

Stability of Quasi-linear Hyperbolic Dissipative Systems

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1. INTRODUCTION

In this work we want to explore the relationship between certain eigenvalue condition for the symbols of first order partial differential operators describing evolution processes and the linear and nonlinear stability of their stationary solutions.

Consider the initial value problem for the following general first order quasi-linear system of equations

$$v_t = P(v, x, t, \nabla)v = \sum_{v=1}^s A_v(v, x, t) \frac{\partial}{\partial x_v} v + B(v, x, t)v,$$

$$v(x, 0) = f(x).$$

Here v is a (column) vector valued function of the real space variables (x_1, \dots, x_s) and time t with components v_1, \dots, v_n . A_v and B are $n \times n$ matrices and $f(x)$ is a vector valued function of the space variables.

We are interested in solutions which are 2π -periodic in all space variables. There is no difficulty to extend the results to the Cauchy problem on the whole x -space. Instead of Fourier series we would use Fourier integrals.

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We shall restrict our considerations to the case

$$u_t = \sum_{v=0}^s (A_{0v} + \varepsilon A_{1v}(x, t, u, \varepsilon)) \frac{\partial}{\partial x_v} u + (B_0 + \varepsilon B_1(x, t, u, \varepsilon))u. \quad (1.1)$$

Here A_{0v}, B_0 are constant matrices and ε is a small parameter. This is, for instance, the case when the stationary solution is constant and we consider the solution close to the steady state.

Assumption 1.1. For every $p = 0, 1, 2, \dots$ and any $c > 0$, there is a constant K_p such that the maximum norm of the p th derivatives of A_{1v}, B_1 with respect to x, t, ε and u are bounded by K_p , provided $|u|_\infty \leq c$. For $f(x)$, the corresponding estimates hold.

DEFINITION 1.1. The system (1.1) is said to satisfy the *stability eigenvalue condition* if there is a constant $\delta > 0$ such that, for all real ω , the eigenvalues λ of the symbol

$$\hat{P}_0(i\omega) + B_0 := i \sum_{v=1}^s A_{0v} \omega_v + B_0 \quad (1.2)$$

satisfy

$$\operatorname{Re} \lambda \leq -\delta. \quad (1.3)$$

We have to define stability for system (1.1).

DEFINITION 1.2. The system (1.1) is stable if, for any f , there exists an ε_0 such that, for $0 \leq \varepsilon \leq \varepsilon_0$, the solutions of (1.1) converge to zero for $t \rightarrow \infty$; and there exists an integer p_0 such that ε_0 depends only on the constants K_p with $p \leq p_0$.¹

In this work we shall look at sufficient conditions under which the stability eigenvalue condition implies stability.

Consider first the constant coefficient case, i.e., set $\varepsilon = 0$ in the above system. In Section 2 we shall prove that it is possible to find a positive definite selfadjoint operator H_0 such that all solutions of the system satisfy

$$\frac{d}{dt}(u, H_0 u) \leq -\delta(u, H_0 u),$$

¹ We have not specified the norm under which that convergence takes place, but we shall be using uniform pointwise convergence.

provided that the problem is well posed in the L_2 sense and the eigenvalue condition is satisfied. In this case the system of equations is a contraction in a new norm.

In Section 3 we consider linear systems with variable coefficients, i.e., the A_{1v} depend on x and t but not on u . The construction of H proceeds via the theory of pseudo-differential operators, i.e., we construct the symbol $\hat{H}(x, t, \omega)$ and define the operator H by

$$Hu = \sum_{\omega} e^{i\langle \omega, x \rangle} \hat{H}(x, t, \omega) \hat{u}(\omega) \quad \text{for all } u = \sum_{\omega} e^{i\langle \omega, x \rangle} \hat{u}(\omega).$$

\hat{H} depends on the symbols

$$\hat{P}_0(i\omega) =: \sum_{v=1}^s A_{0v} i\omega_v, \quad \hat{P}_1(x, t, i\omega) =: \sum_{v=1}^s A_{1v}(x, t) i\omega_v.$$

We need that \hat{H} is a smooth function of all variables. This is only the case if \hat{P}_0, \hat{P}_1 satisfy extra restrictions. For the linear and the nonlinear case, we make one of the following assumptions.

Assumption 1.2. The stability eigenvalue condition is satisfied and the multiplicities of the eigenvalues of $\hat{P}_0(i\omega') + \varepsilon \hat{P}_1(x, t, u, i\omega')$ do not depend on $x, t, u, \omega', \varepsilon$. Also, for every $x, t, u, \omega', \varepsilon$, there is a complete system of eigenvectors.

Assumption 1.3. The stability eigenvalue condition is satisfied and the matrices A_{0v}, B_0 and $A_{1v}, v = 1, \dots, s$, are Hermitian.

Under any of these conditions we can again construct an H -norm and prove that the problem becomes a contraction.

In the last section we consider the nonlinear equations and the main result of this paper is

MAIN THEOREM. *Suppose that Assumption 1.1 and Assumption 1.2 or 1.3 hold. Then, for sufficiently small ε , the problem is a contraction in a suitable H -norm and the system (1.1) is thus stable.*

In the Appendix we relax the eigenvalue condition somewhat.

