A decomposition theorem for singular integral operators on spaces of homogeneous type

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Abstract

Let (X, d, μ) be a space of homogeneous type. Under the assumption $\mu(\{x\}) = 0$ for all $x \in X$, we prove a decomposition theorem for singular integral operators on (X, d, μ). Isotropic Haar expansion gives a representation of the integral operator as a series of simple shifts and rearrangements plus two paraproducts. This yields a UMD-valued $T(1)$ theorem on spaces of homogeneous type.

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1. Introduction

The $T(1)$ theorem for scalar valued singular integral operators on $\mathbb{R}^n$ was initially proved by David and Journé [10] using Fourier analysis methods. It was later extended to operators on spaces of homogeneous type. See Christ [5, Chapter VI], who refers to Coifman (unpublished), and also Christ [4] and Coifman, Jones, Semmes [6]. The structural framework for both proofs is given by Cotlar–Stein theorem on almost orthogonal operators. Consequently, different methods had to be developed to obtain a $T(1)$ theorem for integral operators taking values in general Banach spaces. This was done by T. Figiel [12,13] who introduced a general method of decomposing integral operators into series of basic building blocks. This decomposition arises canonically by expanding the integral kernel along the isotropic Haar system in $\mathbb{R}^n \times \mathbb{R}^n$. Thus proving boundedness of integral operators is reduced to the following problems:

- Verify a priori norm estimates for the building blocks (this is independent of the underlying integral kernel).
- Verify compensating coefficient estimates arising in the isotropic series expansion of the kernel (the decay of the coefficients depends on the size and smoothness of the kernel under investigation).

The basic building blocks isolated by Figiel are simple rearrangements and shifts plus two paraproducts. These rearrangements and shifts act on the Haar system in $\mathbb{R}$. It is important to note that their definition depends expressly on the group structure of the underlying domain $(\mathbb{R}^n, +)$. Figiel's decomposition was applied later to several singular integral operators beyond the Calderón–Zygmund class. These included applications to Dirichlet kernels of generalized Franklin systems [20] and interpolatory estimates arising in the theory of compensated compactness [21].

In the present paper we extend Figiel's decomposition method to the setting of spaces of homogeneous type. Our extension of this method is based on constructing – without recourse to group structure – a suitable class of rearrangement and shift operators that allow us to decompose singular integral operators on (X, d, μ) into a series of basic building blocks that can be analyzed and estimated by combinatorial means. The central result of this paper is the convergence of this operator-series (4.9).

A source of renewed interest in spaces of homogeneous type is the recent development of diffusion wavelets and their multiresolution analysis that was carried out on spaces of homogeneous type by Coifman and Maggioni [7]. We recall further that the vector-valued $T(1)$ theorem on spaces of homogeneous type is an essential first step towards the solution of the open classification problem for the vector valued Banach spaces $H_E^1(X, d, \mu)$. See [24,26].

The organization of the paper is as follows: In Section 2 we review some preliminaries. In Section 3 we construct shifts on spaces of homogeneous type (X, d, μ). In Sections 3.1 and 3.2, we review important properties of dyadic cubes in (X, d, μ). Based on those dyadic cubes, we construct in Section 3.3 martingale differences on X that possess analogous properties to the Haar system in $\mathbb{R}$. We then introduce an isotropic basis in $L^2(X \times X)$ using tensor products $f \otimes g$ of Haar functions in Section 3.4. In Section 3.5 we decompose the collection of all pairs of dyadic cubes into subcollections that fix their relative distance. These subcollections are then decomposed further in Section 3.6 to define injections (shifts). Section 3.7 is then to extract further subcollections that satisfy a certain nestedness condition.

In Section 4 we use the isotropic basis to decompose integral operators. Then according to the decomposition developed in Section 3 we split an operator further into a sum of paraproducts, shift and rearrangement operators (see (4.9)), which, in combination with the L_E^p-boundedness of these parts, is the main result of this article. Here E is a UMD Banach space. Rearrangement and shift operators are defined via the shift of cubes introduced in Section 3.6. In Sections 4.4 and 4.5 we obtain L_E^p-estimates for these rearrangement and shift operators.

In Section 4.6, we prove a UMD-valued $T(1)$ theorem on spaces of homogeneous type for a class of admissible integral kernels strictly larger than the Calderón–Zygmund class (see Definition 4.1 and the remark afterwards).

Related recent developments on $T(1)$. By use of anisotropic kernel expansions, the $T(1)$ theorem was extended in an important series of papers by Nazarov, Treil, Volberg [28–30], David [9] and David, Mattila [11] to subdomains of $\mathbb{R}^n$ that are not necessarily of homogeneous type. Random anisotropic expansions were used in Nazarov, Treil, Volberg [28–30], in Hytonen [16], and in relation to A_2 weights in Hytonen, Pérez, Treil, Volberg [18]. These random expansions depend expressly on the additive group structure of the ambient space $\mathbb{R}^n$ as follows: Random expansions are determined by randomly shifted dyadic filtrations where the shift is performed by adding a random shift parameter x_k to the dyadic intervals of length 2^k, see Nazarov, Treil, Volberg [30, Section 9.1].

Hytonen [16] recently obtained the UMD-valued $T(1)$ theorem for non-homogeneous subdomains of $\mathbb{R}^n$. He based his approach on random, anisotropic dyadic kernel expansions. As in Nazarov, Treil, Volberg [30], the random expansions in Hytonen [16] depend expressly on the additive group structure of $\mathbb{R}^n$. In this sense the vector valued methods of Hytonen [16] don't cover completely the operators on spaces of homogeneous type as treated in the present paper (see also Definition 4.1 and the remark afterwards).

For further recent work based on the random anisotropic expansion on metric measure spaces we refer to Nazarov, Reznikov, Volberg [27] and Hytonen, Makarainen [17]. A proof of the

vector valued $T(1)$ theorem on spaces of homogeneous type can be built by merging the random dyadic systems in spaces of homogeneous type [27,17] with the anisotropic vector valued Haar expansion in Hytonen [16].

