New Characterizations of Some Mean-Values

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The purpose of this paper is to give new characterizations of some mean-values of two positive real numbers. The arithmetic, geometric, and harmonic means of two positive real numbers are the fundamentals of this paper. © 1996 Academic Press, Inc.

1. INTRODUCTION

Throughout this paper, let \( a, b \) be positive real numbers. A mean-value of \( a, b \), denoted by \( M(a, b) \), is defined to be a real-valued function \( M \), which satisfies the following postulates (cf. [2, 7])

\[
\begin{align*}
(P_1) & \quad M: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}; \\
(P_2) & \quad M(a, b) = M(b, a) \text{ (symmetry property)}; \\
(P_3) & \quad M(a, a) = a \text{ (reflexivity property)}. 
\end{align*}
\]

If \( a \) and \( b \) are any two positive real numbers, then their arithmetic, geometric, and harmonic mean-values are denoted by \( A = A(a, b) \), \( G = G(a, b) \), and \( H = H(a, b) \), respectively. For these three mean-values, see...
The formula
\[ A(a, b) H(a, b) = \sqrt{ab} = G(a, b) \] (1)
holds for all positive real numbers \(a, b\).

The formula (1) suggests us to consider the two mean-values
\[ J(a, b) \overset{\text{def}}{=} \sqrt[A]{A(a, b) G(a, b)} = \sqrt{\frac{a + b}{2}} \sqrt{ab}, \] (2)
and
\[ K(a, b) \overset{\text{def}}{=} \sqrt[H]{H(a, b) G(a, b)} = \sqrt{\frac{2ab}{a + b}} \sqrt{ab}. \] (3)

Remark 1. It is easy to prove that each of \(J(a, b)\) and \(K(a, b)\) is a mean-value of \(a, b\). The formula
\[ \sqrt[\text{J}(a, b) K(a, b)]{ab} = \sqrt{ab} = G(a, b) \] (4)
holds for all positive real numbers \(a, b\).

Remark 2. The formula (4) is similar to the formula (1).

In [11] is considered the following mean-value of \(a, b\),
\[ M(a, b; p(r)) \overset{\text{def}}{=} p^{-1}\left(\frac{1}{2\pi} \int_{0}^{2\pi} p(r) \, d\theta\right), \] (5)
where \(p: R^+ \to R\), \(p^n(x)\) is a continuous function in \(R^+\), \(p = p(x)\) is strictly monotonic in \(R^+\), and we denote \(\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}\) by \(r\).

Remark 3. Since the proof that \(M(a, b; p(r))\) is a mean-value of \(a, b\) is easy, we omit it.

In [11] the following theorem was proved, where we denote the arithmetic-geometric mean of Gauss for \(a, b\) (see [5, 7, 9, 11]) by \(U(a, b)\):

**Theorem A.**

(i) \(M(a, b; p(r)) = U(a, b)\) holds for all positive real numbers \(a, b\) iff \(p(r) = A \cdot (1/r) + B\), where \(A \neq 0\), \(B\) are arbitrary real constants.

(ii) \(M(a, b; p(r)) = (a + b)/2\) holds for all positive real numbers \(a, b\) iff \(p(r) = A \log r + B\), where \(A \neq 0\), \(B\) are arbitrary real constants.

(iii) \(M(a, b; p(r)) = \sqrt{ab}\) holds for all positive real numbers \(a, b\) iff \(p(r) = A \cdot (1/r^2) + B\), where \(A \neq 0\), \(B\) are arbitrary real constants.
(iv) $M(a, b; p(r)) = \sqrt{(a^2 + b^2) / 2}$ (the root-mean-square of $a, b$) holds for all positive real numbers $a, b$ iff $p(r) = Ar^2 + B$ where $A(\neq 0), B$ are arbitrary real constants.

(v) There exists no $p(r)$ such that $M(a, b; p(r)) = 2ab/(a + b)$ (the harmonic mean of $a, b$) holds for all positive real numbers $a, b$.

