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# A survey of integrity

K.S. Bagga*, **, L.W. Beineke*

*Indiana University-Purdue University at Fort Wayne, Fort Wayne, IN 46805, USA*

W.D. Goddard

*Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

M.J. Lipman*, *** and R.E. Pippert*

*Indiana University-Purdue University at Fort Wayne, Fort Wayne, IN 46805, USA*

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*Abstract*

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A communication network can be considered to be highly vulnerable to disruption if the destruction of a few elements can result in no member's being able to communicate with very many others. This idea suggests the concept of the integrity of a graph — the minimum sum of the orders of a set of vertices being removed and a largest remaining component. This survey includes results on the integrity of specific families of graphs and combinations of graphs, relationships with other parameters, bounds, computational complexity, and some variations on the concept.

## 1. Introduction

In an analysis of the vulnerability of a communication network to disruption, two quantities (there may be others) that come to mind are (i) the number of elements that are not functioning and (ii) the size of the largest remaining group within which mutual communication can still occur. In particular, in an adversarial relationship, it would be desirable for an opponent's network to be such that the two quantities can be made to be simultaneously small.

The concept of integrity was introduced as a measure of graph vulnerability

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** Current address: Ball State University, Muncie, IN 47306, USA.

*** Current address: Office of Naval Research, Arlington, VA 22217, USA.

in this sense. Formally, the *vertex-integrity* (frequently called just the *integrity*) is

$$I(G) := \min_{X \subset V(G)} \{|X| + m(G-X)\},$$

where $m(H)$ denotes the order of a largest component of $H$. This concept was introduced by Barefoot, Entringer and Swart [7], who discovered many of the early results on the subject. In his thesis [15], Goddard added many results and developed some generalizations.

We note that even though the motivation for the concept was the disruption of communication, the definition does not require that a graph be connected. We also note that the original definition permitted the removal of all vertices, but that is never necessary. As immediate consequences of the definition, we have the following: If $G$ is a graph of order $p$, then $1 \le I(G) \le p$, and if $H$ is any subgraph of $G$, then $I(H) \le I(G)$.

A few further comments on notation are appropriate here. The order of a graph $G$ (that is, the number of vertices) will generally be denoted by $p$, but may also be denoted by $|G|$. As usual, $V$ and $E$ will denote respectively the sets of vertices and edges of $G$, and $X$ will denote a proper subset of $V$. As noted earlier, $m(G)$ equals the largest order among the components of $G$.

The next section is the heart of this survey. It contains results on the integrity of specific families of graphs, on bounds for the integrity, on maximal and minimal graphs of given integrity, on relationships between integrity and other parameters, and on computational complexity. In Section 3 we discuss variations and generalizations of the concept.

## 2. Results on integrity

### 2.1. Basic results

In order to develop some intuition for integrity, we first consider trees. Since the deletion of the center vertex from the star $K_{1,n}$ leaves $n$ isolated vertices, clearly $I(K_{1,n}) = 2$. For a second example, we consider the path $P_p$ with $p$ vertices. If a set of $r$ vertices is removed, then there are $r+1$ or fewer components remaining, and one of them must have at least $(p-r)/(r+1)$ vertices. It follows that

$$I(P_p) \ge \min_r \left\{ r + \frac{p-r}{r+1} \right\}.$$

Now for $x \ge 0$, the function $f(x) = x + (p-x)/(x+1)$ has a minimum value of $2\sqrt{p+1} - 2$. Since the integrity is integer valued, we round this up to get a lower bound. This value can in fact be achieved for each $p$, so we have

$$I(P_p) = \lceil 2\sqrt{p+1} \rceil - 2.$$

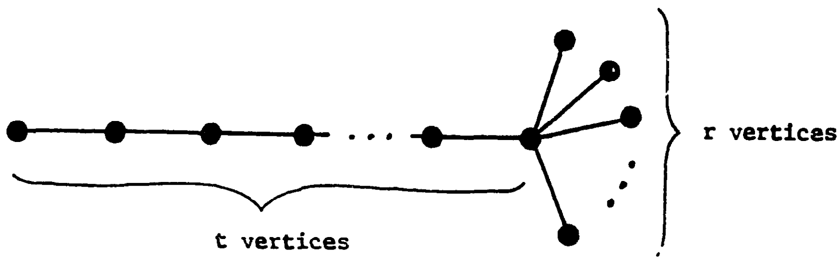


Fig. 1.

Having considered stars and paths, we turn to other trees. Define the *comet* $C_{t,r}$ to be the graph obtained by identifying one end of the path $P_t$ with the center of the star $K_{1,r}$ (see Fig. 1). It is not difficult to show that if one edge is moved from the "head" to the "tail", then the integrity cannot go down and it can go up by at most 1; that is,

$$I(C_{t,r}) \le I(C_{t+1,r-1}) \le I(C_{t,r}) + 1.$$

Since paths and stars are extreme comets, it follows that comets of order $p$ achieve all values between $I(K_{1,p-1})$ and $I(P_p)$. Not surprisingly, this covers the range of values for all trees. Among all graphs of order $p$, the range of the integrity is the set $\{1,2,\dots,p\}$; for nontrivial connected graphs, it is $\{2,3,\dots,p\}$, these values being assumed by graphs obtained by attaching a complete graph to the end of a path.

