Existence theorem of periodic solutions for a delay nonlinear differential equation with piecewise constant arguments

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Abstract

Existence criteria are proved for the periodic solutions of a first order nonlinear differential equation with piecewise constant arguments.
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1. Introduction

Considerable attention has been given to delay differential equations with piecewise constant arguments by several authors including Cooke and Wiener [1], Shah and Wiener [2], Aftabizadeh et al. [3]. This class of differential equations has useful applications in biomedical models of disease that has been developed by Busenerg and Cooke [4]. Studies of such equations were motivated by the fact that they represent a hybrid of discrete and continuous dynamical systems and combine the properties of both differential and differential–difference equation.

On the other hand, properties and solutions of delay differential equations with piecewise constant arguments and piecewise constant time delay have received considerable attention by several authors including Wiener [5], Cooke and Wiener [6]. Wiener and
Cooke [7], Wiener and Debnath [8,9], Gopalsamy et al. [10], Lin and Wang [11], Papaschinopoulos and Schinas [12], Huang [13], Shen and Stavroulakis [14] and Wiener and Heller [15].

Carvalho and Wiener [16] considered periodic solutions of the first-order differential equation with piecewise constant argument

\[ x'(t) = ax(t)(1 - x([t])), \]

where \( a \) is constant and \([\cdot]\) is greatest-integer function. As mentioned by Cooke and Wiener [17], equations such as (1.1) may be treated as semi-discretization of the ordinary logistic equation and solutions of (1.1) exhibit a wide variety of properties of interest.

In this paper, we consider following equation:

\[ x'(t) + f(t, x([t]), x([t - 1]), \ldots, x([t - k]), x(t)) = 0, \]

where \( f \in C(\mathbb{R}^{k+3}, \mathbb{R}) \) and it is periodic function of \( t \) of positive integer period \( \omega \).

The main objective of this paper is to prove several existence criteria for \( \omega\)-periodic solutions of (1.2) by using Mawhin’s continuous theorem. We note that Mawhin’s continuous theorem is usually used to prove the existence of periodic solutions for continuous differential systems, but (1.2) is a semi-discretion differential system. We introduce a technique to such that (1.2) into an integral equation, then using it. Finally, as application of main results of this paper, we lead to some results for the nonlinear difference equations, these results are new and strong interest.

By a solution of (1.2), we mean a function \( x(t) \) which is defined on \( \mathbb{R} \) and which satisfies the following conditions:

(i) \( x(t) \) is continuous on \( \mathbb{R} \);
(ii) The derivative \( x'(t) \) exists at each point \( t \in \mathbb{R} \), with the possible exception of the points \([t]\) \( \in \mathbb{R} \), where one-side derivatives exist;
(iii) Equation (1.2) is satisfied on each interval \([n, n+1) \subset \mathbb{R} \) with integral endpoints.

We also state Mawhin’s continuous theorem (see [18]) in the form

**Theorem 1.1.** Let \( X \) and \( Y \) be two Banach spaces and \( L: \text{dom} L \cap X \to Y \) be a Fredholm mapping of index 0. Suppose that \( \varOmega \) is open bounded in \( X \) and \( N: \bar{\varOmega} \to Y \) is \( L \)-compact on \( \varOmega \). Furthermore, suppose that

(i) For each \( \lambda \in (0, 1) \), \( x \in \partial \varOmega \),

\[ Lx \neq \lambda Nx; \]

(ii) For each \( x \in \partial \varOmega \cap \text{Ker} L, \)

\[ QNx \neq 0, \]

and

\[ \text{deg}(JQN, \varOmega \cap \text{Ker} L, 0) \neq 0, \]

where \( J: \text{Im} Q \to \text{Ker} L \) is isomorphism.

