JOURNAL OF ALGEBRA 172, 78–89 (1995)

On a Generalization of Azumaya's Exact Rings*

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Communicated by Kent R. Fuller

Received March 15, 1993

INTRODUCTION

Azumaya [A] introduced (Artinian) exact rings, and Camillo *et al.* [CFH] adapted this exactness to bimodules (with a composition series). Now we define (arbitrary) exact bimodules. Let R and S be two rings. We call a bimodule ${}_RM_S$ an *exact bimodule* in the case that for any submodules $N < L \leqq {}_RM_S$ if ${}_R(L/N)_S$ is simple then it is balanced. A ring R is called an *exact ring* in the case that the bimodule ${}_RR_R$ is exact. If a bimodule ${}_RM_S$ has a composition series, then our exactness coincides with that given in [CFH]. In particular, if R is either left or right Artinian, then exact rings are just the exact rings studied by Azumaya [A].

This paper consists of three sections. In the first section, some basic properties of exact bimodules and exact rings are obtained; e.g., we observe that exact bimodules and exact rings are preserved under Morita equivalence, and that the n by n upper triangular matrix rings over an exact ring are still exact rings. In the second section, we prove that a Noetherian and locally distributive semiperfect ring R is an exact ring, and that R has a Morita duality if it is linearly compact. In the final section, we show that another class of exact rings, namely linearly compact duo rings, also have Morita duality, which generalizes a recent result of Ánh [A1]. Our generalization is based on a result of [A2] and some ideas of [A1].

Throughout this paper, all rings are associative and have identity elements and all modules are unitary. We freely use terminologies and notations of [AF].

1. EXACT BIMODULES AND EXACT RINGS

Modifying the proof given in [CFH], we note the following

PROPOSITION 1.1. *Let F*: Mod-S → Mod-T *define a Morita equivalence. Then* (1) *a bimodule ${}_RM_S$ is exact if and only if the bimodule ${}_RF(M)_T$ is exact, and* (2) *S is an exact ring if and only if T is an exact ring.*

*This research is supported by National Education Commission of China.

0021-8693/95 $6.00

COROLLARY 1.2. *Let ${}_RM_S$ be a bimodule. If S is a semiperfect ring with a basic idempotent v, then ${}_RM_S$ is exact if and only if the bimodule ${}_RMv_{vSv}$ is exact.*

Proof. Since $\mathrm{Hom}_S(vS, -)$: Mod-$S \to$ Mod-vSv defines a Morita equivalence, the bimodule ${}_RM_S$ is exact if and only if the bimodule ${}_RMv_{vSv} \cong \mathrm{Hom}_S(vS, M)$ is exact by Proposition 1.1.

However, for an arbitrary idempotent e of S, we have the following result which is a generalization of [CFH, Lemma 1.4].

LEMMA 1.3. *Let ${}_RM_S$ be an exact bimodule and e an idempotent of S such that $Me \neq 0$. Then the bimodule ${}_RMe_{eSe}$ is exact too.*

Proof. Let A be a submodule of ${}_RMe_{eSe}$ and B a maximal submodule of A; i.e., ${}_R(A/B)_{eSe}$ is a simple bimodule. Let $L = AS$. Then L is clearly a submodule of ${}_RM_S$ and we have

$$Le = ASe = AeSe = A.$$

Consider the family F of those submodules Y of ${}_RL_S$ which satisfy $Ye = B$; F is non-empty because $Y = BS$ satisfies

$$Ye = BSe = BeSe = B.$$

Let

$$Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq \cdots$$

be an ascending chain of submodules in F. Then $\cup_i Y_i$ is a submodule of ${}_RL_S$ and $(\cup_i Y_i)e = \cup_i Y_i e = B$. Therefore, by Zorn's lemma, there exists a maximal member, say N in F. Let X be a submodule of ${}_RL_S$ which properly contains N. Then $B = Ne \subseteq Xe \subseteq Le = A$, and $B \neq Xe$ by the maximality of N. Since B is a maximal submodule of ${}_RA_{eSe}$ and Xe is clearly a submodule of ${}_RA_{eSe}$, it follows that $Xe = A$ and hence $XeS = AS = L$. This, together with the fact that $XeS \subseteq X \subseteq L$, implies that $X = L$. Thus we know that N is a maximal submodule of ${}_RL_S$, i.e., ${}_R(L/N)_S$ is a simple bimodule. Since ${}_RM_S$ is an exact bimodule, it follows that ${}_R(L/N)_S$ is balanced. Therefore, by [CFH, Lemma 1.4] (or virtually, by what is proved in the latter half of the lemma), the bimodule ${}_R(L/N)e_{eSe}$ is balanced. If we observe that $Le \cap N = Ne$ (since e is an idempotent), we have that

$$(L/N)e = (Le + N)/Ne \cong Le/(Le \cap N) = Le/Ne = A/B$$

as $(R - eSe)$-bimodules. Thus ${}_R(A/B)_{eSe}$ is balanced, which proves that ${}_RMe_{eSe}$ is exact.