In the following theorem, we give the integrity of a variety of families of graphs. These results were first found by Barefoot, Entringer and Swart [7,8].

**Theorem 2.1.** *The integrity of*

(a) *the complete graph* $K_p$ *is* $p$;
(b) *the null graph* $\bar{K}_p$ *is* 1;
(c) *the star* $K_{1,n}$ *is* 2;
(d) *the path* $P_p$ *is* $\lceil 2\sqrt{p+1}\,\rceil - 2$;
(e) *the cycle* $C_p$ *is* $\lceil 2\sqrt{p}\,\rceil - 1$;
(f) *the comet* $C_{p-r,r}$ *is* $I(P_p)$, *if* $r \le \sqrt{p+1} - \frac{5}{4}$; $\lceil 2\sqrt{p-r}\,\rceil - 1$, *otherwise*;
(g) *the complete bipartite graph* $K_{m,n}$ *is* $1 + \min\{m,n\}$;
(h) *any complete multipartite graph of order* $p$ *and largest partite set of order* $r$ *is* $p-r+1$.

We observe that among the graphs missing from this list is the $n$-dimensional cube; in fact, its integrity remains undetermined and this is discussed in Subsection 2.5.4.

The second theorem tells which graphs have integrity near the extremes of the range; only the case $I(G) = p-1$ is nontrivial [19].

**Theorem 2.2.** *Let $G$ be a graph of order $p$.*

(a) $I(G)=1$ *if and only if $G$ is null.*

(b) $I(G)=2$ *if and only if all nontrivial components of $G$ are edges or the only nontrivial component is a star.*

(c) $I(G)=p-1$ *if and only if $G$ is not complete and $\bar{G}$ has girth at least* 5.

(d) $I(G)=p$ *if and only if $G$ is complete.*

### 2.2. *Extremal matters*

In proving results on integrity, one frequently uses a set $X$ of vertices, called an *I-set*, that achieves the integrity; that is, $X$ is an $I$-set if $|X|+m(G-X)=I(G)$. If $X$ is an $I$-set, then it is not difficult to show that

(a) $I(G-X)=m(G-X)$;

(b) *$X$ is a cut set unless $G$ is complete*;

(c) *if $X$ is minimal, then each vertex $v$ in $X$ is a cut vertex of* $G-(X-\{v\})$.

This last observation leads to an alternative formulation of integrity, as stated in our next result; it could be taken as a recursive definition [19].

**Theorem 2.3.** *If $G$ is a nontrivial graph, then*

$$I(G)=\min\{m(G), 1+\min_{v\in V} I(G-v)\}.$$

For nontrivial connected graphs, this can be restated as

$$I(G)=1+\min_{v\in V} I(G-v).$$

Some vertices that achieve this minimum are described in the next result.

**Theorem 2.4.** *If in graph $G$, $v$ is a vertex for which* $\deg v\geq I(G-v)$, *then* $I(G)=1+I(G-v)$.

We observe that this result is best possible in that the bound on the degree cannot in general be reduced. An example is an endvertex $v$ in $P_3$. Furthermore, the converse is not true since all vertices in an $I$-set of $G$ may satisfy $\deg v<I(G-v)$, as in a path.

Graphs that are minimal with respect to a given parameter are generally of interest. Formally, we define a graph $G$ to be *I-minimal* if for every edge $e$ in $G$, $I(G-e)<I(G)$. Note that if $G$ is $I$-minimal, then $I(G-e)=I(G)-1$, and because of the monotonicity property of integrity, $I(H)<I(G)$ for every proper subgraph $H$ of $G$. Clearly, every graph has an $I$-minimal subgraph with the same integrity. Complete graphs are of course $I$-minimal, and in fact $K_2$ is the only connected $I$-minimal graph of integrity 2. Beyond this, little is known about such graphs.

A similar concept is that of an *I-critical* graph: $G$ is such a graph if $I(G-v)<I(G)$ for every vertex $v$. Again, not much study has been made of these graphs. Clearly,

an $I$-critical graph can have no isolated vertices, and an $I$-minimal graph without such vertices must be $I$-critical. Some graphs that are $I$-critical but not $I$-minimal are the cycles of square order.

At the opposite extreme from an $I$-minimal graph is a graph $G$ satisfying $I(G\cup e)>I(G)$ for every edge $e$ in $\bar{G}$. Such graphs are called ***I-maximal***, and they are characterized in [19]; they are joins of complete graphs with the unions of other complete graphs.