Then \( Lx = Nx \) has at least one solution in \( \varOmega \cap \text{dom} L \).
We use that the following conditions, where $D$ and $M$ are positive constants:

(a1) $f(t, x_0, \ldots, x_{k+1}) > 0$ for $t \in \mathbb{R}$ and $x_i \geq D$ ($i = 0, \ldots, k + 1$);

(a2) $f(t, x_0, \ldots, x_{k+1}) < 0$ for $t \in \mathbb{R}$ and $x_i \geq D$ ($i = 0, \ldots, k + 1$);

(b1) $f(t, x_0, \ldots, x_{k+1}) < 0$ for $t \in \mathbb{R}$ and $x_i \leq -D$ ($i = 0, \ldots, k + 1$);

(b2) $f(t, x_0, \ldots, x_{k+1}) > 0$ for $t \in \mathbb{R}$ and $x_i \leq -D$ ($i = 0, \ldots, k + 1$);

(c1) $f(t, x_0, \ldots, x_{k+1}) \geq -M$ for $(t, x_0, \ldots, x_{k+1}) \in \mathbb{R}^{k+3}$;

(c2) $f(t, x_0, \ldots, x_{k+1}) \leq M$ for $(t, x_0, \ldots, x_{k+1}) \in \mathbb{R}^{k+3}$.

2. Main results

Theorem 2.1. If $(a_1)$, $(b_1)$ and $(c_1)$ are satisfied, then $(1.2)$ has an $\omega$-periodic solution.

Theorem 2.2. If $(a_2)$, $(b_2)$ and $(c_1)$ are satisfied, then $(1.2)$ has an $\omega$-periodic solution.

Theorem 2.3. If $(a_1)$, $(b_1)$ and $(c_2)$ are satisfied, then $(1.2)$ has an $\omega$-periodic solution.

Theorem 2.4. If $(a_2)$, $(b_2)$ and $(c_2)$ are satisfied, then $(1.2)$ has an $\omega$-periodic solution.

We only give proof of Theorem 2.1, as Theorems 2.2–2.4 can be proved similarly.

To prove Theorem 2.1, we need preliminaries. It is easy to see that $x(t)$ is an $\omega$-periodic solution of (1) if and only if $x(t)$ is an $\omega$-periodic solution of the following equation:

$$x(t) = x(0) - \int_0^t f(s, x([s]), x([s - 1]), \ldots, x([s - k]), x(s)) \, ds. \quad (2.1)$$

Let

$$X_\omega := \{ x \in C(\mathbb{R}, \mathbb{R}) \mid x(t + \omega) = x(t) \}$$

and endowed with the norm $\|x\|_1 = \max_{0 \leq t \leq \omega} |x(t)|$. Let

$$Y_\omega := \{ y \in C(\mathbb{R}, \mathbb{R}) \mid y(0) = 0, y(t) = \alpha t + h(t) \text{ for some } \alpha \in \mathbb{R}, h \in X_\omega \}$$

and endowed with the norm $\|y\|_2 = |\alpha| + \|h\|_1$. Then both $(X_\omega, \| \cdot \|_1)$ and $(Y_\omega, \| \cdot \|_2)$ are Banach spaces. Define respectively the mappings $L$ and $N$ as

$$L : X_\omega \to Y_\omega, \quad Lx(t) = x(t) - x(0), \quad t \in \mathbb{R}, \quad (2.2)$$

and

$$N : X_\omega \to Y_\omega, \quad Nx(t) = -\int_0^t f(s, x([s]), x([s - 1]), \ldots, x([s - k]), x(s)) \, ds, \quad t \in \mathbb{R}. \quad (2.3)$$

