A notion of derivation tree is introduced for ground term rewriting systems. Using standard tree language theory, new proofs are given for some old results.

1. INTRODUCTION

A ground term rewriting system is a term rewriting system for which the rules do not contain variables. We will show that a natural concept of derivation tree can be defined for these rewriting systems, in such a way that a tree $t_1$ can be (iteratively) rewritten to a tree $t_2$ iff there is a derivation tree for which the "yield" is the pair $(t_1, t_2)$, with an appropriate definition of "yield." Derivations that differ only in the order of independent rule applications, correspond to the same derivation tree. Moreover, the set of derivation trees forms a regular tree language. Thus, the situation is analogous to (and, in fact, generalizes) the situation for context-free grammars. Using this concept of derivation tree, and the well-known closure properties of the regular tree languages, we give a new proof for the main result of [Bra]: the set of trees that can be obtained by (iterated) rewriting of the trees of a regular tree language (using the rules of a ground term rewriting system) is again a regular tree language. Viewing strings as monadic trees in the usual way, the result of [Bra] generalizes the original result of [Büc]: every regular canonical system generates a regular string language (effectively). Thus, we provide in particular a tree language theoretic proof of Büchi’s result on strings. Based on the result of [Bra] we also give a new proof of the following result of [DauTis1, DHLT]. For every ground term rewriting system there exist regular tree languages $L_1, R_1, \ldots, L_n, R_n$ such that a tree $t_1$ can be (iteratively) rewritten to a tree $t_2$ iff $t_2$ can be obtained (in one stroke) from $t_1$ by replacing independent subtrees $u_1, \ldots, u_k$ of $t_1$ by subtrees $v_1, \ldots, v_k$, respectively, where for every $i$ there exists $j$ such that $(u_i, v_i) \in (L_j, R_j)$. In the terminology of [DauTis1, DHLT], every ground term rewriting system can be simulated by a ground tree transducer. This result was used in [DauTis1, DHLT, DauTis2] to give an elegant proof of the decidability of confluence of a ground term rewriting system (also proved in [Oya]), and, more generally, of the decidability of the first-order theory of ground term rewriting. At
the end of the paper we discuss this decidability result, together with the decidability of termination of a ground term rewriting system (shown in [HueLan]).

Derivation trees of ground term rewriting systems were considered before in [Oya, CoqGil], but they seem to be less natural than the ones introduced here, which were inspired by [DauTis1, DHLT].

Recently the result of [Bra] has been extended to more general term rewriting systems, viz., to linear semimonadic systems in [CDGV] and to inverse growing systems in [Jac]. Unfortunately, our notion of derivation tree does not seem to be useful in the proofs of these generalizations. We do, however, generalize the result of [Bra] to "extended" ground term rewriting systems (which may have infinitely many rules, represented by a finite set of pairs of regular tree languages); these contain the left-linear right-ground term rewriting systems.

2. GROUND TERM REWRITING SYSTEMS

We assume the reader is familiar with tree language theory (see, e.g., [GécStel1, GécStec2]), in particular with the notion of a regular (or recognizable) tree language, i.e., a tree language generated by a regular tree grammar (or accepted by a finite tree automaton). For a ranked alphabet \( \Sigma \), the class of regular tree languages over \( \Sigma \) is denoted \( \text{REGT}_\Sigma \). The class of all regular tree languages is denoted \( \text{REGT} \). We will make extensive use of well-known (effective) closure properties of \( \text{REGT} \), such as closure under union, intersection, and complementation (see, e.g., Theorem II.4.2 of [GécStel1]).

For a ranked alphabet \( \Sigma \), the set of all trees (or ground terms) over \( \Sigma \) is denoted \( T_\Sigma \). Trees with variables are not allowed in ground rewriting systems. However, they will be used as a technical tool, in particular to define the context in which ground terms are replaced by other ground terms. Trees with variables are trees over \( \Sigma \cup X \), where \( X = \{ x_1, x_2, x_3, ... \} \) and each variable \( x_i \) is of rank 0. For a tree \( t \in T_{\Sigma \cup X} \) and trees \( t_1, ..., t_k \) \( (k \in \mathbb{N} = \{ 0, 1, 2, ... \}) \), \( t[t_1, ..., t_k] \) denotes the tree obtained from \( t \) by substituting \( t_i \) for every occurrence of \( x_i \), for \( 1 \leq i \leq k \). For \( k \in \mathbb{N} \), a \( k \)-place context is a tree \( c \) over \( \Sigma \cup \{ x_1, ..., x_k \} \) such that every variable from \( \{ x_1, ..., x_k \} \) occurs in \( c \) exactly once. As usual, a tree \( u \) is a subtree of a tree \( t \) if \( t = c[u] \) for some 1-place context \( c \). Intuitively, such a decomposition \( c[u] \) of \( t \) is uniquely determined by a node of \( t \), viz. the root of the (occurrence of the) subtree \( u \) in \( t \). For an example see Fig. 1.

![Fig. 1](image_url)

FIG. 1. A node of \( t \) determines a decomposition \( c[u] \) of \( t \); for \( c = \sigma(x_1, a) \) and \( u = \sigma(b, b) \), \( t = c[u] = \sigma(\sigma(b, b), a) \).
Let \( \Sigma \) be a ranked alphabet. A \textit{ground rewrite system} over \( \Sigma \) is a finite subset \( P \) of \( T_\Sigma \times T_\Sigma \). An element \((u, v)\) of \( P \) is called a rule (or production) of \( P \), and is also written \( u \rightarrow v \). The \textit{rewrite relation} \( \Rightarrow_p \subseteq T_\Sigma \times T_\Sigma \) is defined: for \( s, t \in T_\Sigma \), \( s \Rightarrow_p t \) iff there is a rule \( u \rightarrow v \) of \( P \) and a 1-place context \( c \) such that \( s = c[u] \) and \( t = c[v] \). As usual, \( \Rightarrow \) denotes the reflexive, transitive closure of \( \Rightarrow_p \). The \textit{parallel rewrite relation} \( \Rightarrow_p \subseteq T_\Sigma \times T_\Sigma \) is defined: for \( s, t \in T_\Sigma \), \( s \Rightarrow_p t \) iff there are a \( k \in \mathbb{N} \), a \( k \)-place context \( c \), and rules \( u_1 \rightarrow v_1, \ldots, u_k \rightarrow v_k \) in \( P \), such that \( s = c[u_1, \ldots, u_k] \) and \( t = c[v_1, \ldots, v_k] \). Thus, in a parallel rewrite step any number of rules can be applied to independent subtrees. Note that \( \Rightarrow \subseteq \Rightarrow_p \).