The purpose of this paper is to obtain further characterizations of the means

$$A(a, b) = \frac{a + b}{2}, \quad G(a, b) = \sqrt{ab} \cdot \left(\frac{a^{1/2} + b^{1/2}}{2}\right)^2,$$

$$K(a, b) = \sqrt{\frac{2ab}{a + b} \sqrt{ab}} \quad \text{(see (3))},$$

$$H(a, b) = \frac{2ab}{a + b} \cdot \left(\frac{a^{-1/2} + b^{-1/2}}{2}\right)^{-2},$$

$$J(a, b) = \sqrt{\frac{a + b}{2} \sqrt{ab}} \quad \text{(see (2))}$$

in the spirit of Theorem A (see Theorems 1 and 2 below). To this end, instead of $p(r)$ in Theorem A, we introduce the following two functions of $\theta$ in $-\infty < \theta < +\infty$:

(i) $q(s)$ where $q: R^+ \to R, q''(x)$ is continuous in $R^+, q = q(x)$ is strictly monotonic in $R^+$, and we denote $a\sin^2 \theta + b\cos^2 \theta$ by $s$.

(ii) $u(t)$ where $u: R^+ \to R, u''(x)$ is continuous in $R^+, u = u(x)$ is strictly monotonic in $R^+$, and we denote $(\sin^2 \theta/a + \cos^2 \theta/b)^{-1}$ by $t$.

Then we introduce, instead of (5), the following two mean-values of $a, b$:

$$M(a, b; q(s)) = q^{-1}\left(\frac{1}{2\pi} \int_0^{2\pi} q(s) \, d\theta\right) \quad \text{(6)}$$

and

$$M(a, b; u(t)) = u^{-1}\left(\frac{1}{2\pi} \int_0^{2\pi} u(t) \, d\theta\right). \quad \text{(7)}$$

Remark 4. It is easy to prove that each of $M(a, b; q(s))$ and $M(a, b; u(t))$ is a mean-value of $a, b$. These two mean-values will be discussed in Theorems 1 and 2, respectively.
2. STATEMENT OF THE RESULTS

**THEOREM 1.** (i) \( M(a, b; q(s)) = (a + b)/2 \) holds for all positive real numbers \( a, b \) iff \( q(s) = A \cdot (1/2) + B \), where \( A \neq 0, B \) are arbitrary real constants.

(ii) \( M(a, b; q(s)) = \sqrt{ab} \) holds for all positive real numbers \( a, b \) iff \( q(s) = A \cdot (1/s) + B \), where \( A \neq 0, B \) are arbitrary real constants.

(iii) \( M(a, b; q(s)) = ((a^{1/2} + b^{1/2})/2)^2 \) holds for all positive real numbers \( a, b \) iff \( q(s) = A \log s + B \) where \( A \neq 0, B \) are arbitrary real constants.

(iv) \( M(a, b; q(s)) = \sqrt{(2ab/(a + b))\sqrt{ab}} \) holds for all positive real numbers \( a, b \) iff \( q(s) = A \cdot (1/s^2) + B \) where \( A \neq 0, B \) are arbitrary real constants.

**THEOREM 2.** (i) \( M(a, b; u(t)) = \sqrt{ab} \) holds for all positive real numbers \( a, b \) iff \( u(t) = At + B \) where \( A \neq 0, B \) are arbitrary real constants.

(ii) \( M(a, b; u(t)) = 2ab/(a + b) \) holds for all positive real numbers \( a, b \) iff \( u(t) = A \cdot (1/t) + B \), where \( A \neq 0, B \) are arbitrary real constants.

(iii) \( M(a, b; u(t)) = ((a^{-1/2} + b^{-1/2})/2)^2 \) holds for all positive real numbers \( a, b \) iff \( u(t) = A \cdot \log t + B \) where \( A \neq 0, B \) are arbitrary real constants.

(iv) \( M(a, b; u(t)) = \sqrt{(a + b)/2}\sqrt{ab} \) holds for all positive real numbers \( a, b \) iff \( u(t) = At^2 + B \) where \( A \neq 0, B \) are arbitrary real constants.

**Remark 5.** There are two characterizations for \( A(a, b) = (a + b)/2 \). See Theorem A(ii) and Theorem 1(i). Furthermore, there are three characterizations for \( G(a, b) = \sqrt{ab} \). See Theorem A(iii), Theorem 1(ii), and Theorem 2(i).