Since
\[- \int_0^t f(s, x([s]), x([s - 1]), \ldots, x([s - k]), x(s)) \, ds \]
\[+ \frac{t}{\omega} \int_0^\omega f(s, x([s]), x([s - 1]), \ldots, x([s - k]), x(s)) \, ds \]

is an \( \omega \)-periodic function of \( t \in R \) for any \( x \in X_\omega \), we can see that \( N \) is well-defined operator from \( X_\omega \) to \( Y_\omega \) and is a completely continuous mapping. On the other hand, direct leads to \( \ker L = \{ x \in X_\omega \mid x(t) = x(0) \} \) and \( \text{Im} \, L = Y_\omega \cap X_\omega \). Furthermore, if we define the projections \( P \) and \( Q \) as \( P : X_\omega \rightarrow X_\omega, \ P x(t) = x(0), \) \( Q : Y_\omega \rightarrow Y_\omega, \ Q y(t) = \alpha t \) for \( y(t) = \alpha t + h(t), \ y(0) = 0, \ \alpha \in R, \ h \in X_\omega, \) \( 2.5 \) respectively, then \( X_\omega = \ker P \oplus \ker L \) and \( Y_\omega = \text{Im} \, L \oplus \text{Im} \, Q \). It is easy to see that \( \text{Im} \, L = \{ y \in X_\omega \mid y(0) = 0 \} \) is closed in \( Y_\omega \) and \( \text{codim} \, \text{Im} \, L = \dim \text{Im} \, Q = 1 = \dim \ker L \), thus we know that the following Lemma 2.1 is true.

**Lemma 2.1.** Let \( L \) be defined by (2.2); then \( L \) is a Fredholm operator of index zero.

**Lemma 2.2.** Let \( L \) and \( N \) be defined by (2.2) and (2.3), respectively, suppose \( \Omega \) is an open bounded subset of \( X_\omega \), then \( N \) is \( L \)-compact on \( \bar{\Omega} \).

**Proof.** It is easy to see that for any \( x \in \bar{\Omega}, \)

\[ QNx(t) = - \frac{t}{\omega} \int_0^\omega f(s, x([s]), x([s - 1]), \ldots, x([s - k]), x(s)) \, ds, \] \( 2.6 \)

so

\[ \| QNx \|_2 = \left| \frac{1}{\omega} \int_0^\omega f(s, x([s]), x([s - 1]), \ldots, x([s - k]), x(s)) \, ds \right|. \] \( 2.7 \)

We denote the inverse of the map \( L_{|_{\text{dom} \, L \cap \ker P}} \) \( \text{dom} \, L \cap \ker P \rightarrow Y_\omega \) by \( K_P \), direct leads to

\[ K_P(I - Q)Nx(t) \]
\[= - \int_0^t f(s, x([s]), x([s - 1]), \ldots, x([s - k]), x(s)) \, ds \]
\[+ \frac{t}{\omega} \int_0^\omega f(s, x([s]), x([s - 1]), \ldots, x([s - k]), x(s)) \, ds. \] \( 2.8 \)
By (2.7), we see that $QN(\bar{\Omega})$ is bounded. Noting that (2.8) holds and $N$ is a completely continuous mapping, by using Arzela–Ascoli theorem, we know that

$$K(I - Q)N(\bar{\Omega})$$

is relatively compact, thus $N$ is $L$-compact on $\bar{\Omega}$. The proof of Lemma 2.2 is complete.

**Lemma 2.3.** Suppose $g(t)$ is a real bounded continuous function on $[a, b)$ and $\lim_{x \to b^-} g(t)$ exists. Then there is a point $\xi \in (a, b)$, such that

$$\int_a^b g(s)\,dt = g(\xi)(b - a).$$

(2.9)

**Proof.** In view of the integrating intermediate value theorem in [19], it is easy to see that the result is true. The proof of Lemma 2.3 is complete.

**Lemma 2.4.** Suppose $(a_1)$ and $(b_1)$ hold, and if $x(t) \in X_\omega$ such that

$$\omega \int_0^a f(s, x([s]), x([s - 1]), \ldots, x([s - k]), x(s))\,ds = 0.$$ (2.10)

Then there is $t_1 \in [0, \omega]$, such that $|x(t_1)| < D$.