**Theorem 2.5.** *A noncomplete graph is I-maximal if and only if it is of the form* $K_r+(K_{n_1}\cup\cdots\cup K_{n_t})$, *where* $t\geq 2$, *and if* $n_1\leq n_2\leq\cdots\leq n_t$, *then* $n_{t-1}=n_t\leq n_1+n_2-1$.

There are many other interesting extremal questions about integrity, most of which are still unexplored. One example of such a problem is to find the maximum and minimum integrities for connected graphs (or other families) with given numbers $p$ and $q$ of vertices and edges. As we observed above, among trees (when $q=p-1$) the extremes occur for paths and stars; and when $q=p$, the cycle $C_p$ gives the maximum, while a star with one additional edge clearly achieves the minimum (there are others). In general, this seems to be a hard problem, for as is easily observed from Theorem 2.2(c), just finding the minimum number of edges in a graph with integrity $p-1$ is equivalent to finding the maximum number of edges in a graph with girth at least 5.

### *2.3. Integrity and other parameters*

Certain other parameters provide bounds on the integrity of a graph. For example, since a minimal $I$-set in a noncomplete graph must be a cut set, it is obvious that the integrity of any graph is at least 1 more than the connectivity. However, even though equality sometimes holds here (as, for example, for stars), this lower bound can be improved by using instead the minimum degree. Parameters that will be discussed here include the following:

- $\delta$, the minimum vertex degree;
- $\kappa$, the connectivity;
- $\alpha$, the covering number;
- $\beta$, the independence number;
- $\chi$, the chromatic number;
- $\tau$, the toughness.

The results that follow are due to Goddard and Swart [19,20]. Each of the bounds given in the first theorem is sharp; the second theorem tells when equality holds for three of them.

**Theorem 2.6.** *For any graph* $G$,

(a) $I(G)\leq\alpha(G)+1$;

(b) $I(G)\geq\delta(G)+1$;

(c) $I(G) \geq \min_t \max\{d_t, t-1\}$, *where the degrees of $G$ are $d_1 \geq d_2 \geq \cdots \geq d_p$*;
(d) $I(G) \geq \chi(G)$;
(e) $I(G) \geq (p-\kappa(G))/\beta(G)+\kappa(G)$;
(f) $I(G) \geq 2\sqrt{\tau p}-\tau$, *if $G$ is not complete.*

**Theorem 2.7.** *For any graph $G$,*
(a) $I(G)=\kappa(G)+1$ *if and only if* $\kappa(G)=\alpha(G)$;
(b) $I(G)=\alpha(G)+1$ *if and only if $G$ does not contain $2K_2$ as an induced subgraph*;
(c) $I(G)=\delta(G)+1$ *if and only if $G \cong rK_n$ or $G \cong rK_n+F$ for some graph $F$ satisfying $\delta(F) \geq |G|-(2r-1)n-1$.*

In addition, we note that equality holds in part (d) (and hence also in (c)) of Theorem 2.7 for stars and complete graphs, among others. Inequality (e) is sharp for any graph $G$ satisfying $\kappa(G)=\alpha(G)$ or $\alpha(G)-1$, while (f) is sharp for cycles of length greater than 3.

Comparisons among several vulnerability measures for some specific families of graphs have been made by Barefoot, Entringer and Swart [7] and Goddard and Swart [18].

### *2.4. Unary operations*

The remainder of this section is devoted primarily to results on the integrity of graphs obtained via various operations. In this subsection we concentrate on the operations of taking complements and powers; the next is devoted to binary operations.

#### *2.4.1. Complement*

We begin with the Nordhaus-Gaddum theory as developed in [18]. The following theorem gives lower bounds for the sum and product of the integrity of a graph and its complement.

**Theorem 2.8.** *For any graph $G$,*
(a) $I(G)+I(\bar{G}) \geq p+1$;
(b) $I(G) \cdot I(\bar{G}) \geq p$.

As is frequently the case with results of this type, the bound for the product is simply an algebraic consequence of that for the sum. Here, equality in (b) is attained only when $G$ or $\bar{G}$ is complete. However, there are additional graphs for which equality holds in (a), such as any complete bipartite graph or $K_{2,4}-e$.

Good upper bounds appear to be considerably more difficult to obtain, with the best current results being given in terms of the Ramsey numbers $r(m,n)$—the smallest number such that every graph of that order contains either $m$ mutually adjacent vertices or $n$ independent vertices. Few values of $r(m,n)$ are known.

**Theorem 2.9.** *Let $S_p := 2p+4-\min\{m+n\colon r(m,n)>p\}$. Then for any graph G of order p,*

(a) $I(G)+I(\bar{G}) \le S_p$;

(b) $I(G)\cdot I(\bar{G}) \le \lfloor S_p^2/4 \rfloor$.