3. LEMMAS

In Sections 4 and 5 we shall apply the following three lemmas:

**Lemma 1.** If \( a, b \) are positive real constants, then

(i) \[ \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{a \sin^2 \theta + b \cos^2 \theta} = \frac{1}{\sqrt{ab}} \quad (\text{see [11]}), \]

(ii) \[ \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(a \sin^2 \theta + b \cos^2 \theta)^2} = \frac{1}{2\sqrt{ab}} \left( \frac{1}{a} + \frac{1}{b} \right). \]
(iii) \[ \frac{1}{2\pi} \int_0^{2\pi} \log(a \sin^2 \theta + b \cos^2 \theta) \, d\theta = \log \left( \frac{a^{1/2} + b^{1/2}}{2} \right)^2. \]

hold.

**Proof.** (i) Since the proof is elementary, we omit it. (ii) We apply differentiation under the integral sign for (i).

Differentiating both sides of (i) with respect to \( a \) yields

\[ \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin^2 \theta}{(a \sin^2 \theta + b \cos^2 \theta)^2} \, d\theta = \frac{1}{2ab} \] (8)

for all positive \( a, b \).

Differentiating both sides of (i) with respect to \( b \) yields

\[ \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos^2 \theta}{(a \sin^2 \theta + b \cos^2 \theta)^2} \, d\theta = \frac{1}{2b^2} \] (9)

for all positive \( a, b \).

Adding (8) and (9) side by side yields (ii).

(iii) See [11].

**Lemma 2.** Let \( q: R^+ \to R \). We assume that \( q''(x) \) is continuous in \( R^+ \). If we set

\[ f(a, b) \overset{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} q(s) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} q(a \sin^2 \theta + b \cos^2 \theta) \, d\theta \] (10)

for all positive \( a, b \), then

\[ f_{aa}(c, c) = \frac{3}{8} q''(c), \]

where \( c \) is an arbitrarily fixed positive real number.

**Proof.** The proof follows from differentiation under the integral sign and the formula

\[ \int_0^{2\pi} \sin^4 \theta \, d\theta = (3/4)\pi. \]

**Lemma 3.** Let \( u: R^+ \to R \). We assume that \( u''(x) \) is continuous in \( R^+ \). If we set

\[ f(a, b) \overset{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} u(t) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} u \left( \left( a \sin^2 \theta + b \cos^2 \theta \right)^{-1} \right) \, d\theta \] (11)
for all positive \(a, b\), then
\[
f_{aa}(c, c) = \frac{3}{8} u''(c) - \frac{1}{4c} u'(c),
\]
where \(c\) is an arbitrarily fixed positive real number.

Proof. The proof follows from differentiation under the integral sign and the formula \(\int_0^{2\pi} s^4 \sin \theta \, d\theta = (3/4)\pi\).

4. PROOF OF THEOREM 1

(i) First we shall prove the "only if" part. By hypothesis, we have
\[
M(a, b; q(s)) = \frac{a + b}{2}
\]
for all positive \(a, b\).

By (6) (the definition of \(M(a, b; q(s))\)) and (12) we obtain
\[
 q^{-1}\left(\frac{1}{2\pi} \int_0^{2\pi} q(s) \, d\theta\right) = \frac{a + b}{2},
\]
and so
\[
\frac{1}{2\pi} \int_0^{2\pi} q(s) \, d\theta = q\left(\frac{a + b}{2}\right)
\]
for all positive \(a, b\).

Setting \(f(a, b) = (1/2\pi) \int_0^{2\pi} q(s) \, d\theta\) and using (13) yields
\[
f(a, b) = q\left(\frac{a + b}{2}\right)
\]
for all positive \(a, b\).

Let \(c\) be an arbitrarily fixed positive real number.

Operating on both sides of (14) with \(\partial^2/\partial a^2\) and setting \(a = c, b = c\) in the resulting equality yields
\[
f_{aa}(c, c) = \frac{1}{4} q''(c).
\]

By (15) and Lemma 2 we obtain
\[
\frac{3}{8} q''(c) = \frac{1}{4} q''(c),
\]
and therefore

\[ q''(c) = 0. \]

Since \( c \) was an arbitrarily fixed positive real number, we can replace \( c \) by a positive real variable \( s \) in the above equality. Hence we have

\[ q''(s) = 0 \quad \text{(16)} \]

in \( \mathbb{R}^+ \).

From (16) we obtain

\[ q(s) = As + B \]

in \( \mathbb{R}^+ \), where \( A, B \) are real constants with \( A \neq 0 \).

