



Dimensionality in the Freund–Rubin cosmology

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Abstract

In the n -dimensional Freund–Rubin model with an antisymmetric tensor field of rank $s - 1$, the dimension of the external spacetime we live in must be $\min(s, n - s)$. This result is a generalization of the previous result in the $d = 11$ supergravity case, where $s = 4$.

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The history of Kaluza–Klein models is almost as long as that of General Relativity. The idea of dimensional reduction has been revived many times, for example, in the context of nonabelian gauge theory, extended supergravity, and most recently, M -theory or brane cosmology.

Traditionally, it is assumed that in the Kaluza–Klein models, the n -dimensional spacetime is a product of a s -dimensional manifold M^s and a $(n - s)$ -dimensional manifold M^{n-s} . Many studies have been done to show how to decompose M into the product of an internal and an external space in the classical framework. The key problem is to identify the external spacetime in which we are living. Many works appeal to the Anthropic Principle [1]: there may exist five or more dimensions, however, only in the 4-dimensional

nearly flat spacetime we, the observers, would be able to exist.

In this Letter we shall argue that, in the framework of quantum cosmology, this problem can be solved in some toy models without using the Anthropic Principle.

The quantum state of the universe is described by its wave function Ψ . In the no-boundary universe [3], the wave function is defined by the path integral over all compact manifolds with the argument of the wave function as the only boundary. The main contribution to the path integral comes from the instanton solution. This is the so-called WKB approximation. Therefore, the instanton can be thought as the seed of the universe.

Let us study the following Freund–Rubin toy models [2]. The matter content of the universe is an antisymmetric tensor field $A^{\alpha_1 \dots \alpha_{s-1}}$ of rank $s - 1$. Its field strength is a completely antisymmetric tensor $F^{\alpha_1 \dots \alpha_s}$. If $s = 2$, then the matter field is Maxwell. The

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Lorentzian action can be written as

$$I_{\text{Lorentz}} = \frac{1}{16\pi} \int_M \left(R - 2\Lambda - \frac{8\pi}{s} F^2 \right) + \frac{1}{8\pi} \int_{\partial M} K, \quad (1)$$

where Λ is the cosmological constant, R is the scalar curvature of the spacetime M and K is the extrinsic curvature of its boundary ∂M .

The Einstein equation is

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu} = 8\pi \theta^{\mu\nu}, \quad (2)$$

where the energy–momentum tensor $\theta^{\mu\nu}$ is

$$\theta^{\mu\nu} = F_{\alpha_1 \dots \alpha_{s-1}}{}^{\mu} F^{\alpha_1 \dots \alpha_{s-1} \nu} - \frac{1}{2s} F_{\alpha_1 \dots \alpha_s} F^{\alpha_1 \dots \alpha_s} g^{\mu\nu}. \quad (3)$$

The field equation is

$$g^{-1/2} \partial_\mu (g^{1/2} F^{\mu\alpha_2 \dots \alpha_s}) = 0. \quad (4)$$

We use indices m, \dots for the manifold M^s and \bar{m}, \dots for M^{d-s} , respectively. We assume that M^s and (M^{d-s}) are topologically spheres, and only components of the field F with all unbarred indices can be nonzero. From de Rham cohomology, there exists unique harmonics in S^s [4], i.e., the solution to the field equation (4)

$$F^{\alpha_1 \dots \alpha_s} = \kappa \epsilon^{\alpha_1 \dots \alpha_s} (s! g_s)^{-1/2}, \quad (5)$$

where g_s is the determinant of the metric of M_s , κ is a charge constant. We set κ to be imaginary, for this moment.

We first consider the case $\Lambda = 0$. From above one can derive the scalar curvature for each factor space

$$R_s = \frac{(n-s-1)8\pi\kappa^2}{n-2} \quad (6)$$

and

$$R_{n-s} = -\frac{(s-1)(n-s)8\pi\kappa^2}{s(n-2)}. \quad (7)$$

It appears that the F field behaves as a cosmological constant, which is anisotropic with respect to the factor spaces.

The metrics of the factor spacetimes should be Einstein. The created universe would select the manifolds

with maximum symmetry. This point can be justified in quantum cosmology. As we shall show below, at the WKB level, the relative creation probability of the universe is exponential to the negative of the Euclidean action of the seed instanton. The action is proportional to the product of the volumes of the two factor manifolds. Maximization of the volumes can be realized only by the manifolds with maximum symmetries. Therefore, the instanton metric is a product of $S^s \times S^{n-s}$. The metric signature of $S^s (S^{n-s})$ is negative (positive) definite. This is the instanton version of the Freund–Rubin solution [2].

To obtain the Lorentzian spacetime, one can begin with the S^s metric

$$ds_s^2 = -dt^2 - \frac{\sin^2(L_s t)}{L_s^2} (d\chi^2 + \sin^2 \chi d\Omega_{s-2}^2), \quad (8)$$

where L_s is the radius of the S^s and $d\Omega_{s-2}^2$ represents the unit $(s-2)$ -sphere.