**Proof.** From (2.10) and Lemma 2.3, we have $\xi_i \in (i - 1, i)$ ($i = 1, \ldots, \omega$), such that

$$\sum_{i=1}^\omega f(\xi_i, x(i - 1), x(i - 2), \ldots, x(i - k - 1), x(\xi_i))$$

$$= \sum_{i=1}^\omega \int_{i-1}^i f(s, x([s]), x([s - 1]), \ldots, x([s - k]), x(s))\,ds$$

$$= \int_0^\omega f(s, x([s]), x([s - 1]), \ldots, x([s - k]), x(s))\,ds = 0.$$ (2.11)

If there is one whose absolute value is less than $D$ in $x(i - 1), x(i - 2), \ldots, x(i - k - 1), x(\xi_i)$ ($i = 1, 2, \ldots, \omega$), it is ideal. Or, by $(a_1)$, $(b_1)$ and (2.11), there should be $x(\eta_1)$ and $x(\eta_2)$ in $x(i - 1), x(i - 2), \ldots, x(i - k - 1), x(\xi_i)$ ($i = 1, 2, \ldots, \omega$), such that $x(\eta_1) \geq D$ and $x(\eta_2) \leq -D$. Noting that $x(t)$ is continuous on $R$, in view of the intermediate value theorem, there is $x(\eta_3)$ such that $-D < x(\eta_3) < D$ (here $\eta_1 < \eta_3 < \eta_2$ or $\eta_2 < \eta_3 < \eta_1$). Since $x(t)$ is periodic, there is $t_1 \in [0, \omega]$ such that $|x(t_1)| = |x(\eta_3)| < D$. The proof of Lemma 2.4 is complete.
Now, we consider the following equation:

\[ x(t) - x(0) = -\lambda \int_0^t f(s, x([s]), x([s-1]), \ldots, x([s-k]), x(s)) ds, \quad (2.12) \]

where \( \lambda \in (0, 1) \).

**Lemma 2.5.** Suppose (a1), (b1) and (c1) are satisfied. Then for any \( \omega \)-periodic solution \( x(t) \) of (2.12),

\[ |x(t)| \leq D + 2\omega M, \quad t \in [0, \omega]. \quad (2.13) \]

**Proof.** Let \( x(t) \) be an \( \omega \)-periodic solution of (2.12). By (2.12) we have

\[ \omega \int_0^\omega f(s, x([s]), x([s-1]), \ldots, x([s-k]), x(s)) ds = 0, \quad (2.14) \]

by Lemma 2.4, there is \( t_1 \in [0, \omega] \) such that

\[ |x(t_1)| < D. \quad (2.15) \]

If we write

\[ G_+(t) = \max \{ f(t, x([t]), x([t-1]), \ldots, x([t-k]), x(t)), 0 \} \quad (2.16) \]

and

\[ G_-(t) = \max \{ -f(t, x([t]), x([t-1]), \ldots, x([t-k]), x(t)), 0 \}, \quad (2.17) \]

then \( G_+(t) \) and \( G_-(t) \) are piecewise continuous functions on \( \mathbb{R} \), and that

\[ f(t, x([t]), x([t-1]), \ldots, x([t-k]), x(t)) = G_+(t) - G_-(t) \quad (2.18) \]

and

\[ |f(t, x([t]), x([t-1]), \ldots, x([t-k]), x(t))| = G_+(t) + G_-(t). \quad (2.19) \]

In view of (c1) and (2.17), we have

\[ |G_-(t)| = G_-(t) \leq M, \quad t \in \mathbb{R}. \quad (2.20) \]

Thus

\[ \int_0^\omega G_+(s) ds \leq \omega M. \quad (2.21) \]

From (2.14), (2.18) and (2.21), it yields that

\[ \int_0^\omega G_+(s) ds = \int_0^\omega G_-(s) ds \leq \omega M. \quad (2.22) \]
By (2.19) and (2.22), we know that
\[ \int_0^\omega \left| f(s, x([s]), \ldots, x([s-k]), x(s)) \right| ds \leq 2\omega M. \] (2.23)