To differentiate the above results from our work in the present paper we note the following: Choosing the expansion along *random anisotropic* Haar systems leads to decompositions of integral operators that are *structurally* different from those studied in the present paper. The latter come from expansions along a *deterministic and isotropic* Haar system. As mentioned above, our decomposition (4.9) into simple rearrangements and shifts is the central assertion of our work. It covers a class of singular integral operators strictly larger than the Calderón–Zygmund class (see Definition 4.1 and the remark afterwards). In the context of $\mathbb{R}^n$ the isotropic expansion was used to estimate integral operators beyond the Calderón–Zygmund class arising in vector valued Approximation Theory and in the Calculus of Variations (compensated compactness) (see Kamont, Müller [20] and Lee, Müller, Müller [21]).

2. Martingale preliminaries

In this section, we collect a set of martingale inequalities we use throughout the paper.

2.1. Kahane's contraction principle

We use Kahane's contraction principle in the following form (Kahane [19], Marcus and Pisier [23]).

Theorem 2.1 *(Kahane, contraction principle). Let $e_1, \ldots, e_m$ be elements in a Banach space E and $r_1, \ldots, r_m$ be independent Rademacher functions. If $a_1, \ldots, a_m$ are real numbers with $\sup_{k \leqslant m} |a_k| \leqslant 1$, we have for any $1 \leqslant p < \infty$*

$$\int\limits_0^1 \left\| \sum_{k=1}^m a_k r_k(t) e_k \right\|_E^p dt \leqslant \int\limits_0^1 \left\| \sum_{k=1}^m r_k(t) e_k \right\|_E^p dt.$$

2.2. UMD *spaces*

Definition 2.2. A Banach space E is called a UMD-*space* (*unconditional for martingale differences*), if for every $1 < p < \infty$ there exists a constant β_p such that for every E-valued martingale difference sequence $(d_k)_{k \geqslant 0}$ we have the inequality

$$\left\| \sum_{k=0}^n \varepsilon_k d_k \right\|_{L_E^p} \leqslant \beta_p \left\| \sum_{k=0}^n d_k \right\|_{L_E^p} \tag{2.1}$$

for all sequences ε of numbers in $\{-1, 1\}$ and all $n \in \mathbb{N}$.

Remark.

1. We remark that if there exists one $1 < p < \infty$ with a constant β_p such that (2.1) holds, we have automatically that for all $1 < p < \infty$ there exists a constant β_p with (2.1).

2. Hilbert spaces are UMD-spaces, UMD-spaces are reflexive and the UMD-property is a self dual isomorphic invariant (see for instance Figiel [12], Figiel [13], Figiel and Wojtaszczyk [14] or Burkholder [2]).

2.3. The space BMO

We let $(X, \mathcal{F}, \mu)$ be a probability space and $\{\mathcal{F}_k\}_{k \in \mathbb{N}_0}$ be a sequence of σ-algebras such that $\mathcal{F}$ is generated by the union $\bigcup_k \mathcal{F}_k$. For $f \in L^1(X)$ we introduce the abbreviations

$$\mathbb{E}_k f := \mathbb{E}(f|\mathcal{F}_k) \quad \text{and} \quad \Delta_k := \mathbb{E}_k - \mathbb{E}_{k-1}.$$

Definition 2.3 (*Bounded mean oscillation*). A function $f : X \to \mathbb{R}$ is said to be in BMO$(X, (\mathcal{F}_k))$ if and only if f is in $L^2(X)$ and

$$\|f\|_{\text{BMO}} := \sup_{k \in \mathbb{N}} \left\| \sqrt{\mathbb{E}_k \big(|f - \mathbb{E}_{k-1} f|^2\big)} \right\|_{\infty} < \infty. \tag{2.2}$$

This is a norm, if we factor out the constants.

Remark. Recall that no matter what exponent $1 \leqslant p < \infty$ in (2.2) is chosen instead of 2, the definition leads to the same space BMO$(X, (\mathcal{F}_k))$ with equivalent norms (cf. Garsia [15] or Bourgain [1]).

3. Extracting rearrangements on spaces of homogeneous type

This section contains an extensive combinatorial analysis of dyadic cubes in spaces of homogeneous type. We recall first basic properties of those cubes and of the martingale differences they generate. Thus we construct orthonormal bases in $L^2(X)$ and $L^2(X \times X)$. Next we introduce a coloring on the collection of all dyadic cubes, so that on each monochromatic subcollection there are well-defined rearrangement operators that act like "shifts by q^m units" (Proposition 3.11). The complications in the proof of this proposition are due to the fact that we need to have good quantitative control on the numbers of colors involved. This in turn is dictated by the nature of the kernel operators we treat in Section 4. Theorem 3.17 is the second main result of this section. It provides the combinatorial basis for the norm estimates of the rearrangement operators defined in Section 4.3.

3.1. Definitions

Definition 3.1. Let X be a set. A mapping $d : X \times X \to \mathbb{R}_0^+$ with the properties

1. $d(x, y) = 0 \Leftrightarrow x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \leqslant K(d(x, z) + d(z, y))$ for all $x, y, z \in X$ and some constant $K \geqslant 1$ that is independent of x, y, z,

is called a *quasimetric* and (X, d) is called a *quasimetric space*.

Given a quasimetric d, we define the ball centered at $x \in X$ with radius $r > 0$ as

$$B(x, r) := \{y \in X \colon d(x, y) < r\}.$$

Additionally, a set $A \subset X$ is called *open* if and only if for all $x \in X$ there exists $r > 0$ such that $B(x, r) \subseteq A$.

Definition 3.2. Let (X, d) be a quasimetric space such that every ball in the quasimetric d is open and μ a Borel measure. If there is an $A > 0$ such that

$$0 < \mu\big(B(x, 2r)\big) \leqslant A\mu\big(B(x, r)\big) < \infty, \quad \text{for all } x \in X \text{ and all } r > 0,$$

then (X, d, μ) is called a *space of homogeneous type*. Additionally, if there exist constants b_1, b_2 such that

$$b_1 r \leqslant \mu\big(B(x, r)\big) \leqslant b_2 r$$

for all $x \in X$ and all r with $\mu(\{x\}) < r < \mu(X)$, we call the space of homogeneous type (X, d, μ) *normal*.