**Example 1.** Let \( \Sigma = \{ \sigma, a, b, p, q, r, s \} \), where \( \sigma \) has rank 2 and all other symbols have rank 0. As an example, consider the ground rewrite system \( P = P_1 \cup P_2 \) over \( \Sigma \), where \( P_1 \) consists of the rules

\[
a \rightarrow p, \quad \sigma(p, p) \rightarrow p, \quad \sigma(p, p) \rightarrow q, \quad q \rightarrow \sigma(q, b), \quad q \rightarrow b
\]

and \( P_2 \) consists of the rules

\[
\sigma(b, b) \rightarrow r, \quad \sigma(r, b) \rightarrow r, \quad r \rightarrow s, \quad s \rightarrow \sigma(a, s), \quad s \rightarrow a.
\]

Then, for instance, \( \sigma(\sigma(a, a), a) \rightarrow^{*} \sigma(\sigma(b, b), a) \), as a result of the rewrite steps:

\[
\sigma(\sigma(a, a), a) \rightarrow_{p} \sigma(\sigma(p, a), a) \\
\rightarrow_{p} \sigma(\sigma(p, p), a) \\
\rightarrow_{p} \sigma(q, a) \\
\rightarrow_{p} \sigma(\sigma(q, b), a) \\
\rightarrow_{p} \sigma(\sigma(b, b), a).
\]

And, for instance \( \sigma(\sigma(b, b), s) \Rightarrow_p \sigma(r, \sigma(a, s)) \).

Apart from the usual ground rewrite systems we will also be interested in ground rewrite systems with infinitely many rules that can be represented by regular tree languages. For ground rewrite systems with infinitely many rules the above definitions are valid too. An \textit{extended ground rewrite system} over \( \Sigma \) is a finite subset \( P \) of \( \text{REGT}_\Sigma \times \text{REGT}_\Sigma \). Let \( P' \subseteq T_\Sigma \times T_\Sigma \) be the (ordinary) ground rewrite system consisting of all rules \( u \rightarrow v \) such that \( u \in L \) and \( v \in R \) for some \( (L, R) \in P \). Then, by definition, \( \Rightarrow_{p} \Rightarrow_{p'} \Rightarrow_{p} = \Rightarrow_{p'} \), and the rules of \( P \) are those of \( P' \). Thus, each “regular rule” \((L, R)\), where \( L \) and \( R \) are regular tree languages, abbreviates all rules \( u \rightarrow v \) with \( u \in L \) and \( v \in R \). Note that the rules that are used in a parallel rewrite step of \( P \), are derived from possibly different regular rules. For algorithmic purposes, an extended ground rewrite system is specified by giving regular tree grammars (or finite tree automata) for the regular tree languages involved. Obviously every ground rewrite system is also an extended ground rewrite system.
Consider the extended ground rewrite system \( Q \) over \( A \) containing the two regular rules \((A, C_a)\) and \((C'_b, C_a)\), where \( A \) is the set of all trees over \( \{\sigma, a, b\} \) that contain at least one \( \sigma \), i.e., \( A = T_{\{\sigma, a\}} \setminus \{a\} \), \( C_a \) is the set of all trees \( \sigma(\sigma(\cdots \sigma(b, b) \cdots, b), b) \) with \( n \geq 0 \) symbols \( \sigma \), \( C'_b \) is the same as \( C_b \) except that \( n \neq 1 \), and \( C_a \) is the set of all trees \( \sigma(a, \ldots \sigma(a, a) \cdots) \) with \( n \geq 0 \) symbols \( \sigma \). It is not difficult to see that for all trees \( t_1, t_2 \in T_A \), \( t_1 \overset{\sigma}{\rightarrow}_Q t_2 \) if and only if \( t_1 \overset{\sigma}{\rightarrow}_P t_2 \), where \( P \) is the ground rewrite system of Example 1.

The relation of interest for an extended ground rewrite system \( P \) is the relation \( \overset{\sigma}{\rightarrow}_P \). Whenever we are mainly interested in the parallel rewrite relation \( \Rightarrow_P \), an extended ground rewrite system \( P \) will also be called a ground tree transducer.

A ground tree grammar, introduced in [Bra], where it is called a regular system, is a tuple \( G = \langle A, \Sigma, P, S \rangle \), where \( P \) is a ground rewrite system over \( \Sigma \) and \( S \) is a finite subset of \( T \). The language generated by \( G \) is \( L(G) = \{ t \mid t \in T, t \overset{\sigma}{\rightarrow}_P S \} \). A regular tree grammar is a ground tree grammar \( G = \langle A, \Sigma, P, S \rangle \) such that (1) all elements of \( \Sigma - A \) have rank 0 (and are called nonterminals), (2) the left-hand side of each rule of \( P \) is in \( \Sigma - A \), and (3) \( S \) is a singleton containing one element of \( \Sigma - A \). This is the usual notion of regular tree grammar (see Section II.3 of [GecSte1]). The main result of [Bra] is that for every ground tree grammar an equivalent regular tree grammar can effectively be constructed. In Section 4 we will give a new proof of this result.

Example 2. Let \( A = \{\sigma, a, b\} \), where \( \sigma \) has rank 2 and \( a, b \) have rank 0. Consider the extended ground rewrite system \( Q \) over \( A \) containing the two regular rules \((A, C_a)\) and \((C'_b, C_a)\), where \( A \) is the set of all trees over \( \{\sigma, a\} \) that contain at least one \( \sigma \), i.e., \( A = T_{\{\sigma, a\}} \setminus \{a\} \), \( C_a \) is the set of all trees \( \sigma(\sigma(\cdots \sigma(b, b) \cdots, b), b) \) with \( n \geq 0 \) symbols \( \sigma \), \( C'_b \) is the same as \( C_b \) except that \( n \neq 1 \), and \( C_a \) is the set of all trees \( \sigma(a, \ldots \sigma(a, a) \cdots) \) with \( n \geq 0 \) symbols \( \sigma \). It is not difficult to see that for all trees \( t_1, t_2 \in T_A \), \( t_1 \overset{\sigma}{\rightarrow}_Q t_2 \) if and only if \( t_1 \overset{\sigma}{\rightarrow}_P t_2 \), where \( P \) is the ground rewrite system of Example 1.