In view of (2.12), (2.15), and (2.23), we conclude that for any \( t \in [0, \omega] \),

\[ |x(t)| = \left| x(t_1) - \lambda \int_{t_1}^t f(s, x([s]), \ldots, x([s-k]), x(s)) ds \right| \leq |x(t_1)| + \int_0^\omega \left| f(s, x([s]), \ldots, x([s-k]), x(s)) \right| ds \leq D + 2\omega M. \]

The proof of Lemma 2.4 is complete. \( \square \)

**The proof of Theorem 2.1.** Let \( L, N, P \) and \( Q \) be defined by (2.2)–(2.5), respectively. Set
\[ \Omega := \{ x \in X_\omega \mid \|x\|_1 < \bar{D} \}, \]
where \( \bar{D} \) is fixed which satisfies \( \bar{D} > D + 2\omega M \). It is easy to see that \( \Omega \) be an open bounded subset of \( X_\omega \), and from Lemmas 2.1 and 2.2, we know that \( L \) is a Fredholm mapping of index 0 and \( N \) is \( L \)-compact on \( \bar{\Omega} \). Noting that \( \bar{D} > D + 2\omega M \), by Lemma 2.5, we lead to: for each \( \lambda \in (0, 1) \), \( x \in \partial \Omega \), \( Lx \neq \lambda Nx \). Next we show that a function \( x \in \text{Ker} L \cap \partial \Omega \) must be constant \( x(t) \equiv \bar{D} \) or \( x(t) \equiv -\bar{D} \).

If the isomorphism \( J : \text{Im} Q \rightarrow \text{Ker} L \) is defined by \( J(\alpha t) = \alpha, \alpha \in \mathbb{R} \), then
\[ JQN x(t) = -\frac{1}{\omega} \int_0^\omega f(s, x, \ldots, x, x) ds. \]

In particular, we see that
\[ -\frac{1}{\omega} \int_0^\omega f(s, \bar{D}, \ldots, \bar{D}) ds < 0 \quad \text{and} \quad -\frac{1}{\omega} \int_0^\omega f(s, -\bar{D}, \ldots, -\bar{D}) ds > 0, \]
which shows that
\[ \text{deg}(JQN, \Omega \cap \text{Ker} L, 0) \neq 0. \]

By Theorem 1.1, we find that equation \( Lx = Nx \) has a solution in \( \bar{\Omega} \cap \text{dom} L \), that is to say that (1.2) has an \( \omega \)-periodic solution. The proof of Theorem 1 is complete. \( \square \)
Example 2.1. Consider the equation
\[ x'(t) + \left( x(t) + x\left(\lfloor t \rfloor\right) + x\left(\lfloor t - 1 \rfloor\right) + \sin \pi t \right) \times \exp\left( x(t) + x\left(\lfloor t \rfloor\right) + x\left(\lfloor t - 1 \rfloor\right) - \cos \pi t \right) = 0, \] (2.24)
where \( f(t, x_0, x_1, x_2) = (x_0 + x_1 + x_2 + \sin \pi t) \exp(-x_0 - x_1 - x_2 - \cos \pi t). \) It is easy to verify that all the conditions of Theorem 2.1 are satisfied with \( D > 1/3, \ M = 3 \) and \( \omega = 2, \) hence (2.24) has a 2-periodic solution.

3. Application in difference equations

Consider the following nonlinear difference equation:
\[ y_{n+1} - y_n = F(y_n, y_{n-1}, \ldots, y_{n-k}), \] (3.1)
where \( F \in C(R^{k+1}, R). \)

We remark that there are a lot of good results on existence of \( \omega \)-periodic solutions of (3.1) (see, e.g., [20]), as application of main results of this paper, we lead to some new results for (3.1).