Remark. We note that if (X, d, μ) is a space of homogeneous type, then for all $\lambda > 0$ there exists A_λ, such that

$$\mu\big(B(x, \lambda r)\big) \leqslant A_\lambda \mu\big(B(x, r)\big) \quad \text{for all } x \in X \text{ and all } r > 0.$$

Since for a given quasimetric space (X, d), the balls in X are not necessarily open, we added this condition to the definition. This is the case, if for instance one has a Hölder condition for d: There exist $C < \infty$ and $0 < \beta < 1$ such that for all $x, y, z \in X$ we have

$$\big|d(x, z) - d(y, z)\big| \leqslant C d(x, y)^\beta \max\{d(x, z), d(y, z)\}^{1-\beta}. \tag{3.1}$$

In fact, Macías and Segovia proved in [22] that for every space of homogeneous type there exists an equivalent quasimetric with the desired Hölder property. Here, a quasimetric d' is equivalent to a quasimetric d if there exists a finite constant C such that

$$\frac{1}{C} d(x, y) \leqslant d'(x, y) \leqslant C d(x, y),$$

whenever $x, y \in X$.

Standard assumptions on X. In the following, we always assume that the spaces X we work with are spaces of homogeneous type, equipped with a quasimetric d and a Borel probability measure μ. Additionally we impose the restriction that X is normal and that for all $x \in X$ we have $\mu(\{x\}) = 0$, i.e. we have no isolated points.

3.2. Dyadic cubes

In a space of homogeneous type there are analogues for dyadic cubes in $\mathbb{R}^n$ (see Christ [5] and David [8]).

Theorem 3.3. *Let (X, d, μ) be a space of homogeneous type. Then there exist a system of open sets*

$$\mathcal{A} := \big\{ Q_\alpha^n \subseteq X \mid n \in \mathbb{Z}, \ \alpha \in \mathcal{K}_n \big\},$$

points $z_\alpha^n \in Q_\alpha^n$ and constants $q > 1$, $c_1, c_2, c_3, \eta \in \mathbb{R}^+$, $N \in \mathbb{N}$ such that we have the following properties:

1. *For all $n \in \mathbb{Z}$ we have that $X = \bigcup_{\alpha \in \mathcal{K}_n} Q_\alpha^n$ up to μ-null sets.*
2. *For Q_α^m, Q_β^n with $m \leqslant n$ and $\alpha \in \mathcal{K}_m$ and $\beta \in \mathcal{K}_n$ we have either $Q_\alpha^m \subseteq Q_\beta^n$ or $Q_\alpha^m \cap Q_\beta^n = \emptyset$. That means that the cubes $\{Q_\alpha^n\}$ are nested.*
3. *For each Q_α^n and every $m \geqslant n$ there is exactly one $\beta \in \mathcal{K}_m$ such that $Q_\alpha^n \subseteq Q_\beta^m$.*
4. *For all $n \in \mathbb{Z}$ and for all $\alpha \in \mathcal{K}_n$ we have that $B(z_\alpha^n, c_1 q^n) \subseteq Q_\alpha^n \subseteq B(z_\alpha^n, c_2 q^n)$.*
5. *With*

$$\partial_t Q_\alpha^n := \big\{ x \in Q_\alpha^n \colon d\big(x, X \backslash Q_\alpha^n\big) \leqslant t q^n \big\},$$

   we have

$$\forall n \in \mathbb{Z}, \ \forall \alpha \in \mathcal{K}_n \colon \quad \mu\big(\partial_t Q_\alpha^n\big) < c_3 t^\eta \mu\big(Q_\alpha^n\big).$$

6. *For all $n \in \mathbb{Z}$, the set $\mathcal{K}_n$ is countable.*
7. *For all $n \in \mathbb{Z}$ and all $\alpha \in \mathcal{K}_n$ we have $|\{\beta \in \mathcal{K}_{n-1} \colon Q_\beta^{n-1} \subseteq Q_\alpha^n\}| \leqslant N$.*
8. *For all $n \in \mathbb{Z}$, $\alpha \in \mathcal{K}_n$ there is a subset E of $\mathcal{K}_{n-1}$ with $|E| \leqslant N$ such that*

$$Q_\alpha^n = \bigcup_{\beta \in E} Q_\beta^{n-1} \ \text{up to } \mu\text{-null sets}.$$

Remark. We note that these dyadic cubes were constructed by Christ in [5] and by David in [8] in a slightly different way. We further remark that in the future use of the dyadic cubes, we neglect μ-null sets in points 1 and 8 of Theorem 3.3 and assume equality.

We now collect a few useful definitions, which we will need in the sequel.

Definition 3.4. We let

$$\mathcal{A}_n := \big\{ Q_\alpha^n \colon \alpha \in \mathcal{K}_n \big\},$$

be the set of dyadic cubes with level $n \in \mathbb{Z}$. Furthermore, let $A \in \mathcal{A}_{n+1}$ and choose $A^*(A) \in \mathcal{A}_n$ arbitrarily (but fixed for all subsequent sections) with $A^*(A) \subseteq A$. Then we set

$$\mathcal{E}(A) := \big\{ B \in \mathcal{A}_n \colon B \subseteq A \backslash A^*(A) \big\}.$$

We denote the cardinality $|\mathcal{E}(A)|$ of this set by $N(A)$. Additionally, we define the level of $A \in \mathcal{A}_{n+1}$ as

$$\operatorname{lev} A := n + 1.$$

The unique element $A \in \mathcal{A}_{n+1}$ such that for $Q \in \mathcal{A}_n$ we have $Q \subset A$ will be denoted by

$$\operatorname{pre} Q, \tag{3.2}$$

which indicates that A is the predecessor of Q. Furthermore, we define a subset of dyadic cubes

$$\mathcal{E}(\mathcal{A}) := \bigcup_{A \in \mathcal{A}} \mathcal{E}(A).$$

Remark. Due to point 7 of Theorem 3.3 we have that the cardinality $N(A)$ of $\mathcal{E}(A)$ is bounded by a uniform constant $N - 1$ independent of $A \in \mathcal{A}_{n+1}$.

