)$$

(7.14)

where $J_0(u, v)$ and $J_1(u, v)$ are functionals involving $Q_0$ and $Q_1$, respectively. Let us first consider $J_0$. Let $u - g, v - g \in W^{1,p}(\Omega)$ and $\eta = u - v$. Then, according to (7.10),

$$J_0(u, v) = \int_{\Omega} \frac{dQ_0(v + \theta \nabla \eta)}{d\theta} \cdot \nabla \eta \cdot \nabla \eta \, dv$$

$$\geq \gamma_0 \int_{\Omega} \left| \nabla v + \theta \nabla \eta \right|^{p-2} \theta \left| \nabla \eta \right|^2 \, dv$$

$$\geq \gamma'_0 \int_{\Omega} \left| \nabla \eta \right|^p \, dv \geq \gamma''_0 \left\| u - v \right\|_{W^{1,p}(\Omega)}^p.$$  

(7.15)

Here $\eta = u - v$, and in the last step we have used the fact that

$$\int_0^1 \left| a + \theta b \right|^{p-2} d\theta \geq c_1 \left| b \right|^{p-2} \quad \forall a, b \in \mathbb{R}$$

$$\left\| \eta \right\|_{W^{1,p}(\Omega)} = \left\| \nabla \eta \right\|_{L^p(\Omega)} + \left\| \nabla \eta \right\|_{L^p(\Omega)} \leq c_2 \left\| \nabla \eta \right\|_{L^p(\Omega)}$$

where $c_1$ and $c_2$ are positive constants.

Also,

$$J_1(u, v) = \int_{\Omega} (Q_1(\nabla u, u, X) - Q_1(\nabla v, v, X)) : \nabla(u - v) \, dv$$

$$= \int_{\Omega} \frac{dQ_1(\nabla v + \theta \nabla \eta, v + \theta \eta, X)}{d\theta} \nabla \eta \cdot \nabla \eta \, dv$$

$$= \int_{\Omega} \left[ \frac{\partial Q_1(\nabla w, w, X)}{\partial \nabla w} \nabla \eta \cdot \nabla \eta + \frac{\partial Q_1(\nabla w, w, X)}{\partial w} \cdot \eta \cdot \nabla \eta \right] \, dv$$

$$\geq \beta_1 \int_{\Omega} \left( 1 + \left| v + \theta \eta \right|^{q-1} + \left| \nabla v + \theta \nabla \eta \right|^{q-1} \right) \left| \eta \right| \left| \nabla \eta \right| \, d\theta \, dv$$

$$= \beta_1 \int_{\Omega} \left| w + \theta \eta \right|^{q-1} \left| \eta \right| \left| \nabla \eta \right| \, d\theta \, dv$$

(w = v + \theta \eta, \eta = u - v).$$

(7.16)
Applying Hölder’s inequality, we observe that

\[
\int_\Omega |\nabla v + \theta \nabla \eta|^q - 1 | \eta | | \nabla \eta | \, dv \\
\leq \| \nabla \eta \|_{L^p(\Omega)} \| \eta \|_{L^p(\Omega)} \| \nabla v + \theta \nabla \eta \|_{L^{p/(p-2)}(\Omega)}^{q-1}.
\]

Since \((q - 1) \rho/(\rho - 1) < \rho\) and \(u, v \in B_\rho(0)\),

\[
\| \nabla v + \theta \nabla \eta \|_{L^{p/(p-1)}/(\rho-2)}(\Omega) \leq c_4 (\| \nabla v + \theta \nabla \eta \|_{L^p(\Omega)})^{q-1} \leq c_4 \mu^{q-1}(1 + \theta)^{q-1}
\]

where \(c_4\) is a positive constant. Hence, using Young’s inequality, we have

\[
\int_\Omega \int_0^1 |\nabla v + \theta \nabla \eta|^q - 1 | \eta | | \nabla \eta | \, dv \\
\leq \frac{\epsilon^p}{p} \| \eta \|_{W^{1,p}(\Omega)}^p + (c_2 \mu^{q-1})^{p'} \frac{\eta^{p'}}{\epsilon^{p'}} \| \eta \|_{L^{p'}(\Omega)}^{p'}
\]

\(c_2\) being a positive constant. A similar inequality holds for the remaining term in (7.16). Thus, we arrive at the inequality

\[
J_1(u, v) \leq c_5 \epsilon^p \| u - v \|_{W^{1,p}(\Omega)}^p - c_4 (\epsilon, \mu) \| u - v \|_{L^p(\Omega)}^{p/(p-1)} \tag{7.17}
\]

where \(c_5\) and \(c_4\) are positive constants, \(c_4\) depending continuously on \(\epsilon\) and \(\mu\).

