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On the Theories of Morse and Lusternik–Schnirelman for Open Bounded Sets on Fredholm Hilbert Manifolds

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1. INTRODUCTION

The Morse theory of critical points of a real valued function f defined on a finite dimensional manifold M without boundary was generalized by Palais and Smale to the case where M is a Hilbert manifold without boundary [8, 10]. In particular if all critical points are nondegenerate (and therefore isolated) the well known Morse inequalities between the Betti number R_q of M and the Morse numbers M_q were generalized in an appropriate form (see [8, p. 338, Theorem 7]; for the definition of M_q see Eq. (3.10) of the present paper; see also Remark at the end of Section 3).

On the other hand the Morse theory for real valued functions f defined in the closure $\bar{V}$ of a bounded open set V in a finite dimensional vector space E was generalized to the case where E is a Hilbert space. The case where f satisfies a "regular boundary condition" (stating essentially that at every point of the boundary $\dot{V}$ of V the gradient of f is exteriorly directed, see Assumption 2.4) was treated in [14], and the case of "general boundary conditions" (where f is allowed to have the direction of the interior normal at a finite number of points of $\dot{V}$) was treated in [13]. In either case the statement of the boundary condition required the existence of a unique exterior unit normal, and it was this requirement which motivated the assumption that $\dot{V}$ be a Fredholm manifolds. In the case of the regular boundary condition sufficient conditions for the validity of the Morse relations were given, (see [14, Theorem 8] where f is supposed to be bounded).

The present paper aims at a synthesis of the investigations mentioned in the preceding paragraph with those referred to in the first paragraph by treating the case where the domain of f is the closure $\bar{V}$ of an open bounded connected subset V of a Hilbert–Fredholm–Riemannian manifold M and where a regular boundary condition is satisfied. (Thus critical points on the boundary $\dot{V}$ of V are excluded, and so are corners on $\dot{V}$. Such points are admitted in the investigations of D. Braess concerning the finite dimensional case (see [1]).

Section 2 below is mainly concerned with the geometric background. Under the assumption that the Hilbert manifold M is Fredholm the notion of a hyper-submanifold N of M is introduced in analogy to the notion of an $(n - 1)$-dimensional submanifold of an n-dimensional manifold (Definition 2.2). It is supposed that the boundary $\mathring{V}$ of the open subset V of M is a hyper-submanifold of M and that M is Riemannian. Then a unique exterior unit normal to $\mathring{V}$ can be defined (Definition 2.3), and the regular boundary condition (Assumption 2.4) can be stated. This assumption together with the other basic assumptions of this paper (Assumptions 2.1–2.3) allow us to show that a "gradient line" through a point of V does not intersect the boundary $\mathring{V}$ if followed in the sense of decreasing f (Theorem 2.2).

Section 3 deals with the Morse theory. First a function f defined on a topological space S is considered and under rather general assumptions (Assumption 3.1) critical groups $C_q(c)$ are attached to a critical value c of f. Sufficient conditions for the validity of the Morse relations (3.11) and (3.12) (which latter are written in terms of the ranks of the groups $C_q(c)$) are given (Theorem 3.1 whose proof is essentially the same as the one given by Pitcher [11] in the finite dimensional case). It is then shown that these sufficient conditions are satisfied if $S = \overline{V}$ and if Assumption 3.2 is added to Assumptions 2.1–2.4 (Theorem 3.2).

So far only critical *levels* were considered. The consideration of critical *points* begins with Definition 3.4. Critical groups are attached to critical points and a relation is established between the group $C_q(c)$ and the groups of the critical points at level c provided there are only a finite number of critical points at that level (Theorem 3.3). This finiteness condition is (on account of Lemma 2.8) certainly satisfied if there are althogether only isolated (not necessarily nondegenerate) critical points. Thus in this case all groups $C_q(c)$ are finitely generated and therefore the Morse relations (3.12) hold. For the special case that all critical points are non-degenerate see the Remark following Theorem 3.4.

In Section 4 it is shown that for functions which are bounded from below the main facts of the Lusternik–Schnirelman theory hold under Assumptions 2.1–2.4. The proof consists in verifying that under these assumptions a set of conditions is satisfied which were proved to be sufficient by F. Browder [2, Theorems 2 and 3]. A different proof for assertion (ii) of Theorem 4.1 can be given by generalizing a method employed by Seifert and Threlfall [16, p. 91] in the case of a finite dimensional manifold. This proof requires more assumptions on f but it is more constructive in that it constructs k closed sets covering $\overline{V}$ if there are k critical points by extending the "cylindrical neighborhood" of each critical point. (For the definition of a cylindrical neighborhood see [16, Section 9] in the finite dimensional case, and [14, Section 5] in the Hilbert space case.) Details of this proof will be given in another paper [15a].

For a short survey of the history of the Lusternik–Schnirelman theory we refer the reader to [2, pp. 5 and 6].

We conclude this introduction by listing a few notations used in the sequel: If F is a real valued function with domain S and a areal number then

$$\{f = a\} = \{x \in S \mid f(x) = a\};$$
$$f_a = \{x \in S \mid f(x) < a\}, \qquad \text{and} \qquad \bar{f}_a = \{x \in S \mid f(x) \leqslant a\}.$$

In general an upper bar denotes closure. The distance of the elements x and y in a metric space will be denoted by $\| x, y \|$ while $B(x, a)$ denotes the open ball with center x and radius a. If $b > a$ the $[a, b]$ denotes the closed interval with endpoints a and b while the point set $[a, b] - \{b\}$ is denoted by $[a, b)$. The symbol "$\approx$" between two groups denotes isomorphism. The zero element of a vectorspace will be denoted by θ.

2. The Geometrical Background

Let M be a connected C^r manifold without boundary modelled on a fixed Hilbert space E(r is a positive integer). For the definition of such a manifold as well as for the definition and properties of charts and of an atlas for M we refer the reader to [7] or [8]. Here we recall that a chart for M at a point $x_0 \in M$ is a pair (U, ϕ) where U is an open neighborhood of x_0 and ϕ is a bijection of U onto an open subset of E, and that an atlas A for M is a collection of charts such that the neighborhoods U cover M, with the additional property: if (U, ϕ) and (W, ψ) are two charts in A for which the intersection $U \cap W$ is not empty then the map

$$\psi\phi^{-1}: \phi(U \cap W) \to \psi(U \cap W) \tag{2.1}$$

is a C^r isomorphism, i.e., a one to one map onto admitting continuous (Fréchet) differentials up to and including order r.

Since $r \geqslant 1$ it follows that the differential $d\psi\phi^{-1}(u_0, u)$ of the map (2.1) at a point $u_0 \in \phi(U)$ with "increment" $u \in E$ is (as function of u) a (bounded) linear one to one map of E onto E.

Definition 2.1. M is called a Fredholm manifold if there exists an atlas A for M such that for any two charts (U, ϕ) and (W, ψ) in A

$$d\psi\phi^{-1}(u_0, u) = u + C(u), \tag{2.2}$$

where C is completely continuous. (See [4].) Concerning the question when a given manifold can be "reduced" to a Fredholm manifold we refer the reader to [4, p. 75] and [3, p. 768]. From the statements made there it follows, e.g., that every paracompact manifold modelled on a separable Hilbert space can be so reduced.

Out next goal is to give a definition of a hypersubmanifold of a Fredholm manifold M. From the definition of a submanifold N of M as given in [7, Chapter II, Section 2], and adopted in the present paper, the following fact follows directly; if $y \in N$ then there exists a chart (U, ϕ) for M at y with the property: there exists a direct decomposition

$$E = E^1 + E^2, \tag{2.3}$$

of E into two closed (linear) subspaces, and two sets V^1 and V^2 which are open subsets of E^1 and E^2 resp. such that

$$\phi(U) = V^1 \times V^2, \tag{2.4}$$

and

$$\phi(U \cap N) = V^2 \subset E^2. \tag{2.5}$$

A chart (W, ψ) for N at y is obtained by setting

$$W = U \cap N, \qquad \psi = \phi\,|_W \tag{2.6}$$

and the charts so obtained form an atlas for N.

We now would like to give the following

Definition 2.2. The submanifold N of the Fredholm manifold M is said to be hypersubmanifold of M at the point $y \in N$ if the space E^1 in the decomposition (2.3) is one dimensional. N is called a hypersubmanifold of M if it is a hypersubmanifold at everyone of its points.

However to make this definition legitimate we obviously have to prove the following.

Lemma 2.1. *Let* (W, ψ) *be a chart for* N *at* y *as in* (2.6), *and let* $(\tilde{W}, \tilde{\psi})$ *be another chart for* N *at* y *obtained from a chart* $(\tilde{U}, \tilde{\phi})$ *for* M *at* y *for which there exist a direct decomposition*

$$E = \tilde{E}^1 + \tilde{E}^2, \tag{2.3}$$

and sets $\tilde{V}^i$ *which are open subsets of* $\tilde{E}^i$ $(i = 1, 2)$ *such that* $\tilde{\phi}(\tilde{U}) = \tilde{V}^1 \times \tilde{V}^2$ *and* $\tilde{\phi}(\tilde{U} \cap N) = \tilde{V}^2 \subset \tilde{E}^2$. *It is asserted: if* E^1 *is one dimensional then* $\tilde{E}^1$ *is onedimensional.*

The proof is based on the following.