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Example 3. Consider the ground tree grammar \( G = \langle A, \Sigma, P, S \rangle \), where \( P \) is the ground rewrite system over \( \Sigma \) of Example 1, \( A = \{\sigma, a, b\} \), and \( S = \{s\} \). It can be shown that \( L(G) = L(G') \), where \( G' = \langle A, \Sigma', P, S \rangle \) is the regular tree grammar with \( \Sigma' = \{\sigma, a, b, s\} \) and \( P' \) consists of all rules

\[
s \rightarrow \sigma(a, s), \quad s \rightarrow \sigma(s, b), \quad s \rightarrow a, \quad s \rightarrow b.
\]

An example of an extended ground tree grammar is \( G'' = \langle A, A, Q, \{\sigma(a, a)\} \rangle \), where \( Q \) is the extended ground rewrite system over \( A \) of Example 2. It can be shown that also \( L(G'') = L(G') \).

The main result of [Bra] is a generalization of the following result of [Buc] for strings: every regular canonical system (defined below) generates a regular string language, effectively. In fact, it is well known that strings correspond to trees over a monadic ranked alphabet. A ranked alphabet \( \Sigma \) is monadic if it is of the form...
\[ \Sigma = A \cup \{ e \}, \text{ where } e \text{ is a fixed symbol of rank } 0 \text{ (standing for the empty string) and every element of } A \text{ has rank } 1. \] The string \( a_1 a_2 \cdots a_n \) over the alphabet \( A \) will be identified with the tree \( a_n \cdots (a_2 (a_1 (e))) \) over \( \Sigma \). A regular canonical system is a ground tree grammar \( G = (A, \Sigma, P, S) \) with monadic ranked alphabets \( \Sigma \) and \( A \).

Thus, if \( A = A \cup \{ e \} \), then \( L(G) \subseteq A^* \). Note that, on strings, the rules of \( P \) are Chomsky type 0 rules that are applied to prefixes of the sentential forms only (because the subtrees of a monadic tree are the prefixes of the corresponding string). Since, in the monadic case, a regular tree grammar is the same as a left-linear grammar (with productions of the form \( X \rightarrow Y w \) or \( X \rightarrow w \), where \( X \) and \( Y \) are nonterminals and \( w \) is a terminal string), it should be clear that the result of [Büc] is the monadic case of the result of [Bra].

### 3. DERIVATION TREES

Let \( P \) be an extended ground rewrite system over \( \Sigma \). A derivation of \( P \) is a sequence of trees \( t_1 \rightarrow_P t_2 \rightarrow_P \cdots \rightarrow_P t_n \). The basic idea of this paper is that the derivations of \( P \) can be represented by derivation trees (modulo the interchange of independent derivation steps) and that the derivation trees of \( P \) form a regular tree language. This is similar to the situation for context-free grammars. If \( t \) is a derivation tree of the above derivation, then the “transduction” of \( t \) is the pair of trees \( (t_1, t_n) \); hence, the set of transductions of all derivation trees of \( P \) is the relation \( \rightarrow_P^* \).

This is similar to the fact that the set of yields of derivation trees of a context-free grammar \( G \) is the language generated by \( G \). Thus, in our setting, “transduction” plays the role of “yield.” Also similar to yield, the “transduction” of a tree \( t \) can be defined in a straightforward way for arbitrary trees rather than just derivation trees. We will use a special symbol \( \# \) which, for derivation trees, indicates the application of a rule.

For a ranked alphabet \( \Sigma \) we denote by \( \Sigma \# \) the ranked alphabet \( \Sigma \cup \{ \# \} \), where \( \# \) is a new symbol of rank 2. For a tree \( t \in T_{\Sigma \#} \), the trees \( \text{left}(t) \) and \( \text{right}(t) \) in \( T_{\Sigma} \) are defined recursively, where \( \sigma \) is an element of \( \Sigma \) of rank \( k \geq 0 \), and the \( t_i \) are trees in \( T_{\Sigma \#} \), as

\[
\begin{align*}
\text{left}(\sigma(t_1, \ldots, t_k)) &= \sigma(\text{left}(t_1), \ldots, \text{left}(t_k)), \\
\text{left}(\#(t_1, t_2)) &= \text{left}(t_1), \\
\text{right}(\sigma(t_1, \ldots, t_k)) &= \sigma(\text{right}(t_1), \ldots, \text{right}(t_k)), \\
\text{right}(\#(t_1, t_2)) &= \text{right}(t_2).
\end{align*}
\]

For a tree \( t \in T_{\Sigma \#} \), the transduction of \( t \) is defined as \( \text{trans}(t) = (\text{left}(t), \text{right}(t)) \).

For a tree language \( L \subseteq T_{\Sigma \#} \), the transduction of \( L \) is defined as \( \text{trans}(L) = \{ \text{trans}(t) | t \in L \} \). Note that \( \text{trans}(L) = T_{\Sigma} \times T_{\Sigma} \).

Thus, for a tree \( t \in T_{\Sigma \#} \), \( \text{left}(t) \) (\( \text{right}(t) \)) is obtained from \( t \) by choosing the left (right) subtree of every occurrence of \( \# \). Clearly, both “left” and “right” are linear tree homomorphisms from \( T_{\Sigma \#} \) to \( T_{\Sigma} \) (see, e.g., Section II.4 of [GécSte1] for the
concept of a linear tree homomorphism). Intuitively, \( \text{left}(t) \) can be seen as a part of \( t \), in the sense that the nodes of \( \text{left}(t) \) are a subset of the nodes of \( t \) and the edges of \( \text{left}(t) \) are paths in \( t \), as follows. A node \( x \) of \( t \) is a node of \( \text{left}(t) \) if its label is not \# and, walking from the root of \( t \) to \( x \), at each \#-labeled node the left child is chosen. For two nodes \( x \) and \( y \) of \( t \) that are also nodes of \( \text{left}(t) \), \( y \) is the left (right) child of \( x \) in \( \text{left}(t) \) if \( y \) is a descendant of the left (right) child of \( x \) in \( t \) and, walking from \( x \) to \( y \) in \( t \), all intermediate nodes have label \#.