COROLLARY 1.4. *If R is an exact ring and e is a non-zero idempotent of R then eRe is an exact ring too.*

Now we consider exactness related to Morita duality. A presentation of Morita duality can be found in [AF, Sections 23 and 24] or [X3]. Using a slightly different approach, we generalize [X2, Lemma 4] as follows. (The condition that ${}_RM$ is Noetherian should be added to [X2, Lemma 4 and Theorem 5].)

LEMMA 1.5. *Let ${}_RU_S$ define a Morita duality. If ${}_RM_T$ is an exact bimodule and ${}_RM$ is a Noetherian module, then the bimodule ${}_TM_S^* = \mathrm{Hom}_R(M, U)$ is exact.*

Proof. Suppose $N < L \leqq {}_TM_S^*$ and ${}_T(L/N)_S$ is simple. Then ${}_R(L/N)_T^*$ is a simple bimodule. By the exactness of ${}_RM_T \cong {}_RM_T^{**}$, we see that the simple bimodule ${}_R(L/N)_T^*$ is balanced. Since ${}_R(L/N)^*$ is a factor module of a submodule of the Noetherian module ${}_RM^{**}$, ${}_R(L/N)^*$ is Noetherian too, and then $J(R)(L/N)^*$ is a proper submodule of ${}_R(L/N)_T^*$. Hence $J(R)(L/N)^* = 0$ and ${}_R(L/N)^*$ is semisimple. It follows that ${}_R(L/N)^*$ has finite length, and then $(L/N)_S$ has finite length too. Now let $f \in \mathrm{End}((L/N)_S)$. Then $f^* = \mathrm{Hom}_S(f, U) \in \mathrm{End}({}_R(L/N)^*)$, so there exists a $t \in T$ such that $f^*(x) = xt$ for all $x \in (L/N)^*$. It follows that $f^{**}(y) = ty$ for all $y \in (L/N)^{**}$. Hence $f(z) = tz$ for all $z \in L/N$. By [CFH, Proposition 1.1 (a) $\Rightarrow$ (b)], ${}_T(L/N)_S$ is balanced, and so ${}_TM_S^*$ is an exact bimodule.

COROLLARY 1.6. *Let ${}_RU_S$ define a Morita duality. Then*

(1) if R is exact and left Noetherian then the bimodule ${}_RU_S$ is exact; and

(2) if R is an exact Artinian ring then so is S.

We do not know if the condition that ${}_RM$ is Noetherian in Lemma 1.5 can be weakened to that ${}_RM$ is U-reflexive. Similarly, we do not know if the Noetherian and Artinian conditions in Corollary 1.6 can be deleted.

The following result was observed in [CFH] for a bimodule with a composition series.

PROPOSITION 1.7. *Let $X \leqq {}_RM_S$. Then ${}_RM_S$ is exact if and only if both ${}_RX_S$ and ${}_R(M/X)_S$ are exact.*

Proof. ($\Rightarrow$) Clearly.

($\Leftarrow$) Suppose $N < L \leqq {}_RM_S$ and ${}_R(L/N)_S$ is simple. We have a canonical monomorphism of $(R - S)$-bimodules

$$f: (L \cap X)/(N \cap X) \hookrightarrow L/N$$

and a canonical epimorphism of $(R-S)$-bimodules

$$g: L/N \to (L+X)/(N+X).$$

If $(L \cap X)/(N \cap X) \neq 0$, f is an isomorphism, and then ${}_R((L \cap X)/(N \cap X))_S$ is simple. Since ${}_RX_S$ is exact, ${}_R((L \cap X)/(N \cap X))_S$ is balanced, and so is ${}_R(L/N)_S$.

If $(L + X)/(N + X) \neq 0$, g is an isomorphism, and then ${}_R((L+X)/(N+X))_S$ is simple. Since ${}_R(M/X)_S$ is exact, ${}_R((L+X)/(N+X))_S$ is balanced, and so is ${}_R(L/N)_S$.

If both $L \cap X = N \cap X$ and $L + X = N + X$, the modularity asserts that $L = N$, which is a contradiction.