One can obtain the s -dimensional anti-de Sitter space by an analytic continuation at a $(s-1)$ -dimensional surface where the metric is stationary. One can choose $\chi = \pi/2$ as the surface, set $\Omega = i(\chi - \frac{\pi}{2})$ and obtain the metric with signature $(-, \dots, -, +)$

$$ds_s^2 = -dt^2 - \frac{\sin^2(L_s t)}{L_s^2} (-d\Omega^2 + \cosh^2 \Omega d\Omega_{s-2}^2). \quad (9)$$

Then one can analytically continue the metric through the null surface at $t = 0$ by redefining $\rho = \Omega + \frac{i\pi}{2}$ and get the s -dimensional anti-de Sitter metric

$$ds_s^2 = -dt^2 + \frac{\sin^2(L_s t)}{L_s^2} (d\rho^2 + \sinh^2 \rho d\Omega_{s-2}^2). \quad (10)$$

The obtained Lorentzian spacetime is the product of the s -dimensional anti-de Sitter space, which we consider as the external spacetime, and a S^{n-s} , which is identified as the internal space. The apparent dimension of the spacetime is s [2].

From the same S^s one can also get a s -dimensional hyperboloid by setting $\Sigma = i(t - \frac{\pi}{2L_s})$

$$ds_s^2 = d\Sigma^2 + \frac{\cosh^2(L_s \Sigma)}{L_s^2} (d\rho^2 + \sinh^2 \rho d\Omega_{s-2}^2). \quad (11)$$

One can also obtain the $(n-s)$ -dimensional de Sitter space through a simple analytic continuation from the factor space S^{n-s} as in the 4-dimensional case [3],

and consider the positive definite s -dimensional hyperboloid as the internal space. Then the apparent dimension of the external spacetime becomes $n - s$ [2].

One can appeal to quantum cosmology to discriminate these two possibilities. The relative creation probability of the universe is

$$P = \Psi^* \Psi \approx \exp(-I), \quad (12)$$

where Ψ is the wave function of the configuration at the quantum transition. The configuration is the metric and the matter field at the equator. I is the Euclidean action of the instanton. It is worth emphasizing that the instanton is constructed by joining its south hemisphere and its time reversal, its north hemisphere.

In the Lorentzian regime, the probability of a quantum state is independent of the representation. However, in the Euclidean regime this is not the case. In quantum cosmology, the universe is created from nothing in imaginary time. In the Euclidean regime the total relative probability of finding the universe does not stay constant. In fact, formula (12) can only be meaningful when one uses a right representation for the wave function at the equator. This problem was hidden in the earlier years of research of quantum cosmology. At that stage, only regular instantons were considered as seeds of universe creations.

Now, it is well known that regular instantons are too rare for the creation scenario of a more realistic cosmological model. One has to appeal to the constrained instantons [5]. The right representation can be obtained through a canonical transform from the wrong representation. The wave function subjects to a Fourier transform in the Lorentzian regime. At the WKB level, this corresponds to a Legendre transform, the Legendre term at the equator will change the probability value in Eq. (12). For a regular instanton, one member of any pair of canonical conjugate variables must vanish, so does the Legendre term.

The criterion for the right representation in formula (12) with a constrained instanton is that across the equator the arguments of the wave function must be continuous. This problem was encountered in the problem of quantum creation of magnetic and electric black holes [6]. If one considers the quantum creation of a general charged and rotating black hole, this point is even more critical. It becomes so acute that unless the right configuration is used, one cannot even find a constrained instanton seed [7].

Now, the action (1) is given under the condition that at the boundary ∂M the metric and the tensor field $A^{\alpha_1 \dots \alpha_{s-1}}$ are given. If we assume the external space is the s -dimensional anti-de Sitter space, then the Euclidean action is

$$I = \frac{1}{16\pi} \int_M \left(R - 2\Lambda - \frac{8\pi}{s} F^2 \right) + \frac{1}{8\pi} \int_{\partial M} K, \quad (13)$$

where all quantities are Euclidean and the path of the continuation from the Lorentzian action to the Euclidean action has been such that the sign in front of R term should be positive. Since $R = R_{n-s} + R_s$, the negative value of R_s is crucial for the perturbation calculation around the background of the external spacetime. The right sign is necessary for the primordial fluctuations to take the minimum excitation states allowed by the Heisenberg Uncertainty Principle [8].

The action of the instanton can be evaluated as

$$I = \left(\frac{n-2s}{2s(n-2)} - \frac{1}{2s} \right) \kappa^2 V_s V_{n-s}, \quad (14)$$

where the volumes V_s and V_{n-s} of S^s and S^{n-s} are

$$\frac{2\pi^{(s+1)/2} L_s^s}{\Gamma((s+1)/2)} \quad \text{and} \quad \frac{2\pi^{(n-s+1)/2} L_{n-s}^{n-s}}{\Gamma((n-s+1)/2)},$$

respectively, and L_{n-s} is the radius of the S^{n-s} .