LEMMA 2.2. *If E^2 is hyperspace in E (i.e., a closed linear subspace of codimension 1) then the image of E^2 under a nonsingular linear map $E \to E$ of the form $u + C(u)$ with a completely continuous C is also a hyperspace in E.*

This lemma was proved in [13, Lemma 3.2].

Proof of Lemma 2.1. We consider maps $\tilde{\phi}\phi^{-1}: \phi(U \cap \tilde{U}) \to \tilde{\phi}(U \cap \tilde{U})$, $\tilde{\psi}\psi^{-1}: \psi(W \cap \tilde{W}) \to \tilde{\psi}(W \cap \tilde{W})$, and, at a point $u_1 \in \psi(W \cap \tilde{W})$, their differentials

$$L(u) = d\tilde{\phi}\phi^{-1}(u_1, u), \qquad l(u) = d\tilde{\psi}\psi^{-1}(u_1, u).$$

Then $l(u)$ is a linear map of E^2 onto $\tilde{E}^2$, while $L(u)$ maps E onto E. Now it is easily verified that $\tilde{\psi}\psi^{-1}$ is a restriction of $\tilde{\phi}\phi^{-1}$ from $\phi(U \cap \tilde{U})$, to $\psi(W \cap \tilde{W})$. It is not hard to see (using the definition of a Fréchet differential) that this fact implies that $l(u)$ is the restriction of $L(u)$ from E to E^2. Therefore $L(E^2) = l(E^2) = \tilde{E}^2$. Application of Lemma 2.2 now finishes the proof of Lemma 2.1.

Later on we will deal with real valued functions defined on the closure $\bar{V}$ of an open bounded subset V of M. From now on we will assume that M is a Fredholm manifold and that the boundary $\dot{V}$ of V is a hypersubmanifold of M (cf. Assumption 2.1 below).

The following lemma is a consequence of this assumption.

LEMMA 2.3. *Let (U, ϕ) be a chart for M at a point y of $\dot{V}$ with $\phi(y) = \theta$, and let E^1 and E^2 be as in* (2.3) *with $N = \dot{V}$. Then there exist a unit vector $e^1 \in E^1$ and a positive number ζ_0 of the following property: if $0 < \zeta < \zeta_0$ and if the sets B_ζ^+ and B_ζ^- are defined by*

$$B_\zeta^+ = \{u \in B(\theta, \zeta) \mid \langle u, e^1\rangle > 0\}, \qquad B_\zeta^- = \{u \in B(\theta, \zeta) \mid \langle u, e^1\rangle < 0\}, \tag{2.7}$$

then (i) *the sets $\phi^{-1}(B_\zeta^+)$ and $\phi^{-1}(B_\zeta^-)$ are contained in U, and* (ii) *the points of $\phi^{-1}(B_\zeta^+)$ are exterior to V while $\phi^{-1}(B_\zeta^-) \subset V$.*

Proof. Assertion (i) follows trivially from the fact that $\phi(U)$ is open. Let us then prove the first part of assertion (ii) with a positive $\zeta < \zeta_0$ where ζ_0 satisfies (i). $\phi^{-1}(B(\theta, \zeta))$ is a neighborhood of the boundary point y of V and therefore contains a point y_0 exterior to V. Let $u_0 = \phi(y_0)$. Then $u_0 \in \phi(U) = V_1 \times V_2 \subset E$, and if $\bar{e}^1$ is one of the two unit vectors which span E^1 we have by (2.3) the representation

$$u_0 = \bar{t}_1\bar{e}^1 + e^2, \qquad \bar{t}_1 = \langle u_0, \bar{e}^1\rangle, \qquad e^2 \in V^2 \subset E^2. \tag{2.8}$$

Here $\bar{t}_1 \neq 0$ since otherwise

$$y_0 = \phi^{-1}(u_0) = \phi^{-1}(e^2) \subset \phi^{-1}(V^2) \subset \mathring{V},$$

in contradiction to the fact the y_0 is an exterior point. We now define e^1 by setting $e^1 = \bar{e}^1$ if $\bar{t}_1$ is positive, and $e = -\bar{e}^1$ if $\bar{t}_1$ is negative. Then $\langle u_0 e^1 \rangle > 0$.

With e^1 thus defined we will prove that $(\phi^{-1}(u_1)$ is exterior to V for u_1 in the set B_ζ^+ given by (2.7). Indeed $u_\alpha = (1 - \alpha)\, u_0 + \alpha u_1 \in B_\zeta^+$ for $0 \leqslant \alpha \leqslant 1$ since B_ζ^+ is convex. In particular u_α has no point in common with E^2. From this it follows that $\phi^{-1}(u_\alpha)$ is a continuous curve not intersecting $\mathring{V}$. Since $\phi^{-1}(u_0) = y_0$ is an exterior point so is $\phi^{-1}(u_1)$. This finishes the proof of the first part of assertion (ii), and the second part is proved correspondingly.

Corollary to Lemma 2.3. *Every point x_0 in $\bar{V}$ has a neighborhood (with respect to $\bar{V}$) which is contractible to x_0 on $\bar{V}$. (This corollary will be used in Section 4).*

Proof. Consider first a point $x_0 = y \in \mathring{V}$. Then in the notation of the preceding lemma it follows from that lemma that for ζ small enough the set $\phi^{-1}(B_\zeta)$ is a neighborhood of x_0 with respect to $\bar{V}$. This neighborhood satisfies the requirement of the corollary since $\bar{B}_\zeta$ is contractible on itself to x_0. The proof is still simpler if $x_0 \in V$.

We now recall certain facts concerning tangent spaces to M and N. Our goal is to arrive at a definition of exterior normals at points of $\mathring{V}$.

Let (U_j, ϕ_j) be the charts at a point of M indexed by an index set I, and let u_j denote points in E. It can be verified that an equality of the form

$$u_k = d\phi_k \phi_j^{-1}(u_0, u_j), \qquad u_0 = \phi_j(x_0), \qquad x_0 \in U_j \cap U_k, \tag{2.9}$$

represents an equivalence relation. If (2.9) holds we say that the triples (U_j, ϕ_j, u_j) and (U_k, ϕ_k, u_k) are equivalent or simply that u_j and u_k are equivalent (with respect to the above charts), in symbols: $u_j \sim u_k$.

The tangentspace to M at x, denoted by M_x, is then defined as the set of equivalence classes under the relation (2.9). A point t of M_x is a collection $\{u_j\}_{j \in I}$ of equivalent points. With a natural definition of addition and multiplication by a scalar, M_x becomes a linear space.

We recall the definition of the differential of a map F of M into another manifold M^* modelled on a Hilbert space E^*. Let (U, ϕ) be a chart at $x_0 \in M$, and let (U^*, ϕ^*) be a chart at $y_0 = F(x_0)$. Let $F_{\phi^*\phi}$ be the map of $\phi(U)$ on $\phi^*(U^*)$ defined by $F_{\phi^*\phi} = \phi^* F \phi^{-1}$. Then $dF_{\phi^*\phi}(u_0, u)$ is well defined if $u_0 \in \phi(U)$ and $u \in E$. If $(\tilde{U}, \tilde{\phi})$ is another chart of x_0 it can be shown that

$$dF_{\phi^*\tilde{\phi}}(v_0; v) = dF_{\phi^*\phi}(u_0, u), \tag{2.10}$$

if

$$v_0 = \tilde{\phi}^* \phi^{-1}(u_0) \qquad \text{and if} \qquad v \sim u. \tag{2.11}$$

If $(\tilde{U}^*, \tilde{\phi}^*)$ is another chart at y_0 it can be shown that

$$dF_{\phi^*\phi}(u_0, u) \sim dF_{\tilde{\phi}^*\phi}(u_0; u). \tag{2.12}$$

It follows from (2.10) and (2.11) and the definition of a tangent space that $dF_{\phi^*\phi}(u_0, u)$ induces a linear mapping $t^* = dF(x_0; t)$, called the differential of F at x_0, of M_{x_0} into $M^*_{y_0}$.

Lemma 2.4 below follows immediately from the above definitions applied to the map $F = \phi^{-1}$ of $\phi(U)$ onto $U \subset M$, from (2.11) and from the non-singularity of $d\phi$.

Lemma 2.4. *$u \to t = d\phi^{-1}(u_0; u)$ is a map of E onto M_{x_0}. Moreover $d\phi^{-1}(u_0; u) = d\phi^{*-1}(v_0 v)$ if and only if* (2.11) *holds.*

We now assume that M is a Riemannian manifold. Then for each $x \in M$ the tangent space M_x is a Hilbert space with a scalar product $\langle s, t \rangle_x$. If then (U, ϕ) is a chart at x_0 and $u_0 = \phi(x_0)$ it follows from Lemma 2.4 that

$$\langle s, t \rangle_{x_0} = \langle d\phi^{-1}(u_0; u), d\phi^{-1}(u_0; v) \rangle_{x_0}. \tag{2.12}$$

with u, v uniquely determined by s, t. The right member is a positive definite symmetric form on E and defines therefore a scalar product $\langle u, v \rangle \phi$ on E depending on ϕ. For $\phi = \phi_j$ we write $\langle u, v \rangle_j$ for $\langle u, v \rangle \phi$ if $(U_j, \phi_j)_{j \in I}$ is the family of charts at x_0 indexed by the set I. Let $u = u_j$, $v = v_j$ be the couple satisfying (2.12) for given s, t. Then by (2.12)

$$\langle s, t \rangle_{x_0} = \langle u_j, v_j \rangle \phi_j = \langle u_j, v_j \rangle_j. \tag{2.13}$$

Let now x_0 be a point of $\mathring{V}$. Then by (2.5) and Definition 2.2

$$\phi_j(U \cap N) = V_j^2 \subset E_j^2$$

where E_j^2 is a hyperspace in E. Thus $E = E_j^1 + E_j^2$ for any space E_j^1 spanned by an element e_j^1 of E not in E_j^2. We choose e_j^1 as a unit vector orthogonal to E_j^2 in the metric given by the scalar product $\langle , \rangle_j$ defined in (2.13). We make the choice of e_j^1 unique by the additional requirement that $\phi_j^{-1}(\zeta e_j^1)$ is exterior to V for small enough positive ζ. (This choice is possible by Lemma 2.3.)