COROLLARY 1.8. *Let R be a ring with an R-bimodule ${}_RM_R$. If $R \propto M$ is the trivial extension ring of R by M, then $R \propto M$ is an exact ring if and only if ${}_RM_R$ is an exact bimodule and R is an exact ring.*

COROLLARY 1.9. *Let ${}_RM_S$ be a bimodule. Then the formal triangular matrix ring $T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ is an exact ring if and only if ${}_RM_S$ is an exact bimodule and both R and S are exact rings. In particular, the n by n upper triangular matrix rings over an exact ring are still exact rings.*

2. LOCALLY DISTRIBUTIVE RINGS

In this section, we shall prove two results for locally distributive rings (Theorems 2.6 and 2.9). To do so, we need some preparation. Let S be a semiperfect ring with (Jacobson) radical J. Two idempotents e and f in S are said to be *isomorphic* in case $Se \cong Sf$ as left-S-modules; equivalently, $eS \cong fS$ as right-S-modules. In this event, if ${}_RM_S$ is a bimodule then ${}_RMe \cong {}_RM \otimes_S Se \cong {}_RM \otimes_S Sf \cong {}_RMf$, hence $Me \neq 0$ if and only if $Mf \neq 0$.

LEMMA 2.1. *Let ${}_RM_S$ be a simple bimodule over a ring R and a basic semiperfect ring S. Suppose that M_S is annihilated by the radical J of S: $MJ = 0$. Then there is a primitive idempotent e of S such that $Me \neq 0$, or equivalently $Me = M$. For this e, ${}_RM_{eSe}$ is simple, and it is balanced if and only if ${}_RM_S$ is balanced. Moreover, M is simple even as a left R-module.*

Proof. That M satisfies $MJ = 0$ implies that ${}_RM_S$ can be regarded as a bimodule ${}_RM_{\bar{S}}$ in the natural manner, where $\bar{S} = S/J$. Since S is basic, 1 is a sum of pairwise orthogonal and non-isomorphic primitive idempotents $e_1, \ldots, e_n$ of S. Then, taking cosets mod J, $\bar{1} = \bar{e}_1 + \cdots + \bar{e}_n$ with pairwise orthogonal and non-isomorphic primitive central idempotents

$\bar{e}_1, \ldots, \bar{e}_n$ of $\bar{S}$. Let $Me_i \neq 0$ for some e_i. Then $Me_i = M\bar{e}_i = M\bar{S}\bar{e}_i = M\bar{e}_i\bar{S}$, which is a submodule of the simple bimodule ${}_RM_S$ and so we have $Me_i = M$. From this it follows that $Me_j = Me_ie_j = 0$ for $j \neq i$. We now put $e = e_i$ for simplicity. Then $M = Me = M\bar{e}$ implies that M can be regarded either as a bimodule ${}_RM_{eSe}$ or as a bimodule ${}_RM_{\bar{e}\bar{S}\bar{e}}$. Let x be any element of $M = Me$. Then $x = xe$. Therefore, for every $s \in S$, we have $sx = xes = xese = x\bar{e}\bar{s}\bar{e}$. From these equalities it follows immediately that (a) the bimodules ${}_RM_{eSe}$ and ${}_RM_{\bar{e}\bar{S}\bar{e}}$ are also simple, (b) ${}_RM_S$ is right balanced if and only if ${}_RM_{eSe}$ is right balanced if and only if ${}_RM_{\bar{e}\bar{S}\bar{e}}$ is right balanced, and (c) the endomorphism rings of the right modules M_S, M_{eSe}, and $M_{\bar{e}\bar{S}\bar{e}}$ are exactly the same and therefore ${}_RM_S$ is left balanced if and only if ${}_RM_{eSe}$ is left balanced if and only if ${}_RM_{\bar{e}\bar{S}\bar{e}}$ is left balanced. So assume now that ${}_RM_S$, or equivalently ${}_RM_{\bar{e}\bar{S}\bar{e}}$, is left balanced. Let x be any non-zero element of M. Then, since $\bar{e}\bar{S}\bar{e}$ is a division ring, $x\bar{e}\bar{S}\bar{e}$ is a direct summand of the right module $M_{\bar{e}\bar{S}\bar{e}}$. Let y be any element of M. Then there is an $\bar{e}\bar{S}\bar{e}$-homomorphism $x\bar{e}\bar{S}\bar{e} \to M$ which maps x to y (because $\bar{e}\bar{S}\bar{e}$ is a division ring). So it can be extended to an endomorphism of $M_{\bar{e}\bar{S}\bar{e}}$. Since ${}_RM_{\bar{e}\bar{S}\bar{e}}$ is left balanced, the endomorphism is induced by the left multiplication of an element r of R so that we have $rx = y$. This shows that ${}_RM$ is a simple left module.

