Unbounded solutions of a class of planar systems

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Abstract

In this paper, the unbounded solutions for the following nonlinear planar system:

\[ x' = a^+ y^+ - a^- y^- + f(t), \]
\[ y' = -b^+ x^+ + b^- x^- + g(t), \]

is discussed, where \( a^\pm, b^\pm \) are positive constants satisfying

\[ \frac{1}{\sqrt{a^+ b^+}} + \frac{1}{\sqrt{a^- b^-}} + \frac{1}{\sqrt{a^+ b^-}} + \frac{1}{\sqrt{a^- b^+}} = \frac{4}{\omega}, \]

\( x^\pm = \max\{\pm x, 0\}, \ y^\pm = \max\{\pm y, 0\}, \omega \in \mathbb{R}^+ \setminus \mathbb{Q}, \ f(t), g(t) \in L^\infty[0, 2\pi] \) are \( 2\pi \)-periodic functions.

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1. Introduction

In this paper, we consider the existence of unbounded solutions for the following planar system:

\[ x' = a^+ y^+ - a^- y^- + f(t), \]
\[ y' = -b^+ x^+ + b^- x^- + g(t), \]

(1)
where \(a^\pm, b^\pm\) are positive constants satisfying

\[
\frac{1}{\sqrt{a^+ b^+}} + \frac{1}{\sqrt{a^- b^+}} + \frac{1}{\sqrt{a^+ b^-}} + \frac{1}{\sqrt{a^- b^-}} = \frac{4}{\omega},
\]

(2)

and \(x^\pm = \max\{\pm x, 0\}, y^\pm = \max\{\pm y, 0\}, \omega \in R^+/Q, f(t), g(t) \in L^{\infty}[0, 2\pi]\) are \(2\pi\)-periodic functions.

Let \(a^+ = a^- = 1, b^+ = \alpha, b^- = \beta, f \equiv 0\). Then (1) is equivalent to the following second order differential equation:

\[x'' + \alpha x^+ - \beta x^- = g(t)\]

(3)

with \(\alpha, \beta\) satisfying

\[
\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = \frac{2}{\omega}.
\]

(4)

If there exists \(n \in N\) such that

\[
\frac{2}{n + 1} < \frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} < \frac{2}{n},
\]

(5)

then Fučík [5] proved that (3) has at least one \(2\pi\)-periodic solution under condition (5).

The unboundedness problem of solutions of (3) was recently discussed in [1] in case \(\alpha \neq \beta\) and

\[
\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = \frac{2m}{n},
\]

where \(m, n \in N\).

Let \(C(t)\) be the solution of the following initial value problem:

\[
x'' + \alpha x^+ - \beta x^- = 0, \quad x(0) = 1, \quad x'(0) = 0.
\]

Then it is well known that \(C(t) \in C^2(S^1 =: R/2\pi Z)\) is \(\tau\)-periodic with

\[
\tau = \frac{\pi}{\sqrt{\alpha}} + \frac{\pi}{\sqrt{\beta}}.
\]

Define a \(2\pi\)-periodic function \(\Phi_g\) if \(\tau = 2m\pi/n\), with \(m, n \in N\) by

\[
\Phi_g(\theta) = \frac{2\pi}{\int_0^{2\pi} C\left(\frac{m\theta}{n} + t\right) g(t) dt}, \quad \theta \in S^1 =: R/2\pi Z,
\]

it is proved in [1] that if the set

\[
\Omega = \{\theta \in S^1, \ \Phi_g(\theta) = 0\}
\]

is nonempty and for every \(\theta \in \Omega\), \(\Phi'_g(\theta) \neq 0\), then there exists \(R_0 > 0\) such that every solution \(x(t)\) of (3) with initial value \((x(t_0), x'(t_0))\) such that

\[
x^2(t_0) + (x'(t_0))^2 > R_0^2,
\]
for some $t_0 \in \mathbb{R}$, goes to infinity in the future or in the past. Recently, Wang [8], studied the following differential equation:

$$x'' + f(x)x' + ax^+ - bx^- = p(t), \quad (6)$$

where $a$, $b$ are positive constant satisfying

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = \alpha \in \mathbb{R}^+ \setminus \mathbb{Q}. \quad (7)$$