We observe that again inequality (b) follows from (a). It is known that (a) is sharp for $p \le 10$ except when $p=8$, in which case $S_p = 13$, but 12 is an upper bound. In general, the largest known sum for $I(G)+I(\bar{G})$ when $p>4$ is $\lceil 3p/2 \rceil$, and this is achieved when the cycle $C_p$ is taken to the power $\lfloor p/4 \rfloor$ (see next theorem).

### 2.4.2. Powers

We begin with the integrity of powers of cycles, a result due to Barefoot, Entringer and Swart ([8] corrects some values given in [7]).

**Theorem 2.10.** *For $1 \le k \le p/2$, let $s = \lceil \sqrt{p/k+\frac{1}{4}} - \frac{1}{2} \rceil$. Then*

$$I(C_p^k) = k(s-1) + \lceil p/s \rceil.$$

Since for any graph $G$ and positive integer $i$, $G^i$ is a subgraph of $G^{i+1}$, it follows that

$$I(G) \le I(G^2) \le I(G^3) \le \cdots.$$

Furthermore, if $I(G)=m(G)$, then equality holds throughout. The following partial converse to that fact is established in [17].

**Theorem 2.11.** *If G is a graph for which $I(G^3)=I(G)$, then $I(G)=m(G)$.*

## 2.5. Binary operations

In this subsection we consider results (primarily from [17]) on the integrity of the union, join, composition, and product of two graphs.

### 2.5.1. Union

Clearly, if $G$ is the union of $G_1, G_2, \ldots, G_n$, then $m(G) = \max m(G_i)$. It follows that when $G$ has enough components of the largest order (or enough near to that), then $I(G)=m(G)$. In particular, if $G=nH$ and $n \ge m(H)-1$, then $I(G)=m(H)$. In general, however, we have only elementary bounds for the integrity of unions of graphs, as stated in the following result.

**Theorem 2.12.** *If $G=\bigcup_{i=1}^n G_i$, then*

$$\max_i I(G_i) \le I(G) \le \sum_{i=1}^n I(G_i) - n + 1.$$

It is not difficult to find graphs for which these bounds are sharp. In fact, the two bounds are equal for any graph with just one nontrivial component.

*2.5.2. Join*

In this case, exact results are known.

**Theorem 2.13.** *For any graphs G and H,*

$$I(G+H)=\min\{I(G)+|H|,\ I(H)+|G|\}.$$

**Corollary.** *For any graph G,* $I(G+K_r)=I(G)+r$.

This result also follows from Theorem 2.5.

*2.5.3. Composition*

The *composition* (also known as the lexicographic product) $G_1[G_2]$ of two graphs $G_1$ and $G_2$ has as its vertex set $V(G_1)\times V(G_2)$, with $(u_1,u_2)$ adjacent to $(v_1,v_2)$ if either $u_1$ is adjacent to $v_1$ in $G_1$ or $u_1=v_1$ and $u_2$ is adjacent to $v_2$ in $G_2$. Note that, unlike the union, join, and (Cartesian) product, this operation is not commutative.

**Theorem 2.14.** *For any graphs G and H,*

$$I(G[H])=\min\{I(G)\cdot|H|,\ \alpha(G)\cdot|H|+I(\beta(G)H)\}.$$

**Corollary.** *For any graph G,*

(a) $I(G[K_n])=nI(G)$.

(b) $I(K_n[G])=(n-1)|G|+I(G)$.

*2.5.4. Product*

The (Cartesian) *product* $G_1\times G_2$ of graphs $G_1$ and $G_2$ also has $V(G_1)\times V(G_2)$ as its vertex set, but here $(u_1,u_2)$ is adjacent to $(v_1,v_2)$ if either $u_1=v_1$ and $u_2$ is adjacent to $v_2$ or $u_2=v_2$ and $u_1$ is adjacent to $v_1$. We observe that the product of two graphs is a subgraph of their composition and that it contains copies of each graph equal in number to the order of the other graph. These observations give the rather crude bounds stated in the next theorem.

**Theorem 2.15.** *For any graphs G and H,*

$$\max\{I(|G|\,H),I(|H|\,G)\}\le I(G\times H)\le\min\{I(G[H]),I(H[G])\}.$$

By using results on unions and compositions, various other bounds for the integrity of products can be obtained. Furthermore, when one of the graphs is specified, it may be possible to refine the bounds, as is done for $K_2$ in the following result [2]:

**Theorem 2.16.** *For any graph G,*

$$\tfrac{1}{2}I(G) \le I(K_2 \times G) \le 2I(G).$$

These bounds are sharp; for example, $G = K_{2n}$ achieves the lower bound, while for $m \ne n$, $G = K_{m,n}$ achieves the upper one. Our next two theorems (also from [2]) show that the integrity of prisms and ladders is always within 1 of this upper bound, and equals it about half the time.