In the same way \( \text{right}(t) \) can be viewed as a part of \( t \), see Fig. 2 for an example.

Let \( P \) be an extended ground rewrite system over \( \Sigma \). A derivation tree of \( P \) is a tree \( t \in T_{\Sigma^*} \) such that for every subtree \( \#(t_1, t_2) \) of \( t \), \( \text{right}(t_1) \rightarrow \text{left}(t_2) \) is a rule of \( P \). The set of all derivation trees of \( P \) is denoted \( D_P \).

**Example 4.** Figure 3 shows a derivation tree \( t \) of the ground rewrite system \( P \) of Example 1. Considering the five nodes with label \# in infix order, the rules \( \text{right}(t_1) \rightarrow \text{left}(t_2) \) in \( P \) corresponding to these nodes are \( a \rightarrow p, a \rightarrow p, \sigma(p, p) \rightarrow q, q \rightarrow \sigma(q, b), \) and \( q \rightarrow b \), respectively. Since, as shown in Fig. 2, \( \text{left}(t) = \sigma(\sigma(a, a), a) \) and \( \text{right}(t) = \sigma(\sigma(b, b), a) \), the transduction of \( t \) is the pair \( \text{trans}(t) = (\sigma(\sigma(a, a), a), \sigma(\sigma(b, b), a)) \). In fact, as will be clear from the proof of the next theorem, \( t \) corresponds to the derivation \( \sigma(\sigma(a, a), a) \rightarrow^* \sigma(\sigma(b, b), a) \) given in Example 1.

Figure 3 also shows another derivation tree \( t' \) of \( P \), closely related to \( t \). The (infix order) sequence of rules \( \text{right}(t_1) \rightarrow \text{left}(t_2) \) of \( t' \) is the same as that of \( t \), followed by the rule \( \sigma(b, b) \rightarrow r \).

The main properties of \( D_P \) are that \( \text{trans}(D_P) = \rightarrow^* \) and that \( D_P \) is a regular tree language.

**FIG. 2.** The left and right of a tree \( t \in T_{\Sigma^*} \).
Theorem 1. For every extended ground rewrite system $P$, $\text{trans}(D_P) = \sigma^* P$.

Proof. Let $P$ be an extended ground rewrite system over $\Sigma$. As in the case of context-free grammars, we will associate derivations with derivation trees, and derivation trees with derivations. We start with the former.

To prove the inclusion $\text{trans}(D_P) \subseteq \sigma^* P$, we show the following by structural induction on $t$: if $t \in D_P$, then $\text{left}(t) \rightarrow^* \sigma^* \text{right}(t)$. First, let $t = \_ (t_1, \ldots, t_k)$ with $\_ \not\in \Sigma$. Then, by definition, $\text{left}(t) = \_ (\text{left}(t_1), \ldots, \text{left}(t_k))$ and $\text{right}(t) = \_ (\text{right}(t_1), \ldots, \text{right}(t_k))$. Note that every subtree of $t$ is in $D_P$. Hence, by induction, $\text{left}(t_i) \rightarrow^* \sigma^* \text{right}(t_i)$ for every $1 \leq i \leq k$. This implies that $\text{left}(t) \rightarrow^* \sigma^* \text{right}(t)$. Second, let $t = * (t_1, t_2)$. Then $\text{left}(t) = \text{left}(t_1)$ and $\text{right}(t) = \text{right}(t_2)$. Since $t$ is a derivation tree of $P$, $\text{right}(t_1) \rightarrow \text{left}(t_2)$ is a rule of $P$ and hence $\text{right}(t_1) \rightarrow^* \sigma^* \text{left}(t_2)$. Thus, by induction, $\text{left}(t_1) \rightarrow^* \sigma^* \text{right}(t_1) \rightarrow^* \sigma^* \text{left}(t_2) \rightarrow^* \sigma^* \text{right}(t_2)$. This shows that $\text{left}(t) \rightarrow^* \sigma^* \text{right}(t)$.

Next we show the inclusion $\sigma^* P \subseteq \text{trans}(D_P)$. For this purpose we prove the following: for trees $s_1, \ldots, s_n \in T_\Sigma (n \geq 1)$, if $s_1 \rightarrow P s_2 \rightarrow P \cdots \rightarrow P s_n$ then there exists $t \in D_P$ such that $s_i = \text{left}(t_i)$ and $s_n = \text{right}(t)$. We prove this by induction on the sum of the sizes of $s_1, \ldots, s_n$, distinguishing two cases. In the first case there exists a derivation step $s_i \rightarrow P s_{i+1}$ such that $s_i = s_{i+1}$ is in $P$. By induction there are derivation trees $t_1$ and $t_2$ such that left($t_1$) = $s_1$, right($t_1$) = $s_j$, left($t_2$) = $s_{j+1}$, and right($t_2$) = $s_n$. Hence, $t = \# (t_1, t_2)$ satisfies the requirements. In the second case no $s_i \rightarrow P s_{i+1}$ is in $P$. This means intuitively that the roots of the $s_i$ remain unchanged. Formally it is straightforward to show that there exist $k \geq 0$, $\sigma \in \Sigma$ of rank $k$, and trees $r_{i,j}$ $(1 \leq i \leq n, 1 \leq j \leq k)$ such that $s_i = \sigma (r_{i,1}, \ldots, r_{i,k})$, and $r_{i,j} \rightarrow P r_{i,j+1}$ or $r_{i,j} = r_{i,j+1}$. Thus $t = \sigma (r_{1,1}, \ldots, t_k)$ satisfies the requirements.