PROPOSITION 2.2. *Let ${}_RM_S$ be a simple bimodule over a ring R and a semiperfect ring S. Suppose that either M_S is finitely generated or S is one-sided perfect. Then there is a primitive idempotent e of S such that $Me \neq 0$. For such an e, the bimodule ${}_RMe_{eSe}$ is simple, while ${}_RMe_{eSe}$ is balanced if and only if ${}_RM_S$ is balanced. If f is another primitive idempotent of S then $Mf \neq 0$ if and only if e and f are isomorphic if and only if ${}_RMe \cong {}_RMf$.*

Proof. Since J is an ideal of S, MJ is a submodule of the simple bimodule ${}_RM_S$. So we have either $MJ = M$ or $MJ = 0$. Hence we must have $MJ = 0$ if M_S is finitely generated. If S is right perfect, $MJ \neq M$ by [AF, Lemma 28.3], hence $MJ = 0$ in this case. If S is left perfect, $\mathrm{Soc}(M_S) \neq 0$ by [AF, Theorem 28.4], hence $MJ = 0$ in this case too. We have thus shown that $MJ = 0$ under our assumption. But this means that M can be regarded as a bimodule ${}_RM_{\bar{S}}$ in the natural manner. Since S is semiperfect, $Me \neq 0$ for some primitive idempotent e of S. Then $MeS = M$ since ${}_RM_S$ is simple. Let f be another primitive idempotent of S which is not isomorphic to e. Then $\bar{e}$ and $\bar{f}$ are also non-isomorphic primitive idempotents of $\bar{S}$, which implies that $\bar{e}\bar{S}\bar{f} \cong \mathrm{Hod}_{\bar{S}}(\bar{S}\bar{e}, \bar{S}\bar{f}) = 0$. Thus we know that $Mf = MeSf = M\bar{e}\bar{S}\bar{f} = 0$. It follows from this that if f is a primitive idempotent of S such that $Mf \neq 0$ then e and f are isomorphic. Conversely, if e and f are isomorphic then, as we mentioned at the beginning of this section, we have ${}_RMe \cong {}_RMf$. Clearly, if ${}_RMe \cong {}_RMf$ then $Mf \neq 0$.

Let e be a primitive idempotent of S such that $Me \neq 0$ as above. As is easily seen, there is a basic idempotent v of S such that e is contained in the basic ring vSv, or equivalently $ve = ev = e$. Consider then the simple bimodule ${}_RMv_{vSv}$. By Corollary 1.2, ${}_RM_S$ is balanced if and only if ${}_RMv_{vSv}$ is balanced. On the other hand, since $MJ = 0$, it follows that $MvvJv = 0$. But vJv is the radical of vSv and besides $Mve = Me \neq 0$. Therefore we can apply Lemma 2.1 to Mv and vSv instead of M and S respectively to know that ${}_RMe_{eSe} = {}_RMve_{evSve}$ is simple and ${}_RMe_{eSe}$ is balanced if and only if ${}_RMv_{vSv}$ is balanced. This completes the proof.

PROPOSITION 2.3. *Let ${}_RM_S$ be a bimodule over a ring R and a semiperfect ring S. Suppose that M_S is Noetherian or S is one-sided perfect. Then the following are equivalent:*

(1) *${}_RM_S$ is exact;*

(2) *${}_RMe_{eSe}$ is exact for any primitive idempotent e of S such that $Me \neq 0$;*

(3) *For every primitive idempotent e of S for which $Me \neq 0$ there exists a primitive idempotent f in S such that e is isomorphic to f and ${}_RMf_{fSf}$ is exact.*

Proof. (1) ⇒ (2) by Lemma 1.3. (2) ⇒ (3) is clear. (3) ⇒ (1). Suppose $Y < X \leq {}_RM_S$ and ${}_R(X/Y)_S$ is simple. Then by Proposition 2.2 there exists a primitive idempotent e in S such that $(X/Y)e \neq 0$ whence $Me \neq 0$. Then by the assumption, we can find a primitive idempotent f in S such that e and f are isomorphic, or equivalently, $(X/Y)f \neq 0$ whence $Mf \neq 0$ and ${}_RMf_{fSf}$ is exact. Since $(X/Y)f \neq 0$, ${}_R(X/Y)f_{fSf}$ is simple by Proposition 2.2. But that ${}_R(X/Y)f_{fSf} \cong {}_R(Xf/Yf)_{fSf}$ and ${}_RMf_{fSf}$ is exact implies that ${}_R(X/Y)f_{fSf}$ is balanced. Therefore ${}_R(X/Y)_S$ is balanced again by Proposition 2.2.