To prove stability for time dependent partial differential equations via changing the norm has been done before. For example, in [1] the method was applied to mixed symmetric hyperbolic-parabolic equations which included the Navier–Stokes equations. In that case H was explicitly constructed and not related to an eigenvalue condition. If we make Assumption 1.3, then our H is similar to the H in [1].

2. SYSTEMS WITH CONSTANT COEFFICIENTS

In this section we consider the system

$$y_t = \sum_{v=0}^s A_{0v} \frac{\partial y}{\partial x_v} + B_0 y =: \left(P_0 \left(\frac{\partial}{\partial x} \right) + B_0 \right) y, \tag{2.1}$$

$$y(x, 0) = f(x),$$

with constant coefficients. We are interested in solutions which are 2π -periodic in all space variables. We assume that the problem is well posed in the L_2 sense, i.e., for every T there exists a constant $K(T)$ such that the solutions of (2.1) satisfy the estimate

$$\|y(\cdot, t)\| \leq K(T) \|y(\cdot, 0)\|, \quad 0 \leq t \leq T. \tag{2.2}$$

Here

$$(u, v) = \int_0^{2\pi} \dots \int_0^{2\pi} \langle u, v \rangle dx_1 \dots dx_s, \quad \|u\|^2 = (u, u)$$

denote the usual L_2 scalar product and norm.

One can characterize well posed problems algebraically. Using the Kreiss matrix theorem (see [2], Sec. 2.3), one can prove

THEOREM 2.1. *The problem (2.1) is well posed in the L_2 sense if and only if it is strongly hyperbolic, i.e., the eigenvalues of the symbol*

$$\hat{P}_0(i\omega) = i \sum_{v=1}^s A_{0v} \omega_v, \quad \omega_j \text{ real},$$

are purely imaginary and, for every fixed $\omega' = \omega/|\omega|$, there exists a complete set of eigenvectors t_1, \dots, t_n which is uniformly independent, i.e., there is a constant K such that

$$|T^{-1}| + |T| \leq K, \quad T = (t_1, \dots, t_n).$$

We can expand the solution of (2.1) into a Fourier series

$$y(x, t) = \sum_{\omega} e^{i\langle \omega, x \rangle} \hat{y}(\omega, t). \tag{2.3}$$

The Fourier coefficients are the solution of the Fourier transformed system (2.1)

$$\hat{y}_t = \left(i \sum_{v=0}^s A_{0v} \omega_v + B_0 \right) \hat{y} =: (\hat{P}_0(i\omega) + B_0) \hat{y}. \tag{2.4}$$

We assume that the eigenvalue condition (1.2), (1.3) is satisfied. Then we can find, for every fixed ω , a positive definite Hermitian matrix \hat{H} , a Lyapunov function, such that

$$\begin{aligned} 2 \operatorname{Re} \hat{H}(\omega)(\hat{P}_0(i\omega) + B_0) &=: \hat{H}(\omega)(\hat{P}_0(i\omega) + B_0) + (\hat{P}_0^*(i\omega) + B_0^*) \hat{H}(\omega) \\ &\leq -\delta \hat{H}(\omega). \end{aligned} \quad (2.5)$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial t} \langle \hat{y}(\omega, t), \hat{H}(\omega) \hat{y}(\omega, t) \rangle &= 2 \operatorname{Re} \langle \hat{y}(\omega, t), \hat{H}(\omega)(\hat{P}_0(i\omega) + B_0) \hat{y}(\omega, t) \rangle \\ &\leq -\delta \langle \hat{y}(\omega, t), \hat{H}(\omega) \hat{y}(\omega, t) \rangle. \end{aligned}$$

Thus, for every fixed ω , the transformed system (2.4) is a contraction in the $\hat{H}(\omega)$ -norm.

Using the Kreiss matrix theorem, one can prove (see [2, Sec. 2.3])

THEOREM 2.2. *Assume that the problem (2.1) is well posed in the L_2 sense and that the eigenvalue condition (1.2), (1.3) is satisfied. Then we construct the matrices $\hat{H}(\omega)$ such that they satisfy the uniform inequalities*

$$K_4^{-1} I \leq \hat{H}(\omega) \leq K_4 I. \quad (2.6)$$

Here K_4 does not depend on ω .

We can use $\hat{H}(\omega)$ to define an operator H by

$$Hu = \sum_{\omega} \hat{H}(\omega) \hat{u}(\omega) e^{i\langle \omega, x \rangle}. \quad (2.7)$$

It has the following properties

- (1) H is selfadjoint and

$$K_4^{-1} \|u\|^2 \leq (u, Hu) \leq K_4 \|u\|^2.$$

- (2) $2 \operatorname{Re} H(P_0 + B_0) =: H(P_0 + B_0) + (P_0^* + B_0^*) H \leq -\delta H$.