It is easy to see that the above inductive proofs describe a constructive way of associating derivations with derivation trees, and vice versa. As in the case of context-free grammars, the derivations associated with derivation trees are left-most...
Derivations (where “left-most” is defined in the obvious way). In fact, every node 
with label \# of a derivation tree \( t \) corresponds to the application of a rule, and in 
the corresponding left-most derivation the rules are applied according to the infix 
order of these nodes in \( t \); cf. Example 4. In the other direction, the proof does not 
produce a unique derivation tree; it is, however, unique modulo the associativity of 
\#. This is due to the fact that in a derivation there may be several derivation steps 
that are rules in \( P \). If systematically the left-most such derivation step is always 
taken, then the constructed derivation trees \( t \) have the following property: the left 
child of a node with label \# does not have label \# (cf. Fig. 3). It can be shown 
(but we will not do this here) that, analogous to the case of context-free grammars, 
there is a one-to-one correspondence between derivation trees with the above 
property and left-most derivations. As an example, the derivation tree \( t \) of Fig. 3 
corresponds to the derivation given in Example 1, and the derivation 
tree \( t' \) corresponds to that same derivation extended by \( \sigma(\sigma(b, h), a) \rightarrow_p \sigma(r, a) \). 

Note that there is a one-to-one correspondence between left-most derivations and 
equivalence classes of derivations with respect to the interchange of independent 
derivation steps. Thus, as for context-free grammars, derivation trees (with the 
above property) faithfully represent the “parallelism” in derivations. This does not 
hold for the derivation trees in \([\text{CoqGil}]\). As a simple example, if \( P \) has two rules 
\( a \rightarrow p \) and \( a \rightarrow q \), then both derivations \( \sigma(a, a) \rightarrow_p \sigma(p, a) \rightarrow_p \sigma(p, q) \) and 
\( \sigma(a, a) \rightarrow_p \sigma(a, q) \rightarrow_p \sigma(p, q) \) have the derivation tree \( \sigma(\#, (a, p), \#, (a, q)) \).

Derivation trees can also be constructed incrementally: if \( t \) is a derivation tree 
for a derivation \( s_1 \rightarrow_p s_2 \), and \( s_2 \rightarrow_p s_3 \) is another derivation step, then it is 
straightforward to construct a derivation tree \( t' \) for \( s_1 \rightarrow_p s_2 \rightarrow_p s_3 \), as follows. Sup- 
pose that \( s_2 = c[u] \) and \( s_3 = c[v] \) with \( u \rightarrow v \) in \( P \); thus \( \text{right}(t) = c[u] \). Now it can 
be shown that \( t = c'[u'] \) with \( \text{right}(c') = c \) and \( \text{right}(u') = u \) (and the root label of 
\( u' \) is not \#), and it can be shown that \( t' = c'[\#, (u', v)] \) satisfies the requirements, 
cf. Fig. 3. In fact, if the decomposition \( c[u] \) of \( \text{right}(t) \) is determined by node \( x \) of 
\( \text{right}(t) \), i.e., \( x \) is the root of (the occurrence of) \( u \) in \( \text{right}(t) \), then the decomposition 
c\( u' \) of \( t \) is also determined by \( x \), viewed as a node of \( t \). Recall that, 
intuitively, \( \text{right}(t) \) can be viewed as a part of \( t \), as shown in Fig. 2. Thus, if (in that 
figure) \( x \) is the lowest node of \( \text{right}(t) \) with label \( \sigma \), then the corresponding node 
\( x \) in \( t \) is also the lowest (encircled) node with label \( \sigma \). The decomposition \( c[u] \) is 
shown in Fig. 1, and \( t = c'[u'] \) with \( c' = \sigma(\#, (\sigma(\#, (a, p), \#, (a, p)), \#, (q, x_1)), a) \) 
and \( u' = \sigma(\#, (q, b), b) \).

Up to now we did not use the regularity of the tree languages \( L \) and \( R \) in a 
“regular rule” \((L, R)\) of an extended ground rewrite system. Thus, Theorem 1 holds 
in fact for arbitrary term rewriting systems (with variables), viewed as abbreviations 
of ground term rewriting systems with infinitely many rules, in the obvious way.

The regularity of the set of derivation trees \( D_p \) of an extended ground rewrite 
system \( P \) is an easy exercise in tree language theory.

**Theorem 2.** For every extended ground rewrite system \( P \), \( D_p \) is a regular tree 
language (effectively).

**Proof.** To prove this we use (effective) closure properties of the class of regular 
tree languages. For any tree language \( L \), let \( \text{allsub}(L) \) denote the set of all trees \( t \)
such that every subtree of $t$ is in $L$. It is easy to see that REGT is effectively closed under the allsub operation (see, e.g., Section II.8 of [GécSte1], where “allsub” is denoted “rest”). For a symbol $\sigma$ of rank $k$ and tree languages $L_1, \ldots, L_k$, $\sigma(L_1, \ldots, L_k)$ denotes the set of all trees $\sigma(t_1, \ldots, t_k)$ such that $t_i \in L_i$ for every $1 \leq i \leq k$. It is well known that REGT is effectively closed under these operations (see, e.g., Corollary II.4.12 of [GécSte1]).

Let $P$ be an extended ground rewrite system over $\Sigma$. Define $D'_P$ to be the set of all trees $t \in T_{\Sigma^*}$ such that either the root label of $t$ is in $\Sigma$ or $t = \#(t_1, t_2)$ and $\text{right}(t_1) \rightarrow \text{left}(t_2)$ is a rule of $P$. Then $D_P = \text{allsub}(D'_P)$. Clearly, $D'_P$ is the (finite) union of all tree languages $\sigma(T_{\Sigma^*}, \ldots, T_{\Sigma^*})$, for $\sigma \in \Sigma$, and all tree languages $\#(\text{right}^{-1}(L), \text{left}^{-1}(R))$, for $(L, R) \in P$. The result now follows from the fact that $T_{\Sigma^*}$ and all $L$ and $R$ are regular, from the above closure properties and closure under union, and from the (effective) closure of REGT under inverse tree homomorphisms (see, e.g., Theorem II.4.18 of [GécSte1]). Recall that both “left” and “right” are tree homomorphisms.

These results show that the derivation trees of extended ground rewrite systems have properties similar to those of context-free grammars. In fact, in a sense to be explained now (informally), the former can be viewed as a proper generalization of the latter. With every context-free grammar $G$ one can associate a ground rewrite system $G'$ in a natural (and well-known) way. In fact, $G'$ is a regular tree grammar that has the same nonterminals as $G$ (with the same initial nonterminal), and for every production $A \rightarrow x_1 \cdots x_k$ of $G$ (where $A$ is a nonterminal and each $x_i$ is either a nonterminal or a terminal) $G'$ has a rule $A \rightarrow c_k(x_1, \ldots, x_k)$ where $c_k$ is a (new) terminal symbol of rank $k$ (intuitively standing for the concatenation of $k$ strings) and each $x_i$ has rank 0. It should now be clear that there is a natural one-to-one correspondence between the derivations of $G$ and $G'$ and, thus, a very close one-to-one relationship between the (usual) derivation trees of $G$ and the derivation trees of $G'$ (see the following example). Thus, the derivation trees of ground rewrite systems model the parallelism in derivations in the same way as those of context-free grammars.