Definition 2.3. The element $n(x_0) = \{n_j\}_{j \in I}$ of M_{x_0} where

$$n_j = d\phi_j^{-1}(\phi_j(x_0); e_j^1)$$

is called the exterior normal to $\mathring{V}$ at x_0.

The following theorem shows that a number of properties intuitively expected of an exterior normal actually hold for $n(x_0)$.

THEOREM 2.1. (i) *M_{x_0} is spanned by $n(x_0)$ and $\mathring{V}_{x_0}$*; (ii) *$n(x_0)$ is orthogonal to $\mathring{V}_{x_0}$*; (iii) *for positive ζ small enough $\phi_j^{-1}(\zeta e_j^1)$ is a point of M which is exterior to V while for all real ζ, $d\phi_j^{-1}(\phi(x_0); \zeta e_j^1)$ is the point $\zeta n(x_0)$ of $M_{x'}$.*

For the proof we need the following

LEMMA 2.5. *Let 0 be an open subset of E, and let E^2 be a closed linear subspace of E. Let F be a C^1 map $0 \to E$, and let F^2 be the restriction of F to $0^2 = 0 \cap E^2$. Finally let u_2 be an element of 0^2. Then the restriction to E^2 of the linear map $E \to E$ given by $u \to dF(u_2, u)$ is the map $E^2 \to E^2$ given by $u \to dF^2(u_2, u)$.*

The proof consists in a routine argument based on the definition of a differential. We therefore omit it and proceed to the proof of Theorem 2.1.

Proof of Theorem 2.1. (i) Let $(U_j, \phi_j)_{j\in I}$ be the charts for M at x_0 indexed by I. Then a point of M_{x_0} is a set of points u_j of E which are equivalent under the relation

$$u_k = d\phi_k\phi_j^{-1}(\phi(x_0); u_j). \tag{2.14}$$

If (W_j, ψ_j) is the chart for $\mathring{V}$ at x_0 defined as in (2.6) (with $N = \mathring{V}$), then a point of $\mathring{V}_{x_0}$ is a set of points v_j which are equivalent under the equivalence relation

$$v_k = d\psi_k\psi_j^{-1}(\psi(x_0); v_j). \tag{2.15}$$

We note that the "target space" for all ϕ_j is E, while the target space for ψ_j is E_j^2 since $\psi_j(W_j) = \phi_j(U_j \cap \mathring{V}) \subset V_j^2 \subset E_j^2$.

Now $\psi_k\psi_j^{-1}$ is the restriction of $F = \phi_k\phi_j^{-1}$ from $0 = \phi_j(U_j)$ to $\phi_j(u_j \cap \mathring{V})$. It therefore follows from Lemma 2.5 that $d\psi_k\psi_j^{-1}(u_0; v)$ is the restriction of $d\phi_k\phi_j^{-1}(u_0; u)$ from E to E_j^2. Consequently, if (2.15) holds then (2.14) holds with $u_j = v_j$, $u_k = v_k$, and we thus see that a point of $\mathring{V}_{x_0}$ is also a point of M_{x_0}, in other words $\mathring{V}_{x_0} \subset M_{x_0}$.

Let now $t = \{u_j\}_{j\in I}$ be an arbitrary point of M_{x_0}. But $u_j = \lambda e_j^1 + u_j^2$ with λ real and $u_j^2 \in E_j^2$. Thus $t = \lambda\{e_j^1\} + \{u_j^2\}$. This proves assertion (i) of our theorem since $\{e_j^1\} = n(x_0)$ and $\{u_j^2\} \in \mathring{V}_{x_0}$.

Proof of (ii). Let t be an arbitrary element of $\mathring{V}_{x_0}$. By Lemma 2.4 (with ϕ replaced by ψ_j and E by E_j^2) and again by Lemma 2.5 t is of the form

$$t = d\psi_j^{-1}(\psi_j(x_0); v_j) = d\phi_j^{-1}(\phi_j(x_0); v_j), \qquad v_j \in E_j^2.$$

Therefore by Definition 2.3 and by (2.13)

$$\langle t, n(x_0)\rangle_{x_0} = \langle d\phi_j^{-1}(\phi_j(x_0); v_j), d\phi_j^{-1}(\phi_j(x_0); e_j^1)\rangle_{x_0}$$
$$= \langle v_j , e_j^1\rangle_j .$$

But the scalar product at the right is zero since $v_j \in E_j^2$, and since e_j^1 was chosen to be orthogonal to E_j^2. This proves assertion (ii).

Finally the two assertions of (iii) follow immediately from Definition 2.3 and the paragraph preceding it.

We now recall the definition of the gradient of a C^1 map $f: M_{x_0} \to R$, the reals. Since R may be identified with its tangent space the differential $df(x_0; t)$ is a real valued continuous linear functional on the Hilbert space M_{x_0}. Therefore there exists a unique element $g(x_0) \in M_{x_0}$ such that

$$df(x_0; t) = \langle g(x_0), t\rangle_{x_0} .$$

$g(x_0)$ is called the gradient of f at x_0. (In symbols ∇f, or grad f). If f is C^{k+1} ($k \geqslant 1$, then g is C^k. (For a proof see [8, p. 313].)

The next lemma deals with relations between the gradient on M and the gradient on the linear space E.

LEMMA 2.6. *Let (U, ϕ) be a chart at $x_0 \in M$. We set*

$$h(u) = f\phi^{-1}(u), \gamma(u) = \text{grad } h(u), u \in \phi(U) \subset E. \tag{2.16}$$

Let

$$u_0 = \phi(x_0), \qquad t = d\phi^{-1}(u_0; u). \tag{2.17}$$

Then,

$$\langle \gamma(u_0), u\rangle \phi = \langle g(x_0), t\rangle_{x_0} . \tag{2.18}$$

$$g(x_0) = d\phi^{-1}(u_0; \gamma(u_0), \qquad \gamma(u_0) = d\phi(x_0; g(x_0)) \tag{2.19}$$

$$\| g(x_0)\|_{x_0}^2 = \| \gamma(u)\|_\phi^2 . \tag{2.20}$$

Proof of (2.18). Using (2.17), the definitions involved and the chain rule we see that

$$\langle \gamma(u_0), u\rangle_\phi = dh(u_0; u) = df\phi^{-1}(u_0; u) = df(\phi^{-1}(u_0); d\phi^{-1}(u_0; u))$$
$$= df(x_0; t) = \langle g(x_0), t\rangle_{x_0} .$$

Proof of (2.19). The two assertions of (2.19) are equivalent. We prove the first one. From (2.18), the definition of $\langle,\rangle_\phi$ (given in the paragraph following (2.12)), and from (2.17) we see that

$$\langle g(x_0), t\rangle_{x_0} = \langle \gamma(u_0), u\rangle_\phi = \langle d\phi^{-1}(u_0; \gamma(u_0)), d\phi^{-1}(u_0; u)\rangle_{x_0}$$
$$= \langle d\phi^{-1}(u_0; \gamma(u_0), t\rangle_{x_0} .$$

Since this equality holds for all $t \in M_{x_0}$ it implies the first part of (2.19).

Proof of (2.20). By (2.19) and the definition of $\langle , \rangle_\phi$

$$\| g(x_0) \|_{x_0}^2 = \langle g(x_0), g(x_0) \rangle_{x_0} = d\phi^{-1}(u_0; \gamma(u_0), d\phi^{-1}(u_0; \gamma(u_0)\rangle_{x_0}$$
$$= \langle \gamma(u_0), \gamma(u_0) \rangle_\phi = \| \gamma(u_0) \|_\phi^2 .$$

LEMMA 2.7. *Let (U, ϕ) be a chart for M. Let A be an open interval and let x be a C^1 map $A \to \phi(U)$. For α in A we adopt the usual definition $x'(\alpha) = dx(\alpha; 1)$. Moreover let*

$$\eta(a) = \phi(x(\alpha)). \tag{2.21}$$

Using the notations of the preceding lemma we assert:

(i) *If*

$$x'(\alpha) = -g(x(\alpha)), \tag{2.22}$$

then

$$\eta'(\alpha) = -\gamma(\eta(\alpha)). \tag{2.23}$$

(ii) *If*

$$x'(\alpha) = -g(x(\alpha))/\| g \|^2, \tag{2.24}$$

then

$$\eta'(\alpha) = -\gamma(\eta(\alpha))/\| \gamma \|^2. \tag{2.25}$$

Proof. From (2.21) and the chain rule we see that

$$\eta'(\alpha) = d\phi(x(\alpha); 1) = d\phi(x(\alpha); x'(\alpha)). \tag{2.26}$$

Therefore if (2.22) holds then by (2.19) and (2.21)

$$-\eta'(\alpha) = d\phi(x(\alpha); g(x(\alpha)) = \gamma(\phi(x(\alpha)) = \gamma(\eta)$$

which proves (2.23).