He showed that the Poincaré mapping of Eq. (6) in generalized polar coordinates can be written in the following form:

$$\theta_1 = \theta + 2\alpha \pi + r^{-1} \mu_1(\theta) + O(r^{-2}),$$

$$r_1 = r + \mu_2(\theta) + O(r^{-1}), \quad r \gg 1. \quad (8)$$

Then every solution of (6) with large initial value, that is, for $r_0 \gg 1$, $x(t)$ of (6) with initial value satisfying

$$x^2(t_0) + (x'(t_0))^2 \geq r_0^2$$

goes to infinity in the future or in the past if the following holds:

$$\int_0^{2\pi} \mu_2(\theta) d\theta \neq 0.$$

For more recent boundedness or unboundedness problem of solutions of (1), we refer to [2–6] and references therein. But so far few results have been obtained in the literature if $\alpha$ and $\beta$ satisfy (7) and $\mu_2(\theta) \equiv 0$. In this paper, by applying the well-known Birkhoff ergodic theorem, we obtain some sufficient conditions for the existence of unbounded solutions for Eq. (1).

2. Fučík spectrum and generalized polar coordinates transformation

Let $a^\pm, b^\pm$ be nonzero constants, we consider the Fučík spectrum [5] for the following homogeneous planar system, that is, we find conditions for the existence of nonzero periodic solution to the system

$$x' = a^+ y^+ - a^- y^-,$$

$$y' = -b^+ x^+ + b^- x^- \quad (9)$$

By reversing time if necessary, we can assume $a^+ > 0$. From (9), we see that any nonzero periodic solution of (9) must turn around the origin, hence by the assumption $a^+ > 0$, we get $a^- > 0$. Similarly we get $b^+ > 0, b^- > 0$. Next, we will show $a^\pm, b^\pm$ must satisfy (2). Since (9) is positively homogeneous, that is, if $(x(t), y(t))$ is a solution of (9), then $(kx(t), ky(t))$ is also a solution of (9) for any positive constant $k$, we can assume therefore that $(x(t), y(t))$ is a nonzero periodic solution satisfying $(x(0), y(0)) = (0, 1)$. It
is easy to see this solution turns around the origin clockwise. Let $t_k$ be the least time for $(x(t), y(t))$ stay in the $k$th quadrant. Then in the first quadrant, (9) becomes
\[
x' = a^+ y,
y' = -b^+ x.
\] (10)
Hence, by using $(x(0), y(0)) = (0, 1)$, we obtain
\[
a^+ y^2 + b^+ x^2 = a^+.
\] (11)
Substituting (11) into (10), we obtain
\[
\frac{dy}{dt} = -\sqrt{a^+ b^+ (1 - y^2)}
\] from which we obtain
\[
t_1 = \frac{1}{\sqrt{a^+ b^+}} \int_0^1 \frac{dy}{\sqrt{1 - y^2}} = \frac{\pi}{2\sqrt{a^+ b^+}}.
\]
Similarly, we can prove
\[
t_2 = \frac{\pi}{2\sqrt{a^+ b^+}}, \quad t_3 = \frac{\pi}{2\sqrt{a^- b^-}}, \quad t_4 = \frac{\pi}{2\sqrt{a^- b^-}}.
\]
Combing above results, we obtain
\[
t_1 + t_2 + t_3 + t_4 = \frac{\pi}{2} \left( \frac{1}{\sqrt{a^+ b^+}} + \frac{1}{\sqrt{a^+ b^-}} + \frac{1}{\sqrt{a^- b^+}} + \frac{1}{\sqrt{a^- b^-}} \right).
\] (12)
Therefore it follows from (12) that a necessary and sufficient condition for the $(x(t), y(t))$ to be $(2\pi/\omega)$-periodic is that $t_1 + t_2 + t_3 + t_4 = 2\pi/\omega$, which implies that (2) holds. Moreover, the origin of (9) is a global center and any nonzero solution of (9) has the same period of $2\pi/\omega$.