These properties follow from Parseval's relation

$$\begin{aligned} (v, Hu) &= \sum_{\omega} \langle \hat{v}(\omega), \hat{H}(\omega) \hat{u}(\omega) \rangle \\ &= \sum_{\omega} \langle \hat{H}(\omega) v(\omega), \hat{u}(\omega) \rangle = (Hv, u). \end{aligned}$$

Also,

$$\begin{aligned} K_4^{-1} \|u\|^2 &= K_4^{-1} \sum_{\omega} |\hat{u}(\omega)|^2 \leq \sum_{\omega} \langle \hat{u}(\omega), \hat{H}(\omega) \hat{u}(\omega) \rangle \\ &= (u, Hu) \leq K_4 \|u\|^2 \end{aligned}$$

and

$$\begin{aligned} 2(u, \operatorname{Re} H(P_0 + B_0) u) &= 2 \sum_{\omega} \langle \hat{u}(\omega), \operatorname{Re} \hat{H}(\omega)(\hat{P}_0(i\omega) + B_0) \hat{u}(\omega) \rangle \\ &\leq -\delta \sum_{\omega} \langle \hat{u}(\omega), \hat{H}(\omega) \hat{u}(\omega) \rangle = -\delta(u, Hu). \end{aligned}$$

Thus, we can use H to define a new scalar product by

$$(v, u)_H = (v, Hu), \quad \|u\|_H^2 = (u, u)_H,$$

which is equivalent with the L_2 -norm. The second property gives us

THEOREM 2.3. *If the conditions of Theorem 2.2 are satisfied, then the problem (2.1) is a contraction in the H -norm.*

Proof.

$$\frac{\partial}{\partial t} (y, Hy) = 2 \operatorname{Re}(y, H(P_0 + B_0) y) \leq -\delta(y, Hy).$$

This proves the theorem.

3. LINEAR SYSTEMS WITH VARIABLE COEFFICIENTS

In this section we want to generalize Theorem 2.2 to linear systems

$$\begin{aligned} v_t &= \sum_{v=0}^s (A_{0v} + \varepsilon A_{1v}(x, t)) \frac{\partial v}{\partial x_v} + (B_0 + \varepsilon B_1) v \\ &=: \left(P_0 \left(\frac{\partial}{\partial x} \right) + B_0 + \varepsilon \left(P_1 \left(x, t, \frac{\partial}{\partial x} \right) + B_1 \right) \right) v \end{aligned} \quad (3.1)$$

and show that it is a contraction in a suitable H -norm. We shall construct the H -norm with help of a pseudo-differential operator

$$H(t) = H_0 + S + \varepsilon H_1(t) \quad (3.2)$$

with the following properties.

(1) $H_0, S, H_1(t)$ are bounded selfadjoint operators. H_0 and S do not depend on t . dH_1/dt exists and is also a bounded operator. Thus, there is a constant K such that

$$\|H_0\| + \|S\| + \|H_1(t)\| + \left\| \frac{dH_1}{dt} \right\| \leq K.$$

(2) $H_0 + S$ is positive definite with K such that

$$\|H_0 + S\| + \|(H_0 + S)^{-1}\| \leq K.$$

(3) $2 \operatorname{Re} H_0 P_0 =: H_0 P_0 + P_0^* H_0 \equiv 0$.

(4) $2 \operatorname{Re}(H_0 + S)(P_0 + B_0) = 2 \operatorname{Re}(S P_0 + H_0 B_0) \leq -\delta(H_0 + S)$.

(5) S is a smoothing operator with

$$\|S P_1\| \leq K.$$

(6) $\|\operatorname{Re}(H_0 + \varepsilon H_1(t))(P_0 + \varepsilon P_1)\|$
 $= \varepsilon \|\operatorname{Re}(H_0 P_1 + H_1 P_0 + \varepsilon H_1 P_1)\| \leq \varepsilon K$.

We can prove

THEOREM 3.1. *Assume that there is an operator H of the form (3.2) with the properties (1)–(6). For sufficiently small ε the scalar product (u, Hv) defines a norm which is equivalent with the L_2 -norm and the system (3.1) is a contraction in the H -norm.*

Proof. That (u, Hv) defines a norm which is equivalent with the L_2 -norm follows from properties (1) and (2). Also,

$$\begin{aligned} \frac{\partial}{\partial t} (u, Hu) &= \varepsilon (u, H_{1t} u) + 2 \operatorname{Re}(u, (H_0 + S + \varepsilon H_1)(P_0 + B_0 + \varepsilon(P_1 + B_1)) u) \\ &= \varepsilon (u, H_{1t} u) + 2 \operatorname{Re}(u, (H_0 + S)(P_0 + B_0) u) \\ &\quad + 2 \operatorname{Re}(u, (H_0 + \varepsilon H_1)(P_0 + \varepsilon P_1) u) \\ &\quad + 2\varepsilon \operatorname{Re}(u, (H_0 B_1 + S(P_1 + B_1) + H_1(B_0 + \varepsilon B_1)) u) \\ &\leq -(\delta + \mathcal{O}(\varepsilon)) (u, Hu). \end{aligned}$$

This proves the theorem.

