JOURNAL OF COMBINATORIAL THEORY, Series B 49, 295–298 (1990)

Note

A Note on the Edge-Reconstruction of $K_{1,m}$-Free Graphs

I. KRASIKOV

School of Mathematical Sciences,
Tel-Aviv University, Tel-Aviv, Israel

Communicated by the Editors

Received March 1, 1988

We show that there exists an absolute constant c such that any $K_{1,m}$-free graph with the maximum degree $\Delta > cm(\log m)^{1/2}$ is edge reconstructible. © 1990 Academic Press, Inc.

We follow the basic notation and terminology of [7].

Let G be a finite simple graph. As usual, $V(G)$ and $E(G)$ denote the vertex set and the edge set, respectively, and $\bar{G}$ stands for the complement of G. We use $\beta_0(G)$ for the vertex independence number of G. Denote by ER the class of edge-reconstructivle graphs. All logarithms here are to base 2.

Recently Ellingham, Pyber, and Yu Xing Xing established the edge-reconstruction conjecture for $K_{1,3}$-free graphs [5]. Their proof is based on a very small upper bound, derived from the Nash–Williams lemma, for the maximum degree Δ of a hypothetical counterexample. They pointed out that their method yields that $K_{1,m}$-free graphs with $\Delta \geqslant R(m)$, the Ramsey number, are edge reconstructible. Here we improve this inequality and show that $G \in ER$ whenever $\Delta > cm(\log m)^{1/2}$. In particular, this yields that, for an appropriate constant c, a $K_{1,m}$-free graph with average degree $d > 2\log m + \log\log m + c$, is edge reconstructible. This is slightly sharper than a result obtained in [4, Theorem 2.5].

Suppose G_1 and G_2 are spanning subgraphs of K_n. Let S_n denote the set of bijections ϕ: $V(K_n) \to V(K_n)$. Then the set of *embeddings* of G_1 into G_2 is

$$G_1 \to G_2 = \{\phi \in S_n : \phi(G_1) \subseteq G_2\}.$$

Let G be a spanning subgraph of K_n. Then an *NW-family* (in honour of Nash–Williams) for G is a set F of spanning subraphs of K_n such that:

0095-8956/90 $3.00
Copyright © 1990 by Academic Press, Inc.
All rights of reproduction in any form reserved.

(1) $e(H)=e(G)$ for all $H \in F$;

(2) for any $x \subseteq E(G)$ with $|x|$ even there is an embedding $\phi \in S_n$ such that $E(G) \backslash E(\phi(H))=x$ for some $H \in F$.

Note that by letting $x=\varnothing$ we see that F must contain a graph isomorphic to G. Let $f(G)$ be the size of the smallest NW-family for G.

NASH–WILLIAMS LEMMA [7]. *If $G \notin ER$ then $f(G)=1$.*

In other words, the Nash–Williams lemma states that for a nonreconstructible graph G,

$$\forall x \subseteq E(G),\ |x| \text{ even},\ \exists\, y \subseteq E(\bar{G}) \text{ such that } (G \backslash x) \cup y \cong G.$$

The following lemma and its corollary are straightforward generalizations of results of Lovasz (see [8]) and Muller [6], respectively.

LEMMA 1. *Let Γ and G be a spanning subgraphs of K_n with $\Gamma \subseteq G$. Let F be an NW-family for G. Then*

$$\sum_{H \in F} |\Gamma \to H| \geqslant 2^{e(G)-e(\Gamma)-1}.$$

Proof. Take any $x \subseteq E(G) \backslash E(\Gamma)$ with $|x|$ even. Then for some $H \in F$ and $\phi \in S_n$, $E(G) \backslash E(\Gamma)=x$. But then $\phi^{-1} \in \Gamma \to H$. So for each of the $2^{e(G)-e(\Gamma)-1}$ sets x we have a pair (H, ϕ) with $H \in F$ and $\phi^{-1} \in \Gamma \to H$. The result follows. ∎

COROLLARY 1. *For any graph $e \leqslant \log(2fn!)$.*

Proof. Choose $E(\Gamma)=\varnothing$. ∎

For a vertex $v \in G$ we denote by $N(v)$ the induced subgraph consisting of the vertices adjacent to v. The vertex v will be called the root of $N(v)$. Consider the set $R(G)=\{N(v): \deg v=\Delta\}$. Let $R_m \in R(G)$ be a graph with the minimal number of edges. We shall estimate $f(R_m)$ for a nonreconstructible graph.

