Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
J. Differential Equations 204 (2004) 202–226
Journal of Differential Equations
http://www.elsevier.com/locate/jde

Geometric bounds on the growth rate of null-controllability cost for the heat equation in small time

Luc Miller[a,b,*,1]

[a] Équipe Modal'X, EA 3454, Université Paris X, Bât. G, 200 Av. de la République, 92001 Nanterre, France
[b] Centre de Mathématiques Laurent Schwartz, UMR CNRS 7640, École Polytechnique, 91128 Palaiseau, France

Received July 11, 2003

Abstract

Given a control region Ω on a compact Riemannian manifold M, we consider the heat equation with a source term g localized in Ω. It is known that any initial data in $L^2(M)$ can be steered to 0 in an arbitrarily small time T by applying a suitable control g in $L^2([0,T]\times\Omega)$, and, as T tends to 0, the norm of g grows like $\exp(C/T)$ times the norm of the data. We investigate how C depends on the geometry of Ω. We prove $C\geqslant d^2/4$ where d is the largest distance of a point in M from Ω. When M is a segment of length L controlled at one end, we prove $C\leqslant\alpha_* L^2$ for some $\alpha_*<2$. Moreover, this bound implies $C\leqslant\alpha_* L_\Omega^2$ where L_Ω is the length of the longest generalized geodesic in M which does not intersect Ω. The *control transmutation method* used in proving this last result is of a broader interest.

Keywords: Heat equation; Control cost; Null-controllability; Observability; Small time asymptotics; Multipliers; Entire functions; Transmutation

*Fax: +33-1-69-33-30-19.
E-mail address: miller@daphne.math.polytechnique.fr.
[1] This work was partially supported by the ACI grant "Équation des ondes: oscillations, dispersion et contrôle".

0022-0396/$ - see front matter
doi:10.1016/j.jde.2004.05.007

1. The problem

Let (M, g) be a smooth connected compact n-dimensional Riemannian manifold with metric g and boundary ∂M. When $\partial M \neq \emptyset$, M denotes the interior and $\bar{M} = M \cup \partial M$. Let $\mathrm{dist} : \bar{M}^2 \to \mathbb{R}_+$ denote the distance function. Let Δ denote the (negative) Dirichlet Laplacian on $L^2(M)$ with domain $D(\Delta) = H_0^1(M) \cap H^2(M)$.

Consider a positive control time T, and an open control region Ω. Let $\mathbf{1}_{]0,T[\times\Omega}$ denote the characteristic function of the space–time control region $]0, T[\times\Omega$. The heat equation on M is said to be *null-controllable* (or exactly controllable to zero) in time T by interior controls on Ω if for all $u_0 \in L^2(M)$ there is a control function $g \in L^2(\mathbb{R} \times M)$ such that the solution $u \in C^0([0, \infty), L^2(M))$ of the mixed Dirichlet–Cauchy problem

$$\partial_t u - \Delta u = \mathbf{1}_{]0,T[\times\Omega}\, g \quad \text{in }]0, T[\times M, \quad u = 0 \quad \text{on }]0, T[\times\partial M, \tag{1}$$

with Cauchy data $u = u_0$ at $t = 0$, satisfies $u = 0$ at $t = T$. For a survey on this problem prior to 1978 we refer to [19]. For a recent update, we refer to [26]. Lebeau and Robbiano have proved (in [14] using local Carleman estimates) that there is a continuous linear operator $S : L^2(M) \to C_0^\infty(\mathbb{R} \times M)$ such that $g = Su_0$ yields the null-controllability of the heat equation on M in time T by interior controls on Ω.

The most striking feature of this result is that we may control the heat in arbitrarily small time whatever geometry the control region has. In this paper we address the following question: *How does the geometry of the control region influence the cost of controlling the heat to zero in small time*?

Now, we shall formulate this question more precisely and give references.

Definition 1.1. For all control time T and all control region Ω, the *null-controllability cost* for the heat equation on M is the best constant, denoted $C_{T,\Omega}$, in the estimate

$$||g||_{L^2(\mathbb{R}\times M)} \leqslant C_{T,\Omega} ||u_0||_{L^2(M)}$$

for all initial data u_0 and control g solving the null-controllability problem described above.

By duality (cf. [5]), $C_{T,\Omega}$ is also the best constant in the observation inequality for the homogeneous heat semigroup $t \mapsto e^{t\Delta}$:

$$\forall u_0 \in L^2(M), \quad ||e^{T\Delta} u_0||_{L^2(M)} \leqslant C_{T,\Omega} ||e^{t\Delta} u_0||_{L^2((0,T)\times\Omega)}.$$

Lebeau and Robbiano's result implies the finiteness of the null-controllability cost for the heat equation on M for any control time and any control region. Èmanuilov extended this result to more general parabolic operators in [6] using global Carleman estimates with singular weights. When (M, g) is an open set in Euclidean space, this method was used by Fernández-Cara and Zuazua [8] to obtain the optimal time dependence of the null-controllability cost for small time, i.e.

$$0 < \sup_{\bar{B}_\rho \subset M \setminus \bar{\Omega}} \rho^2/4 \leqslant \liminf_{T \to 0} T \ln C_{T,\Omega} \leqslant \limsup_{T \to 0} T \ln C_{T,\Omega} < +\infty, \tag{2}$$

where the supremum is taken over balls B_ρ of radius ρ. The lower bound is stated in Section 4.1 of Zuazua [26] and it is based on the construction of a "very singular solution of the heat equation in $(0, +\infty) \times \mathbb{R}^n$" used in the proof of Theorem 6.2 in [8]. Note that the method used in Theorem 1 of Lebeau and Robbiano [14] seems to fall short of the optimal time dependence. Actually, using the improved version of Proposition 1 in [14] presented as Proposition 2 in [15], we have only been able to prove that $\limsup_{T \to 0} T^\gamma \ln C_{T,\Omega}$ is finite for all $\gamma > 1$.

Indeed Seidman had already asked how violent fast controls are, and his first answer concerned heat null-controllability from a boundary region $\Gamma \subset \partial M$. In [21], under the condition that the wave equation on M is exactly controllable by controls in Γ in time L, he computes an explicit positive value β such that $\limsup_{T \to 0} T \ln C_{T,\Gamma} \leqslant \beta L^2$ (we give more explanations on this geometric upper bound in Section 2 after Theorem 2.3). The positivity of $\liminf_{T \to 0} T \ln C_{T,\Gamma}$ when M is an interval was subsequently proved by Güichal in [9], ensuring the optimality of Seidman's result with respect to the time dependence. Later, Seidman also addressed finite-dimensional linear systems as well as the Schrödinger and plate equations (cf. the companion paper [16] for more details and references).

2. The results

2.1. Lower bound

Our first result, proved in Section 3, generalizes and improves on the geometric lower bound of Fernández-Cara and Zuazua.

Theorem 2.1. *The null-controllability cost of the heat equation for small time* (*cf. Definition* 1.1) *satisfies the following geometric lower bound*:

$$\liminf_{T \to 0} T \ln C_{T,\Omega} \geqslant \sup_{y \in M} \operatorname{dist}(y, \bar{\Omega})^2/4. \tag{3}$$

As put in [26], such a lower bound follows from the construction of a "very singular solution of the heat equation". Our construction underscores that only a large but finite number of modes is needed. For a short control time $T > 0$, we consider a Dirac mass as far from Ω as possible, we smooth it out by applying the homogeneous heat semigroup for a very short time (εT with small ε) and truncating very large frequencies (larger than $(\varepsilon T)^{-1}$), and finally we take it as initial data in (1). The proof relies on Varadhan's formula for the heat kernel in small time (cf. [25]), which requires very low smoothness assumptions as proved in [17].

We believe that there is no solution of the heat equation which is more singular than the heat kernel and therefore conjecture that this lower bound is also an upper bound, i.e. $\lim_{T\to 0} T \ln C_{T,\Omega} = \sup_{y\in M} \operatorname{dist}(y, \bar{\Omega})^2/4$.

2.2. The segment controlled at one end

Our second result, proved in Section 4, concerns the most simple heat null-controllability problem: the heat equation on a segment controlled at one end through a Dirichlet condition. It is an upper bound of the same type as the lower bound in Theorem 2.1, except that the quite natural rate $\frac{1}{4}$ is replaced by the technical rate (resulting from the complex multiplier Lemma 4.4)

$$\alpha_* = 2\left(\frac{36}{37}\right)^2 < 2. \tag{4}$$

Theorem 2.2. *For any $\alpha > \alpha_*$ defined by* (4), *there exists $C > 0$ such that, for $B = 1$ or $B = \partial_s$, for all $L > 0$, $T \in]0, \inf(\pi, L)^2]$ and $u_0 \in L^2(0, L)$, there is a $g \in L^2(0, T)$ such that the solution $u \in C^0([0, \infty), L^2(0, L))$ of the following heat equation on $[0, L]$ controlled by g from one end*:

$$\partial_t u - \partial_s^2 u = 0 \quad \textit{in }]0, T[\times]0, L[, \quad (Bu)_{\rceil s=0} = 0, \quad u_{\rceil s=L} = g, \quad u_{\rceil t=0} = u_0,$$

satisfies $u = 0$ at $t = T$ and $\|g\|_{L^2(0,T)} \leqslant C e^{\alpha L^2/T} \|u_0\|_{L^2(0,L)}$.

Theorem 3.1 in [21] yields this theorem for $\alpha_* = 4\beta_*$ with $\beta_* \approx 42.86$. This result of Seidman can be improved to $\alpha_* = 8\beta_*$ with $\beta_* \approx 4.17$ using his Theorem 1 in [22]. The value α_* defined by (4) in Theorem 2.2 is the best we obtained yet following the well trodden path of the harmonic analysis of this problem (cf. [19,23] for seminal and recent references). As explained at the end of the previous subsection, we conjecture that $\alpha_* = \frac{1}{4}$ is the optimal rate. The proof of Theorem 2.1 also applies here, so that Theorem 2.2 does not hold with $\alpha_* < \frac{1}{4}$. This theorem is valid for more general linear parabolic equations and boundary conditions as formulated in Theorem 4.1.

2.3. Upper bound under the geodesics condition

Our third result gives a good reason to strive for the best rate α_* in Theorem 2.2. In Section 5, we prove that the upper bound for the null-controllability cost of the heat equation on a segment controlled at one end—the particular case in which the computation are the most explicit—is also an upper bound for the multidimensional case of Eq. (1) under the following *geodesics condition* on the control region: every generalized geodesic in $\bar{M}$ intersects Ω.

In this context, the *generalized geodesics* are continuous trajectories $t \mapsto x(t)$ in $\bar{M}$ which follow geodesic curves at unit speed in M (so that on these intervals $t \mapsto \dot{x}(t)$ is continuous); if they hit ∂M transversely at time t_0, then they reflect as light rays or billiard balls (and $t \mapsto \dot{x}(t)$ is discontinuous at t_0); if they hit ∂M tangentially then either there exists a geodesic in M which continues $t \mapsto (x(t), \dot{x}(t))$ continuously and they branch onto it, or there is no such geodesic curve in M and then they glide at unit speed along the geodesic of ∂M which continues $t \mapsto (x(t), \dot{x}(t))$ continuously until they may branch onto a geodesic in M. For this result and whenever generalized geodesics are mentioned, we make the additional assumptions that they can be uniquely continued at the boundary ∂M (as in [15], to ensure this, we may assume either that ∂M has no contacts of infinite order with its tangents, or that g and ∂M are real analytic), and that Ω is open.