**Theorem 2.17.** (a) *For* $n = 3$ *or* 4, $I(K_2 \times C_n) = 2I(C_n) - 1 = 5$.
(b) *For* $n \ge 5$, *if* $n = r^2 + k$ *with* $0 \le k \le 2r$, *then*

$$I(K_2 \times C_n) = \begin{cases} 2I(C_n) - 1, & \text{if } 1 \le k \le \dfrac{r}{2} \text{ or } r < k \le \dfrac{3r}{2}, \\ 2I(C_n), & \text{otherwise.} \end{cases}$$

**Theorem 2.18.** *For* $n \ge 2$, *if* $n = r^2 + k$ *with* $0 \le k \le 2r$, *then*

$$I(K_2 \times P_n) = \begin{cases} 2I(P_n) - 1, & \text{if } 0 \le k < \dfrac{r}{2} \text{ or } r \le k < \dfrac{3r}{2}, \\ 2I(P_n), & \text{otherwise.} \end{cases}$$

It would be very interesting to learn the integrity of general products of paths— the so-called "grid graphs", for which this last theorem provides a beginning. In fact, the previous two theorems notwithstanding, exact results on products seem difficult to obtain, even for natural families of graphs. One other successful effort yielded the integrity of products of stars [17].

**Theorem 2.19.** *If* $r \le s$, *then*

$$I(K_{1,r} \times K_{1,s}) = \begin{cases} 2r - 1, & \text{if } r = s, \\ 2r, & \text{otherwise.} \end{cases}$$

Even the integrity of the product of complete graphs is complicated. The expression in the next formulation in fact involves the solution of an integer optimization problem.

**Theorem 2.20.** *Let* $2 \le m \le n$. *Then*

$$I(K_m \times K_n) = mn - \max_{1 \le j < m} j \left\lfloor \frac{n(m-j)}{m} \right\rfloor .$$

**Corollary.** (a) $I(K_{2r} \times K_{2s}) = 3rs$.
(b) *If* $2s + 1 \le r^2$, *then* $I(K_{2r} \times K_{2s+1}) = 3rs + 2r$.

(c) *Let* $m=2r+1$, $n\geq r^2$, *and* $n\equiv t \pmod m$ *with* $0\leq t<m$. *Then*

$$I(K_m\times K_n)=\begin{cases} mn-(r+1)\left\lfloor \dfrac{rn}{m}\right\rfloor, & \textit{if } t \textit{ is odd},\\ mn-r\left\lfloor \dfrac{(r+1)n}{m}\right\rfloor, & \textit{if } t \textit{ is even}.\end{cases}$$

We conclude this subsection with another problem involving products: What is the integrity of the $n$-dimensional cube? For some time it was conjectured to be $2^{n-1}+1$, a value obtained by removing a complete partite set of vertices. Quite recently, it was shown [11] that the integrity of the $n$-dimensional cube is $O(2^n \log n/\sqrt{n})$.

### 2.6. *Computational complexity*

As is true of many interesting measures of vulnerability, the determination of integrity is NP-complete. More formally, it was shown by Clark, Entringer and Fellows [13] that the following decision problem is NP-complete:

> **Vertex-integrity.**
> *Input*: A graph $G$ and a positive integer $k$.
> *Question*: Is $I(G)\leq k$?

In fact, the question remains NP-complete even when the input is restricted to planar graphs.

On the other hand, it was shown by Fellows and Stueckle [14] that when $k$ is fixed the decision problem has an $O(p^2)$ algorithm; that is, the following decision problem is $O(p^2)$:

> **$k$-vertex-integrity.**
> *Input*: A graph $G$.
> *Question*: Is $I(G)\leq k$?

This result uses the Robertson–Seymour theory of minors and the fact that the set of graphs $\mathcal{G}_k:=\{G\colon I(G)\leq k\}$ is closed under the operations of replacing a graph with a subgraph of itself and of contracting an edge. (That is, $\mathcal{G}_k$ is a lower ideal in the minor ordering. It is not true that $\mathcal{G}_k$ is a lower ideal in the immersion ordering.) The proof produces an algorithm with a hidden constant associated with $O(p^2)$ that is exponential in $k$. In [13] the obstruction sets for $k\leq 3$ are constructed, and it is proved that for large $k$ the number of obstruction sets is greater than $(1.7)^k$.

A second type of complexity question concerns the computation of $I(G)$ when $G$ is restricted to some specified class of graphs. In particular, we have shown that

$I(G)$ can be computed in time $O(p^3)$ when $G$ is a tree or a cactus (unpublished). These algorithms work from the endblocks of the graph toward the center and have the feature that for a specified $M$ they compute a minimal set $X$ for which $m(G-X) \leq M$. The integrity is then found by repeating the procedure for several values of $M$.

Since the original decision problem is known to be NP-complete for planar graphs but polynomial for trees, it would be interesting to know more about the computational complexity of other families of graphs.