LEMMA 2. *If $G \notin ER$ then $f(R_m) \leqslant (\Delta^2+\Delta+2)/2$.*

Proof. We apply the Nash–Williams lemma to G but choose only those x which are in $E(R_m)$. Let K_r be the complete graph on the vertex set $V(R_m)$. For each $x \subseteq E(R_m)$, $|x| \equiv 0 \pmod 2$, fix $y \subset E(\bar{G})$ such that $\phi_x: G \to G_x=(G \backslash x) \cup y$ is an isomorphism. Observe that $G_x \cong G$ implies $y \subset E(K_r)$, by the minimality of $E(R_m)$.

So, for each x let $R_x = R_m \backslash x \cup y$. Clearly $F = \{R_x\}$ is an NW-family for R, since $R \backslash R_x = x$ and $e(R_x) = e(R_m)$ for each x.

Now, when x is deleted from G and y is added to get G_x, a neighbourhood isomorphic to R_x is created. Since $G \cong G_x$, a neighbourhood isomorphic to R_x must have been destroyed first. This must be $N(u)$ where u is v, adjacent to v, or adjacent to at least two vertices of $N(v) = R_m$. The number of possible vertices u is therefore at most $1 + \Delta + \Delta(\Delta - 1)/2 = (\Delta^2 + \Delta + 2)/2$. Hence each element in F is isomorphic to one of at most $(\Delta^2 + \Delta + 2)/2$ graphs, and therefore $f(R_m) \leqslant (\Delta^2 + \Delta + 2)/2$. ∎

MAIN THEOREM. *Let $G \notin ER$ be a $K_{1,m}$-free graph, then* $\Delta = O(m(\log m)^{1/2})$.

Proof. Let $G \notin ER$ be a $K_{1,m}$-free graph. We suppose that m is large enough to justify all approximations below. Moreover, we also assume that $m = o(\Delta)$.

Define a covering path system of a graph as a spanning set of vertex-disjoint paths. We fix $R_m \subset G$. Choose a covering path system $P \subset R_m$ with the minimal possible number of paths. Since $\beta_0(R_m) < m$ it follows by the Gallai–Milgram theorem [3] that P contains less than m paths. Moreover, as is well known (see [1, p. 275]), $\beta_0 \geqslant n/(d+1)$ for any graph with the average degree d. Thus, $e = e(R_m) > \Delta^2/4m$ for sufficiently large m.

Observe that $|P \to R_m| < \binom{e}{e(P)} 2^m m!$ since for any $\phi \in P \to R_m$ there are at most $\binom{e}{e(P)}$ choices for $E(\phi(P))$ and at most $2^m m!$ ways to embed P in any particular subgraph with $e(P)$ edges.

We have, by Lemmas 1 and 2,

$$2^{e - e(P) - 1} < \frac{\Delta^2 + \Delta + 2}{2} \binom{e}{e(P)} 2^m m!.$$

Using $\binom{a}{b} < (3a/b)^b$, $2^m m! < m^m$, and $e(P) < \Delta$, we get, for sufficiently large Δ, $2^e < (6e/\Delta)^\Delta m^m$. Finally, by Corollary 1, $e < \Delta \log \Delta$ and so,

$$\frac{\Delta^2}{40} < e < \Delta \log(6 \log \Delta) + m \log m,$$

which implies $\Delta = O(m(\log m)^{1/2})$. ∎

COROLLARY 2. *For a suffciently large constant c any $K_{1,m}$-free graph with average degree $d > 2 \log m + \log \log m + c$ is edge reconstructible.*

Proof. The graphs with $d > 2 \log 2\Delta$ are known to be edge reconstructible (Caunter and Nash–Williams, see [2]). ∎

Acknowledgment

I express my thanks to L. Pyber for his suggestion to use a covering path system which enabled me to improve an estimate on Δ originally obtained by the author.

References

1. B. Bollobas, "Random Graphs," Academic Press, Orlando, FL, 1985.
2. J. A. Bondy, The reconstruction of graphs, preprint.
3. T. Gallai and A. N. Milgram, Verallgemeinerung eines Graphen theoretischen Satzes von Redei, *Acta Sci. Math.* **21** (1960), 181–186.
4. Y. Caro, I. Krasikov, and Y. Roditty, Spanning trees and some edge-reconstructible graphs, *Ars Combin.* **20-A** (1985), 109–118.
5. M. N. Ellingham, L. Pyber, and Yu Xing, Claw-free graphs are edge-reconstructible, *J. Graph Theory*, in press.
6. V. Muller, The edge-reconstruction hypothesis is true for graphs with more than $n \log_2 n$ edges, *J. Combin. Theory Ser. B* **22** (1977), 281–283.
7. C. St. J. A. Nash-Williams, The reconstruction problem, *in* "Selected Topics in Graph Theory," pp. 205–236, Academic Press, London, 1978.
8. L. Pyber, The edge reconstruction of hamiltonian graphs, *J. Graph Theory*, in press.