The action is invariant under the gauge transformation

$$A_{\alpha_1 \dots \alpha_{s-1}} \longrightarrow A_{\alpha_1 \dots \alpha_{s-1}} + \partial_{[\alpha_1} A_{\alpha_2 \dots \alpha_{s-1}]}. \quad (15)$$

One can select a gauge such that there is only one nonzero component $A^{2 \dots s}$, where the index 1 associated with the time coordinate is excluded. There is no way to find a gauge in which the field $A_{2 \dots s}$ is regular for the whole manifold S^s using single neighborhood. One can integrate (5) to obtain its value at the equator with the regular condition at the south pole. The field for the north hemisphere can be obtained from the south solution through time reversal and a sign change. This results in a discontinuity across the equator. When we calculate the wave function of the universe, we implicitly fixed the gauge and no freedom is left for the gauge transform. On the other hand, the field strength $F^{\alpha_1 \dots \alpha_s}$ or the canonical momentum $P^{2 \dots s}$ is well defined and continuous. Therefore, the field strength is the right representation.

One can Fourier-transform the wave function $\Psi(h_{ij}, A_{2\dots s})$ to get the wave function $\Psi(h_{ij}, P^{2\dots s})$

$$\Psi(h_{ij}, P^{2\dots s}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iA_{2\dots s}P^{2\dots s}} \Psi(h_{ij}, A_{2\dots s}), \quad (16)$$

where h_{ij} is the metric of the equator. Here $A_{2\dots s}$ is the only degree of freedom of the matter content under the minisuperspace ansatz. $P^{2\dots s}$ is defined as

$$P^{2\dots s} = - \int_{\Sigma} (s-1)! F^{1\dots s}, \quad (17)$$

where Σ denotes the equator.

At the WKB level, the Fourier transform is reduced into a Legendre transform for the action. The Legendre transform introduces an extra term $-2A_{2\dots s}P^{2\dots s}$ to the Euclidean action I , where $A_{2\dots s}$ is evaluated at the south side of the equator. The two sides of the equator is taken account by the factor 2 here. The above calculation is carried out for the equator $t = \pi/2L_S$. However, the true quantum transition should occur at $\chi = \pi/2$. Since these two equators are congruent, the result should be the same. This has also been checked.

It turns out the extra term is

$$I_{\text{Legendre}} = \frac{1}{s} V_s V_{n-s} \kappa^2. \quad (18)$$

Then the total action becomes

$$I_s = \left(\frac{n-2s}{2s(n-2)} + \frac{1}{2s} \right) \kappa^2 V_s V_{n-s}. \quad (19)$$

Now if one uses the same instanton, and analytically continues from the factor space S^{n-s} at its equator to obtain an $(n-s)$ -dimensional de Sitter spacetime, and the internal space is an s -dimensional hyperboloid. Then one still encounters the representation problem of $A_{2\dots s}$. In the context of our argument, it has been implicitly assumed that for the argument of the wave function the gauge is fixed. The singularity or discontinuity is not avoidable. This is compatible with the fact that the instanton is constrained. We know regular instantons are either discrete or of constant action [5]. The action does depend on the parameter κ (see (20) below), therefore the instanton does not qualify as a regular instanton. However, the canonical momentum is zero here, and so is the Legendre term.

By the same argument as earlier for the continuation of the factor space S^s , the Euclidean action should take an extra negative sign, and the total action should be the negative of that in (14),

$$I_{n-s} = - \left(\frac{n-2s}{2s(n-2)} - \frac{1}{2s} \right) \kappa^2 V_s V_{n-s}. \quad (20)$$

From (12), we know the relative creation probability is the exponential to the negative of the Euclidean action, therefore if $2s - n < 0$, then the creation probability of the universe with the s -dimensional external space exponentially dominates that with the $(n-s)$ -dimensional one, that is the apparent dimension is most likely to be s . Otherwise the apparent dimension should be $n-s$. If $2s = n$, then the two possibilities of creations are equally likely.

One may also discuss the case with a real κ . For the case $2s - n < 0$, the universe is a product of an s -dimensional de Sitter space and a $(n-s)$ -dimensional hyperboloid. For the case $2s - n > 0$, the universe is a product of a $(n-s)$ -dimensional anti-de Sitter space and an s -dimensional sphere.

It is noted that the dimension of the external spacetime can never be higher than that of the internal space.

In the $d = 11$ supergravity, under a special ansatz, one can derive the Freund–Rubin model with $n = 11$, $s = 4$. It has been shown that the apparent dimension must be 4 [9].

At this moment, it is instructive to recall the representation problem in quantum creation of a Reissner–Nordström–de Sitter black hole. In the “regular” instanton case, the space is the product $S^2 \times S^2$. The situation can be considered as a special case of the Freund–Rubin toy models with $n = 4$, $s = 2$. If the Maxwell field lives in the internal space, then we obtain a magnetic black hole. For this case the charge, or κ , is real. If the Maxwell field lives in the exterior space (or 2-dimensional de Sitter space in the Lorentzian regime), then the black hole is electric. For this case, the charge, or κ , is imaginary. As we mentioned, the electric or magnetic instanton is not a regular instanton, it is a constrained instanton, therefore one has to use a right representation as we explained above. In the magnetic case the Legendre term is zero. After the Legendre transform, the duality between the electric and magnetic black holes is recovered, as far as the creation probability is concerned [6,7].

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