COROLLARY 2.4. *Let R be a Noetherian semiperfect ring or a one-sided perfect ring. Then the following are equivalent:*

(1) *R is exact;*

(2) *${}_{eRe}eRf_{fRf}$ is exact or 0 for all primitive idempotents e and f of R;*

(3) *there is a basic set of primitive idempotents $e_1, \ldots, e_n$ of R such that ${}_{e_iRe_i}(e_iRe_j)_{e_jRe_j}$ is exact or zero for all $i, j \in \{1, \ldots, n\}$.*

COROLLARY 2.5. *Let ${}_RM_S$ be a bimodule over a ring R and a semiperfect ring S. Suppose that M_S is Noetherian or S is one-sided perfect. Suppose that*

e is a non-zero idempotent of S and $1-e \neq 0$, i.e., $Me \neq 0$ and $M(1-e) \neq 0$. Then ${}_RM_S$ is exact if and only if both ${}_RMe_{eSe}$ and ${}_RM(1-e)_{(1-e)S(1-e)}$ are exact.

Proof. The "only if" is a consequence of Lemma 1.3. So assume that both ${}_RMe_{eSe}$ and ${}_RM(1-e)_{(1-e)S(1-e)}$ are exact. Let $e = e_1 + \cdots + e_k$ and $1 - e = e_{k+1} + \cdots + e_n$ be decompositions of e and $1-e$ as sums of pairwise orthogonal primitive idempotents. Then by Lemma 1.3 again we see that each ${}_R(Me_i)_{e_iSe_i}$ is exact or zero, $1 \leq i \leq n$. Let e be any primitive idempotent of S such that $Me \neq 0$. Then there is an e_i which is isomorphic to e. Then $Me_i \neq 0$ and ${}_R(Me_i)_{e_iSe_i}$ is exact. By Proposition 2.3, ${}_RM_S$ is exact.

A module is called *distributive* in the case that its lattice of submodules is distributive. Fuller [F, Lemma 4] proved that a module ${}_RM$ over a semiperfect ring R is distributive if and only if ${}_{eRe}eM$ is uniserial for each primitive idempotent e of R. A ring R is called *locally distributive* in the case the modules ${}_RR$ and R_R are direct sums of the distributive modules.

It is proved in [CFH] that Artinian locally distributive rings are exact. A slight generalization is given as follows.

THEOREM 2.6. *If R is a Noetherian and locally distributive semiperfect ring, then R is exact.*

Proof. Let e and f be primitive idempotents of R such that $eRf \neq 0$. Suppose $N < L \leq {}_{eRe}eRf_{fRf}$ and ${}_{eRe}(L/N)_{fRf}$ is simple. Since R is Noetherian, both ${}_{eRe}eRf$ and eRf_{fRf} are Noetherian, and hence both ${}_{eRe}(L/N)$ and $(L/N)_{fRf}$ are finitely generated. Therefore we have $eJe(L/N) \neq L/N$, where J and so eJe are radicals of R and eRe respectively. But since $eJe(L/N)$ is a submodule of the simple bimodule ${}_{eRe}(L/N)_{fRf}$, it follows that $eJe(L/N) = 0$. This means that L/N can be regarded as a left module over the division ring $\bar{e}\bar{R}\bar{e} = eRe/eJe$. On the other hand, since ${}_RRf$ is distributive, ${}_{eRe}eRf$ whence ${}_{eRe}(L/N) = {}_{\bar{e}\bar{R}\bar{e}}(L/N)$ is uniserial by [F, Lemma 4]. Since L/N is a vector space over the division ring $\bar{e}\bar{R}\bar{e}$, equivalently the left module ${}_{eRe}(L/N)$ is one-dimensional over $\bar{e}\bar{R}\bar{e}$, or equivalently the left module ${}_{eRe}(L/N)$ is simple. Similarly we can see that $(L/N)_{fRf}$ can be regarded as a right vector space over the division ring $\bar{f}\bar{R}\bar{f} = fRf/fJf$, and indeed L/N is one-dimensional over $\bar{f}\bar{R}\bar{f}$ or equivalently $(L/N)_{fRf}$ is simple. Therefore it follows from [CFH, Lemma 2.1($\Leftarrow$)] that the bimodule ${}_{eRe}(L/N)_{fRf}$ is balanced. Thus ${}_{eRe}eRf_{fRf}$ is exact, and so R is exact by Corollary 2.4.

The next example shows that the one-sided Noetherian condition in Theorem 2.6 is not enough to insure exactness, even for a serial ring with self-duality.