If (2.24) holds we see from (2.26), (2.19) and (2.20) that

$$-\eta'(\alpha) = d\phi(x(\alpha); g(x(\alpha))/\| g \|^2 = \gamma(\phi(x(\alpha))/\| \gamma \|^2.$$

We now state our basic assumptions:

Let M be a connected Hilbert Riemannian C^r manifold ($r \geqslant 1$) without boundary, let V be an open bounded subset of M, and f a real valued function defined on the closure $\overline{V}$ of V.

ASSUMPTION 2.1. M is a Fredholm manifold, and the boundary $\dot{V}$ of V is a hyper-submanifold of M (see Definition 2.2).

ASSUMPTION 2.2. (i) f is not constant in any ball; (ii) the gradient g of f exists and is locally Lipschitz; (iii) if W is a subset of V on which $|f|$ is bounded then $\|g\|$ is bounded on W.

ASSUMPTION 2.3. The Palais–Smale condition is satisfied, i.e., if f is bounded on a subset S of $\bar{V}$ while $\|g\|$ is not bounded away from zero on S then g vanishes in some point of closure $\bar{S}$ of S.

ASSUMPTION 2.4.

$$\langle g(x), n(x)\rangle_x > 0 \qquad \text{for every } x \in \mathring{V}. \tag{2.27}$$

Here $n(x)$ denotes the exterior unit normal to $\mathring{V}$ at the point $x \in \mathring{V}$ (see Definition 2.3).

THEOREM 2.2. *Let f be a real valued C^1 function with domain $\bar{V}$, and let Assumptions 2.1–2.4 be satisfied. Let $x_0 \in V$, and let $x(\alpha)$ be the gradient line through x_0, i.e., the solution of the differential equation*

$$x'(\alpha) = -g(x(\alpha)), \tag{2.28}$$

satisfying the initial condition

$$x(0) = x_0 . \tag{2.29}$$

Then $x(\alpha) \in V$ for all nonnegative α for which $x(\alpha)$ is defined.

Proof. If the assertion were not true there would be an $\alpha_1 > 0$ such that

$$x_1 = x(\alpha_1) \in \mathring{V}, \qquad x(\alpha) \in V \qquad \text{for } 0 \leqslant \alpha < \alpha_1 . \tag{2.30}$$

Let now (U, ϕ) be a chart for M at x_1 (of the type described in (2.3) to (2.6)), let $\eta(\alpha)$ be defined by (2.21), and let

$$u_1 = \phi(x_1) = \phi(x(\alpha_1)) = \eta(\alpha_1). \tag{2.31}$$

Let e^1 be defined as in the paragraph preceding Definition 2.3 (with $\phi_j = \phi$) and let γ be as in Lemma 2.6. We then see from (2.17), (2.18) (with $u = e^1$), from the Definition 2.3 of the exterior normal, and from (2.27) that

$$\langle \gamma(u_1), e^1\rangle_\phi = \langle g(x_1), n(x_1)\rangle_{x_1} > 0,$$

and, taking (2.28) and Lemma 2.7 into account, that

$$\langle \eta'(\alpha_1), e'\rangle_\phi < 0. \tag{2.32}$$

We will now show that, in contradiction to (2.32),

$$\langle\eta'(\alpha_1), e')\rangle_\phi \geqslant 0. \tag{2.33}$$

and thus finish the proof of the theorem. We see from Lemma 2.3 that for positive and small enough $\alpha_1 - \alpha$,

$$\eta(\alpha) = -\lambda e' + e^2, \qquad \lambda = \lambda(\alpha) > 0, \qquad e^2 \in E^2. \tag{2.34}$$

But it follows from (2.30) and (2.31) that $\eta(\alpha_1) \in E^2$, and therefore from (2.34) that $\eta(\alpha) - \eta(\alpha_1) = -\lambda e' + \tilde{e}^2$ with $\lambda > 0$, and with $\tilde{e}^2 \in E^2$. This proves that

$$\langle(\eta(\alpha) - \eta(\alpha_1))/(\alpha - \alpha_1), e'\rangle\phi > 0, \tag{2.35}$$

since e' is orthogonal to E^2 (with respect to the scalar product $\langle,\rangle_\phi$) and since $\alpha - \alpha_1$ is negative. (2.35) obviously implies (2.33).

DEFINITION 2.4. A point $\gamma_0 \in \bar{V}$ is stationary for f if $g(x_0) = 0$. A number c is called a stationary value (or level) for f if $f(\gamma) = c$ for at least one stationary point γ. The set of stationary points will be denoted by Γ and the set of stationary levels by Λ.

LEMMA 2.8. *Let the set W be as in Assumption* 2.2(iii) *and assume it to be open. Then the set $\Gamma \cap \bar{W}$ is compact.*

Since f is bounded on $\Gamma \cap W$ the proof given for assertion (i) of Lemma 2.3 in [14] applies.

LEMMA 2.9. *The set Λ is closed.*

Proof. Let $c_1, c_2, \ldots$ be a convergent sequence of stationary levels, and let

$$c_0 = \lim_{i\to\infty} c_i. \tag{2.36}$$

We have to prove that

$$c_0 \in \Lambda. \tag{2.37}$$

Let γ_i be a stationary point at level c_i. Then on account of (2.36), f is bounded on the set $\{\gamma_i\}$. Since γ_i is an element of the open set V we can choose positive ζ_i such that $|f(x) - f(\gamma_i)| < | 1$ for $x \in B(\gamma_i, \zeta_i) \subset V$. Then obviously f is bounded on the open set $W = \bigcup_i B(\gamma_i, \zeta_i)$. Consequently, by Lemma 2.8, there exists a subsequence $\{\gamma_{n_i}\}$ of the sequence $\{\gamma_i\}$ which converges to a point $\gamma_0 \in \Gamma$. Then we see from (2.36) that

$$f(\gamma_0) = \lim_{i\to\infty} f(\gamma_{n_i}) = \lim_{i\to\infty} c_{n_i} = c_0.$$

This proves (2.37).

LEMMA 2.10. *If the closed interval $[a, b]$ contains no critical levels then there exists a positive m such that*

$$\| g(x)\| > m \qquad \text{for } x \in f^{-1}[a, b]. \tag{2.38}$$

Proof. The lemma is an immediate consequence of Assumption 2.3 since f is bounded in the closed set $f^{-1}[a, b]$.

LEMMA 2.11. *Let a, b and m be as in the preceding lemma. Let $x(\alpha, x_0)$ be the gradient line through x_0, (as defined in the Theorem 2.2). Then*

$$x(\alpha, x_0) \in \bar{f}_a \quad \text{if} \quad x_0 \in \bar{f}_b \qquad \text{and} \qquad \alpha \geqslant T = (b - a/m^2).$$

Proof. If this were not true then $f(T, x_0) > a$ for some $x_0 \in f^{-1}[a, b]$, and for such x_0 we would see from (2.38) that

$$a < f(x_0) + \int_0^T \frac{df}{dt}\, dt = f(x_0) + \int_0^T \left\langle g(x), \frac{dx}{dt} \right\rangle dt$$

$$= f(x_0) - \int_0^T \| g(x)\|^2\, dt < b - m^2 T = a.$$

LEMMA 2.12. *Let $\Gamma(c)$ denote the set of stationary points at level c (which may be the empty set). Let W be an open neighborhood of $\Gamma(c)$. Then there exist real numbers a, b and T with $a < c < b$ and $T > 0$ such that*

$$x(T, x_0) \in \bar{f}_a \cup W \qquad \text{for } x_0 \in \bar{f}_b\,. \tag{2.39}$$

Remark. We note that the existence of deformations having the property asserted for $x(\alpha, x_0)$ in the lemma was proved by Palais [9a] and Browder [2] for Banach manifolds (without boundary) by the use of "pseudo gradients." The proof below is given for completeness sake. It is divided into four steps.

Step 1. It is asserted that there exist a_0, b_0 with $a_0 < c < b_0$ such that

$$\Gamma^0 = \Gamma \cap f^{-1}[a_0\,, b_0] \subset W. \tag{2.40}$$

We will show that there exists a $\bar{d} > 0$ such that (2.40) is true with $a_0 = c - d$, $b_0 = c + d$ if $0 < d < \bar{d}$.

If this were not true there would exist sequence of positive numbers d_ν converging to zero, and a sequencc of points γ_ν with the properties

$$\gamma_\nu \in f^{-1}[\bar{a} - d_\nu\,, c + d_\nu], \qquad \gamma_\nu \in \Gamma - W. \tag{2.41}$$

Since the sequence $f(\gamma_\nu)$ is bounded we may by Lemma 2.8 assume that the γ_ν converge. The limit γ_0 is a stationary point which, by (2.41), lies on the level c. Thus $\gamma_0 \in \Gamma(c) \subset W$, and therefore $\gamma_\nu \in W$ form a certain ν on. This contradicts the second part of (2.41).