We construct the symbol of the pseudo-differential operator (3.2) in the following way. Consider all systems with constant coefficients which we obtain by freezing the coefficients of (3.1) at every point $x = x_0, t = t_0$. We assume that the initial value problem for all these systems is well posed in the L_2 sense and, therefore, we can construct the matrices $\hat{H}(x, t, \omega)$ for every fixed x, t . Now we think of $\hat{H}(x, t, \omega)$ as a symbol of a pseudo-differential operator where x, t are independent variables. Formally, we define the operator H by

$$Hu = \sum_{\omega} e^{i\langle \omega, x \rangle} \hat{H}(x, t, \omega) \hat{u}(\omega).$$

This definition makes sense only if \hat{H} satisfies the usual properties of symbols for pseudo-differential operators. Also, we need the algebra for such operators to prove that (3.1) becomes a contraction. We want to prove

THEOREM 3.2. *Assume that the following conditions hold.*

(a) *There exists a positive definite Hermitian matrix $\tilde{H}_0(\omega')$ which is a smooth function of $\omega' = \omega/|\omega|$ such that*

$$2 \operatorname{Re} \tilde{H}_0(\omega') \hat{P}_0(i\omega) =: \tilde{H}_0(\omega') \hat{P}_0(i\omega) + \hat{P}_0^*(i\omega) \tilde{H}_0(\omega') \equiv 0. \quad (3.3)$$

(b) *For sufficiently large $|\omega|$, there is a Hermitian matrix $\tilde{S} = \tilde{S}(\omega', 1/|\omega|)$ which is a smooth function of ω' and $1/|\omega|$ such that*

$$\begin{aligned} 2 \operatorname{Re} \left(\tilde{H}_0(\omega') + \frac{1}{|\omega|} \tilde{S}(\omega', 1/|\omega|) \right) (|\omega| \hat{P}_0(i\omega') + B_0) \\ \leq -\delta \left(\tilde{H}_0(\omega') + \frac{1}{|\omega|} \tilde{S}(\omega', 1/|\omega|) \right). \end{aligned} \quad (3.4)$$

(c) *There exists a Hermitian matrix $\tilde{H}_1(x, t, \omega')$ which is a smooth function of x, t, ω' such that*

$$2 \operatorname{Re}(\tilde{H}_0(\omega') + \varepsilon \tilde{H}_1(x, t, \omega', \varepsilon))(\hat{P}_0(i\omega') + \varepsilon \hat{P}_1(x, t, i\omega')) = 0. \quad (3.5)$$

Then we can construct the pseudo-differential operator (3.2) which has the properties (1)–(6). Also, there exists an integer p_0 such that the constant K depends only on the first p_0 derivatives of the symbols and of the coefficients of (3.1). Thus, the problem (3.1) is a contraction in the H -norm.

Proof. We construct the symbols for the pseudo-differential operators

$$\begin{aligned} H_0 u &= \sum_{\omega} e^{i\langle \omega, x \rangle} \hat{H}_0(\omega) \hat{u}(\omega), \\ Su &= \sum_{\omega} e^{i\langle \omega, x \rangle} \hat{S}(\omega) \hat{u}(\omega) \\ H_1 u &= \sum_{\omega} e^{i\langle \omega, x \rangle} \hat{H}_1(x, t, \omega) \hat{u}(\omega). \end{aligned} \tag{3.6}$$

$\hat{H}_0(\omega)$, $\hat{S}(\omega)$ do not depend on x, t .

Let $C > 0$ be a constant. Consider the symbol (1.2) for $|\omega| \leq C$. The inequality (1.3) implies (see Lemma 3.2.9 in [2]) that there is a positive definite Hermitian matrix $\tilde{S}^{(1)}(\omega)$ which is a smooth function of ω such that

$$2 \operatorname{Re} \tilde{S}^{(1)}(\omega) (\hat{P}_0(i\omega) + B_0) \leq -\delta \tilde{S}^{(1)}(\omega), \quad |\omega| \leq C + 1.$$

Let $\varphi(|\omega|) \in C^\infty$ be a monotone cut-off function with

$$\varphi(|\omega|) = \begin{cases} 1 & \text{for } |\omega| \geq C + 1 \\ 0 & \text{for } |\omega| \leq C \end{cases}.$$

We define

$$\begin{aligned} \hat{H}_0(\omega) &= \varphi(|\omega|) \tilde{H}_0(\omega'), \\ \hat{S}(\omega) &= \frac{\varphi(|\omega|)}{|\omega|} \tilde{S}(\omega', 1/|\omega|) + (1 - \varphi(|\omega|)) \tilde{S}^{(1)}(\omega). \end{aligned}$$

It follows from (3.3) and (3.4) that, for sufficiently large C , the operators H_0 and S have the properties (1)–(5). The symbol

$$\varphi(|\omega|) (\tilde{H}_0(\omega') + \varepsilon \tilde{H}_1(\omega', x, t))$$

defines a pseudo-differential operator $H_0 + \varepsilon H_{11}$ and the algebra of such operators shows that (see, for example, [3])

$$H_0 + \varepsilon H_1 = H_0 + \frac{\varepsilon}{2} (H_{11} + H_{11}^*) \tag{3.7}$$

has the desired properties (1) and (6) and K can be estimated as required. This proves the theorem.

We shall now give algebraic conditions such that the conditions of Theorem 3.2 are satisfied.

