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Convexity with respect to Hölder mean involving zero-balanced hypergeometric functions ☆

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A B S T R A C T

In this note we investigate the convexity of zero-balanced hypergeometric functions with respect to Hölder mean.

1. Introduction

Let $I \subseteq \mathbb{R}$ be a nondegenerate interval and $M : I^2 \to I$ be a continuous function. We say that M is a *mean* on I if it satisfies the following condition $\min\{r,s\} \leqslant M(r,s) \leqslant \max\{r,s\}$ for all $r, s \in I$, $r \neq s$.

Let $\varphi : I \to \mathbb{R}$ be a strictly monotonic continuous function. The function $M_\varphi : I^2 \to I$, defined by

$$M_\varphi(r,s) := \varphi^{-1}\big(A\big(\varphi(r), \varphi(s)\big)\big)$$

is called the *quasi-arithmetic mean* associated to φ, while the function φ is called *generating function* of the quasi-arithmetic mean M_φ, where $A(r,s) := (r+s)/2$ is the arithmetic mean. For more details about quasi-arithmetic mean readers are referred to the works of J. Aczél [1], Z. Daróczy [9] and J. Matkowski [10]. For any two quasi-arithmetic means M_φ, M_ψ associated to φ, ψ defined on intervals I, J, respectively, a function $f : I \to J$ is called $M_{\varphi,\psi}$*-convex* if it satisfies

$$f\big(M_\varphi^{(\lambda)}(r,s)\big) \leqslant M_\psi^{(\lambda)}\big(f(r), f(s)\big) \tag{1}$$

for all $r, s \in I$ and $\lambda \in [0,1]$, and *strictly* $M_{\varphi,\psi}$*-convex* if the inequality is strict except for $r = s$ or $\lambda = 0, 1$, where

$$M_\varphi^{(\lambda)}(r,s) := \varphi^{-1}\big((1-\lambda)\varphi(r) + \lambda\varphi(s)\big)$$

is the weighted version of M_φ. It can be easily proved (see D. Borwein, J. Borwein, G. Fee and R. Girgensohn [7]) that if ψ is strictly increasing then f is (strictly) $M_{\varphi,\psi}$-convex if and only if $\psi \circ f \circ \varphi^{-1}$ is (strictly) convex in the usual sense on $\varphi(I)$, and if ψ is strictly decreasing then f is (strictly) $M_{\varphi,\psi}$-convex if and only if $\psi \circ f \circ \varphi^{-1}$ is (strictly) concave in the usual sense on $\varphi(I)$. For more details about $M_{\varphi,\psi}$-convex readers are referred to the works of D. Borwein, J. Borwein, G. Fee

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and R. Girgensohn [7], J. Matkowski and J. Rätz [11,12]. Of special interest are the *Hölder means* associated to the function $\varphi_p : (0, \infty) \to \mathbb{R}$, defined by

$$\varphi_p(r) := \begin{cases} r^p, & \text{if } p \neq 0, \\ \log r, & \text{if } p = 0, \end{cases}$$

thus

$$M_{\varphi_p}(r, s) = H_p(r, s) = \begin{cases} [A(r^p, s^p)]^{1/p}, & \text{if } p \neq 0, \\ G(r, s) := \sqrt{rs}, & \text{if } p = 0. \end{cases}$$

For $p = 1$, we get the arithmetic mean $A = H_1$, for $p = 0$ the geometric mean $G = H_0$, and for $p = -1$ the harmonic mean $H = H_{-1}$.

For real numbers $a, b,$ and c with $c \neq 0, -1, -2, \ldots,$ the Gaussian hypergeometric function is defined by

$$F(a, b; c; r) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{r^n}{n!}, \quad r \in (-1, 1). \tag{2}$$

Here $(a, 0) = 1$ for $a \neq 0$, and (a, n) is the shifted factorial function $(a, n) = a(a + 1)(a + 2) \cdots (a + n - 1)$ for $n \geqslant 1$. The zero-balanced Gaussian hypergeometric function is defined by $F(r) := F(a, b; a + b; r)$ for all $a, b > 0$ (see G.D. Anderson, R.W. Barnard, K.C. Richards, M.K. Vamanamurthy and M. Vuorinen [2]).