**Example 5.** Consider the context-free grammar $G$ with productions

$$
A \rightarrow aAB, \quad A \rightarrow adB, \quad B \rightarrow bB, \quad B \rightarrow bb,
$$

generating all strings $a^n b^m$ with $n \geq 1$ and $m \geq 2n$ (assuming that $A$ is the initial nonterminal). Then the ground rewrite system (regular tree grammar) $G'$ has rules

$$
A \rightarrow c_3(a, A, B), \quad A \rightarrow c_3(a, d, B), \quad B \rightarrow c_3(b, B), \quad B \rightarrow c_3(b, b).
$$

Figure 4 shows derivation trees of corresponding derivations of $G$ and $G'$. Clearly, for a derivation tree $t$ of $G$, the corresponding derivation tree $\text{der}(t)$ of $G'$ can be obtained recursively as

$$
\text{der}(A(t_1, t_2, t_3)) = \#(A, c_3(\text{der}(t_1), \text{der}(t_2), \text{der}(t_3)))
$$

$$
\text{der}(B(t_1, t_2)) = \#(B, c_3(\text{der}(t_1), \text{der}(t_2))),
$$

and

$$
\text{der}(x) = x \quad \text{for} \quad x \in \{a, b, d\}.
$$
FIG. 4. Derivation trees of a context-free grammar and the corresponding ground rewrite system.

Thus, “der” is a straightforward linear tree homomorphism. Note that the infix order of the \( \# \)-labeled nodes of \( \text{der}(t) \) corresponds to the (usual) prefix order of the nonterminal nodes of \( t \). Thus, the association between left-most derivations and derivation trees is the same in \( G' \) and \( G \).

4. NEW PROOFS OF OLD RESULTS

Using the known closure properties of \( \text{REGT} \) (as in the proof of Theorem 2), it is now easy to show that the relation \( \rightarrow^P \) preserves regular tree languages.

**Theorem 3.** For every extended ground rewrite system \( P \) and every regular tree language \( R \), \( \rightarrow^P(R) \) is a regular tree language (effectively).

**Proof.** By Theorem 1, \( \rightarrow^P(R) = \{ s_2 \mid s_1 \rightarrow^*_P s_2 \text{ for some } s_1 \in R \} = \{ s_2 \mid (s_1, s_2) \in \text{trans}(D_P) \text{ for some } s_1 \in R \} = \{ \text{right}(t) \mid t \in D_P, \text{left}(t) \in R \} \). Hence \( \rightarrow^P(R) = \text{right}(D_P \cap \text{left}^{-1}(R)) \). Since \( D_P \) is regular by Theorem 2, and since \( \text{REGT} \) is effectively closed under inverse tree homomorphisms, intersection, and linear tree homomorphisms (for the latter, see, e.g., Theorem II.4.16 of \([\text{GecSte1}]\), the result follows.

The language generated by an extended ground tree grammar \( G = (A, \Sigma, P, S) \) is \( L(G) = \rightarrow^*_P(S) \cap T_\Sigma \). Since every finite tree language \( S \) is regular and \( \text{REGT} \) is closed under intersection with \( T_\Sigma \), the (slight extension of the) main result of \([\text{Bra}]\) follows immediately from Theorem 3.

**Theorem 4.** For every extended ground tree grammar \( G \) a regular tree grammar \( G' \) with \( L(G') = L(G) \) can effectively be constructed.

It was shown in Theorem 3.21 of \([\text{Bra}]\) that in a ground tree grammar \( G = (A, \Sigma, P, S) \) one can also allow the set \( S \) to be a regular tree language. This means that, in fact, Theorem 3 was also proved in \([\text{Bra}]\) (for ordinary ground rewrite systems).
Using Theorem 3 (and Theorem 1) it is now straightforward to prove result II.5 of [DauTis1] (see also Proposition 2 of [DHLT], which, however, does not show effectiveness).

**Theorem 5.** For every extended ground rewrite system $P$ a ground tree transducer $Q$ such that $\Rightarrow_p^* = \Rightarrow_Q$, can effectively be constructed.

**Proof.** Let $P$ be an extended ground rewrite system over $\Sigma$. By $\lessdot_p$ we denote the relation $(\Rightarrow_p^-)^{-1}$. It is easy to see that this is the relation $\Rightarrow_p^{-1}$, where $P^{-1}$ is the extended ground rewrite system $\{(R,L) | (L,R) \in P\}$. This shows, by Theorem 3, that $\lessdot_p^*(L)$ is (effectively) regular for every regular tree language $L$.

Define $Q = \{(\lessdot_p^*(L), \lessdot_p^*(R)) | (L,R) \in P\}$. Then $Q$ is a ground tree transducer (effectively), by Theorem 3. We first show that $\Rightarrow_p^* \subseteq \Rightarrow_Q$. Let $s \Rightarrow_p^* s'$. By Theorem 1 there is a tree $t \in D_p$ such that $\text{left}(t) = s$ and $\text{right}(t) = s'$. The derivation tree $t \in T_{\Sigma^*}$ can be decomposed (in a unique way) as $t = c(\#(t_1, t_2), \ldots, \#(t_k, t'_k))$, where $c$ is a $k$-place context (for some $k \geq 0$) that does not contain $\#$, and the $t_i, t'_i$ are derivation trees of $P$ such that $\text{right}(t_i) \rightarrow \text{left}(t'_i)$ is a rule of $P$. Define, for $1 \leq i \leq k$, the trees $p_i = \text{left}(t_i)$ and $p'_i = \text{right}(t'_i)$ over $\Sigma$. Since $t_i, t'_i \in D_p$, it follows from Theorem 1 that $p_i \rightarrow_p \lessdot_p^*(\text{right}(t_i))$ and $p'_i \rightarrow_p \lessdot_p^*(\text{left}(t'_i))$. Hence, since $\text{right}(t_i) \rightarrow \text{left}(t'_i)$ is a rule of $P$, $p_i \rightarrow_p \lessdot_p^* (L)$ and $p'_i \rightarrow_p \lessdot_p^* (R)$ for some $(L,R) \in P$. Thus, $p_i \rightarrow_p p'_i$ is a rule of $Q$. Since clearly $s = \text{left}(t) = c[p_1, \ldots, p_k]$ and $s' = \text{right}(t) = c[p'_1, \ldots, p'_k]$, this shows that $s \Rightarrow_Q s'$.