Combining (7.14), (7.15), and (7.17), we see that if \(\epsilon\) is chosen sufficiently small,

\[
\langle \Lambda(u) - \Lambda(v), u - v \rangle \geq \gamma_0'' \| u - v \|_{W^{1,p}(\Omega)}^p - c_4 (\epsilon, \mu) \| u - v \|_{L^p(\Omega)}^{p/(p-1)}. \tag{7.18}
\]

Thus, the operator \(\Lambda\) does possess property (ii) of Gårding operators. Indeed,

\[
\langle \Lambda(u) - \Lambda(v), u - v \rangle \geq -H(\mu, \| u - v \|_{L^p(\Omega)})
\]

where

\[
H(x, y) = c_4 (\epsilon, x) y^{p/(p-1)}.
\]

It only remains to be shown that \(\Lambda\) is coercive. First, we observe that for \(u - g \in W^{1,p}_0(\Omega)\),

\[
\langle \Lambda(u), u \rangle = \langle \Lambda(0), u \rangle + J_0(u, 0) + J_1(u, 0) \geq \gamma_0'' \| u \|_{W^{1,p}(\Omega)}^p + \langle \Lambda(0), u \rangle + J_1(u, 0).
\]

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However, repeated use of Young’s and Hölder’s inequalities gives

\[
\langle \Lambda(0), u \rangle + J_1(u, 0) = \int_{\Omega} (Q_0(0, X) + Q_1(\nabla u, u, X) : \nabla u \, dv
\]

\[
\leq \int_{\Omega} \left( \gamma_1 |\nabla u| + \beta_0 |\nabla u|^{q+1} + \beta_0 |\nabla u| + \beta_0 |\nabla u| |u|^{q} \right) \, dv
\]

\[
\leq (\gamma_1 + \beta_0) \frac{\varepsilon^p}{p} \|u\|_{W^{1,p}(\Omega)}^p + \gamma_1 \frac{p'}{p} \epsilon \left( \frac{\beta_0 p'}{p} (\text{mes}(\Omega))^{p'} \right.
\]

\[
+ \beta_0 \frac{(q + 1)}{p} \epsilon^{p/(q+1)} \|u\|_{W^{1,p}(\Omega)}^p
\]

\[
+ \beta_0 \frac{\varepsilon^p}{p} \|u\|_{W^{1,p}(\Omega)}^p + \frac{c_6 p'}{p} \epsilon \|u\|_{W^{1,p}(\Omega)}
\]

where \( p'' = p/(p - q - 1) \), \( c_6 \) is a positive constant, and \( \hat{p} = q^{p'} < p \). Hence,

\[
\langle \Lambda(u), u \rangle \geq \left( \gamma_0 - (\gamma_1 + 2\beta_0) \frac{\varepsilon^p}{p} - \beta_0 \frac{(q + 1)}{p} \epsilon^{p/(q+1)} \right) \|u\|_{W^{1,p}(\Omega)}^p
\]

\[
- c_7(\epsilon, p, q, \text{mes } \Omega) - c_8(\epsilon, p, \beta_0) \|u\|_{W^{1,p}(\Omega)}
\]

in which \( c_7 \) and \( c_8 \) are positive constants depending on the indicated arguments. Clearly, by taking \( \epsilon \) sufficiently small, we can find a positive constant \( \tilde{\gamma_0} \) such that

\[
\langle \Lambda(u), u \rangle \geq \tilde{\gamma_0} \|u\|_{W^{1,p}(\Omega)}^p - c_1 - c_8 \|u\|_{W^{1,p}(\Omega)} \cdot \hat{p} < p. \quad (7.19)
\]

Therefore,

\[
\|u\|_{W^{1,p}(\Omega)} \langle \Lambda(u), u \rangle \to +\infty \quad \text{as} \quad \|u\|_{W^{1,p}(\Omega)} \to \infty;
\]

i.e., \( \Lambda \) is coercive. The proof is thus complete.

**Remark.** Again, we remark that, aside from coerciveness and a certain degree of continuity of the type (7.18) was a key step in this proof. Growth conditions (7.11)-(7.12) lead to the coerciveness property (7.19), whereas (7.10) and (7.13) were exploited in the proof of (7.18).

8. CONCLUDING COMMENTS

A key tool in the application of the theory presented here to specific cases is the availability of a nonlinear Gårding-type inequality of the form (4.10) or (7.3). In effect, the existence of an inequality of this type generally allows us to show that the formal operators involved, while not necessarily monotone in the
principal derivatives, differ from a monotone operator in these terms only by an operator that is completely continuous; i.e., an operator that maps sequences converging in the weak topology of its dual. Indeed, conditions (ii)-(v) in the definition of variational Gårding operators are apparently sufficient conditions for complete continuity of that part of the principal part of $A$ which differs from a monotone operator by the terms $-H(\mu, \| u - v \|_{\gamma})$.

It is also interesting to observe that a remarkable condition arises from this analysis which can be met in applications to elasticity problems only if conditions are imposed on the constitutive equations. Suppose that a generalized Gårding inequality of the form (4.8) holds; e.g.

$$\int_\Omega (Q(\nabla u, \mathbf{X}) - Q(\nabla v, \mathbf{X})): \nabla (u - v) \, dv$$

$$\geq F(\| u - v \|_{\mathcal{W}(\Omega)}) - S(\mu, \| u - v \|_{\gamma(\Omega)})$$

where $u, v \in B_\mu(0) \subset \mathcal{W}(\Omega)$ and $F$ satisfies (4.9). Then the formal operator $A$ in this problem is a Gårding operator only if

$$F(\| \eta \|_{\mathcal{W}(\Omega)}) \geq 0 \quad \forall \eta \in B_\mu(0).$$

This requirement imposes constraints on the form of the constitutive functional $Q$.

With the exception of Application 3 in Section 7, the level of generality we have adopted here allows us to circumvent some of the most difficult aspects of constructing existence theorems in elasticity in that we can assume our operators possess certain reasonable properties and then demonstrate that these properties are sufficient for the existence of solutions. The problem of demonstrating that given operators do or do not possess these properties is significantly more difficult. What remains to be done is to enrich the theory with concrete examples in which specific spaces, bounds, and inequalities can be established.

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\section*{References}