Recently many authors proved that the zero-balanced Gaussian hypergeometric function is *MN-convex* when M, N are the arithmetic, geometric, or harmonic means (for details see the works of R. Balasubramanian, S. Ponnusamy and M. Vuorinen [4], G.D. Anderson, M.K. Vamanamurthy and M. Vuorinen [3], Á. Baricz [5] and Y.M. Chu, G.D. Wang, X.H. Zhang and S.L. Qiu [8]). Very recently Á. Baricz [6] generalized these results to the H_p-convexity of the zero-balanced Gaussian hypergeometric functions with respect to Hölder means for $p \in [0, 1]$.

In this note the authors will mainly investigate the $H_{p,q}$-convexity of the zero-balanced Gaussian hypergeometric function for some p and q.

2. Main results

Theorem 1. *For all $a, b > 0$, $p \in (-\infty, 1]$ and $q \in [0, +\infty)$ the hypergeometric function $r \mapsto F(r) := F(a, b; a + b; r)$ defined by* (2) *is $H_{p,q}$-convex on* $(0, 1)$ *with respect to the Hölder means H_p and H_q.*

Proof. In order to establish the $H_{p,q}$-convexity of F we need to show that the function $\varphi_q \circ F \circ \varphi_p^{-1}$ is convex in the usual sense on $\varphi_p((0, 1))$. Let us denote

$$f(r) := \varphi_q \circ F \circ \varphi_p^{-1}(r).$$

The proof is divided into the following four cases.

Case 1. $p = q = 0$. The theorem is proved in Theorem 2.1 of [6].

Case 2. $p = 0$ and $q > 0$. So $f(r) = F(e^r)^q$ for $r \in (-\infty, 0)$. A simple computation shows that

$$f'(r) = qF\big(e^r\big)^{q-1} F'\big(e^r\big)e^r = qf(r)\frac{d \log F(e^r)}{de^r} e^r \geqslant 0. \tag{3}$$

By Lemma 2.1 in [4] due to R. Balasubramanian, S. Ponnusamy and M. Vuorinen, the function F is log-convex on $(0, 1)$, and consequently $r \mapsto (d \log F(e^r))/(de^r)$ is increasing on $(-\infty, 0)$. From (3), we obtain that f is increasing, therefore f' is increasing too as a product of three strictly positive and increasing functions.

Case 3. $p \neq 0$ and $q = 0$. So $f(r) = \log F(r^{1/p})$ for $r \in (0, 1)$ if $0 < p \leqslant 1$ and $r \in (1, +\infty)$ if $p < 0$. Similar to Case 2 we have

$$f'(r) = \frac{1}{p} \frac{d \log F(r^{1/p})}{d(r^{1/p})} r^{1/p-1}.$$

Therefore f' is increasing as a product of two strictly positive and increasing functions if $0 < p \leqslant 1$ and a product of a negative number and two strictly positive and decreasing functions if $p < 0$.

Case 4. $p \neq 0$ and $q > 0$. So $f(r) = F(r^{1/p})^q$ for $r \in (0, 1)$ if $0 < p \leqslant 1$ and $r \in (1, +\infty)$ if $p < 0$. A simple computation gives that

$$f'(r) = \frac{q}{p} f(r) \frac{d \log F(r^{1/p})}{d(r^{1/p})} r^{1/p-1}. \tag{4}$$

Clearly, f is increasing if $0 < p \leqslant 1$ and decreasing if $p < 0$. Therefore, from (4), f' is increasing as a product of three strictly positive and increasing functions if $0 < p \leqslant 1$ and a product of a negative number and three strictly positive and decreasing functions if $p < 0$. $\square$

The next theorem is a slight generalization of above theorem. The proof is similar, so we omit the details.

Theorem 2. *For all $a, b > 0$, $q \in [0, +\infty)$ and $p \in (-\infty, m]$, where $m = 1, 2, \dots,$ the hypergeometric function $r \mapsto F_m(r) := F(a, b; a+b; r^m)$ is $H_{p,q}$-convex on $(0, 1)$ with respect to the Hölder means H_p and H_q. In particular, the complete elliptic integral of the first kind, defined by*

$$\mathcal{K}(r) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - r^2 \sin^2\theta}} = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right),$$

is $H_{p,q}$-convex on $(0, 1)$ with respect to the Hölder means H_p and H_q where $p \in (-\infty, 2]$ and $q \in [0, +\infty)$. For all $\lambda, r, s \in (0, 1)$, one has

$$\mathcal{K}\big(\sqrt{(1-\lambda)r^2 + \lambda s^2}\big) \leqslant \mathcal{K}(r)^{1-\lambda}\mathcal{K}(s)^{\lambda}.$$

The following result is similar to Theorem 2.