EXAMPLE 2.7. Let F be a field. Let $F[[x]]$ be the ring of a formal power series ring of one indeterminate x over F, and let $F((x))$ be the quotient field of $F[[x]]$. Let $R = \begin{bmatrix} F((x)) & F((x)) \\ 0 & F[[x]] \end{bmatrix}$, which is a basic semiperfect ring. By [M] or [M2], R has self-duality. Since $F((x))$ is a uniserial module over $F[[x]]$, one notes that R is a (two-sided) serial ring. In the simple bimodule ${}_{F((x))}F((x))_{F[[x]]}$ we see that the canonical ring homomorphism $F[[x]] \to \mathrm{End}({}_{F((x))}F((x)))$ is not surjective, so the bimodule ${}_{F((x))}F((x))_{F[[x]]}$ is not exact. Hence R is not an exact ring by Corollary 1.9. We note that R is left Noetherian but not right Noetherian.

A bimodule ${}_RM_S$ is called *left* (*right*) *duo* in the case each R-submodule (S-submodule) of M is a bi-submodule, and it is called *duo* in the case it is both left and right duo. Clearly, ${}_RM_S$ is duo if and only if $Rm = mS$ for each $m \in M$. A ring R is called a *left duo* (*right duo*, *duo*) *ring* in the case the bimodule ${}_RR_R$ is left duo (right duo, duo).

LEMMA 2.8. *Let ${}_RM_T$ and $M = Rm = mT$ for some $m \in M$. If ${}_RU_S$ is a duo bimodule then ${}_T\mathrm{Hom}_R(M, U)_S$ is a duo bimodule.*

Proof. Let $f \in {}_T\mathrm{Hom}_R(M, U)_S$ and $t \in T$. There exists an $r \in R$ such that $mt = rm$, and so there exists an $s \in S$ such that $rf(m) = f(m)s$. Hence

$$(tf)(m) = f(mt) = f(rm) = rf(m) = f(m)s = (fs)(m).$$

So $tf = fs$. We have proved that $Tf \subseteq fS$. Similarly, we have $fS \subseteq Tf$.

A module M is called *linearly compact* (l.c.) in the case any finitely solvable congruence $m \equiv m_i \pmod{M_i}$ is solvable, where M_i's are submodules and $m_i \in M$. The class of l.c. modules includes Artinian modules. A ring R is left (right) linearly compact if the module ${}_RR$ (resp., R_R) is linearly compact. It is known (e.g., see [X3, Corollary 3.14]) that a left linearly compact ring is semiperfect. Müller's Theorem [M1] states that a ring R has a (left) Morita duality if and only if R is left l.c. and the minimal injective cogenerator in R-Mod is also l.c. If R is an exact left l.c. ring, we do not know if R has a Morita duality or R is also right l.c. But Azumaya [A] proved that an exact Artinian ring does have a Morita duality.

A locally distributive Artinian ring R is an exact ring, so it has a Morita duality [A]. The first result of the following theorem is a generalization, since an Artinian ring is both l.c. and Noetherian. The second result is a generalization of Fuller [F, Lemma 5] who proved this for locally distributive Artinian rings.

THEOREM 2.9. *Every Noetherian locally distributive l.c. ring R has a Morita duality. Moreover, the injective envelope of each simple left* (*resp., right*) *R-module is distributive.*

Proof. We may assume that R is a basic ring with a basic set of primitive idempotents $e_1, \ldots, e_n$. Let $R_i = e_iRe_i$. By [F, Lemma 4], both ${}_{R_i}(e_iRe_j)$ and $(e_iRe_j)_{R_j}$ are uniserial and Noetherian, hence there exists an $e_{ij} \in e_iRe_j$ such that $e_iRe_j = R_ie_{ij} = e_{ij}R_j$. It is known that the local l.c. Noetherian uniserial ring R_i has a Morita duality (see, e.g., [K, Theorem 7.8]). Now $R = \Sigma_{i,j}R_{ij}$ is a finite normalizing extension over its subring $\Sigma_i R_i$ which has a Morita duality. Therefore by [X3, Theorem 9.1], R has a Morita duality.

Let $J = J(R)$, the radical of R. Let $U_i = E(Re_i/Je_i)$, $i = 1, \ldots, n$. Then, ${}_{R_i}(e_iU_i)$ is the minimal injective cogenerator by [X3, Lemma 4.11] and R_i is a duo ring with a self-duality induced by ${}_{R_i}(e_iU_i)_{R_i}$, so ${}_{R_i}(e_iU_i)_{R_i}$ is a duo bimodule. By Lemma 2.8, $(e_iRe_j)^* = {}_{R_j}\mathrm{Hom}_{R_i}(e_iRe_j, e_iU_i)_{R_i}$ is a duo bimodule. Since ${}_{R_i}(e_iRe_j)$ is uniserial, so is $(e_iRe_j)^*_{R_i}$, and hence ${}_{R_j}(e_iRe_j)^*$ is uniserial. By [X3, Lemma 4.10], ${}_{R_j}(e_iRe_j)^* \cong \mathrm{Hom}_R(Re_j, U_i) \cong e_jU_i$. So ${}_{R_j}(e_jU_i)$ is uniserial, and ${}_RU_i$ is distributive by [F, Lemma 4].