Theorem 2.3. *Let L_Ω be the length of the longest generalized geodesic in $\bar{M}$ which does not intersect Ω. If Theorem* 2.2 *holds for some rate α_* then the null-controllability cost of the heat equation for small time* (*cf. Definition* 1.1) *satisfies the following geometric upper bound*:

$$\limsup_{T \to 0} \; T \ln C_{T,\Omega} \leqslant \alpha_* L_\Omega^2. \tag{5}$$

When comparing this result to the lower bound in Theorem 2.1, one should bear in mind that L_Ω is always greater than $2 \sup_{y \in M} \operatorname{dist}(y, \bar{\Omega})$ (because the length of a generalized geodesic through y which does not intersect Ω is always greater than $2 \operatorname{dist}(y, \bar{\Omega})$) and can be infinitely so. For instance, on the sphere $M = S^n$, if Ω is the complementary set of a tube of radius ε around the equator, then $\sup_{y \in M} \operatorname{dist}(y, \bar{\Omega}) = \varepsilon$ and $L_\Omega = \infty$. If Ω is increased by a tube slice of small thickness δ, then the first length is unchanged while the second length becomes greater than the length of the equator of M minus δ, so that L_Ω is finite yet much greater than $\sup_{y \in M} \operatorname{dist}(y, \bar{\Omega})$ as $\varepsilon \to 0$.

Moreover, as recalled in Section 1, this geodesics condition is by no means necessary for the null-controllability of the heat equation. It is more relevant to the wave equation on M, for which it is a sharp sufficient condition for exact controllability in time T by interior controls on Ω as proved in [1] (cf. Theorem 5.3 for the precise statement). It was later proved in [3] that this condition is also necessary when the characteristic function of $]0, T[\times \Omega$ is replaced by a smooth function θ such that $\{\theta(t,x) \neq 0\} =]0, T[\times \Omega$.

In fact we use the exact controllability of the wave equation to prove our result on the null-controllability of the heat equation. This strategy was already applied by Russell in 1973, but he used a complex analysis detour (cf. [19]). In [21], Seidman applied Russell's method to obtain an upper bound which, taking [1] into account, corresponds to Theorem 2.3 with $\alpha_* = \beta_* \approx 42.86$. Theorem 2.3 improves Seidman's result beyond this slight improvement of the rate α_* insofar as the complex analysis multiplier method he uses does not necessarily allow to reach the optimal α_* in Theorem 2.2.

The *control transmutation method* (cf. [10] for a survey on transmutations in other contexts) introduced in Section 5 relates the null-controllability of the heat equation to the exact controllability of the wave equation in a direct way (as opposed to

Russell's indirect complex analysis method). It is well-known that the geometry of small time asymptotics for the homogeneous heat semigroup $t \mapsto e^{t\Delta}$ on $L^2(M)$ can be understood from the even homogeneous wave group $t \mapsto W(t)$ (i.e. the group defined by $W(t)w_0 = w(t)$ where w solves Eq. (53) with $f = 0$ and Cauchy data $(w, \partial_t w) = (w_0, 0)$ at $t = 0$) through Kannai's formula (cf. [4,12], and Section 6.2 in the book [24]):

$$e^{t\Delta} = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-s^2/(4t)} W(s)\, \mathrm{d}s. \tag{6}$$

Our main idea is to replace the fundamental solution of the heat equation on the line $e^{-s^2/(4t)}/\sqrt{4\pi t}$ appearing in Kannai's formula by some *fundamental controlled solution* of the heat equation on the segment $[-L, L]$ controlled at both ends. We use the one-dimensional Theorem 2.2 to construct this fundamental controlled solution in Section 5.

2.4. Open problems

We shall now survey some questions raised by the results we have presented which we have been unable to answer yet.

To improve the rate α_* in Theorem 2.2 by a complex analysis method, one could use the first method in [7], i.e. compute the null-controllability cost on the half-line $[0, +\infty)$ explicitly by Vandermonde determinants and prove a quantitative version of Schwartz's theorem in [20], i.e. estimate with respect to L the best constant c_L in the following statement: every u in the closed linear hull in $L^2(0, +\infty)$ of the real exponential sums $t \mapsto e^{-k^2 t}$ $(k \in \mathbb{N}^*)$ satisfies $||u||_{L^2(0,+\infty)} \leqslant c_L ||u||_{L^2(0,L)}$.

Theorem 2.3 opens new tracks to improve the upper bound for the null-controllability cost of (1) under the geodesics condition by methods which are not complex analytical. To improve the rate α_* in Theorem 2.2 (or in the multi-dimensional case of Eq. (1) when Ω and M are star-shaped with respect to the same point) one could adapt the variational techniques (e.g. the log convexity method) or the Carleman's inequalities devised to prove unique continuation theorems.

In the general case (without the geodesics condition), one could try to adapt the null-controllability proofs which use Carleman inequalities with phases ϕ to obtain an upper bound similar to the lower bound in Theorem 2.1 in terms of the following distance function $d : d(x, y) = \sup\{\phi(y) - \phi(x)\}$, for all x and y in M, where the supremum is taken over all Lipschitz functions $\phi : M \to \mathbb{R}$ with $|\nabla\phi| \leqslant 1$ almost everywhere. There is a more geometric characterization of d in terms of path of least action (cf. Section 2 of Norris [17]).

3. Lower bound

The purpose of this section is to prove Theorem 2.1.

As in Section 1, let Ω be an open set in the n-dimensional Riemannian manifold M such that $\bar{\Omega}\subset M$. Let $(\omega_j)_{j\in\mathbb{N}^*}$ be a nondecreasing sequence of nonnegative real numbers and $(e_j)_{j\in\mathbb{N}^*}$ be an orthonormal basis of $L^2(M)$ such that e_j is an eigenvector of $-\Delta$ with eigenvalue ω_j^2. The heat kernel k can be defined for all $t>0$ and $(x,y)\in\bar{M}^2$ by $k(t,x,y)=\sum_j \exp(-t\omega_j^2)e_j(y)e_j(x)$. Our main ingredient is Varadhan's formula which says that (cf. Theorem 1.1 in [17] for example):

$$\lim_{t\to 0} t\ln k(t,x,y) = -d(x,y)^2/4 \quad \text{uniformly on compact sets of } \bar{M}^2. \tag{7}$$

We shall also use Weyl's asymptotics for eigenvalues

$$\exists W>0, \quad \#\{j\in\mathbb{N}^* \mid \omega_j\leqslant\omega\}\leqslant W\omega^n \tag{8}$$

and the following consequence of Sobolev's embedding theorem:

$$\exists E>0, \quad \forall j\in\mathbb{N}^*, \quad ||e_j||_{L^\infty}\leqslant E\omega_j^{n/2} \tag{9}$$

(cf. Section 17.5 in [11] for example). The unique continuation property for elliptic operators implies that $Y=\{y\in M\backslash\bar{\Omega} \mid e_1(y)\neq 0\}$ is an open dense set in $M\backslash\bar{\Omega}$, so that the supremum in Theorem 2.1 can be taken over $y\in Y$ instead of $y\in M$.

Let $y\in Y$ and $\alpha<d(y,\bar{\Omega})^2/4$ be fixed from now on. To prove Theorem 2.1 we shall find $A>0$ and, for all $T\in]0,1]$ small enough, some data $u_0^T\in L^2(M)$ such that $||e^{t\Delta}u_0^T||_{L^2((0,T)\times\Omega)}\leqslant Ae^{-\alpha/T}||e^{T\Delta}u_0^T||_{L^2(M)}$. To give further insight into the problem, we shall construct each u_0^T as a linear combination of a finite number of modes e_j only.

Let β be a real number such that $\alpha<\beta<d(y,\bar{\Omega})^2/4$. Since $\bar{\Omega}\times\{y\}$ is compact in $\bar{M}^2$, Varadhan's formula (7) yields real numbers $B>0$ and $\bar{T}\in]0,1]$ such that

$$\forall t\in]0,\bar{T}],\ \forall x\in\bar{\Omega},\ |k(t,x,y)|\leqslant Be^{-\beta/t}. \tag{10}$$

Let $\varepsilon\in]0,1]$ small enough as specified later. For all $T\in]0,\bar{T}/(1+\varepsilon)]$ consider the data $u_0^T(x)=\sum_{\omega_j\leqslant(\varepsilon T)^{-1}}\exp(-\varepsilon T\omega_j^2)e_j(y)e_j(x)$. To estimate the corresponding solution

$$u^T(t,x)=(e^{t\Delta}u_0^T)(x)=\sum_{\omega_j\leqslant(\varepsilon T)^{-1}}\exp(-(\varepsilon T+t)\omega_j^2)e_j(y)e_j(x),$$

we compare it with $k(\varepsilon T+t,x,y)$. Using that the heat semigroup is a contraction on $L^2(M)$, Parseval's identity and (9), we obtain

$$\begin{aligned}\sup_{t\in]0,T]} ||k(\varepsilon T+t,x,y)-u^T(t,x)||_{L^2(M)} &\leqslant ||k(\varepsilon T,x,y)-u_0^T(x)||_{L^2(M)}\\ &= \sum_{\omega_j>(\varepsilon T)^{-1}} |e^{-\varepsilon T\omega_j^2}e_j(y)|^2 \leqslant E\sum_{\omega_j\geqslant(\varepsilon T)^{-1}} e^{-\omega_j}\omega_j^n \leqslant E'\sum_{\omega_j\geqslant(\varepsilon T)^{-1}} e^{-\omega_j/2}\end{aligned}$$

for some $E'>0$. But, Weyl's law (8) yields, for $c\geqslant c_0>0$ and $\gamma\geqslant\gamma_0>0$,

$$\begin{aligned}\sum_{\omega_j\geqslant c} e^{-\gamma\omega_j} &= \sum_{k\in\mathbb{N}^*}\sum_{kc\leqslant\omega_j<(k+1)c} e^{-\gamma\omega_j}\leqslant W\sum_{k\in\mathbb{N}^*}((k+1)c)^n e^{-kc\gamma}\\ &\leqslant W_{\gamma_0}\sum_{k\in\mathbb{N}^*} e^{-kc\gamma}e^{(k+1)c\gamma/4}\\ &= W_{\gamma_0}e^{-c\gamma/2}\sum_{k\in\mathbb{N}} e^{-3kc\gamma/4}\leqslant W_{c_0,\gamma_0}e^{-c\gamma/2},\end{aligned}$$

where W_{γ_0} and W_{c_0,γ_0} are positive real numbers which depend on their indices but not on c and γ. Hence, with $c=(\varepsilon T)^{-1}>1=c_0$ and $\gamma=\gamma_0=\frac{1}{2}$, we obtain

$$\exists B'>0,\quad \forall t\in]0,T]\ \ ||k(\varepsilon T+t,x,y)-u^T(t,x)||_{L^2(M)}\leqslant B'e^{-1/(4\varepsilon T)}.$$

Together with the estimate on $k(\varepsilon T+t,x,y)$ which follows from (10), this estimate yields by the triangle inequality, choosing $\varepsilon<1/(4\beta)$ and setting $B''=|\Omega|^{1/2}B+B'$,

$$||u^T||_{L^2((0,T)\times\Omega)}\leqslant(T|\Omega|)^{1/2}Be^{-\beta/((1+\varepsilon)T)}+T^{1/2}B'e^{-1/(4\varepsilon T)}\leqslant B''e^{-\beta/((1+\varepsilon)T)}.$$

But using Parseval's identity and $y\in Y$, we have

$$||e^{T\Delta}u_0^T||_{L^2(M)}=\left(\sum_{\omega_j\leqslant(\varepsilon T)^{-1}}|e^{-(1+\varepsilon)T\omega_j^2}e_j(y)|^2\right)^{1/2}\geqslant e^{-2\omega_1^2}|e_1(y)|>0.$$

Hence, choosing ε small enough so that $\alpha<\beta/(1+\varepsilon)$ and setting $A=e^{-2\omega_1^2}|e_1(y)|B''$, we have

$$\forall T\in]0,\bar{T}/(1+\varepsilon)],\quad ||u^T||_{L^2((0,T)\times\Omega)}\leqslant Ae^{-\alpha/T}||e^{T\Delta}u_0^T||_{L^2(M)}.$$

Since A does not depend on T, this ends the proof of Theorem 2.1.