Let \( Z \) be the set of integers. By a solution of (3.1) we mean a sequence \( \{y_n\}_{n \in Z}, \) which satisfies Eq. (3.1). A sequence \( \{y_n\}_{n \in Z} \) is said to be with period \( \omega, \) if \( y_{n+\omega} = y_n \) for \( n \in Z. \)

Lemma 3.1. Equation (3.1) has an \( \omega \)-periodic solution if and only if the following differential equation with piecewise constant arguments:
\[ y'(t) = F(y(\lfloor t \rfloor), y(\lfloor t - 1 \rfloor), \ldots, y(\lfloor t - k \rfloor)) \] (3.2)
has an \( \omega \)-periodic solution.

Proof. Let \( y(t) \) be an \( \omega \)-periodic solution of (3.2). It is easy to see that for any \( n \in Z, \)
\[ y'(t) = F(y(n), y(n - 1), \ldots, y(n - k)), \quad n \leq t < n + 1. \] (3.3)
Integrating (3.3) from \( n \) to \( t, \) we have
\[ y(t) - y(n) = F(y(n), y(n - 1), \ldots, y(n - k))(t - n), \quad n \leq t < n + 1. \] (3.4)
Since \( \lim_{t \to (n+1)^+} y(t) = y(n+1), \) we see further that
\[ y(n + 1) - y(n) = F(y(n), y(n - 1), \ldots, y(n - k)), \quad n \in Z. \] (3.5)
If we now let \( y_n = y(n) \) for \( n \in Z, \) then \( \{y_n\}_{n \in Z} \) is an \( \omega \)-periodic solution of (3.1).

Conversely, let \( \{y_n\}_{n \in Z} \) be an \( \omega \)-periodic solution of (3.1). Set \( y(n) = y_n \) for \( n \in Z, \) and let the function \( y(t) \) on each interval \([n, n + 1)\) be defined by (3.4). Then it is not difficult to check that this function is an \( \omega \)-periodic solution of (3.2). The proof of Lemma 3.1 is complete. \( \square \)

Some of the following additional conditions will be needed, where \( D \) and \( M \) are positive constants:
\( (\alpha_1) \quad F(x_0, \ldots, x_k) > 0 \) for \( x_i \geq D \ (i = 0, \ldots, k); \)

\( (\alpha_2) \quad F(x_0, \ldots, x_k) < 0 \) for \( x_i \geq D \ (i = 0, \ldots, k); \)

\( (\beta_1) \quad F(x_0, \ldots, x_k) < 0 \) for \( x_i \leq -D \ (i = 0, \ldots, k); \)

\( (\beta_2) \quad F(x_0, \ldots, x_k) > 0 \) for \( x_i \leq -D \ (i = 0, \ldots, k); \)

\( (\gamma_1) \quad F(x_0, \ldots, x_k) \geq -M \) for \( (x_0, \ldots, x_k) \in R^{k+1}; \)

\( (\gamma_2) \quad F(x_0, \ldots, x_k) \leq M \) for \( (x_0, \ldots, x_k) \in R^{k+1}. \)

As immediate corollaries of Theorems 2.1–2.4 and Lemma 3.1, the following results are true.

**Corollary 3.1.** If \( (\alpha_1), (\beta_1) \) and \( (\gamma_1) \) are satisfied, then (3.1) has an \( \omega \)-periodic solution.

**Corollary 3.2.** If \( (\alpha_2), (\beta_2) \) and \( (\gamma_1) \) are satisfied, then (3.1) has an \( \omega \)-periodic solution.

**Corollary 3.3.** If \( (\alpha_1), (\beta_1) \) and \( (\gamma_2) \) are satisfied, then (3.1) has an \( \omega \)-periodic solution.

**Corollary 3.4.** If \( (\alpha_2), (\beta_2) \) and \( (\gamma_2) \) are satisfied, then (3.1) has an \( \omega \)-periodic solution.

**References**