## 3. Variations and generalizations

### 3.1. Edge-integrity

In the most significant variation of integrity, edges rather than vertices are destroyed. Formally, the *edge-integrity* of a graph $G$ is defined as

$$I'(G) := \min_{S \subseteq E} \{|S| + m(G-S)\}.$$

This definition may appear questionable since it involves adding numbers of vertices and edges. An obvious alternative is to weight the elements in some fashion in order to make them more comparable to each other, but it is not obvious what the weighting factors should be. In fact, the original version leads to a number of interesting results, and for that reason, as well as its simplicity, it continues to be used.

Like the vertex-integrity, this concept was introduced by Barefoot, Entringer and Swart [7], and they discovered a number of its basic properties. The edge-integrity of a graph with $p$ vertices is always between 1 and $p$, and it is always at least as large as the vertex-integrity. Exploring edge-integrity in its fullness would take us too far afield from the main topic of this survey. For that reason and for space considerations, we will only consider some highlights here, and refer the reader to [5] for a fuller survey.

The following theorem gives the edge-integrity of some elementary families of graphs.

**Theorem 3.1.** *The edge-integrity of*
(a) *the complete graph* $K_p$ *is* $p$;
(b) *the null graph* $\bar{K}_p$ *is* 1;
(c) *the star* $K_{1,n}$ *is* $n+1$;
(d) *the path* $P_p$ *is* $\lceil 2\sqrt{p} \rceil - 1$;
(e) *the cycle* $C_p$ *is* $\lceil 2\sqrt{p} \rceil$ *for* $p \geq 4$;
(f) *the comet* $C_{p-r,r}$ *is* $I'(P_p)$, *if* $p-r \leq \sqrt{p}-1$; $r+1+\lceil p/(r+1) \rceil$, *otherwise*;
(g) *the complete bipartite graph* $K_{m,n}$ *is* $m+n$;
(h) *any complete multipartite graph of order* $p$ *is* $p$;
(i) *the n-dimensional cube* $Q_n$ *is* $2^n$.

As with the vertex-integrity, paths and stars attain the extremes of the edge-integrity among all trees, but now they are at the opposite ends. The edge-integrity of trees and tree-like graphs has received considerable attention; see, for example, [3,6,8].

The reader may have observed that unlike the vertex-integrity, the edge-integrity of many graphs equals their order, including such sparse graphs as the $n$-cubes. Because such graphs have the greatest possible edge-integrity, they have been called *honest*. If a graph is not honest, then, except for the path $P_4$, its complement is honest [4,21]. But in fact, almost all graphs are honest, since every graph of diameter 2 is known to be honest [1], and almost all graphs have diameter 2. Additional connections between the edge-integrity and diameter have also been explored [9].

Most of the questions that have been asked about vertex-integrity can also be asked in the edge case; see, for example, [1,5,7,10,15,21]. As with the vertex-integrity, it is the case that the computation of the edge-integrity is NP-complete [13,14], but is known to be polynomial for some families such as trees [3,6].

### 3.2. Mean integrity

The mean integrity is a variation that uses the average order, instead of the maximum order, of a component in a graph. Formally, we let $p_v(G)$ denote the order of the component of $G$ containing vertex $v$ and define the *average component order* to be

$$\bar{m}(G) := \frac{1}{p} \sum_{v \in V} p_v(G)$$

(in contrast to the maximum component order $m(G) = \max_{v \in V} p_v(G)$). The *mean integrity* is then defined analogously to the integrity as

$$J(G) := \min_{X \subset V} \{|X| + \bar{m}(G-X)\}.$$

This concept was introduced by Chartrand, Kapoor, McKee and Oellermann [12], and the results given here are taken from that paper. Clearly, for any graph $G$, $J(G) \le I(G)$, and the following theorem implies that there are some elementary graphs for which the two parameters are equal.

**Theorem 3.2.** *Let $G$ be a graph of order $p$.*

(a) *$J(G) = p$ if and only if $G$ is complete.*

(b) *$J(G) = 1$ if and only if $G$ is null.*

(c) *$J(K_{m,n}) = 1 + \min\{m, n\}$.*

(d) *If $G$ is a complete multipartite graph with largest partite set of order $r$, then $J(G) = p - r + 1$.*

However, the following result shows that the range of values of the mean-integrity function is considerably less restricted than that of integrity.

**Theorem 3.3.** *For every rational number $r \geq 1$, there exists a graph with mean integrity $r$.*

It is therefore not surprising that there exist pairs of graphs with equal integrity but different mean integrity, and vice versa.

The mean integrity is related to certain other parameters in the following way:

$$1+\kappa(G) \leq 1+\delta(G) \leq J(G) \leq I(G) \leq 1+\alpha(G).$$

In particular, we note that if $G$ is $n$-connected, then $J(G) \geq n+1$. Chartrand et al. [12] established the following partial converse for graphs with high mean integrity (between $p-2$ and $p-1$).