4. The segment controlled at one end

In this section, we prove Theorem 2.2 for a more general linear parabolic equation on a segment controlled at one end (in particular, it proves that Theorem 2.2 is true for the heat equation on a segment with any Riemannian metric). We follow [7] quite closely.

For a positive a control time T, we consider the following mixed Dirichlet-Cauchy problem on the space segment $[0,X]$:

$$\partial_t u=\partial_x(p(x)\partial_x u)+q(x)u\quad\text{for }\ (t,x)\in]0,T[\times]0,X[,\tag{11}$$

$$(a_0+b_0\partial_x)u_{\rceil x=0}=0,\quad (a_1+b_1\partial_x)u_{\rceil x=X}=g,\quad u_{\rceil t=0}=u_0,\tag{12}$$

$$a_0^2 + b_0^2 = a_1^2 + b_1^2 = 1, \quad 0 < p \in C^2([0,X]), \quad q \in C^0([0,X]). \tag{13}$$

With assumptions (13), the operator A on $L^2(0,X)$ with domain $D(A)$ defined by

$$(Au)(x) = \partial_x(p(x)\partial_x u(x)) + q(x)u(x),$$

$$D(A) = H^2(0,X) \cap \{(a_0 + b_0\partial_x)u_{\rceil x=0} = (a_1 + b_1\partial_x)u_{\rceil x=X} = 0\}$$

is self-adjoint and has a sequence $\{-\lambda_n\}_{n\in\mathbb{N}^*}$ of increasing eigenvalues and an orthonormal Hilbert basis $\{e_n\}_{n\in\mathbb{N}^*}$ in $L^2(0,X)$ of corresponding eigenfunctions, i.e.

$$\forall n \in \mathbb{N}^*, \quad -Ae_n = \lambda_n e_n \text{ and } \lambda_n < \lambda_{n+1}.$$

Moreover, (13) ensures the following eigenvalues asymptotics (cf. [7]):

$$\exists \nu \in \mathbb{R}, \ \lambda_n = \frac{\pi^2}{L^2}(n+\nu)^2 + O(1) \text{ as } n \to \infty, \quad \text{where} \quad L = \int_0^X \sqrt{p(x)}\, dx. \tag{14}$$

Theorem 4.1. *For any* $\alpha > \alpha_*$ *defined by* (4), *there exists* $C > 0$ *such that*, *for any coefficients* (13), *for all* $T \in]0, \inf(\pi, L)^2]$ *and* $u_0 \in L^2(0,X)$ *there is a control* $g \in L^2(0,T)$ *such that the solution* $u \in C^0([0,\infty), L^2(0,X))$ *of* (11) *and* (12) *satisfies* $u = 0$ *at* $t = T$ *and* $\|g\|_{L^2(0,T)} \leqslant Ce^{\alpha L^2/T}\|u_0\|_{L^2(0,X)}$.

As in [7], the proof applies to the slightly more general eigenvalue asymptotics $\lambda_n = \frac{\pi^2}{L^2}(n+\nu) + o(n)$. We divide the proof of this theorem in three steps.

4.1. *Reduction to positive eigenvalues, to a segment of p-length* $L = \pi$, *and to the control window* $]-T/2, T/2[$

As a first step, we reduce the problem to the case $\lambda_1 > 0$ by the multiplier $t \mapsto \exp(\lambda t)$, to the case $L = \pi$ by the time rescaling $t \mapsto \sigma t$ with $\sigma = (\pi/L)^2$, and to the time interval $[-T/2, T/2]$ by the time translation $t \mapsto t - T/2$.

The function u satisfies $\partial_t u = Au$ and $(a_1 + b_1\partial_x)u_{\rceil x=X} = g$ if and only if $\tilde{u}(t,x) = \exp(\lambda t)u(t,x)$ satisfies $\partial_t\tilde{u} = \tilde{A}\tilde{u}$ and $(a_1 + b_1\partial_x)\tilde{u}_{\rceil x=X} = \tilde{g}$ with $\tilde{A} = A + \lambda$ and $\tilde{g}(t) = \exp(\lambda t)g(t)$. For any $\lambda > -\lambda_1$, the lowest eigenvalue of $\tilde{A} \geqslant \lambda_1 + \lambda > 0$ is positive. In $\tilde{A}$, q is changed into $q + \lambda$ and p is unchanged so that L is unchanged. Moreover $\|g\|_{L^2(0,T)} \leqslant \exp(\lambda T/2)\|\tilde{g}\|_{L^2(0,T)}$ so that $\|\tilde{g}\|_{L^2(0,T)} \leqslant \tilde{C}e^{\alpha L^2/T}\|u_0\|_{L^2(0,X)}$ implies the estimate in Theorem 4.1 with $C = \tilde{C}\exp(\lambda\pi/2)$. This proves the reduction to positive eigenvalues.

We now prove the second reduction. Assume the theorem is true when L takes the value $\tilde{L} = \pi$. Given $L > 0$ and $T \in]0, \inf(\pi, L)^2]$ we set $\tilde{T} = \sigma^2 T \in]0, \tilde{L}^2]$ and $\tilde{A} = \sigma^2 A$, where $\sigma = (\pi/L)^2$. By applying the theorem to $\tilde{A}$ on $]0, \tilde{T}[$, we obtain

$||\tilde{g}||_{L^2(0,\tilde{T})} \leqslant \tilde{C}e^{\tilde{\alpha}\tilde{L}^2/\tilde{T}}||u_0||_{L^2(0,X)}$. The function $g(t) = \tilde{g}(\sigma t)$ is a control for the solution $u(t,x) = \tilde{u}(\sigma t, x)$ of $\partial_t u = Au$ on $]0,T[$ at the cost $||g||_{L^2(0,T)} = ||\tilde{g}||_{L^2(0,T)}L/\pi$. Since $T \leqslant \pi^2$ implies $L/\pi \leqslant (L^2/T)^{1/2}$, for all $\alpha > \tilde{\alpha}$ there is a C such that for all $L > 0$ and $T \in]0, \inf(\pi, L)^2]$: $\tilde{C}e^{\tilde{\alpha}\tilde{L}^2/\tilde{T}}L/\pi \leqslant Ce^{\alpha L^2/T}$. Therefore g satisfies the estimate in Theorem 4.1.

These two reductions allow us to assume from now on $\lambda_1 > 0$ and $L = \pi$. Making a weaker assumption on the remainder term in (14), we shall only use the following spectral assumption:

$$\forall n \in \mathbb{N}^*, \quad 0 < \lambda_n < \lambda_{n+1} \text{ and } \exists \nu \in \mathbb{R},\ \lambda_n = (n+\nu)^2 + o(n) \text{ as } n \to \infty. \tag{15}$$

It is obvious that Theorem 4.1 is invariant by time translations and we shall prove it for the control window $]-T/2, T/2[$ instead of $]0,T[$.

4.2. *Spectral reduction to a problem in complex analysis*

In this second step, we recall that the control g in this theorem can be obtained as a series expansion into a Riesz sequence $\{g_n\}_{n\in\mathbb{N}^*}$ in $L^2(-T/2, T/2)$ which is bi-orthogonal to the sequence $\{\exp(-\lambda_n t)\}_{n\in\mathbb{N}^*}$. We also recall how the Paley–Wiener theorem reduces the construction of such biorthogonal functions to the construction of entire functions with zeros and growth conditions (this well-known method in complex analysis is the second method in [7] called the Fourier transform method there). Our estimate on the control cost $||g||_{L^2(-T/2,T/2)}$ relies on a good estimate of $||g_n||_{L^2(-T/2,T/2)}$ as T tends to zero. This additional difficulty was first taken care of by Seidman [22] for $\lambda_n = in^2$ and it was recently overcome for more general sequences in [23]. Our contribution is a slight improvement on the estimates of Seidman and his collaborators in our less general setting.

In terms of the coordinates $c = (c_j)_{j\in\mathbb{N}^*}$ of u_0 in the Hilbert basis $(e_j)_{j\in\mathbb{N}^*}$, the controllability problem in Theorem 4.1 is equivalent to the following moment problem (by straightforward integration by parts, cf. [7]):

$$\int_{-T/2}^{T/2} e^{-\lambda_n(T/2-t)}\gamma_n g(t)\, dt = -e^{-\lambda_n T}c_n,$$

where $\gamma_n = e_n(X)p(X)/b_1$ if $b_1 \neq 0$ and $\gamma_n = -e_n{}'(X)p(X)/a_1$ if $b_1 = 0$. In both cases, the asymptotic expansion of e_n yields that $(|\gamma_n|)$ is bounded from below by some positive constant γ. If $\{g_n\}_{n\in\mathbb{N}^*}$ in $L^2(-T/2, T/2)$ is a sequence which is bi-orthogonal to the sequence $\{\exp(-\lambda_n t)\}_{n\in\mathbb{N}^*}$, i.e.

$$\int_{-T/2}^{T/2} g_n(t)e^{-\lambda_n t}\, dt = 1 \text{ and } \forall k \in \mathbb{N}^*,\ k \neq n, \int_{-T/2}^{T/2} g_n(t)e^{-\lambda_k t}\, dt = 0, \tag{16}$$

then $g(t) = -\sum_{n=1}^{\infty} \frac{c_n}{\gamma_n} e^{-\lambda_n T/2} g_n(-t)$ is a formal solution to this moment problem. The following theorem in complex analysis allows to construct a bi-orthogonal sequence such that this series converges and yields a good estimate of $||g||_{L^2(-T/2,T/2)}$ as T tends to zero.