(ii) First we shall prove the “only if” part. By hypothesis and by (6) we obtain

\[ \frac{1}{2\pi} \int_0^{2\pi} q(s) \, d\theta = q(\sqrt{ab}) \]

for all positive \( a, b \).

Operating on both sides of the above equality with \( \frac{\partial^2}{\partial a^2} \), setting \( a = c, b = c \), where \( c \) is an arbitrarily fixed positive real number, in the resulting equality, using Lemma 2, observing that

\[ \left( \frac{\partial^2}{\partial a^2} q(\sqrt{ab}) \right)_{a=c,b=c} = \frac{1}{4} q''(c) - \frac{1}{4c} q'(c) \]

and simplifying the resulting equality yields

\[ q''(c) + \frac{2}{c} q'(c) = 0. \]

Replacing \( c \) by a positive real variable \( s \) in the above equality yields

\[ q''(s) + \frac{2}{s} q'(s) = 0 \quad \text{(17)} \]

in \( \mathbb{R}^+ \).
Solving the differential equation (17) yields

\[ q(s) = A \cdot \frac{1}{s} + B \]

in \( R^+ \) where \( A, B \) are real constants with \( A \neq 0 \).

Second we shall prove the "if" part.

By using \( q^{-1}(s) = A/(s - B) \), \( s = a \sin^2 \theta + b \cos^2 \theta \), and Lemma 1(i), after some calculations we obtain

\[
q^{-1}\left(\frac{1}{2\pi} \int_0^{2\pi} q(s) \, d\theta\right) = q^{-1}\left(\frac{1}{2\pi} \int_0^{2\pi} \left( A - \frac{B}{s} \right) \, d\theta\right)
\]

\[
= q^{-1}\left(\frac{A}{2\pi} \int_0^{2\pi} \frac{d\theta}{a \sin^2 \theta + b \cos^2 \theta} + B\right)
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{d\theta/(a \sin^2 \theta + b \cos^2 \theta)} = \sqrt{ab}
\]

for all positive \( a, b \).

Hence, by (6) we obtain

\[ M(a, b; q(s)) = \sqrt{ab} \]

for all positive \( a, b \).

(iii) First we shall prove the "only if" part.

By hypothesis and by (6) we obtain

\[ \frac{1}{2\pi} \int_0^{2\pi} q(s) \, d\theta = q\left(\left(\frac{a^{1/2} + b^{1/2}}{2}\right)^2\right) \]

for all positive \( a, b \).

Operating on both sides of the above equality with \( \partial^2/\partial a^2 \), setting \( a = c, b = c \), where \( c \) is an arbitrarily fixed positive real number, in the resulting equality, using Lemma 2, observing that

\[
\left(\frac{\partial^2}{\partial a^2} q\left(\left(\frac{a^{1/2} + b^{1/2}}{2}\right)^2\right)\right)_{a=c, b=c} = \frac{1}{4} q''(c) - \frac{1}{8c} q'(c)
\]

and simplifying the resulting equality yields

\[ q''(s) + \frac{1}{s} q'(s) = 0, \quad (18) \]

where \( s \) is a positive real variable.
Solving the differential equation (18) yields

\[ q(s) = A \log s + B \]

in \( R^+ \) where \( A, B \) are real constants with \( A \neq 0 \).

Second we shall prove the “if” part.

By using \( q^{-1}(s) = \exp((s - B)/A) \), \( s = a \sin^2 \theta + b \cos^2 \theta \) and Lemma 1(iii), after some calculations, we obtain

\[
q^{-1}\left( \frac{1}{2\pi} \int_0^{2\pi} q(s) \, d\theta \right) = q^{-1}\left( \frac{1}{2\pi} \int_0^{2\pi} (A \log s + B) \, d\theta \right)
\]

\[
= q^{-1}\left( \frac{A}{2\pi} \int_0^{2\pi} \log(a \sin^2 \theta + b \cos^2 \theta) \, d\theta + B \right)
\]

\[
= q^{-1}\left( A \log\left( \frac{a^{1/2} + b^{1/2}}{2} \right)^2 + B \right)
\]

\[
= \exp\left( \log\left( \frac{a^{1/2} + b^{1/2}}{2} \right)^2 \right)
\]

\[
= \left( \frac{a^{1/2} + b^{1/2}}{2} \right)^2
\]

for all positive \( a, b \).