Let $(S(t), C(t))$ be the solution of the following initial value problem:
\[
x' = a^+ y^+ - a^- y^-,
y' = -b^+ x^+ + b^- x^-,
x(0) = 0, \quad y(0) = 1.
\]
Then it is easy to verify the following equation:
\[
a^+ (C^+(t))^2 + a^- (C^-(t))^2 + b^+ (S^+(t))^2 + b^- (S^-(t))^2 \equiv a^+, \quad \forall t \in \mathbb{R}.
\] (13)
For $r > 0, \theta \pmod{2\pi}$, we introduce the following generalized polar coordinates transformation $T: (x, y) \rightarrow (r, \theta)$ as
\[
T: \quad x = dr S \left( \frac{\theta}{\omega} \right), \quad y = dr C \left( \frac{\theta}{\omega} \right),
\] (14)
where $d = \omega/a^+$, then system (1) is changed into the following form:
\[
\theta' = \omega + r^{-1}(t) \left( C\left(\frac{\theta}{\omega}\right) f(t) - S\left(\frac{\theta}{\omega}\right) g(t) \right), \\
r' = \frac{1}{\omega} \left( S'\left(\frac{\theta}{\omega}\right) g(t) - C'\left(\frac{\theta}{\omega}\right) f(t) \right). 
\] (15)

For \( r_0 \gg 1, \theta_0 \in \mathbb{R}, t \in [0, 2\pi], \) let \((\theta(t), r(t)) = (\theta(t; \theta_0, r_0), r(t; \theta_0, r_0))\) be the solution of (15) satisfying the initial value \((\theta(0), r(0)) = (\theta_0, r_0)\).

Then \((\theta(t), r(t))\) has the following expression:

\[
\theta(t) = \theta_0 + \omega t + \int_0^t r^{-1}(\tau) \left[ C\left(\frac{\theta}{\omega} + \tau\right) f(\tau) - S\left(\frac{\theta}{\omega} + \tau\right) g(\tau) \right] d\tau, \\
r(t) = r_0 + \frac{1}{\omega} \int_0^t \left[ S'\left(\frac{\theta}{\omega} + \tau\right) g(\tau) - C'\left(\frac{\theta}{\omega} + \tau\right) f(\tau) \right] d\tau. 
\] (16)

From (16), we obtain

\[
r^{-1}(t) = r_0^{-1} + O(r_0^{-2}), \quad \forall t \in [0, 2\pi]. 
\] (17)

Going back to (15), we obtain

\[
\theta(t) = \theta_0 + \omega t + r_0^{-1} \lambda_1(t, \theta_0) + O(r_0^{-2}), \\
r(t) = r_0 + \mu_0(t, \theta_0) + O(r_0^{-1}), 
\] (18)

where

\[
\lambda_1(t, \theta) = \int_0^t \left[ C\left(\frac{\theta}{\omega} + \tau\right) f(\tau) - S\left(\frac{\theta}{\omega} + \tau\right) g(\tau) \right] d\tau 
\] (19)

and

\[
\mu_0(t, \theta) = \frac{1}{\omega} \int_0^t \left[ S'\left(\frac{\theta}{\omega} + \tau\right) g(\tau) - C'\left(\frac{\theta}{\omega} + \tau\right) f(\tau) \right] d\tau. 
\] (20)

Substituting above expressions into (16), we obtain

\[
r^{-1}(t) = r_0^{-1} + r_0^{-2} \mu_0(t, \theta_0) + O(r_0^{-3}). 
\] (21)

and then substituting (21) into (16).