Theorem 4.2. *Let α_* be defined by* (4). *Let $\{\lambda_n\}_{n\in\mathbb{N}^*}$ be a sequence of real numbers satisfying* (15). *For all $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that, for all $\tau \in]0,1]$ and $n \in \mathbb{N}^*$, there is an entire function G_n satisfying*

$$G_n \text{ is of exponential type } \tau, \text{ i.e. } \limsup_{r\to+\infty} r^{-1} \sup_{|z|=r} \ln|G_n(z)| \leqslant \tau, \tag{17}$$

$$G_n(i\lambda_n) = 1 \quad \text{and} \quad \forall k \in \mathbb{N}^*,\ k \neq n,\ G_n(i\lambda_k) = 0, \tag{18}$$

$$||G_n||_{L^2} = \left(\int_{-\infty}^{+\infty} |G_n(x)|^2\, dx \right)^{1/2} \leqslant C_\varepsilon e^{\varepsilon\sqrt{\lambda_n}} e^{\alpha_*(\pi+2\varepsilon)^2/(2\tau)}. \tag{19}$$

According to the Paley–Wiener theorem (1934), (17) implies that the function $x \mapsto G_n(x)$ is the unitary Fourier transform of a function $t \mapsto g_n(t)$ in $L^2(\mathbb{R})$ supported in $[-\tau, \tau]$. With $\tau = T/2$, this yields

$$G_n(x) = \frac{1}{\sqrt{2\pi}} \int_{-T/2}^{T/2} g_n(t) e^{-itx}\, dt \quad \text{and} \quad ||g_n||_{L^2} = ||G_n||_{L^2}. \tag{20}$$

Hence (18) implies (16) and (19) implies that the series defining g converges with

$$||g||_{L^2} \leqslant \sum_{n=1}^{\infty} \left| \frac{c_n}{\gamma_n} \right| e^{-\lambda_n T/2} ||g_n||_{L^2} \leqslant ||u_0||_{L^2} \frac{C_\varepsilon}{\gamma} e^{\alpha_*(\pi+\varepsilon)^2/T} \left(\sum_{n=1}^{\infty} e^{-\lambda_n T} e^{2\varepsilon\sqrt{\lambda_n}} \right)^{1/2}.$$

Since as $T \to 0$ we have

$$\sum_{n=1}^{\infty} e^{-\lambda_n T} e^{2\varepsilon\sqrt{\lambda_n}} \leqslant e^{2\varepsilon^2/T} \sum_{n=1}^{\infty} e^{-\lambda_n T/2} \sim e^{2\varepsilon^2/T} (T/2)^{-1/2} \Gamma(1/2)/2 \leqslant C'_\varepsilon e^{3\varepsilon^2/T},$$

this implies $||g||_{L^2(-T/2,T/2)} \leqslant C_\alpha e^{\alpha\pi^2/T} ||u_0||_{L^2(0,X)}$, with $\alpha = \alpha_*(1 + 2\varepsilon/\pi)^2 + 3\varepsilon^2/\pi^2$ and $C_\alpha = C_\varepsilon C'_\varepsilon/\gamma$. Since $\alpha \to \alpha_*$ as $\varepsilon \to 0$, this completes the proof that Theorem 4.2 implies Theorem 4.1.

4.3. Complex analysis multipliers

In this subsection, we shall prove Theorem 4.2 by the following classical method in complex analysis (cf. Section 14 in [18] for a concise account with references, and the two volumes [13] for an extensive monograph on multipliers): for all $n \in \mathbb{N}^*$ and small

$\tau > 0$, we shall form an infinite product F_n normalized by $F_n(i\lambda_n) = 1$ with zeros at $i\lambda_k$ for every positive integer $k \neq n$, and construct a multiplier M_n of exponential type τ with fast decay at infinity on the real axis so that $G_n = M_n F_n$ is in L^2 on the real axis. At infinity, it is well known that the growth of $z \mapsto \ln|F_n(z)|$ can be bounded from above by a power of $|z|$ which is inverse to that of $n \mapsto |i\lambda_n| \sim n^2$ (cf. Theorem 2.9.5 in [2]) we prove that our $\ln F_n$ is essentially bounded by $z \mapsto \pi\sqrt{|z|} + o(\sqrt{\lambda_n})$ where the constant π is optimal (cf. Remark 4.5). Therefore M_n has to be essentially bounded by $C_n(\tau)\exp(-\pi\sqrt{|x|})$ on the real axis, for some constant $C_n(\tau) > 0$. The key point (as in [21], Theorem 1 in [22] and Theorem 2 in [23]) is to construct a multiplier M_n such that $C_n(\tau)$ has the smallest growth as τ tends to 0. The following two lemmas give the key to the construction of F_n and M_n respectively.

Lemma 4.3. *Let $\{\lambda_n\}_{n\in\mathbb{N}^*}$ be a sequence of real numbers satisfying* (15). *For all $\varepsilon > 0$ there is a $A_\varepsilon > 0$ such that, for all $n \in \mathbb{N}^*$, the entire function f_n defined by $f_n(z) = \prod_{k\neq n}\left(1 - \frac{z}{\lambda_k}\right)$ satisfies*

$$\ln|f_n(z)| \leqslant (\pi + \varepsilon)\sqrt{|z|} + A_\varepsilon, \tag{21}$$

$$|\ln|f_n(\lambda_n)|| \leqslant \varepsilon\sqrt{\lambda_n} + A_\varepsilon. \tag{22}$$

Proof. For every $n \in \mathbb{N}^*$, we introduce the counting function of the sequence $\{\lambda_k\}_{k\in\mathbb{N}^*\setminus\{n\}}$

$$N_n(r) = \#\{k \in \mathbb{N}^*\setminus\{n\} \mid \lambda_k \leqslant r\}.$$

From (15) we have $N_0 - 1 \leqslant N_n \leqslant N_0$ and $\sqrt{\lambda_n} = n + v + o(1)$. Since $\lambda_k \leqslant r < \lambda_{k+1}$ implies $\sqrt{\lambda_k} - k \leqslant \sqrt{r} - N_0(r) \leqslant \sqrt{\lambda_{k+1}} - (k+1) + 1$, we deduce $|\sqrt{r} - N_n(r) - v| \leqslant 2 + o(1)$. The proof uses assumption (15) through the estimates of the increments $\Lambda_n := \lambda_{n+1} - \lambda_n$ and $\Delta_n := \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n}$ and their increments:

$$\lambda_n = n^2 + 2vn + o(n), \quad \Lambda_n = 2n + o(n), \quad \Lambda_n - \Lambda_{n-1} = o(n), \tag{23}$$

$$\sqrt{\lambda_n} = n + v + o(1), \quad \Delta_n = 1 + o(1), \quad \Delta_n - \Delta_{n-1} = o(1), \tag{24}$$

$$\forall r \in]0, \lambda_1[, N_n(r) = 0, \quad \exists A > 0, \ \forall r, \ |\sqrt{r} - N_n(r)| \leqslant A. \tag{25}$$

We shall use repeatedly that for any real sequence $\{r_n\}_{n\in\mathbb{N}^*}$

$$r_n = o(1) \Rightarrow \left|\ln\left(1 + \frac{r_n}{1 + o(1)}\right)\right| = |r_n|(1 + o(1)). \tag{26}$$

To prove (21), we estimate the left-hand side in terms of N_n:

$$\begin{aligned}\ln|f_n(z)| &\leqslant \sum_{k\neq n} \ln\left(1+\frac{|z|}{\lambda_k}\right) = \int_0^\infty \ln\left(1+\frac{|z|}{r}\right) dN_n(r)\\ &= \int_0^\infty N_n(r)\frac{|z|}{|z|+r}\frac{dr}{r} = \int_0^\infty \frac{N_n(|z|s)}{1+s}\frac{ds}{s}.\end{aligned}$$

To estimate this last integral we use (25) and the integral computations:

$$\int_0^\infty \frac{\sqrt{s}}{1+s}\frac{ds}{s} = \int_0^\infty \frac{2\,dr}{1+r^2} = \pi, \quad \int_{\frac{\lambda_1}{|z|}}^\infty \frac{ds}{s(1+s)} = \left[\ln\left|\frac{s}{1+s}\right|\right]_{\frac{\lambda_1}{|z|}}^\infty = \ln\left(1+\frac{|z|}{\lambda_1}\right).$$

Thus we obtain $\ln|f_n(z)|\leqslant\pi\sqrt{|z|}+A\ln(1+\frac{|z|}{\lambda_1})$, so that, for all $\varepsilon>0$ there is a $A'_\varepsilon>0$ such that $\ln|f_n(z)|\leqslant(\pi+\varepsilon)\sqrt{|z|}+A'_\varepsilon$.

To prove (22), we estimate the left-hand side in terms of N_n:

$$\begin{aligned}\ln|f_n(\lambda_n)| &= \sum_{k<n} \ln\left(\frac{\lambda_n}{\lambda_k}-1\right) + \sum_{k>n} \ln\left(1-\frac{\lambda_n}{\lambda_k}\right)\\ &= \int_{\lambda_1^-}^{\lambda_{n-1}^+} \ln\left(\frac{\lambda_n}{r}-1\right) dN_n(r) + \int_{\lambda_{n+1}^-}^\infty \ln\left(1-\frac{\lambda_n}{r}\right) dN_n(r).\end{aligned}$$

Integrating by parts yields $\ln|f_n(\lambda_n)| = I_n + B_n$ with

$$\begin{aligned}I_n &= \int_{\lambda_1^-}^{\lambda_{n-1}^+} N_n(r)\frac{\lambda_n}{\lambda_n - r}\frac{dr}{r} + \int_{\lambda_{n+1}^-}^\infty N_n(r)\frac{\lambda_n}{\lambda_n-r}\frac{dr}{r},\\ B_n &= \left[N_n(r)\ln\left(\frac{\lambda_n}{r}-1\right)\right]_{\lambda_1^-}^{\lambda_{n-1}^+} + \left[N_n(r)\ln\left(1-\frac{\lambda_n}{r}\right)\right]_{\lambda_{n+1}^-}^\infty.\end{aligned}$$

To estimate the boundary term B_n, we first simplify its expression using $N_n(\lambda_1^-)=0$ and $N_n(\lambda_{n-1}^+)=N_n(\lambda_{n+1}^-)=n-1$, then we sort out the increments $\Lambda_n=\lambda_{n+1}-\lambda_n$, and finally we use (23) and (26)

$$\begin{aligned}B_n &= (n-1)\left[\ln\left(\frac{\lambda_n}{\lambda_{n-1}}-1\right) - \ln\left(1-\frac{\lambda_n}{\lambda_{n+1}}\right)\right]\\ &= (n-1)\left[\ln\left(1-\frac{\Lambda_n-\Lambda_{n-1}}{\Lambda_n}\right) + \ln\left(1+\frac{\Lambda_n+\Lambda_{n-1}}{\lambda_{n-1}}\right)\right]\\ &= (n-1)\left[\frac{o(n)}{2n}(1+o(1)) + \frac{4n+o(n)}{n^2}(1+o(1))\right] = o(1).\end{aligned}$$