In the above theorem, if we let ${}_RU = \oplus_{i=1}^n U_i$ and $S = \mathrm{End}({}_RU)$, then we know from Theorem 2.6, Corollary 1.4, and Theorem 2.9 that if R is an exact ring, the exact bimodule ${}_RU_S$ defines a Morita duality. By [AF, Theorem 24.5(3)], we see that the right S-modules fS and $E(fS/fN)$ are distributive for each primitive idempotent f in S, where $N = J(S)$, the radical of S. We do not know whether or not S is locally distributive. This is so if R is Artinian [FX].

3. Linearly Compact Duo Rings

If ${}_RM_S$ is a bimodule and $M = Rm = mS$ for some $m \in M$, we note that ${}_RM_S$ must be balanced. Hence any duo bimodule is an exact bimodule; in particular, every duo ring is an exact ring. Since each one-sided ideal in a duo ring is two-sided, a duo ring is left l.c. if and only if it is right l.c., so we simply speak of l.c. duo rings. If D is a non-commutative division ring then the power series ring $D[[x]]$ is a non-commutative non-Artinian l.c. duo ring.

Ánh [A1] has recently shown that each l.c. commutative ring has a Morita duality, thus giving an affirmative answer to a long-open problem of Müller [M1] and Zelinsky [Z]. In [X1] the author proved that an Artinian duo ring R has a Morita self-duality if $J(R)$ is a direct sum of colocal ideals. Here a module is *colocal* if it has a simple essential socle. In this final section, we generalize the above two results by establishing the following

Theorem 3.1. *Let R be a linearly compact duo ring. Then* (1) *R has a Morita duality, and* (2) *if $J(R)$ is a direct sum of colocal ideals then R has a Morita self-duality.*

Since each idempotent in a duo ring is central [X3, Lemma 12.2], a l.c. duo ring is a finite direct product of local l.c. duo rings. Therefore we assume in this section that

$$\begin{cases} R \text{ is a local l.c. duo ring with radical } J, \text{ and} \\ {}_RU = {}_RE(R/J) \text{ is the minimal injective cogenerator in } R\text{-Mod.} \end{cases}$$

LEMMA 3.2. *If the module ${}_RR$ is colocal, R is a (two-sided) PF-ring; i.e., the bimodule ${}_RR_R$ defines a Morita duality.*

Proof. Since R is duo, $\mathrm{Soc}(R_R) = \mathrm{Soc}({}_RR)$, which is a simple essential ideal of R. By [A2, Corollary 2], R is a PF-ring.

LEMMA 3.3. $\mathbf{r}_U\mathbf{l}_R(u_1, \ldots, u_n) = \sum_{i=1}^{n} Ru_i$, *where $u_1, \ldots, u_n \in U$.*

Proof. If $0 \neq u \in U$, we have $Ru \cong R/\mathbf{l}_R(u)$. Since Ru is colocal, $R/\mathbf{l}_R(u)$ is a PF-ring by Lemma 3.2. As an $R/\mathbf{l}_R(u)$-module, Ru is an injective submodule of $\mathbf{r}_U\mathbf{l}_R(u)$ which is colocal. We conclude that $Ru = \mathbf{r}_U\mathbf{l}_R(u)$ for each $u \in U$. Therefore

$$\begin{aligned} \mathbf{r}_U\mathbf{l}_R(u_1, \ldots, u_n) & \\ &= \mathbf{r}_U(\mathbf{l}_R(u_1) \cap \cdots \cap \mathbf{l}_R(u_n)) \\ &= \mathbf{r}_U\mathbf{l}_R(u_1) + \cdots + \mathbf{r}_U\mathbf{l}_R(u_n) \qquad (\text{since } {}_RU \text{ is injective}) \\ &= Ru_1 + \cdots + Ru_n. \end{aligned}$$

If V is a submodule of ${}_RU$, we let $F(V)$ denote the set of all finite subsets of V.