**Theorem 3.4.** *If $n$ is a positive integer for which*

$$J(G) > p-2+\frac{2}{p-n+1},$$

*then $G$ is $n$-connected.*

**Corollary.** *For any graph $G$,*

$$1+\delta(G) \leq J(G) \leq p-2+\frac{2}{p-\kappa(G)}.$$

We conclude with another lower bound.

**Theorem 3.5.** *Let $G$ be a graph with $p$ vertices and $q$ edges and let*

$$n := \lfloor (p-\tfrac{1}{2}) - \sqrt{(p-\tfrac{1}{2})^2-2q} \rfloor. \qquad (*)$$

*Then $J(G) \geq n+1$.*

This result is sharp since $J(K_n+\bar{K}_{p-n})=n+1$, and $n$, $p$, and $q$ satisfy (*).

### *3.3. The general schema*

In [16], Goddard develops a schema for the construction of graphical parameters that emulate integrity in their behavior. From a given parameter $\psi$, define a new parameter $\Psi$, induced by $\psi$, as follows:

$$\Psi(G) := \min_{X \subset V} \{|X| + \psi(G-X)\}.$$

Integrity (when $\psi(H)=m(H)$) and mean integrity (when $\psi(H)=\bar{m}(H)$) are clearly two examples of such functions. Another is provided by connectivity, for which

$$\psi(H) = \begin{cases} |H|-1, & \text{if } H \text{ is connected,} \\ 0, & \text{if } H \text{ is disconnected.} \end{cases}$$

More will be said about this type of inducing function later, but first we look at some results that are natural extensions of results on integrity.

*3.3.1. Basic properties*

We observe that if $\Psi$ is induced by $\psi$, then for any graph $G$, $\Psi(G) \leq \psi(G)$. Also, $\Psi(K_1) = \psi(K_1)$, which we denote by $\psi_0$. The following theorem gives analogues to certain results from Section 2 that hold when the inducing parameter satisfies one elementary property. There are other analogues that we do not bother to restate here.

**Theorem 3.6.** *Let $\psi$ be such that $\psi(G) \geq \Delta(G) + \psi_0$ for every graph $G$.*

(a) *If $G = H + K_r$, then $\Psi(G) = \Psi(H) + r$, and, in particular,*

$$\Psi(K_p) = p - 1 + \psi_0.$$

(b) *If the degrees of the vertices of $G$ are $d_1 \geq d_2 \geq \cdots \geq d_p$, then*

$$\Psi(G) \geq \psi_0 + \min_t \max\{d_t, t-1\}.$$

(c) *For any graph $G$,*

$$\Psi(G) + \Psi(\bar{G}) \geq p - 1 + 2\psi_0,$$

*and*

$$\Psi(G) \cdot \Psi(\bar{G}) \geq p\psi_0.$$

*3.3.2. Relations between parameters*

We now turn to questions involving more than one parameter in the general schema. Our first result in this area is an elementary observation: If $\psi$ and $\theta$ induce the parameters $\Psi$ and $\Theta$ respectively, and if $\psi(G) \leq \theta(G)$ for all $G$, then $\Psi(G) \leq \Theta(G)$ for all $G$.

For our next results we make another definition. Given a graph parameter $\mu$, a graph $G$ is called *$\mu$-acritical* if $G$ is trivial or for every vertex $v$, $\mu(G - v) \geq \mu(G)$. We denote the family of acritical graphs of $\mu$ by $\mathscr{A}(\mu)$. The following theorem determines the parameters which fit the general schema.

**Theorem 3.7.** *Let $\mu$ and $\psi$ be graphical parameters, with $\mu$ integer valued. Together, the following two conditions are necessary and sufficient for $\mu = \Psi$:*

(1) *For all $G$, $\mu(G) \leq \psi(G)$, with equality if $G \in \mathscr{A}(\mu)$.*

(2) *For all $G$, $\mu(G - v) \geq \mu(G) - 1$ for each vertex $v$.*

It is also shown in [16] that any family $\mathscr{C}$ of graphs that contains $K_1$ is the acritical family of some parameter in the schema. To this end, the *recognizer $\psi_{\mathscr{C}}$ of such a family $\mathscr{C}$* is defined to be

$$\psi_{\mathscr{C}}(G) = \begin{cases} |G| - 1, & \text{if } G \notin \mathscr{C}, \\ 0, & \text{if } G \in \mathscr{C}. \end{cases}$$

The following result is then immediate:

**Theorem 3.8.** *If $\psi_{\mathscr{C}}$ is the recognizer for a family $\mathscr{C}$ of graphs containing $K_1$, then for the induced parameter $\Psi_{\mathscr{C}}$, $\mathscr{A}(\Psi_{\mathscr{C}}) = \mathscr{C}$.*

We note that for nonacritical graphs, the recognizer could take any sufficiently large value instead of the order minus one; this choice is made in general because it fits with integrity. In an earlier observation, connectivity was given as an example in this form, with the acritical family being all disconnected graphs and $K_1$. The vertex covering number has as its acritical family all the null graphs. No such elementary description is known for the family of $I$-acritical graphs.