Hence, by (6) we obtain

\[
M(a, b; q(s)) = \left( \frac{a^{1/2} + b^{1/2}}{2} \right)^2
\]

for all positive \( a, b \).

(iv) First we shall prove the “only if” part.

By hypothesis and by (6) we obtain

\[
\frac{1}{2\pi} \int_0^{2\pi} q(s) \, d\theta = \sqrt{\frac{2ab}{a + b \sqrt{ab}}}
\]

for all positive \( a, b \).

Operating on both sides of the above equality with \( \frac{\partial^2}{\partial a^2} \), setting \( a = c, b = c \), where \( c \) is an arbitrarily fixed positive real number, in the resulting equality, using Lemma 2, observing that

\[
\left( \frac{\partial^2}{\partial a^2} q\left( \sqrt{\frac{2ab}{a + b \sqrt{ab}}} \right) \right)_{a=c, b=c} = \frac{1}{4} q''(c) - \frac{3}{8c} q'(c)
\]
and simplifying the resulting equality yields
\[ q''(s) + \frac{3}{s} q'(s) = 0 \] (19)
in \( R^+ \).

Solving the differential equation (19) yields
\[ q(s) = A \cdot \frac{1}{s^2} + B \]
in \( R^+ \) where \( A, B \) are real constants with \( A \neq 0 \).

Second we shall prove the "if" part.

By using \( q^{-1}(s) = \sqrt[2]{A/(s-B)} \), \( s = a \sin^2 \theta + b \cos^2 \theta \), and Lemma 1(ii), after some calculations, we obtain
\[
q^{-1}\left( \frac{1}{2\pi} \int_0^{2\pi} q(s) \, d\theta \right) = q^{-1}\left( \frac{1}{2\pi} \int_0^{2\pi} \left( A \cdot \frac{1}{s^2} + B \right) \, d\theta \right)
= q^{-1}\left( \frac{A}{2\pi} \int_0^{2\pi} \frac{d\theta}{(a \sin^2 \theta + b \cos^2 \theta)^2} + B \right)
= \sqrt{\frac{1}{(1/2\pi) \int_0^{2\pi} (d\theta/(a \sin^2 \theta + b \cos^2 \theta)^2)}}
= \sqrt{\frac{2ab}{a + b \sqrt{ab}}}
\]
for all positive \( a, b \).

Hence, by (6) we obtain
\[ M(a, b; q(s)) = \sqrt{\frac{2ab}{a + b \sqrt{ab}}} \]
for all positive \( a, b \).

5. PROOF OF THEOREM 2

(i) First we shall prove the "only if" part.

By hypothesis we have
\[ M(a, b; u(t)) = \sqrt{ab} \] (20)
for all positive \( a, b \).
By (7) (the definition of $M(a, b; u(t))$) and (20) we obtain

$$u^{-1}\left(\frac{1}{2\pi} \int_0^{2\pi} u(t) \, d\theta\right) = \sqrt{ab},$$

and so

$$\frac{1}{2\pi} \int_0^{2\pi} u(t) \, d\theta = u(\sqrt{ab}) \quad (21)$$

for all positive $a, b$.

Setting $f(a, b) = (1/2\pi)\int_0^{2\pi} u(t) \, d\theta$ and using (21) yields

$$f(a, b) = u(\sqrt{ab}). \quad (22)$$

for all positive $a, b$.

Operating on both sides of (22) with $\partial^2 / \partial a^2$ and setting $a = c, b = c$, where $c$ is an arbitrarily fixed positive real number, in the resulting equality yields

$$f_{aa}(c, c) = \frac{1}{4} u''(c) - \frac{1}{4c} u'(c). \quad (23)$$

By (23) and Lemma 3 we obtain

$$\frac{3}{8} u''(c) - \frac{1}{4c} u'(c) = \frac{1}{4} u''(c) - \frac{1}{4c} u'(c),$$

and so

$$u''(c) = 0.$$

Replacing $c$ by a positive real variable $t$ in the above equality yields

$$u''(t) = 0 \quad (24)$$

in $R^+$. From (24) we have

$$u(t) = At + B$$

in $R^+$ where $A, B$ are real constants with $A \neq 0$.