Now we estimate the integral term I_n. Performing the change of variable $r = \lambda_n s$ and using (25) yields: $|I_n - \sqrt{\lambda_n} J_n| \leqslant A K_n$ with

$$J_n = \int_{\frac{\lambda_1^-}{\lambda_n}}^{\frac{\lambda_{n-1}^+}{\lambda_n}} \frac{ds}{(1-s)\sqrt{s}} + \int_{\frac{\lambda_{n+1}^-}{\lambda_n}}^{\infty} \frac{ds}{(1-s)\sqrt{s}},$$

$$K_n = \int_{\frac{\lambda_1^-}{\lambda_n}}^{\frac{\lambda_{n-1}^+}{\lambda_n}} \frac{ds}{(1-s)s} + \int_{\frac{\lambda_{n+1}^-}{\lambda_n}}^{\infty} \frac{ds}{(s-1)s}.$$

The term K_n is readily computed and estimated using (23)

$$K_n = \left[\ln \frac{s}{1-s}\right]_{\frac{\lambda_1^-}{\lambda_n}}^{\frac{\lambda_{n-1}^+}{\lambda_n}} + \left[\ln \frac{s-1}{s}\right]_{\frac{\lambda_{n+1}^-}{\lambda_n}}^{\infty} = \ln \frac{\lambda_{n+1}}{\Lambda_n} + \ln \frac{\lambda_{n-1}}{\Lambda_{n-1}} + \ln\left(\lambda_n\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_n}\right)\right)$$
$$= 2\ln \frac{n^2 + O(n)}{2n + o(n)} + 2\ln\sqrt{\lambda_n} + O(1) = o(\sqrt{\lambda_n}).$$

We compute J_n after a change of variable, and estimate it by (24) and (26) after sorting out the increments $\Delta_n = \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n}$

$$J_n = \int_{\frac{\sqrt{\lambda_1}^-}{\sqrt{\lambda_n}}}^{\frac{\sqrt{\lambda_{n-1}}^+}{\sqrt{\lambda_n}}} \frac{2\,dr}{r^2-1} + \int_{\frac{\sqrt{\lambda_{n+1}}^-}{\sqrt{\lambda_n}}}^{\infty} \frac{2\,dr}{r^2-1} = \left[\ln \frac{1-r}{r+1}\right]_{\frac{\sqrt{\lambda_1}^-}{\sqrt{\lambda_n}}}^{\frac{\sqrt{\lambda_{n-1}}^+}{\sqrt{\lambda_n}}} + \left[\ln \frac{r-1}{r+1}\right]_{\frac{\sqrt{\lambda_{n+1}}^-}{\sqrt{\lambda_n}}}^{\infty}$$
$$= \ln\left(\frac{\Delta_{n-1}}{\Delta_n}\right) + \ln\left(\frac{\sqrt{\lambda_{n+1}} + \sqrt{\lambda_n}}{\sqrt{\lambda_n} + \sqrt{\lambda_{n-1}}}\right) - \ln\left(\frac{\sqrt{\lambda_n} - \sqrt{\lambda_1}}{\sqrt{\lambda_n} + \sqrt{\lambda_1}}\right)$$
$$= \ln\left(1 - \frac{\Delta_n - \Delta_{n-1}}{\Delta_n}\right) + \ln\left(1 + \frac{\Delta_n + \Delta_{n-1}}{\sqrt{\lambda_n} + \sqrt{\lambda_{n-1}}}\right) - \ln\left(1 - \frac{2\sqrt{\lambda_1}}{\sqrt{\lambda_n} + \sqrt{\lambda_1}}\right)$$
$$= o(1)(1 + o(1)) + \frac{2 + o(1)}{2n}(1 + o(1)) + \frac{O(1)}{n}(1 + o(1)) = o(1).$$

Plugging the estimates $K_n = o(\sqrt{\lambda_n})$ and $J_n = o(1)$ into $|I_n - \sqrt{\lambda_n} J_n| \leqslant A K_n$ yields $I_n = o(\sqrt{\lambda_n})$. Plugging this estimate and $B_n = o(1)$ into $\ln|f_n(\lambda_n)| = I_n + B_n$ yields $\ln|f_n(\lambda_n)| = o(\sqrt{\lambda_n})$, which completes the proof of (22). □

Lemma 4.4. *Let α_* be defined by* (4). *For all $d > 0$ there is a $D > 0$ such that for all $\tau > 0$, there is an even entire function M of exponential type τ satisfying*: $M(0) = 1$ *and*

$$\forall x > 0, \quad \ln|M(x)| \leqslant \frac{\alpha_* d^2}{2\tau} + D - d\sqrt{x} \quad \textit{and} \quad |M(ix)| \geqslant 1. \tag{27}$$

Proof. Following Ingham and many others since 1934 (cf. Section 14 in [18] for theorems and references) we seek a multiplier M of small exponential type decaying rapidly along the real axis in the following form:

$$M(z) = \prod_{n \in \mathbb{N}} \mathrm{sinc}\left(\frac{z}{a_n}\right) \quad \text{where } \mathrm{sinc}(0) = 1, \ \forall z \in \mathbb{C}^*, \ \mathrm{sinc}(z) = \frac{\sin(z)}{z} \tag{28}$$

and where $\{a_n\}_{n \in \mathbb{N}}$ is a non decreasing sequence of positive real numbers such that $\tau_M = \sum_{n \in \mathbb{N}} \frac{1}{a_n} < \infty$. Since the cardinal sine function sinc is an even entire function of exponential type 1 satisfying $\mathrm{sinc}(0) = 1$ and $\mathrm{sinc}(ix) = \sinh(x)/x \geqslant 1$ for all $x > 0$, (28) defines an even entire function M of exponential type τ_M satisfying $M(0) = 1$ and $|M(ix)| \geqslant 1$ for all $x > 0$.

We define $\{a_n\}_{n \in \mathbb{N}}$ by the slope A of its counting function N and its first term a_0 (to be chosen large enough)

$$N(r) := \sum_{|a_n| \leqslant r} 1 = [A\sqrt{u}] \text{ for } r \geqslant 2 \quad \text{and} \quad a_0 \geqslant A^{-2},$$

where $[x]$ denotes as usual the greatest integer smaller or equal to the real number x. The exponential type τ_M of M is easily bounded from above by $\tau = 2A/\sqrt{a_0}$

$$\tau_M := \sum_{n \in \mathbb{N}} \frac{1}{a_n} = \int_0^\infty \frac{dN(r)}{r} = \int_0^\infty \frac{N(r)}{r^2}\, dr \leqslant \int_{a_0}^\infty \frac{A\sqrt{r}}{r^2}\, dr = \frac{2A}{\sqrt{a_0}} =: \tau$$

and we are left with estimating the decay of

$$\ln|M(x)| = \int_{a_0^-}^\infty f\left(\frac{x}{r}\right) dN(r) \quad \text{where } f(\theta) = \ln \mathrm{sinc}(\theta) = \ln \frac{\sin(\theta)}{\theta}. \tag{29}$$

We shall choose A such that, for all $a_0 \geqslant A^{-2}$, $\ln|M(x)| \leqslant -d\sqrt{x} + O(1)$ as $x \to +\infty$, and then prove that: $\ln|M(x)| \leqslant \alpha_* d^2/(2\tau) - d\sqrt{x} + O(1)$ as $\tau \to 0$ (equivalently $a_0 \to +\infty$) uniformly in $x > 0$.

For $x > a_0$ we take advantage of the boundedness of sine through the estimate $f(\theta) \leqslant -\ln|\theta|$ for $|\theta| \leqslant 1$, by splitting the integral in (29) into the two terms:

$$I = \int_{a_0^-}^x f\left(\frac{x}{r}\right) dN(r) \leqslant \int_{a_0^-}^x \ln\left|\frac{r}{x}\right| dN(r) = -\int_{a_0}^x N(r) \frac{dr}{r}$$

$$J = \int_x^\infty f\left(\frac{x}{r}\right) dN(r) = \int_0^1 f'(\theta) N\left(\frac{x}{\theta}\right) d\theta - f(1)N(x)$$

where right-hand sides were integrated by parts and $\theta = x/r$. Now we plug in the basic estimate on N: $A\sqrt{r} - 1 \leqslant N(r) \leqslant A\sqrt{r}$ for $r \geqslant 2$. The first term is now estimated by

$$I \leqslant -A \int_{a_0}^x \frac{dr}{\sqrt{r}} + \int_{a_0}^x \frac{dr}{r} = -2A(\sqrt{x} - \sqrt{a_0}) + \ln x - \ln a_0. \tag{30}$$

To estimate the second term, we first observe that the Hadamard factorization of the cardinal sine function $\operatorname{sinc}(\pi z) = \prod_{n\in\mathbb{N}^*}\left(1-\frac{z^2}{n^2}\right)$ and the Taylor expansion of the logarithm at 1 imply

$$f(\theta) = -\sum_{k\in\mathbb{N}^*}\frac{\zeta(2k)}{k}\left(\frac{\theta}{\pi}\right)^{2k} \text{ for } |\theta|<1, \quad \text{where } \zeta(s) = \sum_{n\in\mathbb{N}^*}\frac{1}{n^s}.$$

The second term is now estimated by

$$\begin{aligned} J &\leqslant \int_0^1 f'(\theta)\left(\frac{A\sqrt{x}}{\sqrt{\theta}}-1\right)d\theta - f(1)A\sqrt{x} \\ &= A\sqrt{x}\left(\int_0^1 f'(\theta)\frac{d\theta}{\sqrt{\theta}} - f(1)\right) - f(1) \\ &= -A\sqrt{x}\sum_{k\in\mathbb{N}^*}\left(\frac{2k}{2k-\frac{1}{2}}-1\right)\frac{\zeta(2k)}{k\pi^{2k}} - f(1) = -A\Sigma^*\sqrt{x} - f(1), \end{aligned} \tag{31}$$

where the series for f was differentiated, multiplied and integrated term by term, and $\Sigma^* = \sum_{k\in\mathbb{N}^*}\frac{1}{k(4k-1)}\frac{\zeta(2k)}{\pi^{2k}}$. Putting (30) and (31) together yields

$$\forall x > a_0, \quad \ln|M(x)| \leqslant -(2+\Sigma^*)A\sqrt{x} + \ln x - f(1) + 2A\sqrt{a_0},$$

so that, for all $d > (2+\Sigma^*)A$ there is a D_1 such that

$$\forall d > (2+\Sigma^*)A, \exists D_1 > 0, \quad \forall x > a_0, \quad \ln|M(x)| \leqslant 2A\sqrt{a_0} - d\sqrt{x} + D_1. \tag{32}$$

Since $|\operatorname{sinc}|$ is bounded by 1: for all x, $\ln|M(x)| \leqslant 0$. Moreover $d > 2A$, so that (32) implies

$$\forall a_0 \geqslant A^{-2}, \forall x > 0, \quad \ln|M(x)| \leqslant d\sqrt{a_0} - d\sqrt{x} + D_1. \tag{33}$$

Since $d > (2+\Sigma^*)A$ and $\tau = 2A/\sqrt{a_0}$, this proves

$$\forall \tau \leqslant 2A^2, \forall x > 0, \quad \ln|M(x)| \leqslant \frac{\alpha_1 d^2}{2\tau} - d\sqrt{x} + D_1 \tag{34}$$

with $\alpha_1 = 4/(2+\Sigma_*)$.