3.3. Martingale differences

Let (X, d, μ) be a space of homogeneous type with $\mu(X) = 1$. Then we have $X = Q_1^0$, $\mathcal{K}_0 = \{1\}$, $\mathcal{A}_0 = \{X\}$ and $\mathcal{K}_n = \emptyset$ for all $n \in \mathbb{N}$. We use then dyadic cubes to build an orthonormal basis in $L^2(X, d, \mu)$ consisting of martingale differences. Fix $n \in -\mathbb{N}$, $A \in \mathcal{A}_{n+1}$ and enumerate the elements in $\mathcal{E}(A)$ in the way that $\mathcal{E}(A) = \{Q_1, \ldots, Q_{N(A)}\}$. Additionally we set $Q_{N(A)+1} := A^*(A)$. We define the following functions, supported on A.

Definition 3.5. We define for $1 \leqslant k \leqslant N(A)$ and $x \in X$

$$d_{Q_k}(x) := c_{Q_k} \begin{cases} 0, & \text{if } x \in \bigcup_{j=1}^{k-1} Q_j \cup (X \backslash A), \\ \sum_{j=k+1}^{N(A)+1} \mu(Q_j), & \text{if } x \in Q_k, \\ -\mu(Q_k), & \text{if } x \in \bigcup_{j=k+1}^{N(A)+1} Q_j, \end{cases}$$

where we choose c_{Q_k} such that

$$\|d_{Q_k}\|_2 = 1. \tag{3.3}$$

Remark. The functions defined in Definition 3.5 are obviously a martingale difference sequence. We record here also that these martingale differences are just the result of the Gram Schmidt orthogonalization process applied to the indicator functions

$$1_A, 1_{Q_1}, \ldots, 1_{Q_{N(A)}} \tag{3.4}$$
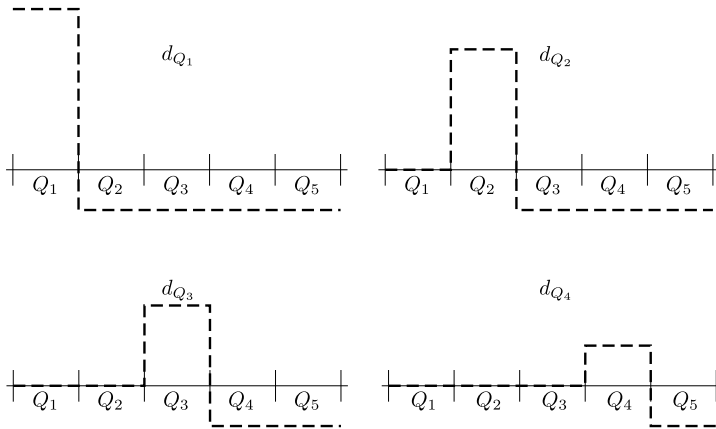
(see Fig. 1).

Fig. 1. Schematic plots of functions in Definition 3.5, where N is set to 5.

Now we enumerate all the functions d_Q, $Q \in \mathcal{E}(A)$ where $A \in \mathcal{A}_{n+1}$, $n \in -\mathbb{N}$ in a canonical way, we set

$$d_0 := 1_X$$

and get the functions that are a basis in the constant functions on $\{Q_1, \ldots, Q_{N(X)}, A^*(X)\}$, where $Q_i \in \mathcal{E}(X)$ for $1 \leqslant i \leqslant N(X)$ and set

$$d_1 = d_{Q_1}, \ldots, d_{N(X)} = d_{Q_{N(X)}}.$$

We continue with this procedure on every Q_i, so we get an enumeration of all functions d_Q, $Q \in \mathcal{E}(A)$ for $A \in \mathcal{A}_{n+1}$, $n \in -\mathbb{N}$ such that the order is preserved in the following way

$$k \leqslant j \quad \Rightarrow \quad \operatorname{lev} R \geqslant \operatorname{lev} Q \quad \text{for } d_k = d_R \text{ and } d_j = d_Q.$$

We refer to the functions d_Q as Haar functions. According to this enumeration we define σ-algebras:

$$\mathcal{F}_i := \sigma(d_0, \ldots, d_i) \quad \text{for } i \in \mathbb{N}_0.$$

With respect to this filtration, the collection $\{d_k\}_{k \in \mathbb{N}}$ is a martingale difference sequence, since we have $\mathbb{E}(d_k | \mathcal{F}_{k-1}) = 0$ for every $k \in \mathbb{N}$. Another important sequence of σ-algebras that we need later is a suitable subsequence of the σ-algebras just created. We set

$$\mathcal{F}_k^{\text{lev}} := \sigma(\mathcal{A}_{-k}) \quad \text{for } k \in \mathbb{N}_0, \tag{3.5}$$

where the superscript lev should indicate that $\mathcal{F}_k^{\text{lev}}$ is the σ-algebra generated by all dyadic cubes of level $-k$.

As in the case $X = \mathbb{R}$ with the standard Haar functions, we have that the L^∞ norm of an L^2 normalized Haar function d_Q is (approximately) $\mu(Q)^{-1/2}$, which is a simple consequence of Theorem 3.3 and the normality of X.

Lemma 3.6. *There exists a constant $c < \infty$ depending only on X such that*

$$c^{-1}\mu(Q)^{-1/2} \leqslant \|d_Q\|_\infty \leqslant c\mu(Q)^{-1/2} \quad \text{for all } Q \in \mathcal{E}(\mathcal{A}).$$

Another simple consequence of Theorem 3.3 is

Lemma 3.7. $\bigcup_{l=1}^{\infty} \mathcal{F}_l$ *generates the Borel σ-algebra on X.*

Remark. If E is a UMD-space, it is in particular reflexive and thus satisfies the Radon–Nikodym property. So, the martingale convergence theorem (see Chatterji [3]) and the above lemma yield that for $f \in L_E^p(X)$ we have that

$$\lim_{k \to \infty} \big\|\mathbb{E}(f|\mathcal{F}_k) - f\big\|_{L_E^p(X)} = 0$$

for all $1 \leqslant p < \infty$. So we get for every $f \in L_E^p(X)$ a unique series expansion

$$f = \sum_{k=0}^{\infty} a_k d_k, \quad a_k \in E,$$

which converges unconditionally in $L_E^p(X)$ for $1 < p < \infty$. In particular for $p = 2$ and $E = \mathbb{R}$, $(d_k)_{k \in \mathbb{N}}$ is an orthonormal basis.