Second we shall prove the "if" part.
By using \( u^{-1}(t) = (t - B)/A, t = (\sin^2 \theta/a + \cos^2 \theta/b)^{-1} \), after some calculations, we obtain

\[
\begin{align*}
    u^{-1}\left( \frac{1}{2\pi} \int_0^{2\pi} u(t) \, d\theta \right) &= u^{-1}\left( \frac{1}{2\pi} \int_0^{2\pi} (At + B) \, d\theta \right) \\
    &= u^{-1}\left( \frac{A}{2\pi} \int_0^{2\pi} \left( \frac{\sin^2 \theta}{a} + \frac{\cos^2 \theta}{b} \right)^{-1} \, d\theta + B \right) \\
    &= \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\sin^2 \theta/a + \cos^2 \theta/b}
\end{align*}
\]

for all positive \( a, b \).

Replacing \( a, b \) by \( 1/a, 1/b \), respectively, in Lemma 1(i) yields

\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\sin^2 \theta/a + \cos^2 \theta/b} = \sqrt{ab}
\]

for all positive \( a, b \).

From the above two equalities we obtain

\[
\begin{align*}
    u^{-1}\left( \frac{1}{2\pi} \int_0^{2\pi} u(t) \, d\theta \right) &= \sqrt{ab}.
\end{align*}
\]

Hence, by (7) we get

\[
M(a, b; u(t)) = \sqrt{ab}
\]

for all positive \( a, b \).

(ii) First we shall prove the "only if" part.

By hypothesis and by (7) we obtain

\[
\begin{align*}
    \frac{1}{2\pi} \int_0^{2\pi} u(t) \, d\theta &= u\left( \frac{2ab}{a + b} \right)
\end{align*}
\]

for all positive \( a, b \).

Operating on both sides of the above equality with \( \partial^2/\partial a^2 \), setting \( a = c, b = c \), where \( c \) is an arbitrarily fixed positive real number, in the resulting equality, using Lemma 3, observing that

\[
\left( \frac{\partial^2}{\partial a^2} u\left( \frac{2ab}{a + b} \right) \right)_{a=c, b=c} = \frac{1}{4} u''(c) - \frac{1}{2c} u'(c)
\]

and simplifying the resulting equality yields

\[
\begin{align*}
    u''(t) + \frac{2}{t} u'(t) &= 0, \quad (25)
\end{align*}
\]

where \( t \) is a positive real variable.
Solving the differential equation (25) in $R^+$ yields

$$u(t) = A \cdot \frac{1}{t} + B$$

in $R^+$ where $A, B$ are real constants with $A \neq 0$.

Second we shall prove the “if” part. By using $u^{-1}(t) = A/(t - B)$, $t = (\sin^2 \theta/a + \cos^2 \theta/b)^{-1}$, after some calculations, we obtain

$$u^{-1}\left(\frac{1}{2\pi \int_0^{2\pi} u(t) \, d\theta}\right) = \left(\frac{1}{2\pi \int_0^{2\pi} \left(\frac{\sin^2 \theta}{a} + \frac{\cos^2 \theta}{b}\right) \, d\theta}\right)^{-1}. \quad (26)$$

By (26) and the formulae $\int_0^{2\pi} \sin^2 \theta \, d\theta = \int_0^{2\pi} \cos^2 \theta \, d\theta = \pi$ we get

$$u^{-1}\left(\frac{1}{2\pi \int_0^{2\pi} u(t) \, d\theta}\right) = \frac{2ab}{a + b}$$

for all positive $a, b$.

Hence, by (7) we obtain

$$M(a, b; u(t)) = \frac{2ab}{a + b}$$

for all positive $a, b$.

(iii) First we shall prove the “only if” part.

By hypothesis and by (7) we obtain

$$\frac{1}{2\pi \int_0^{2\pi} u(t) \, d\theta} = u\left(\left(\frac{a^{-1/2} + b^{-1/2}}{2}\right)^{-2}\right)$$

for all positive $a, b$.