LEMMA 3.4. *If V is a submodule of ${}_RU$, then for each $u \in U$ we have the equality $(\bigcap_{N \in F(V)} \mathbf{l}_R(N))u = \bigcap_{N \in F(V)} (\mathbf{l}_R(N)u)$.*

Proof. The inclusion "$\subseteq$" is obvious. Now let $w = r_N u \in \bigcap_{N \in F(V)} (\mathbf{l}_R(N)u)$ where $r_N \in \mathbf{l}_R(N)$ for each $N \in F(V)$. If $N_1, \ldots, N_n$ are finitely many elements of $V(V)$, then $N = \bigcup_{i=1}^{n} N_i \in F(V)$ and $r_N \in \mathbf{l}_R(N) \subseteq \mathbf{l}_R(N_i)$ for each i. Hence $r_N - r_{N_i} \in \mathbf{l}_R(N_i)$ and $(r_N - r_{N_i})u = w - w = 0$ for each i. This shows that the system of the congruences $r \equiv r_N \pmod{\mathbf{l}_{\mathbf{l}_R(N)}(u)}$ is finitely solvable. Since R is l.c., there is an $r \in R$ such that $r - r_N \in \mathbf{l}_{\mathbf{l}_R(N)}(u)$ for each $N \in F(V)$. Hence $r - r_N \in \mathbf{l}_R(N)$ for each N, so $r \in \bigcap_{N \in F(V)} \mathbf{l}_R(N)$. And also $w = r_N u = ru \in (\bigcap_{N \in F(V)} \mathbf{l}_R(N))u$.

LEMMA 3.5. *If V is a submodule of ${}_RU$ then $\mathbf{r}_U\mathbf{l}_R(V) = V$.*

Proof. Let $u \in \mathbf{r}_U\mathbf{l}_R(V)$. Since

$$\mathbf{l}_R(V) = \bigcap_{v \in V} \mathbf{l}_R(v) = \bigcap_{N \in F(V)} \mathbf{l}_R(N),$$

by Lemma 3.4 we have

$$0 = \mathbf{l}_R(V)u = \Big(\bigcap_{N \in F(V)} \mathbf{l}_R(N)\Big)u = \bigcap_{N \in F(V)} (\mathbf{l}_R(N)u).$$

Since ${}_RU$ has a simple essential socle, we must have $\mathbf{l}_R(N)u = 0$ for some $N = \{v_1, \dots, v_n\} \in F(V)$. Then by Lemma 3.3, $u \in \mathbf{r}_U\mathbf{l}_R(v_1, \dots, v_n) = \Sigma_{i=1}^n Rv_i \subseteq V$.

LEMMA 3.6. *${}_RU$ is a l.c. module.*

Proof. If $u \equiv u_i \pmod{U_i}$ is a finitely solvable system of congruences of ${}_RU$, then the mapping

$$f\colon \sum_i \mathbf{l}_R(U_i) \to U \qquad \text{via } \sum r_i \mapsto \sum r_i u_i$$

is well-defined and an R-homomorphism. The injectivity of ${}_RU$ implies that there exists a $u \in U$ such that $f(r) = ru$ for all $r \in \Sigma_i \mathbf{l}_R(U_i)$. In particular, if $r \in \mathbf{l}_R(U_i)$, then $ru = f(r) = ru_i$. Hence $u - u_i \in \mathbf{r}_U\mathbf{l}_R(U_i) = U_i$ (Lemma 3.5) for each i. This proves that ${}_RU$ is l.c.

The following result follows from Müller's theorem [M1], which we mentioned in Section 2.

PROPOSITION 3.7. *If $S = \mathrm{End}({}_RU)$, then ${}_RU_S$ defines a Morita duality.*

Remark 3.8. Using Lemma 3.3 and [AF, Theorem 24.5], we have $Ru = \mathbf{r}_U\mathbf{l}_R(u) = \mathbf{r}_U\mathbf{l}_R(uS) = uS$, for each $u \in U$. Hence the bimodule ${}_RU_S$ is duo. By [1, Theorem 24.5] again, S is a right duo ring. We conjecture that S is a duo ring.

Proof of Theorem 3.1. We may assume that R is local. By Proposition 3.7 and Remark 3.8, the duo bimodule ${}_RU_S$ defines a Morita duality and S is a right duo ring. To prove the second assertion, let $J = \oplus_{i=1}^n I_i$ be a direct sum of colocal ideals. Then there exist $u_1, \dots, u_n \in U$ such that $U_S/\mathrm{Soc}(U_S) = \oplus_{i=1}^n \bar{u}_i S$. Hence $U = \Sigma_{i=1}^n u_i S = \Sigma_{i=1}^n Ru_i$. We may assume that $n > 1$. (If $n = 1$, R is a PF-ring by Lemma 3.2.) Now applying the proof of [X1, Theorem 3], we conclude that $R \cong S$.

Acknowledgments

Parts of the paper were completed while the author visited the University of Iowa in the fall of 1992. The author sincerely thanks Professor Kent R. Fuller for his helpful suggestions, and the referee for many improvements and simplifications.

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