3.4. Isotropic basis in $L^2(X \times X)$

Next we introduce an isotropic orthogonal basis in $L^2(X \times X)$. Here, the word isotropic means that for an element $f \otimes g$ of this basis (here, $f \otimes g(x, y) := f(x)g(y)$ is the standard tensor product of two functions), the support looks like a square and not like a rectangle. Most of the notation used in the sequel was introduced in Definition 3.4. Let $\varepsilon \in \{0, 1\}$. For $Q \in \mathcal{E}(A)$ and $A \in \mathcal{A}$ we define

$$d_Q^{(\varepsilon)} := d_Q \quad \text{for } \varepsilon = 1, \quad \text{and} \quad d_Q^{(\varepsilon)} := \frac{1_A}{\sqrt{\mu(A)}} \quad \text{for } \varepsilon = 0.$$

Note that the function $d_Q^{(0)}$ is L^2-normalized as is $d_Q^{(1)}$. With these settings, we define the collection of functions on $X \times X$:

$$Z := \{1_X \otimes 1_X\}$$
$$\cup \big\{d_Q^{(\varepsilon_1)} \otimes d_R^{(\varepsilon_2)} \colon Q, R \in \mathcal{E}(\mathcal{A}), \text{ lev } Q = \text{lev } R, \ \varepsilon = (\varepsilon_1, \varepsilon_2) \in \{0, 1\}^2 \backslash \{(0, 0)\}\big\}. \quad (3.6)$$

Explicitly, up to constants, the three groups in (3.6) have the form

$$\big\{d_Q \otimes d_R \colon A, B \in \mathcal{A}_{n+1}, \ Q \in \mathcal{E}(A), \ R \in \mathcal{E}(B), \ n \in -\mathbb{N}\big\}, \quad (3.7)$$

$$\big\{d_Q \otimes 1_B \colon A, B \in \mathcal{A}_{n+1}, \ Q \in \mathcal{E}(A), \ n \in -\mathbb{N}\big\}, \quad (3.8)$$

$$\big\{1_A \otimes d_R \colon A, B \in \mathcal{A}_{n+1}, \ R \in \mathcal{E}(B), \ n \in -\mathbb{N}\big\}. \quad (3.9)$$

The system Z forms an orthonormal basis in $L^2(X \times X)$ and this result follows from the well-known classical

Lemma 3.8. *If $\{e_k\}_{k=1}^\infty$ is an orthogonal basis in $L^2(X, \mathcal{F}, \mu)$, then $\{e_k \otimes e_j\}_{k,j=1}^\infty$ is an orthogonal basis in $L^2(X \times X, \mathcal{F} \otimes \mathcal{F}, \mu \otimes \mu)$.*

Lemma 3.9. *Z is an orthonormal basis in $L^2(X \times X)$.*

Proof. Since the verification of orthonormality is a straightforward calculation, we proceed with showing the basis property. Since we know from Lemma 3.8 that the set

$$\{d_S \otimes d_T \colon\ S \in \mathcal{A}_{m+1},\ T \in \mathcal{A}_{n+1},\ n, m \in -\mathbb{N}\} \quad \text{with } d_X := 1_X$$

is an orthogonal basis in $L^2(X \times X)$, we have to show that each $d_S \otimes d_T$ can be decomposed in a finite linear combination of functions of the form (3.7)–(3.9). To do that, we need the following identities:

$$1_U = 1_{A^*(U)} + \sum_{V \in \mathcal{E}(U)} 1_V, \quad U \in \mathcal{A}_{m+1}, \tag{3.10}$$

$$d_R = c_1 1_{A^*(B)} + \sum_{V \in \mathcal{E}(B)} c_V 1_V, \quad R \in \mathcal{E}(B),\ c_1, c_V \in \mathbb{R} \text{ for } V \in \mathcal{E}(B). \tag{3.11}$$

We then have four cases:

1. Let $d_S = 1_X$, $d_T = 1_X$, then clearly $d_S \otimes d_T \in Z$.
2. $d_S = 1_X$, $T \in \mathcal{A}_n, n \in -\mathbb{N}$, $B \in \mathcal{A}_{n+1}$ with $T \in \mathcal{E}(B)$. Then we get recursively from (3.10), that 1_X is a finite linear combination of functions of the form 1_C, where $C \in \mathcal{A}_{n+1}$. With (3.9), we see that $1_X \otimes d_T \in \lin Z$.
3. Analogously, we treat the case $d_T = 1_X$ and $d_S \neq 1_X$.
4. $S \in \mathcal{A}_n, T \in \mathcal{A}_m, m, n \in -\mathbb{N}$, $S \in \mathcal{E}(A), T \in \mathcal{E}(B), A \in \mathcal{A}_{n+1}, B \in \mathcal{A}_{m+1}$. If $m = n$, we see from (3.7) that $d_S \otimes d_T \in Z$. Without loss of generality we now assume that $m > n$ and we decompose d_T in the form (3.11). Additionally, if $m > n + 1$, we proceed recursively with (3.10) and get from (3.8) that $d_S \otimes d_T \in \lin Z$. $\square$

3.5. Dyadic annuli

Recall that $\mathcal{A}_n$ is the set of dyadic cubes of level n for $n \in -\mathbb{N}_0$ and $\mathcal{A}_0$ consists only of the whole space X and the size of cubes *decreases* with *decreasing* index n. We now introduce the set of all pairs of dyadic cubes of the same level

$$\mathcal{C} := \big\{(A, B)\colon\ A, B \in \mathcal{A}_n,\ n \in -\mathbb{N}_0\big\}$$

and its decomposition into annuli $\mathcal{C} = \bigcup_{m=0}^\infty \mathcal{C}_m$, where

$$\mathcal{C}_m = \big\{(A, B) \in \mathcal{C}\colon\ q^{m-1+\operatorname{lev} A} \leqslant d(A, B) < q^{m+\operatorname{lev} A}\big\} \quad \text{for } m \in \mathbb{N}$$

and

$$\mathcal{C}_0 = \left\{ (A, B) \in \mathcal{C} \colon d(A, B) < q^{\operatorname{lev} A} \right\}.$$

Recall that $\operatorname{lev} A$ denotes the level of A (that is if $A \in \mathcal{A}_n$, then $\operatorname{lev} A = n$) and q is the constant from Theorem 3.3 that determines the growth factor of the cubes in each level. This definition can be interpreted in the following way: Given $A \in \mathcal{A}_n$, we draw an annulus around A with inner radius $q^{m-1+\operatorname{lev} A}$ and outer radius $q^{m+\operatorname{lev} A}$ and take all pairs (A, B) such that B has no point inside the smaller circle and B has at least one point inside the larger circle. It is crucial that the annulus grows with the size of A.