For large $|\omega|$, we can consider the second matrix in (3.9) as a small perturbation of the first. Therefore, (again, see [2]) there is a smooth transformation $T_1(\omega', 1/|\omega|)$ such that

$$\begin{aligned} & \left(I + \frac{1}{|\omega|} T_1 \right)^{-1} T_0^{-1} (|\omega| P_0(i\omega') + B_0) T_0 \left(I + \frac{1}{|\omega|} T_1 \right) \\ &= |\omega| \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_r \end{pmatrix} + \begin{pmatrix} T_{01}^{-1} \tilde{B}_{11} T_{01} + \frac{1}{|\omega|} \tilde{\tilde{B}}_{11} & & 0 \\ & \ddots & \\ 0 & & T_{0r}^{-1} \tilde{B}_{rr} T_{0r} + \frac{1}{|\omega|} \tilde{\tilde{B}}_{rr} \end{pmatrix}. \end{aligned}$$

By assumption, the eigenvalues of \tilde{B}_{jj} have negative real parts. Therefore, we can choose T_{0j} such that

$$2 \operatorname{Re}(T_{0j}^{-1} \tilde{B}_{jj} T_{0j}) \leq -\frac{3\delta}{2} I, \quad |\omega' - \omega'_0| \quad \text{sufficiently small.}$$

(Again, see Lemma 3.2.9 in [2].) Thus,

$$\tilde{H}_0 = (T_0^{-1})^* T_0$$

and, for sufficiently large $|\omega|$,

$$\tilde{H}_0 + \frac{1}{|\omega|} \tilde{S} = \left(\left(\left(I + \frac{1}{|\omega|} T_1 \right) T_0 \right)^{-1} \right)^* \left(\left(I + \frac{1}{|\omega|} T_1 \right) T_0 \right)^{-1}$$

satisfies (3.3) and (3.4). By the usual partition of unity argument, we can construct \tilde{H}_0 and \tilde{S} for all ω' and conditions (a) and (b) in Theorem 3.2 hold.

We now consider the matrix (symbol)

$$\hat{P}_0(i\omega') + \varepsilon \hat{P}_1(x, t, i\omega'). \quad (3.10)$$

As the eigenvalues of (3.10) are purely imaginary and their multiplicity does not change, we can find a smooth transformation $T_2(x, t, \omega', \varepsilon)$ such that

$$\begin{aligned} & (I + \varepsilon T_2)^{-1} T_0^{-1} (\hat{P}_0 + \varepsilon \hat{P}_1) T_0 (I + \varepsilon T_2) \\ &= \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_r \end{pmatrix} + \varepsilon \begin{pmatrix} \tilde{A}_1 & & 0 \\ & \ddots & \\ 0 & & \tilde{A}_r \end{pmatrix}. \end{aligned}$$

Here $\tilde{A}_j = \tilde{\lambda}_j I$ and T_2 is a smooth function of all variables. The matrix

$$\tilde{H}_0 + \varepsilon \tilde{H}_1 = (T^{-1})^* T^{-1}, \quad T = T_0(I + \varepsilon T_2),$$

has the property (3.5) and condition (c) in Theorem 3.2 hold. Therefore, Theorem 3.3 follows from Theorem 3.2.

We consider now the symmetric systems (3.1), i.e., those satisfying Assumption 1.3. In this case the stability eigenvalue condition, for $\omega = 0$, implies that

$$\operatorname{Re} B_0 \leq -\delta I, \tag{3.11}$$

and therefore

$$\operatorname{Re}(u, (P_0 + B_0) u) \leq -\delta(u, u).$$

Thus, we can show that (3.1) is a contraction in the usual L_2 -norm ($H = I$). In the Appendix we shall relax the eigenvalue condition to some cases where (3.11) does not hold. Therefore we give here a proof which does not depend on (3.11).

THEOREM 3.4. *Assume that the coefficients $A_{0j}, A_{1j}, j = 1, 2, \dots, s$, and B_0 but not necessarily B_1 are Hermitian matrices. Assume also that the eigenvalue condition (1.3) holds. Then, the results of Theorem 3.3 are valid.*

Before we give a proof of the last theorem, we will prove

THEOREM 3.5. *Assume that, for sufficiently large $|\omega|$, there is a Hermitian matrix $\tilde{H}(\omega) = I + (1/|\omega|) \tilde{S}$ where $\tilde{S} = \tilde{S}(\omega', 1/|\omega|)$ is a smooth function of ω' and $1/|\omega|$ such that*

$$2 \operatorname{Re} \tilde{H}(\omega)(|\omega| \hat{P}_0(i\omega') + B_0) \leq -\delta \tilde{H}(\omega). \tag{3.12}$$

Then, for sufficiently small ε , the system (3.1) is a contraction.

Proof. The proof proceeds similarly to the proof of Theorem 3.2. It is much simpler, because in this case we construct a time independent pseudo-differential operator of the form

$$H = I + S$$

which has the properties of Theorem 3.1.

Proof of Theorem 3.4. Consider the symbol $|\omega| \hat{P}_0(i\omega') + B_0$ for large $|\omega|$. Let $\omega' = \omega'_0$ be fixed. Since the coefficients A_{0j} are Hermitian, there is a unitary transformation such that

$$\begin{aligned}
& U^*(\omega'_0)(|\omega| \hat{P}_0(i\omega'_0) + B_0) U(\omega'_0) \\
&= i |\omega| \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_r \end{pmatrix} + \begin{pmatrix} \tilde{B}_{11} & \tilde{B}_{12} & \cdots & \tilde{B}_{1r} \\ \tilde{B}_{12}^* & \tilde{B}_{22} & \cdots & \tilde{B}_{2r} \\ \vdots & & \ddots & \vdots \\ \tilde{B}_{1r}^* & \cdots & \tilde{B}_{r-1r}^* & \tilde{B}_{rr} \end{pmatrix} \quad (3.13)
\end{aligned}$$