Theorem 3. *If $a, b, p > 0$, $m = 1, 2, \dots,$ and $q \geqslant p/m$, then $r \mapsto f_m(r) := F(a, b; a+b; r^m) - 1$ is $H_{p,q}$-convex on $(0, 1)$ with respect to the Hölder means H_p and H_q. In particular for $m = 2$ the function $f_2(r) := 2\mathcal{K}(r)/\pi - 1$ is $H_{p,p/2}$-convex on $(0, 1)$ with respect to means H_p and $H_{p/2}$, i.e. for all $\lambda, r, s \in (0, 1)$ and $p > 0$ one has*

$$\frac{2}{\pi}\mathcal{K}\big(\big[(1-\lambda)r^p + \lambda s^p\big]^{1/p}\big) \leqslant 1 + \left[(1-\lambda)\left(\frac{2}{\pi}\mathcal{K}(r) - 1\right)^{p/2} + \lambda\left(\frac{2}{\pi}\mathcal{K}(s) - 1\right)^{p/2}\right]^{2/p}.$$

Proof. We just need to show that $r \mapsto f_m(r^{1/p})^q$ is convex on $(0, 1)$. Let us denote

$$\gamma(r) := \big[F\big(r^{m/p}\big) - 1\big]^q.$$

A simple computation shows that

$$\gamma'(r) = \frac{mq}{p}\left[\frac{F(x) - 1}{x^{p/(mq)}}\right]^q \frac{xF'(x)}{F(x) - 1}, \quad \text{where } x = r^{m/p}.$$

So we need only to prove that the function

$$x \mapsto \left[\frac{F(x) - 1}{x^{p/(mq)}}\right]^q \frac{xF'(x)}{F(x) - 1}$$

is strictly increasing. From the proof of Theorem 2.3 in [6] the function $x \mapsto xF'(x)/(F(x) - 1)$ is strictly increasing. It is sufficient to show that $x \mapsto (F(x) - 1)/(x^{p/(mq)})$ is strictly increasing. By differentiation we have that

$$x^{\frac{p}{mq}+1}\frac{d}{dx}\left(\frac{F(x) - 1}{x^{p/(mq)}}\right) = xF'(x) - \frac{p}{mq}\big(F(x) - 1\big) = \sum_{n=1}^{+\infty}\left(n - \frac{p}{mq}\right)\frac{(a, n)(b, n)}{(c, n)}\frac{x^n}{n!},$$

which is positive because by assumption $p/(mq) \leqslant 1 \leqslant n$. □

The following result gives sufficient conditions for a differentiable log-convex function to be convex with respect to Hölder means.

Theorem 4. *Let $f : I \subseteq [0, \infty) \to [0, \infty)$ be a differentiable function.*

(1) *If the function f is increasing and log-convex, then f is $H_{p,q}$-convex with respect to Hölder means H_p and H_q for $-\infty < p \leqslant 1$ and $q \geqslant 0$.*
(2) *If the function f is decreasing and log-convex, then f is $H_{p,q}$-convex with respect to Hölder means H_p and H_q for $p \geqslant 1$ and $q \geqslant 0$.*

Proof. For part (1), the proof is similar to that of Theorem 1, so we omit the details.

For part (2), let us denote $\gamma(r) := \varphi_q \circ f \circ \varphi_p^{-1}(r)$. For $q = 0$, $\gamma(r) = \log f(r^{1/p})$. By differentiation, we have that

$$\gamma'(r) = -\frac{1}{p}\left[-\frac{d\log f(r^{1/p})}{d(r^{1/p})}\right]r^{(1/p)-1},$$

which is increasing as a product of a negative number and two strictly positive and decreasing functions. For $q > 0$, $\gamma(r) = f(r^{1/p})^q$, and

$$\gamma'(r) = -\frac{q}{p}f\big(r^{1/p}\big)^q\left[-\frac{d\log f(r^{1/p})}{d(r^{1/p})}\right]r^{(1/p)-1},$$

which is increasing as a product of a negative number and three strictly positive and decreasing functions. □

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