We note here that an experienced reader can easily give the above proof without the use of derivation trees, i.e., without the use of Theorem 1: if $s \Rightarrow_p^* s'$ then, obviously, there exist a $k$-place context $c$ and trees $p_i, p'_i, u_i, u'_i$ such that $s = c[p_1, \ldots, p_k]$, $s' = c[p'_1, \ldots, p'_k]$, $p_i \rightarrow_p u_i \rightarrow_p u'_i \rightarrow_p p'_i$, and $u_i \rightarrow u'_i$ in $P$ (which shows $s \Rightarrow_Q s'$). However, the above proof illustrates that derivation trees can be used to give precise formal proofs of such obvious statements.

The proof of the inclusion $\Rightarrow_Q \subseteq \Rightarrow_p^*$ is even easier. Given a context $c$ and trees $p_i \rightarrow_p \lessdot_p^* (L)$ and $p'_i \rightarrow_p \lessdot_p^* (R)$ (for some $(L,R) \in P$) depending on $i$ such that $s = c[p_1, \ldots, p_k]$ and $s' = c[p'_1, \ldots, p'_k]$ are derivations $p_i \rightarrow_p u_i \rightarrow_p u'_i \rightarrow_p p'_i$ such that $u_i \rightarrow u'_i$ is a rule of $P$. Hence $p_i \rightarrow_p p'_i$ and so $s = c[p_1, \ldots, p_k] \rightarrow_p c[p'_1, \ldots, p'_k] = s'$. Note that if $t_i$ and $t'_i$ are derivation trees of $p_i \rightarrow_p u_i$ and $u'_i \rightarrow_p p'_i$, respectively, then $c[\#(t_1, t_2), \ldots, \#(t_k, t'_k)]$ is a derivation tree of $s \rightarrow_p^* s'$.

Theorem 5 is equivalent to saying that the class of ground tree transductions is closed under star (as shown in [DauTis1, DHLT]), because for every ground tree transducer $P$, $\Rightarrow_p^* = \Rightarrow_p^*$. It is rather obvious that ground tree transducers also have “derivation trees.”

**Lemma 6.** For every ground tree transducer $P$ over $\Sigma$ there is a regular tree language $D$ over $\Sigma$ such that $\text{trans}(D) = \Rightarrow_p$ (effectively).

**Proof.** Define $D$ to be the set of all trees $t \in T_{\Sigma^*}$ such that for every subtree $\#(t_1, t_2)$ of $t$, $(t_1, t_2) \in (L,R)$ for some $(L,R) \in P$. Note that in such a tree there are no nested occurrences of $\#$. It should be clear that $\text{trans}(D) = \Rightarrow_p$. To show that $D$ is (effectively) regular, let $D'$ be the regular tree language that is the union of all $\#(L,R)$ for $(L,R) \in P$. Then $D$ is the set of all trees $c[t_1, \ldots, t_k]$, where $c$ is a
$k$-place context (for some $k \geq 0$) and $t_i \in D'$. From this it easily follows that $D$ is regular (to be precise, $D = T_{\Sigma \cup \{s_1\}} \cdot s_1D'$, see, e.g., Theorem II.4.6 of [Géc-sSte1]).

This lemma gives us the following result, by exactly the same proof as the one of Theorem 3.

**Theorem 7.** For every ground tree transducer $P$ and every regular tree language $R$, $\Rightarrow_P(R)$ is a regular tree language (effectively).

Note that Theorem 7 is easy to prove directly and that Theorem 3 follows immediately from Theorems 5 and 7, as shown in Proposition 3.2 of [CoqGil], and in Lemmas 3.6 and 3.7 of [Fül-Vág] for the case that $\Rightarrow$ is a congruence. Thus, Theorems 3 and 5 are quite closely related.

As observed in Section 2, the above results hold in particular for monadic ranked alphabets, in which case they concern strings rather than trees: ground rewrite systems correspond to the regular canonical systems of [Büc] (which are Chomsky-type 0 grammars of which the productions are applied to prefixes of the sentential forms only), and regular tree languages correspond to regular string languages. Thus, Theorem 4 expresses the (extended version of the) main result of [Büc]: for every regular canonical system an equivalent left-linear grammar (or right-linear grammar, or finite automaton) can effectively be constructed (for other proofs see, e.g., Section 2.3 of [Sal], or see [FraPag]). This result is equivalent with the well-known fact that the possible contents of a pushdown automaton form a regular language (see [Gre] and, e.g., p. 335 of [Har]). The results of [Büc] and [Bra] were rediscovered in [DauTis1, DHLT] (in the sense of the above-mentioned close relationship between Theorems 3 and 5), in [Fül-Vág], and in [FraPag]. Complexity issues are considered in [CoqGil, FraPag, Vág]. Theorem 3 is generalized to linear semi-monadic term rewriting systems in Theorem 5.1 of [CDGV] and to (the slightly more general) inverses of growing term rewriting systems in [Jac]; it is not clear whether the notion of derivation tree is relevant to these generalizations.

In the remainder of this section we discuss confluence and termination of ground rewrite systems.