Here

$$A_j = \lambda_j I$$

represent the different eigenvalues according to their multiplicity. Since \tilde{B}_{jj} are also Hermitian, we can assume that they are diagonal. Otherwise, we apply a block-diagonal unitary transformation to (3.13). For large $|\omega|$, we consider the B -matrix in (3.13) a small perturbation of $i |\omega| A$. Therefore, we can construct a transformation $I + (1/|\omega|) T(\omega'_0)$ such that

$$\begin{aligned}
& \left(I + \frac{1}{|\omega|} T(\omega'_0) \right)^{-1} U^*(\omega'_0)(|\omega| \hat{P}_0(i\omega'_0) + B_0) U(\omega'_0) \left(I + \frac{1}{|\omega|} T(\omega'_0) \right) \\
&= i |\omega| \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_r \end{pmatrix} + \begin{pmatrix} \tilde{B}_{11} & & 0 \\ & \ddots & \\ 0 & & \tilde{B}_{rr} \end{pmatrix} + \frac{1}{|\omega|} \tilde{\tilde{B}} \\
&=: i |\omega| A + \tilde{B} + \frac{1}{|\omega|} \tilde{\tilde{B}}.
\end{aligned}$$

The eigenvalue condition guarantees that $\tilde{B}_{jj} \leq -\delta I$ for all j and, for sufficiently large $|\omega|$,

$$2 \operatorname{Re} \left(i |\omega| A + \tilde{B} + \frac{1}{|\omega|} \tilde{\tilde{B}} \right) \leq -\frac{3}{2} \delta I.$$

We shall now show that there is a neighborhood of ω'_0 where the matrix $\tilde{H}(\omega)$ of (3.12) is given by

$$\begin{aligned}
\tilde{H}(\omega) &= U(\omega'_0) \left(I + \frac{1}{|\omega|} T^*(\omega'_0) \right)^{-1} \left(I + \frac{1}{|\omega|} T(\omega'_0) \right)^{-1} U^*(\omega'_0) \\
&=: I + \frac{1}{|\omega|} \tilde{\mathcal{S}} \left(\omega'_0, \frac{1}{|\omega|} \right).
\end{aligned}$$

We have

$$\begin{aligned}
 & 2 \operatorname{Re} \tilde{H}(\omega)(|\omega| P_0(i\omega') + B_0) \\
 &= 2 \operatorname{Re} \tilde{H}(\omega')(|\omega| P_0(i\omega'_0)) + B_0) + |\omega| 2 \operatorname{Re} \tilde{H}(\omega) P_0((i(\omega' - \omega'_0))) \\
 &\leq -\frac{3}{2} \delta \tilde{H}(\omega) + |\omega| \cdot \frac{1}{|\omega|} 2 \operatorname{Re} \tilde{S}\left(\omega'_0, \frac{1}{|\omega|}\right) P_0(i(\omega' - \omega'_0)) \\
 &\leq \left(-\frac{3}{2} \delta + \operatorname{const.} |\omega' - \omega'_0|\right) \tilde{H}(\omega).
 \end{aligned}$$

Thus, for sufficiently small $|\omega' - \omega'_0|$, the inequality (3.12) holds. With help of the usual partition of unity argument (see again Lemma 3.2.9 of [2]), we can construct $\tilde{H}(\omega)$ for all ω' and the theorem follows from Theorem 3.5.

4. NONLINEAR SYSTEMS

In this section we consider the nonlinear system (1.1). We start with the case that $A_{0\nu}, A_{1\nu}, \nu = 1, \dots, s$; are Hermitian matrices and

$$\operatorname{Re} B_0 \leq -\delta. \tag{4.1}$$

Our arguments follow closely the arguments in [2, Chapter 5, 6] and we assume that the readers are familiar with them.

We shall derive a priori estimates and shall use the following notations: $j = (j_1, \dots, j_s), j_\nu$ natural numbers, denotes a multi-index, $|j| = \sum j_\nu, D^j = \partial^{j_1}/\partial x_1^{j_1} \dots \partial^{j_s}/\partial x_s^{j_s}$ denote the space derivatives and

$$\|u\|_p^2 = \sum_{|j| \leq p} \|D^j u\|^2$$

denotes the derivative norm of order p .

To begin with, we assume that $\varepsilon = 0$ and derive estimates for

$$\begin{aligned}
 \frac{\partial u}{\partial t} &= \left(P_0 \left(\frac{\partial}{\partial x} \right) + B_0 \right) u, \\
 u(x, 0) &= f(x).
 \end{aligned} \tag{4.2}$$

Differentiating (4.2) gives us

$$(D^j u)_t = P_0 \left(\frac{\partial}{\partial x} \right) D^j u + B_0 D^j u.$$

Therefore, by (4.1),

$$\begin{aligned} \frac{\partial}{\partial t} \|D^j u\|^2 &= 2 \operatorname{Re} \left(D^j u, P_0 \left(\frac{\partial}{\partial x} \right) D^j u \right) + 2 \operatorname{Re}(D^j u, B_0 D^j u) \\ &= 2 \operatorname{Re}(D^j u, B_0 D^j u) \leq -2\delta \|D^j u\|^2. \end{aligned}$$