Step 2. The set Γ^0 defined by (2.40) is compact, and the set $\bar{V} - W$ is closed. Therefore these two sets which by (2.40) are disjoint have a positive distance $5d_0$. Let now $\gamma_0 \in \Gamma^0$ and let $\zeta(\gamma_0)$ be a number such that

$$0 < \zeta(\gamma_0) < d_0, \tag{2.42}$$

$$|f(x) - f(\gamma_0)| < 1 \qquad \text{for } x \in B(\gamma_0, \zeta(\gamma_0)), \tag{2.43}$$

$$g(x) \text{ satisfies a Lipschitz condition in } B(\gamma_0, 3\zeta(\gamma_0)). \tag{2.44}$$

(See Assumption 2.2).

Since Γ^0 is compact there exist a finite number of points γ^σ ($\sigma = 1, 2,..., s$) in Γ^0 such that the balls $B(\gamma^\sigma, \zeta(\gamma^\sigma))$ cover Γ^0. We set

$$W^j = \bigcup_\sigma B(\gamma^\sigma, j\zeta(\gamma^\sigma)) \qquad \text{for } j = 1, 2, 3. \tag{2.45}$$

Then

$$\Gamma^0 \subset W^1 \subset W^2 \subset W^3 \subset W. \tag{2.46}$$

We claim: there exists a positive T such that

$$x(\alpha, x_0) \in W \qquad \text{for } 0 \leqslant \alpha \leqslant T \quad \text{and} \quad x_0 \in \overline{W^2}. \tag{2.47}$$

For the proof we note first that as a consequence of (2.44) g is bounded in W^3, say

$$\| g(x) \| < M \qquad \text{for } x_0 \in W^3. \tag{2.48}$$

Let now $x_0 \in W^2$. Then

$$\| x - \gamma^\sigma \| < 2\zeta(\gamma^\sigma) \tag{2.49}$$

for at least one σ. For such σ we have obviously

$$B(x_0, \zeta(\gamma^\sigma)) \subset B(\gamma^\sigma, 3\zeta(\gamma^\sigma)) \subset W^3 \subset W. \tag{2.50}$$

It follows from this inclusion in conjunction with (2.44) that g satisfies a Lipschitz condition in $B(x_0, \zeta(\gamma^\sigma))$. Moreover (2.48) is satisfied for $x \in B(x_0, \zeta(\gamma^\sigma))$. Therefore the local existence theorem for differential equations allows us to conclude that $x(\alpha, x_0)$ is defined at least for $|\alpha| < \zeta(\gamma^\sigma)/M$, and that for such α, $x(\alpha, x_0) \in B(x_0, \zeta(\gamma^\sigma)) \subset W^2$ (cf. 2.50)). This shows that the assertion (2.47) is satisfied with

$$T = \min_{\sigma=1,\ldots,s} \zeta(\gamma^\sigma). \tag{2.51}$$

Step 3. We assert the existence of a positive m' such that

$$\| g(x) \| > m_1 \qquad \text{for } x \in S^1 = f^{-1}[a_0, b_0] - W^2. \tag{2.52}$$

Indeed f is bounded on the closed set S^1. Therefore if the assertion were not true, Assumption 2.3, (2.40) and the inclusion (2.46) would imply the existence of a point $\bar{y} \in S' \cap \Gamma \subset \Gamma^0 \subset W^2$. This contradicts the fact that by definition the intersection $S^1 \cap W^2$ is empty.

Step 4. Let T and m_1 be as in (2.51) and (2.52) resp. Let a_0, b_0 be as in Step 1. Finally let, a, b be a couple of numbers satisfying

$$a_0 \leqslant a < b \leqslant b_0 \tag{2.53}$$

and

$$b - a < Tm_1^2. \tag{2.54}$$

With this choice of a, b, and T we will show that (2.39) is satisfied. To this end we write f_b as the union of three sets:

$$f_b = f_a \cup \{f^{-1}[a, b] \cap \overline{W}^2\} \cup \{f^{-1}[a, b] \cap \overline{V} - W^2\} \tag{2.55}$$

If $x_0 \in f_a$ then (2.39) is obviously satisfied since $f(x(\alpha, x_0))$ is non increasing in α. If x_0 is a point in the second summand at the right of (2.55) then, by (2.53), $x_0 \in f^{-1}[a_0, b_0] \cap \overline{W}^2$, and our assertion (2.39) is satisfied by (2.47).

Finally let x_0 be an element of the third summand in (2.55). Suppose first that

$$x(\alpha, x_0) \subset \overline{V} - W^2 \qquad \text{for } 0 \leqslant \alpha \leqslant T. \tag{2.56}$$

We then show that (2.39) holds by proving that

$$f(x(T, x_0)) \leqslant a. \tag{2.57}$$

If this inequality were not true we could conclude from the monotonicity of $f(x(\alpha, x_0)$ and from (2.53) that

$$b_0 \geqslant b \geqslant f(x_0) \geqslant fx(\alpha, x_0)) \geqslant f(x(T, x_0)) > a \geqslant a_0 \qquad \text{for } 0 \leqslant \alpha \leqslant T.$$

Thus, by (2.56), $x(\alpha, x_0)) \in S^1$ (cf. (2.52), and (2.52) holds with $x = x(\alpha, x_0)$ for these α. Then

$$\begin{aligned} a < f(x(T, x_0)) &= f(x_0) + \int_0^T \frac{df}{d\tau}\, d\tau \\ &= f(x_0) - \int_0^T \| g \|^2 \, d\tau < b - m_1{}^2 T < a \qquad \text{(cf. (2.54))}. \end{aligned}$$

This contradiction proves (2.39) if (2.56) is satisfied.

Suppose now that (2.56) is not true. Then since by assumption

$$x(0, x_0) = x_0 \subset \bar{V} - W^2,$$

there must be an α_1 such that

$$0 < \alpha_1 \leqslant T, \qquad x(\alpha, x_0) \begin{cases} \notin \bar{W}^2 & \text{for } 0 \leqslant \alpha < \alpha_1, \\ \in \bar{W}^2 & \text{for } \alpha = \alpha_1. \end{cases} \tag{2.58}$$

Let now $x_1 = x(\alpha_1, x_0)$ and let $\bar{x}(\beta) = \bar{x}(\beta, x_1)$ be the gradient line defined by $d\bar{x}/d\beta = -g(\bar{x}(\beta))$, $\bar{x}(0, x_1) = x_1$. It then follows from (2.47) with $x(\alpha, x_0)$ replaced by $\bar{x}(\beta, x_1)$ that $\bar{x}(\beta, x_1) \in W$ for $0 \leqslant \beta \leqslant T$. Therefore $x(\alpha, x_0) \in W$ for $\alpha_1 \leqslant \alpha \leqslant \alpha_1 + T$ since $x(\alpha_1 + \beta, x_0) = \bar{x}(\beta, x_1)$. This proves that $x(T, x_0) \in W$ since $\alpha_1 \leqslant T$.

LEMMA 2.13. *If the half open interval $[a, c)$ contains no stationary values then $\bar{f}_a$ is a deformation retract of f_c.*

Proof. For $x_0 \in f_c - \bar{f}_a$ let $x(\alpha, x_0)$ be the solution of

$$\frac{dx}{d\alpha} = -(f(x_0) - a)\, g(x)/\|g\|^2, \qquad x(0, x_0) = x_0,$$

and let

$$\delta(\alpha, x_0) \begin{cases} = x(\alpha, x_0), & \text{for } x_0 \in f_c - \bar{f}_a, \\ = x_0, & \text{for } x_0 \in \bar{f}_a \end{cases} \qquad 0 \leqslant \alpha \leqslant 1.$$

It is then easily seen that $\delta(\alpha, x_0)$ retracts f_c onto $\bar{f}_a$ (see, e.g., [14, Lemma 3.4]).

We conclude this section by proving the following theorem needed in Section 4.

THEOREM 2.3. *$\bar{V}$ is an ANR, i.e. an absolute neighborhood retract in the class of metrizable spaces.*

The proof is based on the following lemma.

LEMMA 2.14. *Let Y be an ANR. Let Y_1 and Y_2 be closed subspaces of Y whose union is Y and whose intersection is an ANR. Then Y_1 and Y_2 are ANR's.*

This lemma is proved in [6, Proposition 9.1, p. 47 in conjunction with Theorems 3.1, p. 83 and 3.2, p. 84]).

To apply this lemma to the proof of Theorem 2.4 we recall that every metric Banach manifold is an ANR (see [9, Corollary, p. 3]). Thus $Y = M$ and $\dot{V}$ are ANR. Setting $Y_1 = \bar{V}$, $Y_2 = M - V$ we see that the lemma implies the theorem.

3. The Morse Theory

Let f be a real valued continuous function defined on a topological space S.

Definition 3.1. A real number c is called a critical value (or level) for f if for no two numbers a, b with $a < c < b$ the set $\bar{f}_b$ can be deformed into the set $\bar{f}_a$. The set of critical values will be denoted by Λ_0.

Assumption 3.1. (α) A finite interval contains at most a finite number of cirtical values. (β) If the half open interval $[a, b)$ contains no critical values then f_b can be deformed into $\bar{f}_a$.

For reference sake we state the following obvious consequence of Assumption 3.1 as a lemma;

Lemma 3.1. *If the closed interval* $[a, b]$ *contains no critical values then* $\bar{f}_b$ *can be deformed into* $\bar{f}_a$.