For $x < a_0$, we can also use the better estimate

$$\begin{aligned}\ln|M(x)| &\leqslant \int_{a_0}^{\infty} f\left(\frac{x}{r}\right) dN(r) = \int_0^{x/a_0} f'(\theta) N\left(\frac{x}{\theta}\right) d\theta \\ &\leqslant A\sqrt{x} \int_0^{x/a_0} f'(\theta) \frac{d\theta}{\sqrt{\theta}} - f\left(\frac{x}{a_0}\right) \\ &\leqslant -A\sqrt{a_0} \sum_{k \in \mathbb{N}^*} \frac{4k\zeta(2k)}{k(4k-1)} \left(\frac{x}{a_0 \pi}\right)^{2k} - f(1). \end{aligned} \tag{35}$$

If we keep only the first term (i.e. $k = 1$) of the series in (32) and (35), we get that for all $d > (2 + \frac{1}{3}\frac{\zeta(2)}{\pi^2})A$ there is a D_2 such that

$$\begin{aligned} &\forall x > a_0, \quad \ln|M(x)| \leqslant 2A\sqrt{a_0} - d\sqrt{x} + D_2, \\ &\forall x < a_0, \quad \ln|M(x)| \leqslant -A\sqrt{a_0} \frac{4\zeta(2)}{3\pi^2} \left(\frac{x}{a_0}\right)^2 - f(1). \end{aligned} \tag{36}$$

Now, for all $x < a_0$

$$\ln|M(x)| - 2A\sqrt{a_0} + d\sqrt{x} \leqslant A\sqrt{a_0} F\left(\frac{x}{a_0}\right)$$

with $F(X) = -2 + (2 + \varepsilon)\sqrt{X} + \frac{1}{3}\frac{\zeta(2)}{\pi^2}(\sqrt{X} - 4X^2) = -2 + (37/18 + \varepsilon)\sqrt{X} - 2X^2/9$ and $\varepsilon = d/A - (2 + \frac{1}{3}\frac{\zeta(2)}{\pi^2}) > 0$. Since F is increasing on $[0, 1]$ and $F(1) = \varepsilon - \frac{1}{6}$, choosing A so that $\varepsilon < \frac{1}{6}$, yields that $\ln|M(x)| - 2A\sqrt{a_0} + d\sqrt{x} \leqslant 0$, for all $x < a_0$. Together with (36), this proves

$$\forall x > 0, \quad \ln|M(x)| \leqslant 2A\sqrt{a_0} - d\sqrt{x} + D_2. \tag{37}$$

Since $d > (2 + \frac{1}{3}\frac{\zeta(2)}{\pi^2})A = 37A/18$ and $\tau = 2A/\sqrt{a_0}$, this proves

$$\forall \tau \leqslant 2A^2, \forall x > 0, \quad \ln|M(x)| \leqslant \frac{\alpha_2 d^2}{2\tau} - d\sqrt{x} + D_2 \tag{38}$$

with $\alpha_2 = 2(36/37)^2$.

Eqs. (34) and (38) complete the proof of the Lemma 4.4 with $\alpha_* = \min\{\alpha_1, \alpha_2\}$. Since we have checked on a computer that $\alpha_1 > \alpha_2$, we decided to state the lemma with $\alpha_* = \alpha_2$, i.e. (4). $\square$

To prove Theorem 4.2, we use Lemmas 4.3 and 4.4 with $d = \pi + 2\varepsilon$ and define

$$G_n = F_n M_n \quad \text{with} \quad F_n(z) = f_n(-iz)/f_n(\lambda_n) \quad \text{and} \quad M_n(z) = M(z)/M(i\lambda_n).$$

Thanks to Lemma 4.3, the entire function F_n satisfies

$$F_n(i\lambda_n) = 1 \text{ and } \forall k \in \mathbb{N}^*,\ k \neq n,\ F_n(i\lambda_k) = 0, \tag{39}$$

$$\ln|F_n(z)| \leqslant (\pi + \varepsilon)\sqrt{|z|} + \varepsilon\sqrt{\lambda_n} + 2A_\varepsilon, \tag{40}$$

where (39) is an obvious consequence of the definitions of f_n and F_n, and (40) is a consequence of estimates (21) and (22).

Thanks to Lemma 4.4, there is a $D_\varepsilon > 0$ such that the entire function M_n is of exponential type τ and satisfies

$$M_n(i\lambda_n) = 1, \tag{41}$$

$$\forall x \in \mathbb{R}, \quad \ln|M_n(x)| \leqslant \frac{\alpha_* d^2}{2\tau} + D_\varepsilon - d\sqrt{|x|}, \tag{42}$$

where (41) is an obvious consequence of the definitions of M and M_n, and (42) is a consequence of (27) since M is even.

The entire function G_n has the same exponential type as M_n since (40) implies that the exponential type of F_n is 0. Hence (17) holds. Putting (39) and (41) together yields (18). Since $d = \pi + 2\varepsilon$, (40) and (42) imply

$$\forall x \in \mathbb{R}, \quad \ln|G_n(x)| \leqslant D_\varepsilon + 2A_\varepsilon - \varepsilon\sqrt{|x|} + \varepsilon\sqrt{\lambda_n} + \frac{\alpha_* d^2}{2\tau}.$$

Hence (19) holds with $C_\varepsilon = e^{D_\varepsilon + 2A_\varepsilon}\left(\int_{-\infty}^{+\infty} e^{-2\varepsilon\sqrt{|x|}}\,dx\right)^{1/2}$. Theorem 4.2 is proved. $\square$

Remark 4.5. Under assumption (15), Lemma 3 in [23] (which applies to much more general sequences) proves that $F_n(z) = \prod_{k \neq n}\left[1 - \left(\frac{z-\lambda_n}{\lambda_k - \lambda_n}\right)^2\right]$ satisfies (39) and $\ln|F_n(\lambda_n + z)| \leqslant 2\pi\sqrt{|z|}$, hence $\ln|F_n(z)| \leqslant 2\pi\sqrt{|z|} + O(\sqrt{\lambda_n})$. In (40), the estimate $O(\sqrt{\lambda_n})$ improves to $o(\sqrt{\lambda_n})$ and the constant 2π improves to the optimal π (optimality can be deduced from Theorem 4.1.1 in [2]).

Seidman obtained Lemma 4.4 for $\alpha_* = \beta_*$ with $\beta_* \approx 42.86$ in the proof of Theorem 3.1 in [21]. His later Theorem 1 in [22] improves the rate to $\alpha_* = 2\beta_*$ with $\beta_* \approx 4.17$. Theorem 2 in [23], which applies to much more general spectral sequences, yields Lemma 4.4 for $\alpha_* = 24$. The argument used in Section 3 can be used to prove that Lemma 4.4 does not hold for $\alpha_* < \frac{1}{4}$. It would be interesting to determine the smallest value of α_* for which it holds.

5. Upper bound under the geodesics condition

In this section, we prove Theorem 2.3 in three steps. $\mathcal{D}'(\mathcal{O})$ denotes the space of distributions on the open set $\mathcal{O}$ endowed with the weak topology and $\mathcal{M}(\mathcal{O})$ denotes the subspace of Radon measures on $\mathcal{O}$.

5.1. The segment controlled at both ends

In a first step, we prove that the upper bound for the null-controllability cost of the heat equation on the segment $[0, L]$ controlled at one end is the same as the null-controllability cost of the heat equation on the twofold segment $[-L, L]$ controlled at both ends.

Given a time $T > 0$ and a length $L > 0$, we denote by D (respectively, N) some continuous operator from $L^2(0, L)$ to $L^2(0, T)$ allowing to control to zero in time T the heat equation on $[0, L]$ with zero Dirichlet (respectively, Neumann) condition at 0 by a Dirichlet control at L. More precisely, for all $u_0 \in L^2(0, L)$ the solution $u \in C^0([0, \infty), L^2(0, L))$, denoted by $u = S_D u_0$ (respectively $u = S_N u_0$), of the Cauchy problem in Theorem 2.2 with $B = 1$ (respectively $B = \partial_s$) and $g = Du_0$ (respectively $g = Nu_0$) satisfies $u = 0$ at $t = T$.

Proposition 5.1. *For any time $T > 0$ and any length $L > 0$, there is a continuous operator K from $L^2(-L, L)$ to $L^2(0, T)^2$ allowing to control to zero in time T the heat equation on $[-L, L]$ by Dirichlet controls at both ends at the same cost as D and N, i.e. for all $v_0 \in L^2(-L, L)$ the solution $v \in C^0([0, \infty), L^2(-L, L))$ of*

$$\partial_t v - \partial_s^2 v = 0 \quad \text{in }]0, T[\times] - L, L[, \quad (v_{\rceil s=-L}, v_{\rceil s=L}) = Kv_0, \quad v_{\rceil t=0} = v_0 \quad (43)$$

satisfies $v = 0$ at $t = T$ and $\|K\| \leqslant \sup(\|D\|, \|N\|)$.

Proof. Given $v_0 \in L^2(-L, L)$, we decompose it in odd and even parts: $v_0 = v_{0,\mathrm{odd}} + v_{0,\mathrm{even}}$. We denote by $u_{0,\mathrm{odd}}$ and $u_{0,\mathrm{even}}$ the restrictions of $v_{0,\mathrm{odd}}$ and $v_{0,\mathrm{even}}$ to $[0, L]$. We denote by $f = Du_{0,\mathrm{odd}}$ and $g = Nu_{0,\mathrm{even}}$ the corresponding controls. We denote by $u_{\mathrm{odd}} = S_D u_{0,\mathrm{odd}}$ and $u_{\mathrm{even}} = S_N u_{0,\mathrm{even}}$ the corresponding solutions.

We define $v \in L^2([0, T] \times [-L, L])$ by $v(t, \pm s) = u_{\mathrm{even}}(t, s) \pm u_{\mathrm{odd}}(t, s)$ for $s \geqslant 0$. Since

$$(\partial_t - \partial_s^2) u_{\mathrm{even}} = (\partial_t - \partial_s^2) u_{\mathrm{odd}} = 0 \text{ in } \mathcal{D}'(]0, T[\times]0, L[),$$

we have, denoting the Dirac mass at $s = 0$ by $\delta_s \in \mathcal{D}'(\mathbb{R})$,

$$(\partial_t - \partial_s^2) v = 2u_{\mathrm{odd}}(t, 0) \otimes \delta_s{}'(0) + 2\partial_s u_{\mathrm{even}}(t, 0) \otimes \delta_s(0).$$

But $u_{\mathrm{odd}}(t, 0) = \partial_s u_{\mathrm{even}}(t, 0) = 0$ by the definition of D and N. Hence $(\partial_t - \partial_s^2) v = 0$. Moreover $v(0, s) = v_0(s)$, $v(T, s) = 0$, $v(t, L) = g(t) + f(t)$, $v(t, -L) = g(t) - f(t)$.

Therefore, setting $Kv_0 = (g - f, g + f)$ yields an operator K satisfying the null-controllability property required.