Continuing in this way, we obtain the following approximate expressions:

\[
r(t) = r_0 + \mu_0(t, \theta_0) + \lambda_1(t, \theta_0)r_0^{-1} + \mu_2(t, \theta_0)r_0^{-2} + O(r_0^{-3}), \\
\theta(t) = \theta_0 + \omega t + \lambda_1(t, \theta_0)r_0^{-1} + \lambda_2(t, \theta_0)r_0^{-2} + O(r_0^{-3}). 
\] (22)

Let

\[
S = S\left(\frac{\theta}{\omega} + \cdot\right), \quad C = C\left(\frac{\theta}{\omega} + \cdot\right), \quad f = f(\cdot), \quad g = g(\cdot). 
\]
Substituting (22) into (16), we obtain the following recursive formulas:

\[
\mu_1(t, \theta) = \frac{1}{\omega} \int_0^t [S''g - C''f] \lambda_1 d\tau,
\]

\[
\lambda_2(t, \theta) = \int_0^t \left[ \frac{1}{\omega} (C'f - S'g) \lambda_1 - (Cf - Sg)\mu_0 \right] d\tau,
\]

\[
\mu_2(t, \theta) = \frac{1}{\omega^2} \left\{ \int_0^t [S''g - C''f] \lambda_2 d\tau + \frac{1}{2\omega} \int_0^t [S'''g - C'''f] \lambda_1^2 d\tau \right\},
\]

\[
\lambda_3(t, \theta) = \int_0^t \left[ \frac{1}{\omega} (C'f - S'g) \lambda_2 + \frac{1}{2\omega^2} (C''f - S''g) \lambda_1^2 \right.
\]

\[- \frac{1}{\omega} (C'f - S'g) \lambda_1 \mu_0 + (Cf - Sg)(\mu_0^2 - \mu_1) \bigg] d\tau,
\]

where

\[
\lambda_k = \lambda_k(\cdot, \theta), \quad k = 1, 2,
\]

\[
\mu_m = \mu_m(\cdot, \theta), \quad m = 0, 1.
\]

If we define \( r_1 = r(2\pi), \theta_1 = \theta(2\pi), \lambda_k(\theta) = \lambda_k(2\pi, \theta), \mu_{k-1}(\theta) = \mu_{k-1}(2\pi, \theta), k = 1, 2, 3, \) in (19)–(23), then we obtain the following approximate expansions:

\[
r_1 = r + \mu_0(\theta) + \mu_2(\theta)r^{-1} + \mu_2(\theta)r^{-2} + O(r^{-3}),
\]

\[
\theta_1 = \theta + 2\omega\pi + \lambda_1(\theta)r^{-1} + \lambda_2(\theta)r^{-2} + \lambda_3(\theta)r^{-3} + O(r^{-4}).
\]

Moreover, we have

**Lemma 1.** The following equalities hold:

\[
\mu_0(\theta) = -\frac{1}{\omega} \int_0^{2\pi} C(\theta/\omega + t)f(t) - S(\theta/\omega + t)g(t) dt,
\]

\[
\lambda_1(\theta) = \int_0^{2\pi} \left[ C(\theta/\omega + t)f(t) - S(\theta/\omega + t)g(t) \right] dt,
\]

\[
\mu_1(\theta) = -\frac{1}{\omega^2} \int_0^{2\pi} \left[ C''(\theta/\omega + t)f(t) - S''(\theta/\omega + t)g(t) \right] dt
\]

\[- \int_0^t \left( C(\theta/\omega + \tau)f(\tau) - S(\theta/\omega + \tau)g(\tau) \right) d\tau dt,
\]
\[ \dot{\lambda}_2(\theta) = \lambda_1(\theta)\lambda'_1(\theta), \]  
(30)  
\[ \mu_2(\theta) = -\frac{1}{\omega^2} \left[ \int_0^{2\pi} (C'' f - S'' g) \frac{d\tau}{\tau} \int_0^t (C f - S g) d\tau dt \right. \]  
\[ + \left. \frac{1}{2} \int_0^{2\pi} (C'' f - S'' g) \left( \int_0^t (C f - S g) d\tau \right)^2 dt \right], \]  
(31)  
\[ \dot{\lambda}_3(\theta) = -\frac{1}{2\omega^2} \int_0^{2\pi} (C'' f - S'' g) \left( \int_0^t (C f - S g) d\tau \right)^2 dt \]  
\[ + \lambda_1(\theta) \left( (\lambda'_1(\theta))^2 - \mu_1(\theta) \right). \]  
(32)