If $B \supset A$ is a couple of subsets of S, and q a nonnegative integer then $H_q(B, A)$ will denote the qth singular homology group of the couple (B, A). The coefficient group will always be supported to be a principal ideal ring.

Lemma 3.2. *If* $[a, b]$ *contains no critical values then*

$$H_q(\bar{f}_b, \bar{f}_a) = 0 \qquad \textit{for all } q. \tag{3.1}$$

This is an immediate consequence of Lemma 3.1 and well known properties of the homology groups.

Lemma 3.3. *If for all intervals* $[a, b]$ *containing the real number* r *the homology group* $H_q(\bar{f}_b, \bar{f}_a)$ *is different from zero for some* q *then* r *is a critical value.*

Proof. If r is not critical then by Assumption 3.1(α) there exists an interval $[a, b]$ containing r and no critical values. For such interval (3.1) holds by Lemma 3.2 for all q. This obviously proves the lemma.

Lemma 3.4. *Let* c *be a critical value. Then* $H_q(\bar{f}_b, \bar{f}_a)$ *does not change as long as* c *is the only critical value in* $[a, b]$.

Proof. We have to show that

$$H_q(\bar{f}_b, \bar{f}_a) \approx H_q(\bar{f}_\beta, \bar{f}_\alpha), \tag{3.2}$$

if the intervals $[a, b]$ and $[\alpha, \beta]$ both satisfy the condition of the lemma. It is easy to see that we may assume $b \geqslant \beta > c > \alpha \geqslant a$. We will first show that

$$H_q(\bar{f}_b, \bar{f}_a) \approx H_q(\bar{f}_b, \bar{f}_\alpha). \tag{3.3}$$

Now the interval $[a, \alpha]$ contains no critical value. This fact implies by Lemma 3.2 that

$$H_q(\bar{f}_\alpha, \bar{f}_a) = 0. \tag{3.4}$$

It also implies that $\bar{f}_\alpha$ is empty if $\bar{f}_a$ is empty since, by Lemma 3.1, $\bar{f}_\alpha$ can be deformed into $\bar{f}_a$. Therefore in this special case the assertion (3.3) reduces to $H_q(\bar{f}_b) \approx H_q(\bar{f}_b)$. But if $\bar{f}_a$ is not empty then it is well known that (3.4) implies (3.3) (see [5, 1.8.1]).

This proves (3.3), and the isomorphism

$$H_q(\bar{f}_b, \bar{f}_a) \approx H_q(\bar{f}_\beta, \bar{f}_\alpha), \tag{3.5}$$

is proved in a similar way. But (3.3) and (3.5) imply (3.2).

Lemma 3.4 allows us to make the following definition.

Definition 3.2. The qth critical group $C_q(c)$ at the critical level c of f is defined by $C_q(c) = H_q(\bar{f}_b, \bar{f}_a)$ where $a < c < b$ and where c is the only critical level in $[a, b]$.

Since the coefficient group G of the homology theory is supposed to be a principal ideal ring. The classical decomposition theorems hold if G is finitely generated. If G is not necessarily finitely generated we have the following definition.

Definition 3.3. Let T be the torsion submodule of G. Then the rank $\zeta(G)$ of G is defined as follows: if the quotient module G/T is not finitely generated then $\zeta(g) = \infty$; if G/T is finitely generated then $\zeta(G) = \zeta(G/T)$, i.e., the number of elements in a base of the (free) group G/T.

Lemma 3.5. *If $\zeta(G)$ is finite then we have the direct decomposition*

$$G = F \dotplus T, \tag{3.6}$$

where T is as above and where F is a free module. Moreover

$$\zeta(G) = \zeta(F). \tag{3.7}$$

The proof follows easily from a well-known lemma (see, e.g., [5, p. 133, Lemma 6.3]).

We now introduce notations which will be used in Theorem 3.1 below. Let $a < b$ be two number which are not critical values and let $c_1 < c_2 < \cdots c_N$ be the critical values in $[a, b]$. Moreover let $a_0, a_1, \ldots, a_N$ be numbers such that

$$a \leqslant a_0 < c_1 < a_1 < \cdots a_{\alpha-1} < c_\alpha < a_\alpha < \cdots a_{N-1} < c_N < a_N \leqslant b. \tag{3.8}$$

In addition we set

$$A = \bar{f}_a, \qquad B = \bar{f}_b, \qquad A_\alpha = \bar{f}_{a_\alpha},$$
$$C_q^{\alpha} = C_q(c_a) = H_q(A_\alpha, A_{\alpha-1}), \tag{3.9}$$

$$M_q^{\alpha} = \zeta(C_q^{\alpha}), \qquad M_q = \sum_{\alpha=1}^{N} M_q^{\alpha},$$
$$R_q(a, b) = \zeta(H_q(B, A)), \qquad \alpha = 1, 2, \ldots, N. \tag{3.10}$$

THEOREM 3.1. *For the validity of the inequality*

$$M_q(a, b) \geqslant R_q(a, b), \tag{3.11}$$

each of the following three conditions is sufficient

(i) $\zeta(H_q(A_\alpha, A_0) < \infty$, $\alpha = 1, 2, \ldots, N$.

(ii) *the critical groups C_q^{α} are finitely generated,*

(iii) *the coefficient group G is a field.*

Moreover if (i′) *is condition* (i) *with the additional proviso that*

(iv) $M_q^{\alpha} = \zeta(C_q^{\alpha}) < \infty$, $\alpha = 1, \ldots, N$

then each of the conditions (i′) *and* (ii) *is sufficient for the validity of the inequality*

$$\sum_{\beta=0}^{q} (-1)^{q-\beta} M_q(a, b) \geqslant \sum_{\beta=0}^{q} (-1)^{q-\beta} R_q(a, b). \tag{3.12}$$

Remark. If f is bounded then for small enough a and large enough b, $R_q(a, b)$ is the qth Betti number of S and the "Morse numbers" M_q are independent of a and b. Thus inequalities (3.11) and (3.12) are in this case the classical Morse inequalities.

Proof of Theorem 3.1. If one of the numbers $M_q^1, \ldots, M_q^N$ is infinite then $M_q = \infty$ by (3.10), and (3.11) is trivially satisfied. Therefore for the proof of this inequality we may make the additional assumption (IV), and thus replace (i) by (i′). But under the latter condition the proof of (3.11) and (3.12) inequalities is the same as the one given by Pitcher for the finite dimensional case ([11, Section 11]) and is therefore omitted.

To give the proof under condition (ii) it will now be sufficient to prove that this condition implies (i′). Since (iv) is obviously implied by (ii) the proof will be finished if we verify that (ii) implies (i) by showing that the groups $H_q(A_\alpha, A_0)$ are finitely generated. This is done by induction in

α: $H_q(A_1, A_0)$ is certainly finitely generated since this group is C_q^1. We assume that $H_q(A_{\alpha-1}, A_0)$ is finitely generated and consider the part

$$H_q(A_{\alpha-1}, A_0) \xrightarrow{i^*} H_q(A_\alpha, A_0) \xrightarrow{j^*} H_q(A_\alpha, A_{\alpha-1}),$$

of the homology sequence for the triple $A_0 \subset A_{\alpha-1} \subset A_\alpha$. Here the two extreme groups are finitely generated, the one at the left by induction assumption, the one at the right because it is the critical group C_q^α. From this and the exactness of the sequence it follows that the kernel K of the map j^* as well as the group $H_q(A_\alpha, A_0)/K$ are finitely generated. This obviously implies that $H_q(A_\alpha, A_0)$ is finitely generated. This completes the proof of the theorem since (iii) and (iv) together imply (ii).

We now return to the situation of Section 2 by setting $S = \bar{V}$.

THEOREM 3.2. *Theorem* 3.1 *is valid if* $S = \bar{V}$ *and if Assumption* 3.1 *is replaced by Assumptions* 2.1–2.4 *and the additional*

ASSUMPTOIN 3.2. A finite interval contains at most a finite number of stationary values (cf. Definition 2.4).

In fact Assumptions 2.1–2.4 and 3.2 imply Assumption 3.1 as the following two lemma show.

LEMMA 3.6. *A critical value is a stationary value, i.e.,* $\Lambda_0 \subset \Lambda$.

Proof. We show: if c is not a stationary value then c is not critical. Indeed by Assumption 3.2 there correspond to a nonstationary c two numbers a, b with $a < c < b$ such that $[a, b]$ contains no stationary values. By Lemma 2.11 the set $\bar{f}_b$ can be deformed into the set $\bar{f}_a$ but this implies that c is not critical (cf. Definition 3.1).

LEMMA 3.7. *The set* Λ_0 *of critical values satisfies Assumption* 3.1.

Proof. Assumption 3.2 together with Lemma 3.6 show that the (α) part of Assumption 3.1 is satisfied. The (β) part follows immediately from Lemma 2.13 if the interval $[a, b)$ which by assumption is free of critical levels is also free of stationary levels. Suppose now $[a, b)$ contains stationary values. By Assumption 3.2 there are only a finite number, say $s_1 > s_2 > \cdots > s_r$. Since the s_i are not critical values there exist a_i, b_i such that $a_i < s_i < b_i$ and such that $\bar{f}_{b_i}$ can be deformed into $\bar{f}_{a_i}$. Obviously we can choose the a_i and b_i in such a way that in addition

$$b > b_r > s_r > a_r > b_{r-1} > s_{r-1} > a_{r-1} > \cdots > b_1 > s_1 > a_1 > a$$

(with an obvious modification if a is a stationary value). Now there are no stationary values in $[b_r, b)$. Therefore f_b can be deformed into $\bar{f}_{b_r}$ (again by Lemma 2.13). But $\bar{f}_{b_r}$ can be deformed into f_{a_r}. Going on this way we obtain deformations whose product deforms f_b into $\bar{f}_a$.