Adding these inequalities for all j with $|j| \leq p$ we obtain, for any p ,

$$\frac{\partial}{\partial t} \|u\|_p^2 \leq -2\delta \|u\|_p^2, \quad (4.3)$$

i.e.,

$$\|u(\cdot, t)\|_p^2 \leq e^{-2\delta t} \|u(\cdot, 0)\|_p^2.$$

Now we consider the nonlinear system (1.1). We derive an estimate for $p \geq s + 2$. Local existence causes no difficulty, it has been known for a long time. There exists an interval $0 \leq t \leq T$, $T > 0$, where the solution exists and

$$\|u(\cdot, t)\|_p^2 \leq 2 \|u(\cdot, 0)\|_p^2. \quad (4.4)$$

There are two possibilities:

$$\text{Either } T = \infty \quad \text{or} \quad T < \infty \quad \text{and} \quad \|u(\cdot, T)\|_p^2 = 2 \|u(\cdot, 0)\|_p^2. \quad (4.5)$$

We shall now prove that $T = \infty$ for sufficiently small ε_0 and that the initial value problem is a contraction (see [2, Section 6.4.1])

We differentiate (1.1) and obtain

$$(D^j u)_t = \left(P_0 \left(\frac{\partial}{\partial x} \right) + B_0 \right) (D^j u) + \varepsilon P_1 \left(x, t, u, \frac{\partial}{\partial x} \right) (D^j u) + \varepsilon R_j,$$

where R_j denote lower order terms. Therefore,

$$\begin{aligned} \|D^j u\|_t^2 &= 2 \operatorname{Re} \left(D^j u, \left(P_0 \left(\frac{\partial}{\partial x} \right) + B_0 \right) D^j u \right) \\ &\quad + 2 \operatorname{Re} \varepsilon \left(D^j u, P_1 \left(x, t, u, \frac{\partial}{\partial x} \right) D^j u \right) + 2\varepsilon \operatorname{Re}(D^j u, R_j). \end{aligned} \quad (4.6)$$

By (4.1),

$$2 \operatorname{Re} \left(D^j u, \left(P_0 \left(\frac{\partial}{\partial x} \right) + B_0 \right) D^j u \right) \leq -2\delta \|D^j u\|^2. \quad (4.7)$$

Integration by parts gives us

$$\begin{aligned} \operatorname{Re} \left(D^j u, P_1 \left(x, t, u, \frac{\partial}{\partial x} \right) D^j u \right) &= -\frac{1}{2} \sum_{v=1}^s \left(D^j u, \frac{\partial A_{1v}}{\partial x_v} D^j u \right) \\ &\leq \operatorname{const.} \quad K_1 \left(1 + \sum_{v=1}^s \left| \frac{\partial u}{\partial x^v} \right|_{\infty} \right) \|D^j u\|^2 \\ &\leq M_1 \|u\|_p^2. \end{aligned} \tag{4.8}$$

The M_j are polynomials in $\|u\|_p$ of degree $|j|$ whose coefficients depend only on the constants K_0, \dots, K_j of Assumption 1.1. Using Sobolev inequalities we find bounds

$$\|(D^j u, R_j)\| \leq M_j \|u\|_j^2.$$

Adding all these inequalities gives us

$$\frac{\partial}{\partial t} \|u\|_p^2 \leq -2\delta \|u\|_p^2 + \varepsilon M \|u\|_p^2, \quad M = \max_j M_j. \tag{4.9}$$

Thus, for all ε with $0 \leq \varepsilon \leq \varepsilon_0$ with ε_0 sufficiently small, we have

$$\frac{\partial}{\partial t} \|u\|_p^2 \leq -\delta \|u\|_p^2.$$

Therefore, $T = \infty$ and the initial value problem is a contraction.

We now consider the general case. We assume that the assumptions of Theorem 3.3 or 3.4 are satisfied. Again, we begin with the case that $\varepsilon = 0$. Then there is a pseudo-differential operator

$$\tilde{H} = H_0 + S$$

which defines a norm that is equivalent with the L_2 -norm such that

$$\frac{\partial}{\partial t} \|u\|_{\tilde{H}}^2 = \frac{\partial}{\partial t} (u, \tilde{H}u) = 2 \operatorname{Re}(u, \tilde{H}(P_0 + B_0)u) \leq -\delta(u, u)_{\tilde{H}}.$$

Thus,

$$\|u(\cdot, t)\|_{\tilde{H}}^2 \leq e^{-\delta t} \|u(\cdot, 0)\|_{\tilde{H}}^2$$

and, therefore, also

$$\|u(\cdot, t)\|_{\tilde{H}, p}^2 \leq e^{-\delta t} \|u(\cdot, 0)\|_{\tilde{H}, p}^2, \quad \|u\|_{\tilde{H}, p}^2 = \sum_{|j| \leq p} \|D^j u\|_{\tilde{H}}^2.$$

Now we consider the general system (1.1). We proceed in the same way as in the previous case and derive estimates for $p \geq s + 2$. The only difference is that we derive the estimates in the H -norm.