3.6. Extracting rearrangements – further decomposition of annuli

The aim of this section is to extract (as few as possible) subcollections $\mathcal{C}_{m,i}$ from $\mathcal{C}_m$ such that for each $(A, B) \in \mathcal{C}_{m,i}$ we have that B is uniquely determined by A and A is uniquely determined by B. The benefit of this decomposition is that on $\mathcal{C}_{m,i}$ we can define an injective mapping τ such that $B = \tau(A)$ (see Definition 3.12). We start with the following observation:

Lemma 3.10. *There exists a constant M_0 independent of n and m, such that for $A \in \mathcal{A}_n$ there are at most $M_0 q^m$ elements $B \in \mathcal{A}_n$ with $(A, B) \in \mathcal{C}_m$.*

So, roughly speaking, in an annulus of level m around A, there are at most q^m cubes of the same size as A. This lemma is easily proved using the properties of dyadic cubes in Theorem 3.3 and the normality of X.

Remark. The same argument shows that for each $C > 0$ there exists a constant M_0 s.t. for $A \in \mathcal{A}_n$ we have at most M_0 elements $B \in \mathcal{A}_n$ with

$$d(A, B) \leqslant C q^n.$$

Proposition 3.11. *Let $M_1 := 2M_0$ with M_0 from Lemma 3.10. Then we have for all $m \in \mathbb{N}_0$ that the collection $\mathcal{C}_m \subseteq \mathcal{A} \times \mathcal{A}$ admits a decomposition as*

$$\mathcal{C}_m = \mathcal{C}_{m,1} \cup \cdots \cup \mathcal{C}_{m,M_1 q^m}$$

so that each of the collections $\mathcal{C}_{m,i}$, $1 \leqslant i \leqslant M_1 q^m$ satisfies the two conditions:

1. *For $B \in \mathcal{A}$ there exists at most one $A \in \mathcal{A}$ with $(A, B) \in \mathcal{C}_{m,i}$.*
2. *For $A \in \mathcal{A}$ there exists at most one $B \in \mathcal{A}$ with $(A, B) \in \mathcal{C}_{m,i}$.*

Remark. For the applications in Section 4 it is important that $\mathcal{C}_m$ is decomposed in $M_1 q^m$ sub-collections (and not more). For instance the estimate q^{2m} would be much simpler to obtain, but would not allow us to treat singular integral operators.

Proof. Step 1. Idea of the proof:

Let $Q \in \mathcal{A}$. Then we define the ring collection of Q:

$$\mathcal{O}_m(Q) := \{R \in \mathcal{A}: (Q, R) \in \mathcal{C}_m\}.$$

We will show that there exist $I(Q) \subseteq \{1, \ldots, M_1 q^m\} =: I$ and an enumeration of the dyadic cubes in $\mathcal{O}_m(Q)$ such that

$$\mathcal{O}_m(Q) = \{R_i(Q): i \in I(Q)\}$$

and we have the following property:

$$\forall Q, Q' \in \mathcal{A}, \ Q \neq Q', \ \forall j \in I(Q) \cap I(Q'): \quad R_j(Q) \neq R_j(Q'). \tag{3.12}$$

Then we can define the decomposition

$$\mathcal{A}_{m,i} = \{Q \in \mathcal{A}: i \in I(Q)\} \quad \text{and} \quad \mathcal{C}_{m,i} = \{(Q, R_i(Q)): Q \in \mathcal{A}_{m,i}\}.$$

We thus obtain

$$\mathcal{C}_m = \mathcal{C}_{m,1} \cup \cdots \cup \mathcal{C}_{m,M_1 q^m}$$

and the desired properties hold.

Step 2. Construction of the enumeration:

Let $\mathcal{A} = \{Q^{(k)}: k \in \mathbb{N}\}$ be an enumeration of all dyadic cubes. We proceed by induction over k. For $k = 1$ choose $I(Q^{(1)}) = \{1, \ldots, |\mathcal{O}_m(Q^{(1)})|\}$ and select any enumeration of the cubes $\mathcal{O}_m(Q^{(1)})$. Observe that with Lemma 3.10 we have that $|\mathcal{O}_m(Q^{(1)})| \leqslant M_0 q^m$. Now let $k \in \mathbb{N}$ and assume we have constructed

$$I(Q^{(1)}), \ldots, I(Q^{(k)})$$

with

$$\mathcal{O}_m(Q^{(l)}) = \{R_i(Q^{(l)}): i \in I(Q^{(l)})\} \quad \text{for } l \leqslant k$$

such that the following holds

$$\forall Q, Q' \in \{Q^{(1)}, \ldots, Q^{(k)}\}, \ Q \neq Q', \ \forall j \in I(Q) \cap I(Q'): \quad R_j(Q) \neq R_j(Q').$$

We will now construct $I(Q^{(k+1)})$. To do this we first set

$$\{R^{(1)}, \ldots, R^{(M_*)}\} = \mathcal{O}_m(Q^{(k+1)}), \quad \text{where } M_* \leqslant M_0 q^m.$$