**Proof.** By using (16)–(22) and integration by parts, we obtain above equations. \( \square \)

**Lemma 2.** From above equalities, we can prove the following relations:

\[ \mu_0(\theta) = -\lambda'_1(\theta), \]  
\[ \mu_2(\theta) = \lambda'_3(\theta) - 2\lambda_1(\theta)\lambda'_1(\theta)\lambda''_1(\theta) - \left( \lambda'_1(\theta) \right)^3 + \lambda'_1(\theta)\mu_1(\theta) + \lambda_1(\theta)\mu'_1(\theta). \]

3. Unbounded motions of planar mappings

In this section, we adopt the notations used in [1]. Given \( \sigma > 0 \), let the set \( E_\sigma \) be the exterior of the open ball \( B_\sigma \) centered at the origin and of radius \( \sigma \), that is, \( E_\sigma = \mathbb{R}^2 - B_\sigma \), then \( E_\sigma = \{(\theta, r) \mid r \geq \sigma \} \). Define \( S^1 = \mathbb{R} \setminus 2\pi \mathbb{Z} \), then the points in \( S^1 \) are defined by \( \bar{\theta} = \theta + 2k\pi, \ k \in \mathbb{Z}, \ \theta \in \mathbb{R} \), and the group distance in \( S^1 \) is defined by \( \| \bar{\theta} \| = \min \{ |\theta + 2k\pi| \mid k \in \mathbb{Z} \} \).

Let \( \bar{P} : E_\sigma \to \mathbb{R}^2 \) be a mapping that is one to one and continuous. We assume that its lift, denoted by \( P \), can be expressed in the following form:

\[ \theta_1 = \theta + 2\omega \pi + \lambda_k(\theta)r^{-k} + F_k(r, \theta), \]  
\[ r_1 = r + \mu_m(\theta) r^{-m} + G_m(r, \theta) \]  
(33)

for \( r \geq \sigma, \theta \in S^1 \) and \( \lambda_k, \mu_m \in C(S^1), k \geq 1, m \geq 0, F_k = O(r^{-(k+1)}), G_m = O(r^{-(m+1)}) \) are continuous and \( 2\pi \)-periodic in \( \theta \). Given a point \( (\theta_0, r_0) \in E_\sigma \), let \( \{(\theta_k, r_k)\}_{k \in \mathbb{Z}} \) be the unique solution of the initial value problem for the following difference equation:

\[ (\theta_{k+1}, r_{k+1}) = P(\theta_k, r_k). \]
This solution is defined in a maximal interval

\[ I = \{ k \in \mathbb{Z} \mid k_a < k < k_b \}, \]

where \( k_a, k_b \) are certain numbers in the set \( \mathbb{Z} \cup \{ +\infty, -\infty \} \) satisfying

\[-\infty \leq k_a < 0 < k_b \leq +\infty.\]

The solution \( \{(\theta_k, r_k)\} \) is said to be defined in the future if \( k_b = +\infty \) and is said to be defined in the past if \( k_a = -\infty \).