To finish the proof we estimate its cost $||K||$. Taking the Euclidean norm for $Kv_0 = (g - f, g + f)$, we have $||Kv_0||^2_{L^2(0,T)^2} = 2||f||^2_{L^2(0,T)} + 2||g||^2_{L^2(0,T)}$. Since $f = Du_{0,\mathrm{odd}}$ and $g = Nu_{0,\mathrm{even}}$, setting $C = \sup(||D||, ||N||)$ we have

$$||Kv_0||^2_{L^2(0,T)^2} \leqslant 2C^2(||u_{0,\mathrm{odd}}||^2_{L^2(0,L)} + ||u_{0,\mathrm{even}}||^2_{L^2(0,L)}). \tag{44}$$

Moreover, since $2u_{0,\mathrm{odd}}(s) = v_0(s) - v_0(-s)$ and $2u_{0,\mathrm{even}}(s) = v_0(s) + v_0(-s)$ for $s \in [0, L]$, we have

$$||2u_{0,\mathrm{odd}}||^2_{L^2(0,L)} = ||v_0||^2_{L^2(-L,L)} - 2\int_0^L v_0(s)v_0(-s)\, ds, \tag{45}$$

$$||2u_{0,\mathrm{even}}||^2_{L^2(0,L)} = ||v_0||^2_{L^2(-L,L)} + 2\int_0^L v_0(s)v_0(-s)\, ds. \tag{46}$$

Eqs. (44)–(46) imply $||Kv_0||_{L^2(0,T)^2} \leqslant C||v_0||_{L^2(-L,L)}$. $\square$

5.2. The fundamental controlled solution

In a second step we construct a "fundamental controlled solution" v of the heat equation on the segment controlled by Dirichlet conditions at both ends.

Proposition 5.2. *If Theorem 2.2 holds for some rate α_*, then for any $\alpha > \alpha_*$, there exists $A > 0$ such that for all $L > 0$ and $T \in]0, \inf(\pi/2, L)^2]$ there is a $v \in C^0([0, T], \mathcal{M}(] - L, L[))$ satisfying*

$$\partial_t v - \partial_s^2 v = 0 \quad \textit{in } \mathcal{D}'(]0, T[\times] - L, L[), \tag{47}$$

$$v\rceil_{t=0} = \delta \quad \textit{and} \quad v\rceil_{t=T} = 0, \tag{48}$$

$$||v||_{L^2(]0,T[\times]-L,L[)} \leqslant Ae^{\alpha L^2/T}. \tag{49}$$

We shall sometimes refer to a function v satisfying the above requirements as a fundamental controlled solution on $]0, T[\times] - L, L[$ at cost (A, α).

Proof. We first reduce the problem to the case $L = \pi/2$ using the rescaling $(t, s) \mapsto (\sigma^2 t, \sigma s)$, $\sigma > 0$ with $\sigma = \pi/(2L)$. Given $L > 0$ and $T \in]0, \inf(\pi/2, L)^2]$, we set $\tilde{L} = \pi/2$ and $\tilde{T} = \sigma^2 T \in]0, \tilde{L}^2]$. Let $\tilde{v}$ be a fundamental controlled solution on $]0, \tilde{T}[\times] - \tilde{L}, \tilde{L}[$ at cost $(\tilde{A}, \tilde{\alpha})$. Setting $v(t, s) = \sigma\tilde{v}(\sigma^2 t, \sigma s)$ defines a fundamental controlled solution v on $]0,T[\times]-L,L[$ at cost $(\tilde{A}/\sqrt{\sigma},\tilde{\alpha})$. Since $T\leqslant\tilde{L}^2$, we have $\tilde{A}/\sqrt{\sigma}\leqslant\tilde{A}(L^2/T)^{1/4}$. Hence for all $\alpha>\tilde{\alpha}$ there is an $A>0$ such that v is also a fundamental controlled solution on $]0,T[\times]-L,L[$ at cost (A,α). Therefore, it is enough to prove Proposition 5.2 in the particular case $L=\pi/2$.

We assume Theorem 2.2 holds for some rate α_*. Let $\tilde{\alpha}>\tilde{\alpha}_*>\alpha_*$, $L=\tilde{L}=\pi/2$ and $\tilde{T}\in]0,\tilde{L}^2]$ be fixed from now on. We set $\alpha=(1-\varepsilon)\tilde{\alpha}_*$ and $T=(1-\varepsilon)\tilde{T}$ where $\varepsilon\in]0,1[$ is chosen such that $\alpha>\alpha_*$. Applying Theorem 2.2 once with $B=1$ and once with $B=\partial_s$, and then applying Proposition 5.1 yields a $C>0$ independent of $\tilde{T}$ such that

$$||K||\leqslant\sup(||D||,||N||)\leqslant Ce^{\alpha L^2/T}=Ce^{\tilde{\alpha}_*\tilde{L}^2/\tilde{T}}. \tag{50}$$

We define $\tilde{v}\in C^0([0,\tilde{T}],M(]-\tilde{L},\tilde{L}[))$ as the solution of

$$\partial_t\tilde{v}-\partial_s^2\tilde{v}=0\quad\text{in }]0,\tilde{T}[\times]-\tilde{L},\tilde{L}[,\quad(\tilde{v}_{\rceil s=-\tilde{L}},\tilde{v}_{\rceil s=\tilde{L}})=b,\quad\tilde{v}_{\rceil t=0}=\delta,$$

where the control $b\in L^2(0,\tilde{T})^2$ is defined by $b(t)=0$ for $t\leqslant\varepsilon\tilde{T}$ and by $b(\varepsilon\tilde{T}+t')=K(\tilde{v}_{\rceil t=\varepsilon T})(t')$ for $t'\in]0,T[$. Note that $v_0=\tilde{v}_{\rceil t=\varepsilon T}$ is just the Dirac mass at the origin smoothed out by the homogeneous heat semigroup during a time $\varepsilon\tilde{T}$, so that $v_0\in L^2(-L,L)$. Moreover $\varepsilon\tilde{T}+T=\tilde{T}$ and $v(t,s)=\tilde{v}(\varepsilon\tilde{T}+t,s)$ is the solution of (43), so that $\tilde{v}_{\rceil t=\tilde{T}}=v_{\rceil t=T}=0$.

To finish the proof that $\tilde{v}$ is a fundamental controlled solution on $]0,\tilde{T}[\times]-\tilde{L},\tilde{L}[$, we estimate its $L^2(]0,\tilde{T}[\times]-\tilde{L},\tilde{L}[)$ norm which we abbreviate as $||\tilde{v}||_{\tilde{T},\tilde{L}}$. Setting $e_j(s)=\sin(j(s+\pi/2))/\sqrt{2/\pi}$ defines an orthonormal basis $(e_j)_{j\in\mathbb{N}^*}$ of $L^2(]-\tilde{L},\tilde{L}[)$ such that e_j is an eigenvector of $-\Delta_s$ with eigenvalue j^2. In the weak topology, the Dirac mass can be decomposed in this basis as $\delta(s)=\sum_j e_j(0)e_j(s)$. Note that the sequence $(e_j(0))_{j\in\mathbb{N}^*}$ is bounded. For $t\in]0,\tilde{T}]$, we introduce the coordinates $(\tilde{v}_j(t))_{j\in\mathbb{N}^*}$ of $\tilde{v}(t,\cdot)\in L^2(]-\tilde{L},\tilde{L}[)$ in the Hilbert basis $(e_j)_{j\in\mathbb{N}^*}$. Using these coordinates and abbreviating the $L^2(]0,\tilde{T}[)$ norm as $||\cdot||_{\tilde{T}}$, the function $\tilde{v}$ and its norm write

$$\tilde{v}(t,s)=\sum_j\tilde{v}_j(t)e_j(s)\quad\text{and}\quad||\tilde{v}||^2_{\tilde{T},\tilde{L}}=\int_0^{\tilde{T}}\sum_j|\tilde{v}_j(t)|^2\,dt=\sum_j||\tilde{v}_j||^2_{\tilde{T}}. \tag{51}$$

As in [7], these coordinates can be computed by $\tilde{v}_j(0)=e_j(0)$ and

$$\tilde{v}_j(t)=e^{-j^2t}\tilde{v}_j(0)+\int_0^t e^{-j^2(t-t')}(e_j'(-\tilde{L})\tilde{v}(t',-\tilde{L})-e_j'(\tilde{L})\tilde{v}(t',\tilde{L}))\,dt'. \tag{52}$$

Using Young's inequality to estimate the second term of the right-hand side, we have (since $\tilde{T}<4$, $|e_j'(\pm\tilde{L})|=|\tilde{v}_j(0)|=\sqrt{2/\pi}<1$)

$$\begin{aligned}||\tilde{v}_j||_{\tilde{T}}&\leqslant|\tilde{v}_j(0)|||e^{-j^2t}||_{\tilde{T}}+||e^{-j^2t}||_{L^1(]0,\tilde{T}[)}(|e_j'(-\tilde{L})|||\tilde{v}(t',-\tilde{L})||_{\tilde{T}}+|e_j'(\tilde{L})|||\tilde{v}(t',\tilde{L})||_{\tilde{T}})\\&\leqslant\frac{4}{j}(1+||\tilde{v}(t',-\tilde{L})||_{\tilde{T}}+||\tilde{v}(t',\bar{L})||_{\tilde{T}}).\end{aligned}$$

Hence Eq. (51) implies

$$||\tilde{v}||^2_{\tilde{T},\tilde{L}} \leqslant (1 + ||\tilde{v}(t', -\tilde{L})||^2_{\tilde{T}} + ||\tilde{v}(t', \tilde{L})||^2_{\tilde{T}}) \sum_j \frac{4^3}{j^2}$$
$$= \frac{4^3\pi^2}{6}\left(1 + ||Kv_0||^2_{L^2(]0,\tilde{T}[)}\right).$$

But there is an $A' > 0$ independent of $\varepsilon\tilde{T} < 1$ such that:

$$||v_0||^2_{L^2(]-\tilde{L},\tilde{L}[)} = \sum_j |\tilde{v}_j(\varepsilon\tilde{T})|^2 \leqslant \sum_j e^{-2j^2\varepsilon\tilde{T}} \leqslant \frac{A'}{\sqrt{\varepsilon\tilde{T}}}.$$

Hence Eq. (50) yields a $C' > 0$ independent of $\tilde{T}$ such that

$$||\tilde{v}||_{\tilde{T},\tilde{L}} \leqslant \frac{8\pi}{\sqrt{6}}(1 + 2\sqrt{\pi}||K||\,||v_0||_{\tilde{L}}) \leqslant \frac{C'}{\sqrt{\tilde{T}}} e^{\tilde{\alpha}_* \tilde{L}^2/\tilde{T}}.$$

Since $\tilde{\alpha} > \tilde{\alpha}_*$, there is an $\tilde{A} > 0$ independent of $\tilde{T}$ such that: $||\tilde{v}||^2_{\tilde{T},\tilde{L}} \leqslant \tilde{A} e^{\tilde{\alpha}_* \tilde{L}^2/\tilde{T}}$. This completes the proof that $\tilde{v}$ is a fundamental controlled solution on $]0,\tilde{T}[\times]-\tilde{L},\tilde{L}[$ at cost $(\tilde{A}, \tilde{\alpha})$. $\square$