Operating on both sides of the above equality with $\frac{\partial^2}{\partial a^2}$, setting $a = c$, $b = c$, where $c$ is an arbitrarily fixed positive real number, in the resulting equality, using Lemma 3, observing that

$$\left(\frac{\partial^2}{\partial a^2} u\left(\frac{a^{-1/2} + b^{-1/2}}{2}\right)^{-2}\right)_{a=c, b=c} = \frac{1}{4} u''(c) - \frac{3}{8c} u'(c)$$

and simplifying the resulting equality yields

$$u''(t) + \frac{1}{t} u'(t) = 0, \quad (27)$$

where $t$ is a positive real variable.
Solving the differential equation (27) in $R^+$ yields

$$u(t) = A \log t + B$$

in $R^+$ where $A, B$ are real constants with $A \neq 0$.

Second we shall prove the "if" part.

By using $u^{-1}(t) = \exp((t - B)/A)$, $t = (\sin^2 \theta/a + \cos^2 \theta/b)^{-1}$, after some calculations, we obtain

$$u^{-1}\left( \frac{1}{2\pi} \int_0^{2\pi} u(t) \, d\theta \right) = \exp\left( -\frac{1}{2\pi} \int_0^{2\pi} \log \left( \frac{\sin^2 \theta}{a} + \frac{\cos^2 \theta}{b} \right) \, d\theta \right)$$

(28)

for all positive $a, b$.

Replacing $a, b$ by $1/a, 1/b$, respectively, in Lemma 1, (iii) yields

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left( \frac{\sin^2 \theta}{a} + \frac{\cos^2 \theta}{b} \right) \, d\theta = \log \left( \frac{a^{-1/2} + b^{-1/2}}{2} \right)^2$$

(29)

for all positive $a, b$.

By (28), (29) we obtain

$$u^{-1}\left( \frac{1}{2\pi} \int_0^{2\pi} u(t) \, d\theta \right) = \left( \frac{a^{-1/2} + b^{-1/2}}{2} \right)^{-2}$$

for all positive $a, b$.

Hence, by (7) we obtain

$$M(a, b; u(t)) = \left( \frac{a^{-1/2} + b^{-1/2}}{2} \right)^{-2}$$

for all positive $a, b$.

(iv) First we shall prove the "only if" part.

By hypothesis and by (7) we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} u(t) \, d\theta = u \left( \sqrt{\frac{a + b}{2}}\right)$$

(30)

for all positive $a, b$.

Operating on both sides of (30) with $\frac{\partial^2}{\partial a^2}$, setting $a = c, b = c$, where $c$ is an arbitrarily fixed positive real number, in the resulting equality, using Lemma 3, observing that

$$\left( \frac{\partial^2}{\partial a^2} \left( u \left( \sqrt{\frac{a + b}{2}}\right) \right) \right)_{a=c, b=c} = \frac{1}{4} u''(c) - \frac{1}{8c} u'(c)$$
and simplifying the resulting equality yields

$$u''(t) - \frac{1}{t}u'(t) = 0$$  \hspace{1cm} (31)$$

in $\mathbb{R}^+$. 

Solving the differential equation (31) in $\mathbb{R}^+$ yields

$$u(t) = At^2 + B$$

in $\mathbb{R}^+$ where $A, B$ are real constants with $A \neq 0$.

Second we shall prove the "if" part.

By using $u^{-1}(t) = \sqrt{(t - B)/A}$, $t = (\sin^2 \theta/a + \cos^2 \theta/b)^{-1}$, after some calculations, we obtain

$$u^{-1}\left(\frac{1}{2\pi} \int_0^{2\pi} u(t) \, dt\right) = \sqrt{\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(\sin^2 \theta/a + \cos^2 \theta/b)^2}}$$  \hspace{1cm} (32)$$

for all positive $a, b$.

Replacing $a, b$ by $1/a, 1/b$, respectively, in Lemma 1, (ii) yields

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(\sin^2 \theta/a + \cos^2 \theta/b)^2} = \frac{a + b}{2}\sqrt{ab}$$  \hspace{1cm} (33)$$

for all positive $a, b$.

By (32), (33) we obtain

$$u^{-1}\left(\frac{1}{2\pi} \int_0^{2\pi} u(t) \, d\theta\right) = \sqrt{\frac{a + b}{2}\sqrt{ab}}$$

for all positive $a, b$.

Hence, by (7) we obtain

$$M(a, b; u(t)) = \sqrt{\frac{a + b}{2}\sqrt{ab}}$$

for all positive $a, b$.

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REFERENCES