Step 2a. We start a second induction and begin with $R^{(1)}$. We will define the index $\text{ind}_{Q^{(k+1)}} R^{(1)}$ of $R^{(1)}$ in the enumeration $\mathcal{O}_m(Q^{(k+1)})$ as follows. We put

$$V(R^{(1)}) = \{Q' \in \{Q^{(1)}, \ldots, Q^{(k)}\}: R^{(1)} \in \mathcal{O}_m(Q')\},$$

so $V(R^{(1)})$ contains the cubes Q' for which $R^{(1)}$ is in their ring collection $\mathcal{O}_m(Q')$. Now, since $V(R^{(1)}) \subseteq \mathcal{O}_m(R^{(1)})$, we have an estimate for the cardinality of $V(R^{(1)})$:

$$\left|V(R^{(1)})\right| \leqslant M_0 q^m. \tag{3.13}$$

For $Q' \in V(R^{(1)})$ we already defined the indices $\operatorname{ind}_{Q'} R^{(1)} \in I$. Next we let

$$L(R^{(1)}) = \left\{ \underset{Q'}{\operatorname{ind}}\, R^{(1)}: Q' \in V(R^{(1)}) \right\}$$

the indices of $R^{(1)}$ in the enumeration of Q'. According to (3.13), we have

$$\left|L(R^{(1)})\right| \leqslant M_0 q^m$$

and $|I| = M_1 q^m$. For the reduced index set, defined as

$$I^{\mathrm{red}} = I \backslash L(R^{(1)}),$$

we have

$$\left|I^{\mathrm{red}}\right| \geqslant M_1 q^m - M_0 q^m.$$

In particular, we have $I^{\mathrm{red}} \neq \emptyset$. So we select any element in I^{red} to be the index of $R^{(1)}$ for $Q^{(k+1)}$:

$$\underset{Q^{(k+1)}}{\operatorname{ind}}\, R^{(1)} \in I^{\mathrm{red}}.$$

Thus the beginning of the second induction is completed.

Step 2b. Next we fix $j < M_* \leqslant M_0 q^m$. We now assume that we already defined

$$\underset{Q^{(k+1)}}{\operatorname{ind}}\, R^{(1)}, \ldots, \underset{Q^{(k+1)}}{\operatorname{ind}}\, R^{(j)},$$

so we pick $R^{(j+1)} \in \mathcal{O}_m(Q^{(k+1)})$. As in the beginning of the induction, we set

$$V(R^{(j+1)}) = \{Q' \in \{Q^{(1)}, \ldots, Q^{(k)}\}: R^{(j+1)} \in \mathcal{O}_m(Q')\}.$$

We again have $V(R^{(j+1)}) \subseteq \mathcal{O}_m(R^{(j+1)})$ and thus an estimate for the cardinality

$$\left|V(R^{(j+1)})\right| \leqslant M_0 q^m.$$

Next let

$$L\big(R^{(j+1)}\big) = \Big\{\operatorname*{ind}_{Q'} R^{(j+1)} \colon \ Q' \in V\big(R^{(j+1)}\big)\Big\}$$

be the indices of $R^{(j+1)}$ in the enumeration of Q'. Since $|L(R^{(j+1)})| \leqslant M_0 q^m$, we have for the reduced index set

$$I^{\text{red}} = I \backslash \Big(L\big(R^{(j+1)}\big) \cup \Big\{ \operatorname*{ind}_{Q^{(k+1)}} R^{(1)}, \dots, \operatorname*{ind}_{Q^{(k+1)}} R^{(j)} \Big\} \Big)$$

an estimate for the cardinality

$$\big| I^{\text{red}} \big| > M_1 q^m - M_0 q^m - M_* \geqslant (M_1 - 2M_0)q^m,$$

so we have due to the definition of M_1 that $I^{\text{red}} \neq \emptyset$. We finally select then the index $\operatorname*{ind}_{Q^{(k+1)}} R^{(j+1)}$ to be any element from the reduced index set I^{red}.

Step 3. We summarize and set

$$R_i\big(Q^{(k+1)}\big) = R^{(j)} \quad \text{iff} \quad i = \operatorname*{ind}_{Q^{(k+1)}} R^{(j)}$$

and the index set

$$I\big(Q^{(k+1)}\big) = \Big\{ \operatorname*{ind}_{Q^{(k+1)}} \big(R^{(j)}\big) \colon \ R^{(j)} \in \mathcal{O}_m\big(Q^{(k+1)}\big) \Big\}.$$

It follows from the construction Step 2 that the enumeration R and the index sets $I(Q^{(k)})$ have the desired property (3.12). $\quad\square$

For $1 \leqslant i \leqslant M_1 q^m$ we recall the meaning of $\mathcal{A}_{m,i} \subseteq \mathcal{A}$, which was defined in the previous proof, as

$$\mathcal{A}_{m,i} = \big\{ A \in \mathcal{A} \colon \exists B \in \mathcal{A}, \ \text{such that} \ (A, B) \in \mathcal{C}_{m,i} \big\}.$$

Due to Proposition 3.11, we can define an injective mapping τ on $\mathcal{A}_{m,i}$:

Definition 3.12. We define

$$\tau : \mathcal{A}_{m,i} \to \mathcal{A},$$

$$A \mapsto \tau(A)$$

through the relation

$$\tau(A) = B \quad \text{iff} \quad (A, B) \in \mathcal{C}_{m,i}.$$

Additionally we get an inverse of τ on $\tau(\mathcal{A}_{m,i})$

$$\tau^{-1}(B) = A \quad \text{iff} \quad (A, B) \in \mathcal{C}_{m,i}.$$

3.7. Decomposition of $\mathcal{C}_{m,i}$ using arithmetic progressions

Proposition 3.13. *For all $C > 0$ there is a constant M that depends only on C and the space of homogeneous type X such that we have the decomposition*

$$\mathcal{C}_{m,i} = \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_M,$$

with the property that for all $1 \leqslant l \leqslant M$, $n \in -\mathbb{N}$, and all disjoint A_1, A_2 in $\mathcal{A}_n$ with

$$\big(A_1, \tau(A_1)\big) \in \mathcal{G}_l \quad and \quad \big(A_2, \tau(A_2)\big) \in \mathcal{G}_l$$

the following separation of these sets holds

$$d\big(\tau^i(A_1), \tau^j(A_2)\big) > Cq^n \quad for\ all\ i,\ j \in \{0, 1\}. \tag{3.14}$$

Here, $\tau^0(A) := A$ and $\tau^1(A) := \tau(A)$.