LEMMA 3.8. *$C_q(c) \approx H_q(\bar{f}_b, f_c)$ if c is the only critical value in $[c, b]$.*

This is an immediate consequence of the preceding lemma and Definition 3.2 together with the deformation invariance of the homology groups.

Remark. In Theorems 3.1 and 3.2 only *critical* levels are considered. But if we were to define critical groups $C_q(c)$ (in analogy to Definition 3.2) for *stationary* but not critical levels c then it is easily seen from Definition 3.1 that these $C_q(c)$ are zero groups. Therefore $M_q(c) = \zeta(C_q(c)) = 0$. Thus there would be no change in the inequalities (3.11) and (3.12) if all stationary levels were taken into account, i.e., if Λ_0 is replaced by Λ.

We now consider critical *points.*

DEFINITION 3.4. The point $y_0 \in \bar{V}$ is called a critical point of f is for no neighborhood W of y_0 the set $f_{c_0} \cap W \cup \{y_0\}$, can be deformed into the set $f_{c_0} \cap W$ where $c_0 = f(y_0)$. The critical point y_0 is called isolated if there exists a neighborhood of y_0 containing no other critical point.

LEMMA 3.9. *If y_0 is critical then y_0 is stationary.*

Proof. Suppose y_0 is not stationary and let $c_0 = f(y_0)$. Then $g(y_0) \neq \theta$. Therefore there exists a neighborhood W of y_0 in which $\| g(x)\|$ is bounded from below by $m = \| g(y_0)\|/2$. On account of this fact it is easy to construct a deformation deforming $f_{c_0} \cap W \cup \{y_0\}$ into $f_{c_0} \cap W$ by using the gradientline through y_0 (cf. (2.22)).

LEMMA 3.10. *Let W and W_1 be open neighborhoods of the isolated critical point y_0. Suppose that W and W_1 contain no other critical point. Then, with $c_0 = f(y_0)$,*

$$H_q(f_{c_0} \cap W_1 \cup \{y_0\}, f_{c_0} \cap W_1) \approx H_q(f_{c_0} \cap W \cup \{y_0\}, f_{c_0} \cap W).$$

Proof. We may assume that $W \supset W_1$ (otherwise consider $W_1 \cap W$). Then the lemma follows by excising the set $U = (W - W_1) \cap f_{c_0}$ from the couple at the right member.

This lemma allows us to state the following.

DEFINITION 3.5. The qth critical group $C_q(y_0)$ of the isolated critical point y_0 is defined by

$$C_q(y_0) = H_q(f_{c_0} \cap W \cup \{y_0\}, f_{c_0} \cap W),$$

where W is an open neighborhood of γ_0 containing no other critical point and where $c_0 = f(\gamma_0)$.

A *remark* analogous to the one following Lemma 3.8 can be made concerning the definition made concerning the definition of groups attached to an isolated stationary but not critical point.

THEOREM 3.3. *Let the assumptions of Theorem* 3.2 *be satisfied. Suppose that* c *is a critical level at which there are only a finite number of critical points, say* $\gamma_1, \gamma_2, \ldots, \gamma_r$. *Let* $b > c$ *be such that the interval* $(c, b]$ *contains no critical values. Then*

$$C_q(c) \approx H_q(\bar{f}_c, f_c) \approx \sum_{i=1}^{r} C_q(\gamma_i), \tag{3.13}$$

where Σ *denotes the direct sum.*

For the proof we need Lemmas 3.11 and 3.12.

LEMMA 3.11. *Let the assumptions of Theorem* 3.3 *be satisfied. Let* $x(\alpha, x_0)$ *satisfy*

$$\begin{aligned} \frac{dx}{d\alpha} &= -(f(x_0) - c)\, g(x)/\|g(x)\|^2, \\ x(0, x_0) &= x_0, \qquad x_0 \in f^{-1}(c, b]. \end{aligned} \tag{3.14}$$

Then

(i) $\dfrac{df(x)}{d\alpha} = -(f(x_0) - c)$,

(ii) $c < f(x(\alpha, x_0)) \leqslant b$ *for* $0 \leqslant \alpha < 1$,

(iii) $\lim_{\alpha \to 1^-} f(x(\alpha, x_0)) = c$,

(iv) $\lim_{\alpha \to 1^-} x(\alpha, x_0)$ *exists.*

For assertions (i)–(iii) *the assumption that the critical set at level* c *is finite is not necessary.*

Proof. The elementary proof of assertions (i)–(iii) may be found in [12, Lemma 5.3]. We turn to the proof of (iv) which is an modification suited to the present situation of the proof given in [12, Theorem 5.1)].

Let $\zeta(\alpha)$ be the distance of the point $x(\alpha, x_0)$ to the critical set $\Gamma(c)$ at level c, and let

$$\bar{\zeta} = \inf_{0 \leqslant \alpha < 1} \zeta(\alpha).$$

We distinguish two cases.

Case I. $\bar{\zeta} > 0$. In this case there exists a positive constant m such that

$$\|g(x)\| > m \qquad \text{for } x \in S = \{x = x(\alpha, x_0) \mid 0 \leqslant \alpha < 1\}. \tag{3.15}$$

Indeed otherwise we would from Assumption 2.3 and the boundedness of f on S (quaranteed by (ii)) conclude the existence of a stationary point γ_0 in the closure of S. There would then be a sequence $\alpha_1, \alpha_2, \ldots$ such that

$$\lim_{\nu\to\infty} x_\nu = \gamma_0 \qquad \text{if} \quad x_\nu = x(\alpha_\nu, x_0). \tag{3.16}$$

Now because of our assumption $\bar{\zeta} > 0$ the point γ_0 cannot belong to $\Gamma(c)$. Thus $\gamma_0 \in \Gamma - \Gamma(c)$. But this is also impossible. For by Assumption 3.2 there exists a $d > 0$ such that

$$\Gamma - \Gamma(c) \subset f_{c-d} \quad \bigcup \{f \geqslant b + d\}. \tag{3.17}$$

But it follows from (3.16) and assertion (ii) above that $c \leqslant f(\gamma_0) \leqslant b$.

Thus the existence of an m satisfying (3.15) is established. From this in conjunction with (3.14) we see that for $0 < \alpha' < \alpha'' < 1$

$$\|x(\alpha'', x_0) - x(\alpha', x_0)\| = \left\| \int_{\alpha'}^{\alpha''} x' \, d\alpha \right\| \leqslant |f(x_0) - c)| \, (\alpha'' - \alpha')/m$$
$$\leqslant (b - c)\,(\alpha'' - \alpha')/m.$$

By Cauchy's principle this implies the validity of assertion (iv).

Case II. $\bar{\zeta} = 0$. Then there exists a convergent sequence $\{\alpha_i\}$ with

$$0 \leqslant \alpha_i < 1 \qquad \text{and} \qquad \lim_{i\to\infty} \alpha_i = \alpha_0 \leqslant 1$$

such that the sequence $x(\alpha_i, x_0)$ converges to one of the points $\gamma_1, \gamma_2, \ldots, \gamma_r$, say to γ_1:

$$\lim_{i\to\infty} x(\alpha_i, x) = \gamma_1 . \tag{3.18}$$

We must have

$$\alpha_0 = \lim_{i\to\infty} \alpha_i = 1. \tag{3.19}$$

For otherwise $0 \leqslant \alpha_0 < 1$, and by assertion (ii) above,

$$\lim_{i\to\infty} f(x(\alpha_i, x_0)) = f(x(\alpha_0, x_0) > c.$$

But by (3.18) this limit equals $f(\gamma_1) = c$.

Thus (3.19) is true, and we may assume that the α_i are monotone increasing. We now prove assertion (iv) by showing that

$$\lim_{\alpha\to 1^-} x(\alpha, x) = \gamma_1 . \tag{3.20}$$

To do this we first exhibit a positive ζ_0 of the following property: if $0 < \zeta_1 < \zeta_0$ then there exists a positive constant $m = m(\zeta_1, \zeta_0)$ such that

$$\| g(x)\| > m \qquad \text{for } x \in B(\gamma_1, \zeta_0) - B(\gamma_1, \zeta_1). \tag{3.21}$$

We choose for ζ_0 a positive number satisfying

$$\zeta_0 < \min_{i=2,\ldots,r} \| \gamma_i - \gamma_1 \|/2, \qquad \text{and} \qquad | f(x) - f(\gamma_1)| < \min(d/2, b - c)$$

for $x \in B(\gamma_1, \zeta_0)$ where d is as in (3.17). Since then $c - d/2 < f(x) < b$ for $x \in B(\gamma_1, \zeta_0)$ it is (cf. (3.17)) easily verified that the closure of the set appearing in (3.21) contains no point of $\Gamma = \Gamma(c) \cup (\Gamma - \Gamma(c))$. The existence of an m of the asserted property follows then from Assumption 2.3.