5.3. *The transmutation of waves into heat*

In a third step, we perform a transmutation of an exact control for the wave equation into a null-control for the heat equation. Our transmutation formula can be regarded as the analogue of Kannai's formula (6) where the kernel $e^{-s^2/(4t)}/\sqrt{4\pi t}$, which is the fundamental solution of the heat equation on the line, is replaced by the fundamental controlled solution that we have constructed in the previous step. To ensure existence of an exact control for the wave equation we use the geodesics condition of Bardos–Lebeau–Rauch (already mentioned above Theorem 2.3):

Theorem 5.3 (Bardos et al. [1]). *If $L > L_\Omega$ then for all $(w_0, w_1) \in H^1_0(M) \times L^2(M)$ and all $(w_2, w_3) \in H^1_0(M) \times L^2(M)$ there is a control function $f \in L^2(\mathbb{R}_+ \times M)$ such that the solution $w \in C^0(\mathbb{R}_+, H^1_0(M)) \cap C^1(\mathbb{R}_+, L^2(M))$ of the mixed Dirichlet–Cauchy problem (n.b. the time variable is denoted by s here)*

$$\partial_s^2 w - \Delta w = \mathbf{1}_{]0,L[\times\Omega} f \quad \text{in } \mathbb{R}_+ \times M, \quad w = 0 \quad \text{on } \mathbb{R}_+ \times \partial M, \tag{53}$$

with Cauchy data $(w, \partial_s w) = (w_0, w_1)$ at $s = 0$, satisfies $(w, \partial_s w) = (w_2, w_3)$ at $s = L$. Moreover, the operator $S_W : \left(H^1_0(M) \times L^2(M)\right)^2 \to L^2(\mathbb{R}_+ \times M)$ defined by $S_W((w_0, w_1), (w_2, w_3)) = f$ is continuous.

We assume that Theorem 2.2 holds for some rate α_*. Let $\alpha > \alpha_*$, $T \in]0, \inf(1, L^2_\Omega)[$ and $L > L_\Omega$ be fixed from now on. Let $A > 0$ and $v \in L^2(]0,T[\times]-L,L[)$ be the corresponding constant and fundamental controlled solution given by Proposition 5.2. We define $\underline{v}\in L^2(\mathbb{R}^2)$ as the extension of v by zero, i.e. $\underline{v}(t,s)=v(t,s)$ on $]0,T[\times]-L,L[$ and $\underline{v}$ is zero everywhere else. It inherits from v the following properties:

$$\partial_t \underline{v} - \partial_s^2 \underline{v} = 0 \quad \text{in } \mathcal{D}'(]0,+\infty[\times]-L,L[), \tag{54}$$

$$\underline{v}\rceil_{t=0} = \delta \text{ and } \underline{v}\rceil_{t=T} = 0, \tag{55}$$

$$||\underline{v}||_{L^2(]0,+\infty[\times\mathbb{R})} \leqslant A e^{\alpha L^2/T}. \tag{56}$$

Let $u_0\in H_0^1(M)$ be an initial data for the heat equation (1). Let w and f be the corresponding solution and control function for the wave equation obtained by applying Theorem 5.3 with $w_0=u_0$ and $w_1=w_2=w_3=0$. We define $\underline{w}\in L^2(\mathbb{R};H_0^1(M))$ and $\underline{f}\in L^2(\mathbb{R}\times M)$ as the extensions of w and f by reflection with respect to $s=0$, i.e. $\underline{w}(s,x)=w(s,x)=\underline{w}(-s,x)$ and $\underline{f}(s,x)=f(s,x)=\underline{f}(-s,x)$ on $\mathbb{R}_+\times M$. Since $w_1=0$, Eq. (53) imply

$$\partial_s^2 \underline{w} - \Delta \underline{w} = \mathbf{1}_{]-L,L[\times\Omega}\underline{f} \quad \text{in } \mathcal{D}'(\mathbb{R}\times M), \quad \underline{w}=0 \quad \text{on } \mathbb{R}\times\partial M. \tag{57}$$

The main idea of our proof is to use $\underline{v}$ as a kernel to transmute $\underline{w}$ and $\underline{f}$ into a solution u and a control g for (1). Since $\underline{v}\in L^2(\mathbb{R}^2)$, $\underline{w}\in L^2(\mathbb{R};H_0^1(M))$ and $\underline{f}\in L^2(\mathbb{R}\times M)$, the transmutation formulas

$$u(t,x) = \int_{\mathbb{R}} \underline{v}(t,s)\underline{w}(s,x)\,ds \quad \text{and} \quad g(t,x) = \int_{\mathbb{R}} \underline{v}(t,s)\underline{f}(s,x)\,ds, \tag{58}$$

define functions $u\in L^2(\mathbb{R};H_0^1(M))$ and $g\in L^2(\mathbb{R}\times M)$. Since $\underline{w}(s,x)=\partial_s\underline{w}(s,x)=0$ for $|s|=L$, Eqs. (57) and (54) imply

$$\partial_t u - \Delta u = \underset{]0,T[\times\Omega}{\mathbf{1}} g \quad \text{in } \mathcal{D}'(]0,+\infty[\times M) \text{ and } \quad u=0 \quad \text{on }]0,T[\times\partial M. \tag{59}$$

The property (55) of $\underline{v}$ implies

$$u\rceil_{t=0} = u_0 \text{ and } u\rceil_{t=T} = 0. \tag{60}$$

Setting $C=\sqrt{2}A||S_W||$, Cauchy–Schwarz inequality with respect to s, estimate (56) and $||\underline{f}||^2_{L^2(\mathbb{R}\times M)} = 2||S_W((u_0,0),(0,0))||^2_{L^2(\mathbb{R}_+\times M)}$ imply

$$||g||_{L^2(\mathbb{R}\times M)} \leqslant ||\underline{v}||_{L^2(\mathbb{R}^2)} ||\underline{f}||_{L^2(\mathbb{R}\times M)} \leqslant C e^{\alpha L^2/T} ||u_0||_{H_0^1(M)}. \tag{61}$$

We have proved that for all $\alpha>\alpha_*$ there is a $C>0$ such that for all $u_0\in H_0^1(M)$, $T\in]0,\min\{1,L_\Omega^2\}[$ and $L>L_\Omega$, there is a control g which solves the null-controllability problem (59), (60), at a cost so estimated in (61). The same property holds for the space of data $L^2(M)$ instead of $H_0^1(M)$, since $||e^{\varepsilon T\Delta}u_0||_{H_0^1(M)} \leqslant ||u_0||_{L^2(M)} C_0/\sqrt{\varepsilon T}$ with $\varepsilon \in]0,1[$ and $C_0 = ||(1+\lambda)e^{-2\sqrt{\lambda}}||^{1/2}_{L^\infty(\mathbb{R})}$. Therefore $\limsup_{T\to 0} T \ln C_{T,\Omega} \leqslant \alpha L^2$. Letting α and L tend, respectively, to α_* and L_Ω in this estimate completes the proof of (5).

References

[1] C. Bardos, G. Lebeau, J. Rauch, Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary, SIAM J. Control Optim. 30 (5) (1992) 1024–1065.
[2] R.P. Boas Jr., Entire Functions, Academic Press, New York, 1954.
[3] N. Burq, P. Gérard, Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes, C. R. Acad. Sci., Paris, Ser. I, Math. 25 (7) (1997) 749–752.
[4] J. Cheeger, M. Gromov, M. Taylor, Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds, J. Differential Geom. 17 (1) (1982) 15–53.
[5] S. Dolecki, D.L. Russell, A general theory of observation and control, SIAM J. Control Optim. 15 (2) (1977) 185–220.
[6] O.Y. Èmanuilov, Controllability of parabolic equations, Mat. Sb. 186 (6) (1995) 109–132.
[7] H.O. Fattorini, D.L. Russell, Exact controllability theorems for linear parabolic equations in one space dimension, Arch. Rational Mech. Anal. 43 (1971) 272–292.
[8] E. Fernández-Cara, E. Zuazua, The cost of approximate controllability for heat equations: the linear case, Adv. Differential Equations 5 (4–6) (2000) 465–514.
[9] E.N. Güichal, A lower bound of the norm of the control operator for the heat equation, J. Math. Anal. Appl. 110 (2) (1985) 519–527.
[10] R. Hersh, The method of transmutations, in: J.A. Goldstein (Ed.), Partial differential equations and related topics (Program, Tulane University, New Orleans, LA, 1974), Lecture Notes in Mathematics, Vol. 446, Springer, Berlin, 1975, pp. 264–282.
[11] L. Hörmander, The Analysis of Linear Partial Differential Operators, Vol. III, Springer, Berlin, 1985.
[12] Y. Kannai, Off diagonal short time asymptotics for fundamental solutions of diffusion equations, Comm. Partial Differential Equations 2 (8) (1977) 781–830.
[13] P. Koosis, The logarithmic integral. I & II, Cambridge Studies in Advanced Mathematics, Vol. 12,21 Cambridge University Press, Cambridge, 1988, 1992.
[14] G. Lebeau, L. Robbiano, Contrôle exact de l'équation de la chaleur, Comm. Partial Differential Equations 20 (1–2) (1995) 335–356.
[15] G. Lebeau, E. Zuazua, Null-controllability of a system of linear thermoelasticity, Arch. Rational Mech. Anal. 141 (4) (1998) 297–329.
[16] L. Miller, How violent are fast controls for Schrödinger and plates vibrations?, Arch. Ration. Mech. Anal. 172 (3) (2004) 429–456.
[17] J.R. Norris, Heat kernel asymptotics and the distance function in Lipschitz Riemannian manifolds, Acta Math. 179 (1) (1997) 79–103.
[18] R.M. Redheffer, Completeness of sets of complex exponentials, Adv. in Math. 24 1 (1977) 1–62.
[19] D.L. Russell, Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions, SIAM Rev. 20 (4) (1978) 639–739.
[20] L. Schwartz, Étude des sommes d'exponentielles réelles, Actualités Sci. Ind., Vol. 959, Hermann et Cie., Paris, 1943.
[21] T.I. Seidman, Two results on exact boundary control of parabolic equations, Appl. Math. Optim. 11 (2) (1984) 145–152.
[22] T.I. Seidman, The coefficient map for certain exponential sums, Nederl. Akad. Wetensch. Indag. Math. 48 (4) (1986) 463–478.

[23] T.I. Seidman, S.A. Avdonin, S.A. Ivanov, The "window problem" for series of complex exponentials, J. Fourier Anal. Appl. 6 (3) (2000) 233–254.
[24] M.E. Taylor, Partial differential equations. I, Applied Mathematical Sciences, Vol. 115, Springer-Verlag, New York, 1996, Basic theory.
[25] S.R.S. Varadhan, On the behavior of the fundamental solution of the heat equation with variable coefficients, Comm. Pure Appl. Math. 20 (1967) 431–455.
[26] E. Zuazua, Some results and open problems on the controllability of linear and semilinear heat equations, in: F. Colombini, C. Zuily (Eds.), Carleman estimates and applications to uniqueness and control theory (Cortona, 1999), Progr. Nonlinear Differential Equations Appl., Vol. 46, Birkhäuser, Boston, Boston, MA, 2001, pp. 191–211.