*3.3.3. Variations on the general schema*

More general classes of parameters can be generated by modifying the function being minimized. For a graphical parameter $\varphi$ and a function $f$ of two variables, form

$$\Phi_f(G) := \min_{X \subset V} f(|X|, \varphi(G-X)).$$

For example, the toughness of $G$ is defined as

$$\tau(G) := \min\{|X|/k(G-X)\}$$

where $k(H)$ is the number of components of $H$, and the minimum is taken over all cut sets of $G$ (this must be modified if $G$ is complete).

Further examples of this generalized schema are limited primarily by the interest they generate. Some may be of special interest because they provide good models or detect differences that are deemed important.

Finally, we mention that the general schema has a natural edge analogue:

$$\Psi'(G) := \min_{S \subseteq E} \{|S| + \psi(G-S)\},$$

with many corresponding results. In particular, it is clear that the edge-integrity is induced by $m$ (the order of a largest component).

## References


[1] K.S. Bagga, L.W. Beineke, M.J. Lipman and R.E. Pippert, On the edge-integrity of graphs, Congr. Numer. 60 (1987) 141–144.

[2] K.S. Bagga, L.W. Beineke, M.J. Lipman and R.E. Pippert, The integrity of the prism (Preliminary report), Abstracts Amer. Math. Soc. 10 (1989) 12.

[3] K.S. Bagga, L.W. Beineke, M.J. Lipman and R.E. Pippert, Some bounds and an algorithm for the edge-integrity of trees, J. Combin. Math. Combin. Comput., to appear.

[4] K.S. Bagga, L.W. Beineke, M.J. Lipman and R.E. Pippert, On the honesty of graph complements, Discrete Appl. Math., to appear.

[5] K.S. Bagga, L.W. Beineke, M.J. Lipman and R.E. Pippert, A survey of edge-integrity, Discrete Math., to appear.

[6] K.S. Bagga, L.W. Beineke, M.J. Lipman, R.E. Pippert and R.L. Sedlmeyer, A good algorithm for the computation of the edge-integrity of trees, Congr. Numer. 67 (1988) 225-232.

[7] C.A. Barefoot, R. Entringer and H. Swart, Vulnerability in graphs—A comparative survey, J. Combin. Math. Combin. Comput. 1 (1987) 12-22.

[8] C.A. Barefoot, R. Entringer and H. Swart, Integrity of trees and powers of cycles, Congr. Numer. 58 (1987) 103-114.

[9] L.W. Beineke, W.D. Goddard and M.J. Lipman, Edge-integrity and the diameter of a graph, in preparation.

[10] L.W. Beineke, W.D. Goddard and M.J. Lipman, On the edge-integrity of graph products, in preparation.

[11] L.W. Beineke, W.D. Goddard, P. Hamburger, D.J. Kleitman, M.J. Lipman and R.E. Pippert, The integrity of the cube is small, J. Combin. Math. Combin. Comput. 9 (1991) 191-193.

[12] G. Chartrand, S.F. Kapoor, T.A. McKee and O.R. Oellermann, The mean integrity of a graph, in: V.R. Kulli, ed., Recent Studies in Graph Theory (Vishwa International Publications, Gulbarga, 1989).

[13] L.H. Clark, R.C. Entringer and M.R. Fellows, Computational complexity of integrity, J. Combin. Math. Combin. Comput. 2 (1987) 179-191.

[14] M.R. Fellows and S. Stueckle, The immersion order, forbidden subgraphs and the complexity of network integrity, J. Combin. Math. Combin. Comput. 6 (1989) 23-32.

[15] W. Goddard, On the vulnerability of graphs, Ph.D. thesis, University of Natal, Durban, S.A. (1989).

[16] W. Goddard, Measures of vulnerability—The integrity family, submitted.

[17] W. Goddard and H.C. Swart, On the integrity of combinations of graphs, J. Combin. Math. Combin. Comput. 4 (1988) 3-18.

[18] W. Goddard and H.C. Swart, On some extremal problems in connectivity, in: Y. Alavi et al., eds., Proceedings of the Sixth International Conference on the Theory and Applications of Graphs (Wiley, New York) 535-551.

[19] W. Goddard and H.C. Swart, Integrity in graphs: Bounds and basics, J. Combin. Math. Combin. Comput. 7 (1990) 139-151.

[20] W. Goddard and H.C. Swart, On the toughness of a graph, Quaestiones Math. 13 (1990) 217-232.

[21] R. Laskar, S. Stueckle and B. Piazza, On the edge-integrity of some graphs and their complements, Discrete Math., to appear.