An extended ground rewrite system $P$ over $\Sigma$ is **confluent** if for all trees $t, u, v \in T_\Sigma$ with $t \Rightarrow u$ and $t \Rightarrow v$, there is a tree $w \in T_\Sigma$ such that $u \Rightarrow w$ and $v \Rightarrow w$. In [DauTis1, DHLT, DauTis2] it is shown on the basis of Theorem 5 that confluence is decidable for extended ground rewrite systems. The nicest proof is the one in [DauTis2], where it is even shown that the first-order theory of extended ground rewrite systems is decidable. This first-order theory includes properties such as confluence (as should be clear from the above standard definition) and unique normalization (i.e., for every $s \in T_\Sigma$ there is a unique $t \in T_\Sigma$ such that $s \Rightarrow t$ and there is no $u \in T_\Sigma$ with $t \Rightarrow u$). The essence of the proof in [DauTis2] is that for every extended ground rewrite system $P$, $\Rightarrow$ is a so-called binary RR relation (introduced in [DauTis2] and shown to have a decidable first-order theory). In view of Theorem 5, it is in fact proved that every ground tree transduction is a binary RR relation. We now wish to convince the reader who is
Let Non denote the set of all trees $t \in T_{\Sigma^*}$ that do not have nested occurrences of $\#$ (i.e., for every subtree $\#(t_1, t_2)$ of $t$, $t_1$ and $t_2$ are in $T_{\Sigma^*}$).

**Lemma 8.** For every regular tree language $R$ over $\Sigma^*$ there is (effectively) a regular tree language $R'$ over $\Sigma^*$ such that $\text{trans}(R') = \text{trans}(R)$ and $R' \subseteq \text{Non}$.

**Proof.** Let “prune” be the mapping from $T_{\Sigma^*}$ to $T_{\Sigma^*}$, defined recursively as follows, where $\_ \in \Sigma$ of rank $k$ and $t_i \in T_{\Sigma^*}$:

\[
\text{prune}(\_ ((t_1, ..., t_k))) = \_ (\text{prune}(t_1), ..., \text{prune}(t_k)),
\]

\[
\text{prune}(\#(t_1, t_2)) = \# (\text{left}(t_1), \text{right}(t_2)).
\]

Clearly, for every $t \in T_{\Sigma^*}$, $\text{trans} (\text{prune}(t)) = \text{trans}(t)$, and $\#$ is not nested in $\text{prune}(t)$. Thus, $\text{trans}(\text{prune}(R)) = \text{trans}(R)$ and $\text{prune}(R) \subseteq \text{Non}$. From the recursive definition of “prune” (and “left” and “right”) it is immediate that “prune” is a linear top-down tree transduction (see, e.g., Chap. IV of [GécStel], where top-down tree transducers are called root-to-frontier tree transducers). Since REGT is effectively closed under linear top-down tree transductions (see Corollary IV.6.6 of [GécStel]), $\text{prune}(R)$ is regular. Thus, $R' = \text{prune}(R)$ satisfies the requirements.

We observe here that for every extended ground rewrite system $P$, $\text{prune}(D_P)$ is the set of derivation trees (as defined in the proof of Lemma 6) of the ground tree transducer $Q$ defined in the proof of Theorem 5.

Tree transductions of the form $\text{trans}(R)$, where $R$ is a regular tree language consisting of trees that do not have nested occurrences of $\#$, are just a slight generalization of ground tree transductions (cf. the proof of Lemma 6). Thus, it is straightforward to generalize the proof of the lemma in Section 5 of [DauTis2], which shows that every ground tree transduction is a binary RR relation, to a proof that every transduction $\text{trans}(R)$ with $R \subseteq \text{Non}$ is a binary RR relation. Together with Lemma 8, this gives the following proposition.

**Proposition 9.** For every regular tree language $R$ over $\Sigma^*$, $\text{trans}(R)$ is a binary RR relation (effectively).

Clearly, Proposition 9 and Theorems 1 and 2 imply that $\rightarrow_P^*$ is a binary RR relation for every extended ground rewrite system $P$.

Finally we discuss the termination problem of extended ground rewrite systems. Termination does not seem to be expressible in the first-order theory of ground rewriting (cf. [DauTis2]). However, its decidability is much easier to show than that of confluence. An extended ground rewrite system $P$ is finitely terminating or noetherian if there does not exist an infinite derivation $t_1 \rightarrow_P t_2 \rightarrow_P t_3 \rightarrow_P \cdots$. As mentioned in [HueOpp], decidability of the noetherian property for (ordinary) ground rewrite systems is shown in [HueLan]; a proof that uses ground tree transducers is given in V.3 of [DauTis1]. The result is generalized to right-ground term rewriting systems in [Der].
Theorem 10. It is decidable for an extended ground rewrite system $P$ whether or not $P$ is finitely terminating.

Proof. Let $P$ be an extended ground rewrite system over $\Sigma$. We say that a tree $t \in T_\Sigma$ is nonterminating if there is an infinite derivation that starts with $t$. It is straightforward to prove (by structural induction on $t$) that if $t$ is nonterminating then there exist a 1-place context $c$ and a rule $u \rightarrow v$ of $P$ such that $t \rightarrow^* P c[u]$ and $v$ is nonterminating. Thus, if $P$ is not finitely terminating, then there is an infinite sequence $u_1 \rightarrow v_1, u_2 \rightarrow v_2, u_3 \rightarrow v_3, \ldots$ of rules of $P$ such that for every $i, v_i \rightarrow^* P c[u_{i+1}]$ for some 1-place context $c$. Now let $(L_i, R_i) \in P$ with $u_i \in L_i$ and $v_i \in R_i$. Since $P$ is finite, there exist $i < j$ such that $(L_i, R_i) = (L_j, R_j)$. This shows the existence of a rule $u \rightarrow v$ (viz. $u_i \rightarrow v_j$) such that $v \rightarrow^* P c[u]$ for some context $c$. In the other direction, the existence of such a rule clearly implies that $P$ is not finitely terminating. Thus, $P$ is finitely terminating iff $\rightarrow^* P (R) \cap \text{exsub}(L) = \emptyset$ for all $(L, R) \in P$, where $\text{exsub}(L)$ is the set of all trees that have at least one subtree in $L$. By Theorem 3, $\rightarrow^* P (R)$ is a regular tree language. Clearly, $\text{exsub}(L)$ is a regular tree language (e.g., $\text{exsub}(L) = T_\Sigma \setminus \text{allsub}(T_\Sigma \setminus L)$ and REGT is closed under complementation; for “allsub” see the proof of Theorem 2). Hence, by the closure of REGT under intersection, $\rightarrow^* P (R) \cap \text{exsub}(L)$ is a regular tree language. Thus, since the emptiness problem is decidable for regular tree languages (see, e.g., Theorem II.10.2 of [GécSté]), the above property can be decided.

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