Local existence is again no difficulty. Thus, there is an interval $0 \leq t \leq T$, $T > 0$ where

$$\|u(\cdot, t)\|_{\tilde{H}, p}^2 \leq 2 \|u(\cdot, 0)\|_{\tilde{H}, p}^2.$$

For T , the alternative (4.5) holds. In this interval we can estimate the solution and its derivatives up to order $p - [s/2] - 1$ in the maximum norm in terms of $\|u(\cdot, 0)\|_{\tilde{H}, p}^2$. Thus, we can think of the system (1.1) as a linear system and construct the pseudo-differential operator (3.2) and estimate the solution and its derivatives in the H -norm which differs from the \tilde{H} -norm only by terms of order ε . The symbol depends on the solution but, by Theorem 3.2, if p is sufficiently large, then the constant K in Theorem 3.1 can also be estimated in terms of $\|u(\cdot, 0)\|_{\tilde{H}, p}^2$. The rest of the proof proceeds as before. We differentiate (1.1) with respect to the space derivatives and obtain in the H -norm

$$\begin{aligned} \frac{\partial}{\partial t} (D^j u, HD^j u) &= \varepsilon(u, H_{1,t} u) + (D^j u, H((P_0 + B_0 + \varepsilon(P_1 + B_1)) D^j u) \\ &\quad + 2\varepsilon \operatorname{Re}(D^j u, HR_j). \end{aligned}$$

Using Theorem 3.1, we obtain the inequality (4.9) but now in the H -norm. Thus, we have proved the Main Theorem of Section 1.

APPENDIX

We want to relax the stability eigenvalue condition for the cases when some of the eigenvalues of B_0 have zero real part. We do this only for the 2π -periodic case.

DEFINITION A.1. The system (1.1) is said to satisfy the *relaxed stability eigenvalue condition* if the following conditions hold.

(1) There is a constant $\delta > 0$ such that, the eigenvalues λ of the symbol $\hat{P}(i\omega) + B_0$ satisfy

$$\operatorname{Re} \lambda \leq -\delta \tag{A.1}$$

for all $\omega = (\omega_1, \dots, \omega_s) \neq 0$, ω_j integer.

(2) The eigenvalues $\lambda(0)$ of B_0 satisfy

$$\text{Either } \operatorname{Re} \lambda \leq -\delta \quad \text{or} \quad \lambda = 0. \tag{A.2}$$

Also, if the multiplicity of the zero eigenvalue is r , then there are r linearly independent eigenvectors connected with $\lambda = 0$.

(3) The nullspace of B_1 contains the nullspace of B_0 .

We can find a nonsingular transformation S such that

$$S^{-1}B_0S = \begin{pmatrix} B_{01} & 0 \\ 0 & 0 \end{pmatrix}, \quad B_{01} \text{ nonsingular.} \tag{A.3}$$

If B_0 is symmetric, we can choose S to be unitary. Therefore, we can assume that B_0 already has the form (A.3). Then, by the third part of the assumption, B_1 has the form

$$B_1 = \begin{pmatrix} B_{11} & 0 \\ B_{12} & 0 \end{pmatrix}. \tag{A.4}$$

Let

$$u(x, t) = \begin{pmatrix} \hat{u}^I(0, t) \\ \hat{u}^{II}(0, t) \end{pmatrix} + \sum_{\omega \neq 0} e^{i\langle \omega, x \rangle} \hat{u}(\omega, t).$$

Here the partition of $\hat{u}(0, t)$ corresponds to that of B_0, B_1 . Denote by Q the projection

$$Qu(x, t) = \begin{pmatrix} 0 \\ \hat{u}^{II}(0, t) \end{pmatrix}.$$

Using the notation

$$Qu(x, t) =: u^{(0)}(t), \quad (I - Q)u(x, t) =: v(x, t),$$

we can write the system (1.1) as

$$\begin{aligned} u_t^{(0)} &= Q(P_0 + B_0)(u^{(0)} + v) + \varepsilon Q(P_1 + B_1)(u^{(0)} + v) \\ &= \varepsilon Q(P_1 + B_1)v. \end{aligned} \tag{A.5}$$

$$\begin{aligned} v_t &= (I - Q)(P_0 + B_0)(u^{(0)} + v) + \varepsilon(I - Q)(P_1 + B_1)(u^{(0)} + v) \\ &= (P_0 + B_0)v + \varepsilon(I - Q)(P_1 + B_1)v. \end{aligned} \tag{A.6}$$

As before, we need only to consider linear systems. Then (A.6) decouples completely from (A.5). It is a system on the subspace $(I - Q)L_2$. Our results

tell us that, for sufficiently small ε , it is a contraction and v converges exponentially to zero. Since

$$u^{(0)}(t) = \varepsilon \int_0^t Q(P_1 + B_1) v(x, \xi) d\xi + u^{(0)}(0),$$

it follows that also $u^{(0)}(t)$ converges for $t \rightarrow \infty$.

We summarize the results of the appendix in the following theorem.

THEOREM A.1. *Suppose that assumption 1.1 and assumption 1.2 or assumption 1.3 hold but with the stability eigenvalue condition replaced by the relaxed stability eigenvalue condition. Then, for sufficiently small ε , the problem is a contraction, in a suitable H -norm, for the nontrivial part v of the solution of (1.1) and $u^{(0)} \rightarrow \text{const.}$ when $t \rightarrow \infty$. Thus, the system (1.1) is stable in this generalized sense.*

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