**Proposition 1.** Assume the above conditions hold and

\[ 2\pi \int_0^\mu_m(\theta) \, d\theta > 0. \]

Then there exists \( R_0 > \sigma \), such that if \( r_0 \geq R_0 \), the orbit \( \{(\theta_n, r_n)\} \) of (33) with initial value \( (\theta_0, r_0) \) is defined in the future and satisfies

\[ \lim_{n \to +\infty} r_n = +\infty. \]

**Proof.** By induction one can prove for each \( n \in M \),

\[ \theta_n = \theta_0 + 2n\omega \pi + \left( \sum_{i=0}^{n-1} \lambda_k(\theta_0 + 2i\omega \pi) \right) r_0^{-k} + O(r_0^{-(k+1)}), \]

\[ r_n = r_0 + \left( \sum_{i=0}^{n-1} \mu_m(\theta_0 + 2i\omega \pi) \right) r_0^{-m} + O(r_0^{-(m+1)}). \]

(34)

Define a transformation \( T : S^1 \to S^1 \) as \( T(\theta) = \theta + 2\omega \pi \). Since \( \omega \in \mathbb{R}^+ / \mathbb{Q} \), \( T \) is ergodic. It follows from the Birkhoff ergodic theorem [7, Theorem 1.14] that

\[ \lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu_m(\theta + 2i\omega \pi) = \lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu_m(T^i \theta) = \frac{1}{2\pi} \int_0^{2\pi} \mu_m(\theta) \, d\theta > 0 \]

for almost every \( \theta \in S^1 \). Next we show that

\[ \lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu_m(\theta + 2i\omega \pi) = \frac{1}{2\pi} \int_0^{2\pi} \mu_m(\theta) \, d\theta > 0 \]

holds uniformly in \( \theta \in S^1 \).

In fact, we can assume \( S^1 = [0, 2\pi] \). Let \( I \) be a subset of \([0, 2\pi]\) with measure 0 such that for each \( \theta \in [0, 2\pi] \setminus I \) the above limit hold. This means that for each \( \varepsilon > 0 \), there exists \( n_\varepsilon \in \mathbb{N} \) such that

\[ \left| \frac{1}{n} \sum_{k=0}^{n-1} \mu_m(\theta + 2k\omega \pi) - \mu_m \right| < \frac{\varepsilon}{2}. \]
where
\[ \bar{\mu}_m = \frac{1}{2\pi} \int_0^{2\pi} \mu_m(\theta) d\theta. \]

By the continuity of \( \mu_m \) and the compactness of \([0, 2\pi]\), there exists \( \delta > 0 \) such that for any \( \theta, \theta' \in [0, 2\pi] \) with \( |\theta - \theta'| < \delta \), we have
\[ |\mu_m(\theta) - \mu_m(\theta')| < \frac{\varepsilon}{2}. \]

For any \( \theta_0 \in I \), there exists \( \theta \in [0, 2\pi] \setminus I \) such that \( |\theta - \theta_0| = |(\theta + 2k\omega\pi) - (\theta_0 + 2k\omega\pi)| < \delta \) and hence for \( n \geq n_\varepsilon \), we have
\[
\frac{1}{n} \left| \sum_{k=0}^{n-1} \mu_m(\theta_0 + 2k\omega\pi) - \frac{\mu_m}{n} \right|
\leq \frac{1}{n} \left| \sum_{k=0}^{n-1} \mu_m(\theta + 2k\omega\pi) - \frac{\mu_m}{n} \right|
+ \frac{1}{n} \left| \sum_{k=0}^{n-1} \mu_m(\theta + 2k\omega\pi) - \bar{\mu}_m \right|
< \frac{1}{n} \varepsilon + \frac{\varepsilon}{2} = \varepsilon.
\]

This shows that the uniform convergence claims.