To prove the assertion (3.20) we now make the assumption that it is false. Then there exists a positive δ and a monotone increasing sequence $\{\alpha_j'\}$ such that

$$\| x(\alpha_j', x_0), \gamma_1 \| \geqslant \delta \tag{3.22}$$

and

$$\lim_{j\to\infty} \alpha_j' = 1. \tag{3.23}$$

Let now β be a positive number such that $g(x)$ is Lipschitz in $B(\gamma_1, 5\beta)$ (cf. Assumption 2.2) and such that

$$0 < 5\beta < \min(\delta, \zeta_0). \tag{3.24}$$

Then (3.21) is satisfied if we choose

$$\zeta_1 = \beta. \tag{3.25}$$

Now by (3.18) there exists an integer n_0 such that

$$x(\alpha_n, x_0) \in B(\gamma_1, \beta) \qquad \text{for } n \geqslant n_0 . \tag{3.26}$$

On the other hand, by (3.23), (3.22) and (3.24) there corresponds to each $n > n_0$ an integer $n' = n'(n)$ such that

$$\alpha_{n_0} < \alpha_n < \alpha'_{n'} < 1 \tag{3.27}$$

and

$$\| x(\alpha'_{n'}, x_0), \gamma_1 \| \geqslant 5\beta. \tag{3.28}$$

The proof of (3.20) will be finished by showing that for some $n' = n'(n)$ with $n > n_0$

$$x(\alpha'_{n'}, x_0) \in B(\gamma_1, 4\beta), \tag{3.29}$$

in contradiction to (3.28). To this end we note first that the relations (3.26)–(3.28) imply the existence of an $\alpha_n{}^*$ for which

$$0 < \alpha_{n_0} < \alpha_n < \alpha_n{}^* < \alpha'_{n'} < 1 \tag{3.30}$$

and

$$x(\alpha_n{}^*, x_0) \in \dot{B}(\gamma_1, 3\beta) \tag{3.31}$$

where $\dot{B}$ denotes the boundary of B. Then by (3.30) and (3.19)

$$\lim_{n\to\infty} \alpha_n{}^* = 1. \tag{3.32}$$

We now consider the ball $B_n = B(x(\alpha_n{}^*, x_0), \beta)$. Obviously

$$B_n \subset B(\gamma_1, 5\beta) = B(\gamma_1, \beta).$$

It therefore follows from our first assumptions on β and from (3.21) that the right member of (3.14) is Lipschitz in B_n and there bounded by a constant independent of n. If M is such a constant it follows from the definition of $x(\alpha, x_0)$ as solution of the differential equation (3.14) that

$$\| x(\alpha, x_0), x(\alpha_n{}^*, x_0) \| < M \mid \alpha - \alpha_n{}^* \mid, \tag{3.33}$$

if

$$\mid \alpha - \alpha_n{}^* \mid < \beta/M. \tag{3.34}$$

We now choose a fixed $n > n_0$ such that $0 < 1 - \alpha_n{}^* < \beta/M$. This choice is possible by (3.32) and (3.30). It then follows from (3.30) that (3.34), and therefore (3.33), is satisfied with $\alpha = \alpha'_{n'}$. Thus $x(\alpha'_{n'}, x_0) \in B_n$ which obviously implies (3.29).

LEMMA 3.12. *Under the assumptions of Lemma* 3.11 *the set* $\bar{f}_c$ *is a deformation retract of* $\bar{f}_b$.

Proof. Let $x(\alpha, x_0)$ be as in Lemma 3.11 and let

$$\delta(x_0, \alpha) = \begin{cases} x(\alpha, x_0), \quad \text{if} \quad x_0, & \text{if} \quad x_0 \in \bar{f}_b - f_c, \quad 0 \leqslant \alpha < 1, \\ \lim\limits_{\alpha\to 1-} x(\alpha, x_0), & \text{if} \quad x_0 \in \bar{f}_b - f_c, \quad \alpha = 1, \\ x_0, & \text{if} \quad x_0 \in f_c, \quad 0 \leqslant \alpha \leqslant 1. \end{cases}$$

$\delta(x_0, \alpha)$ obviously retracts $\bar{f}_b$ onto $\bar{f}_c$, (for the continuity of $\delta(x_0, \alpha)$ cf. the appendix in [14]).

We now turn to the proof of Theorem 3.3. From Lemma 3.12 and Definition 3.2 we see that $C_q(c) \approx H_q(\mathring{f}_c, \mathring{f}_a)$ if a is such that c is the only critical level in $[a, b]$. From this the first part of (3.13) follows since f_c can be deformed into $\mathring{f}_a$ (cf. Lemma 3.7).

Now the set $\mathring{f}_c$ can be deformed into the set $f_c \cup \Gamma(c)$ by the deformation given by the solution of (2.22). Therefore

$$C_q(c) \approx H_q(\mathring{f}_c, f_c) \approx H_q(f_c \cup \Gamma(c), f_c). \tag{3.35}$$

For $i = 1, 2,..., r$ let now B_i be a ball with center γ_i in whose closure γ_i is the only critical point, (for the proof of the existence of such a ball cf. the argument for the existence of ζ_0 in the paragraph following (3.21)). If we let $W = \bigcup_i B_i$ and excise the set $f_c \cup \Gamma(c) - W$ from the couple at the right of (3.35) we see from (3.35) and the excision theorem [5, VIII, 9.1] that

$$C_q(c) \approx H_q(f_c \cap W \cup \Gamma(c), f_c \cap W).$$

But the group at the right is isomorphic to the direct sum of the groups $H_q(f_c \cap B_i \cup \gamma_i, f_c \cap B_i)$ as is seen from the addition theorem [5, I, 13.2] and the definition of W. By Definition 3.4 this proves the second part of assertion (3.13).

We now turn to a discussion of the Morse inequalities (3.12) in the case that all critical points are isolated.

THEOREM 3.4. *Using the notations introduced in the paragraph immediately preceding Theorem* 3.1 *we suppose that the critical set $\Gamma(c_\alpha)$ at level c_α consists of a finite number of points $\gamma_\alpha{}^i$ ($i = 1, 2,..., r_\alpha$, $\alpha = 1, 2,..., N$). Let $m_\alpha{}^i$ denote the rank of $C_q(\gamma_\alpha{}^i)$ (see Definition* (3.5)). *We suppose moreover that the groups $C_q(\gamma_\alpha{}^i)$ are finitely generated. Then the Morse inequalities* (3.12) *hold with*

$$M_q = \sum_{\alpha=1}^{N} \sum_{i=1}^{r_\alpha} m_\alpha{}^i.$$

Proof. The theorem is an immediate consequence of Theorems 3.2 and 3.3.

Remark. Suppose all $\gamma_\alpha{}^i$ are non degenerate. (For the definition of non degeneracy and of the index of a nondegenerate critical point see, e.g., [8, p. 307]). Then the critical group $C_q(\gamma_\alpha{}^i)$ is isomorphic to the coefficient group if q equals the index of $\gamma_\alpha{}^i$, and 0 otherwise as proved in [8, p. 336] (for a different proof see [15; Theorem 2.1 and Corollary to Theorem 2.2]). It follows that the conclusion of the preceding theorem is valid in this case. It follows moreover that M_q equals the number of critical points of index q.

Cf. [8, p. 338] where the Morse relations are proved in the case of non-degeneracy if the manifold is without boundary and the coefficient group is a field.

For another case in which the $C_q(\gamma_\alpha{}^i)$ are finitely generated see [14, Theorem 7.3].

4. A Lusternik–Schnirelman Theorem

We first recall some basic definitions. Let A be a subset of the topological space X. Then cat (A, X), the category of A with respect to X is defined as follows: $\text{cat}(A, X) = 1$ if A in contractible on X to a point of X; $\text{cat}(A, X) = k$ if k is the smallest integer such that A can be covered by k closed sets each of which is of category 1 with respect to X; if no such k exists then $\text{cat}(A, X) = \infty$.

For positive integer $k \leqslant \text{cat}(X, X)$ and real valued f with domain X the Lusternik–Schnirelman number $m_k = m_k(f, X)$ is defined as follows: let S_k be the family of those subsets A of X for which $\text{cat}(A, X) \geqslant k$. Then

$$m_k = \inf_{A \in S} \sup_{x \in A} f(x).$$

Theorem 4.1. *Let f and V satisfy the Assumptions* 2.1–2.4. *In addition f is supposed to be bounded below. Then*

(i) *each finite m_k is a stationary value of f,*

(ii) *the number of stationary points is not smaller that* $\text{cat}(X, X)$,

(iii) *if $m_k = m_{k+1} = \cdots m_{k+n}$ and m_k is finite then* $\text{cat}(\Gamma(m_k), X) \geqslant n + 1$. *Here $\Gamma(m_k)$ denotes the set of stationary points at level m_k.*

Proof. The following facts were proved earlier or follow directly from the definitions involved; $\bar{V}$ is a metrizable absolute neighborhood retract (Theorem 2.3). The intersection of the set of stationary points with $f^{-1}[a, b]$ where $[a, b]$ is a finite interval is compact (Lemma 2.8). Each point of $\bar{V}$ has a neighborhood contractible to that point (Corollary to Lemma 2.3). But these facts together with Lemmas 2.11 and 2.12 are known to ensure the validity of our assertion (See [2, Theorems 2 and 3].)

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