It follows from above analysis that there exists an integer \( p \in N \) and a constant \( \delta > 0 \) such that
\[ \frac{1}{p} \sum_{i=0}^{p-1} \mu_m(\theta + 2i\omega\pi) \geq \delta > 0 \]
for all \( \theta \in S^1 \). Therefore for \( r_0 \gg 1 \), it follows from (34) that
\[
r_p = r_0 + \frac{1}{p} \sum_{i=0}^{p-1} \mu_m(\theta_0 + 2i\omega\pi) r_0^{-m} + O(r_0^{-(m-1)}) \geq r_0 + (p-1)\delta r_0^{-m}.
\]
\[
r_p = r_0 + \frac{1}{p} \sum_{i=0}^{p-1} \mu_m(\theta_0 + 2i\omega\pi) r_0^{-m} + O(r_0^{-(m-1)}) \leq r_0 + (p+1)\delta r_0^{-m}.
\]

Inductively, we obtain for \( n \in N \),
\[ r_{np} \geq r_0 + n(p-1)\delta r_0^{-m}, \]
and
\[ r_{np} \leq r_0 + n(p+1)\delta r_0^{-m}. \]

Which implies that the solution \( r_{np} \) is defined in the future and satisfies
\[ \lim_{n \to +\infty} r_{np} = +\infty. \]
Moreover, it is not difficult to show that for each \( j \in \{1, 2, \ldots, p - 1\} \), we have
\[
\lim_{n \to +\infty} r_{np+j} = +\infty.
\]
Therefore we see that \( r_n \) is defined in the future and
\[
\lim_{n \to +\infty} r_n = +\infty.
\]

Similarly we can prove the following

**Proposition 2.** If the above conditions hold and
\[
2\pi \int_0^{2\pi} \mu_m(\theta) \, d\theta < 0,
\]
then there exists \( R_0 > \sigma \) such that if \( r_0 \geq R_0 \), the orbit \( (\theta_n, r_n) \) is defined in the past and satisfies
\[
\lim_{n \to -\infty} r_n = +\infty.
\]

We state now the main results of this paper.

**Theorem 1.** Let the assumptions on \( f, g, \alpha, \beta \) in Section 1 be satisfied and let the functions \( \lambda_k(\theta), \mu_{k-1}(\theta) \), \( k = 1, 2, 3 \), be given by (26)–(32). Suppose the following assumption holds:
\[
\mu_0(\theta) \equiv 0, \quad \int_0^{2\pi} \mu_1(\theta) \, d\theta \neq 0.
\]
Then there exists \( R_0 > 0 \) such that every solutions \( x(t) \) of (1) with initial value \((x(0), x'(0))\) such that
\[
x^2(0) + (x'(0))^2 \geq R_0^2,
\]
goes to infinity in the future or in the past.

**Proof.** Under the generalized polar coordinates transformation (14), the Poincaré mapping of (1) is equivalent to the following system:
\[
\begin{align*}
\theta_1 &= \theta_0 + 2\omega \pi + \lambda_1(\theta_0)r_0^{-1} + \lambda_2(\theta_0)r_0^{-2} + O(r_0^{-3}), \\
r_1 &= r_0 + \mu_0(\theta_0) + \mu_1(\theta_0)r_0^{-1} + O(r_0^{-2}).
\end{align*}
\]
Now Theorem 1 follows from Propositions 1 and 2 for \( \lambda'_1(\theta) = \mu_0(\theta) \equiv 0 \) and \( \mu_m(\theta) = \mu_1(\theta) \).

**Remark.** Since \( S \in C^2(R) \), for \( k \geq 3 \) the functions \( S^{(k)} \) and \( C^{(k-1)} \) in (28)–(32) are not defined in some finite number of points, but as they are bounded, the functions of \( \lambda_j \) and \( \mu_{j-1} \) for \( j \geq 2 \) are well defined.
Example. Consider the following second order differential equation:
\[ x'' + \alpha x^+ - \beta x^- = p(t), \]
where \( \alpha \neq \beta \) satisfy (4), \( p(t) \in L^\infty[0, 2\pi] \) is \( 2\pi \)-periodic and piecewise constant. Then numerical calculation shows that we can choose \( p(t) \) such that
\[ \mu_0(\theta) \equiv 0 \]
and
\[ \int_0^{2\pi} \mu_1(\theta) d\theta \neq 0. \]
In this case, Theorem 1 implies that all solutions of with large initial values goes to infinity either in future or in the past.

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