Proof. Let $\{(Q^{(k)}, \tau(Q^{(k)})): k \in \mathbb{N}\}$ be an enumeration of $\mathcal{C}_{m,i}$. Initialize the collections $\mathcal{G}_1, \ldots, \mathcal{G}_M$ as empty. For $k \in \mathbb{N}$, we inductively add $(Q^{(k)}, \tau(Q^{(k)}))$ to $\mathcal{G}_r$ for

$$r := \min\big\{i \in \mathbb{N}: \text{for all } \big(A_1, \tau(A_1)\big) \in \mathcal{G}_i \text{ we have (3.14) with } A_2 \text{ replaced by } Q^{(k)}\big\}.$$

Thus for $(A, \tau(A)) \in \mathcal{G}_{L+1}$ we have $(A, \tau(A)) \notin \mathcal{G}_l$ for all $l \leqslant L$ and so we have that for all $l \leqslant L$ there exists a pair $(A_l^0, \tau(A_l^0)) \in \mathcal{G}_l$ such that one of the four expressions

$$d\big(A, A_l^0\big), \qquad d\big(A, \tau(A_l^0)\big), \qquad d\big(\tau(A), A_l^0\big), \qquad d\big(\tau(A), \tau(A_l^0)\big)$$

is $\leqslant Cq^n$. According to the properties of the collection $\mathcal{C}_{m,i}$, the sets in the collection $\{A_l^0\}_{l=1}^L$ as well as the sets $\{\tau(A_l^0)\}_{l=1}^L$ are disjoint. So the remark after Lemma 3.10 yields that L can't be greater than $4(M_0 + 1)$ with M_0 depending only on C and on the space of homogeneous type X. This proves the proposition. $\quad\square$

We cannot guarantee that a dyadic A cube divides into $N(A) \geqslant 2$ subcubes, but nevertheless we have as a consequence of the normality of X:

Lemma 3.14. *There exists a constant L such that for every $l \geqslant L$ we have that $A \in \mathcal{A}_n$, $B \in \mathcal{A}_{n-l}$ imply that $A \neq B$.*

We now fix $\mathcal{G} = \mathcal{G}_l$ in the decomposition of Proposition 3.13 for some $l \leqslant M$ and introduce levels using arithmetic progressions. We set

$$\mathcal{A}_{\mathcal{G}} := \big\{A \in \mathcal{A}_{m,i}: \big(A, \tau(A)\big) \in \mathcal{G}\big\}$$

and the levels

$$\mathcal{L}_r = \mathcal{A}_{\mathcal{G}} \cap \bigcup_{l=0}^{\infty} \mathcal{A}_{-l \cdot L(m+1)-r}, \quad \text{where } 0 \leqslant r \leqslant L(m+1) - 1 \tag{3.15}$$

and $L \in \mathbb{N}$ is chosen in such a way that the condition of Lemma 3.14 is satisfied. We will later (in Section 3.7.4) give additional conditions on the constant L. Given a set $A \in \mathcal{L}_r$ we now define appropriate predecessors.

Definition 3.15. If $A \in \mathcal{A}_{-l \cdot L(m+1)-r}$, we define the *arithmetic predecessor*

$$\widetilde{A} \tag{3.16}$$

to be the unique element in $\mathcal{A}_{-(l-1) \cdot L(m+1)-r}$, such that $\widetilde{A} \supset A$.

This works only if $l \geqslant 1$. If $l = 0$, we simply set $\widetilde{A} := X$. We remark that for $A \in \mathcal{L}_r$ we have obviously

$$\widetilde{A} \in \bigcup_{l=0}^{\infty} \mathcal{A}_{-l \cdot L(m+1)-r} \quad \text{or} \quad \widetilde{A} = X,$$

but not necessarily that $\widetilde{A} \in \mathcal{A}_{\mathcal{G}}$, hence $\widetilde{A}$ need not be in $\mathcal{L}_r$.

Note that the dyadic predecessor of A, denoted pre A, defined in (3.2) does not coincide with the arithmetic predecessor $\widetilde{A}$ defined above.

Definition 3.16. Let $\mathcal{Z}$ be a collection of sets. $\mathcal{Z}$ is said to be *nested*, if for all $A, B \in \mathcal{Z}$ we have that either

$$A \cap B = \emptyset \quad \text{or} \quad A \subseteq B \quad \text{or} \quad B \subseteq A$$

holds.

The main result of this section is the following combinatorial theorem. It is the foundation of our work in the subsequent sections. It translates into norm estimates for rearrangement and shift operators in Section 4.3. The significance of Theorem 3.17 can be seen by examining the proof of T. Figiel [12]. To anticipate the notation used in the following theorem, we note that $\mathcal{H}$ will be the collection of cubes A such that $\tau(A)$ has the same arithmetic predecessor as A. $\mathcal{I}$ will be the collection of cubes A such that both A and $\tau(A)$ are well inside their arithmetic predecessors and the collection $\mathcal{J}$ consists of the rest, where we again divide into the cases where either A or $\tau(A)$ or both of them lie near the boundary of their arithmetic predecessors and call the corresponding collections $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$ respectively.

Theorem 3.17. *For $r \leqslant L(m+1) - 1$ and $\mathcal{L} = \mathcal{L}_r$ defined by (3.15), then for $\mathcal{L}$ there exists a decomposition*

$$\mathcal{L} = \mathcal{H} \cup \mathcal{I} \cup \mathcal{J},$$

such that:

1. *The collection*

$$\left\{ A, \tau(A), A \cup \tau(A) \colon A \in \mathcal{H} \right\}$$

   is nested.

2. $\mathcal{I}$ *admits a decomposition as* $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$, *so that the two collections*

$$\{A, \tau(A), A \cup \tau(A): A \in \mathcal{I}_j\} \quad \text{for } j \in \{1, 2\}$$

   are nested.
3. $\mathcal{J}$ *admits a decomposition as* $\mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3$ *such that we have*:
   (a) *There exists an injection* $\gamma_1 : \mathcal{J}_1 \cup \mathcal{J}_3 \to \mathcal{A}$ *such that the collection*

$$\{A, \gamma_1(A), A \cup \gamma_1(A): A \in \mathcal{J}_1\}$$

   is nested and in addition we have for $A \in \mathcal{J}_1$

$$\gamma_1(A) \subseteq \widetilde{A}, \quad d(\gamma_1(A), \complement \widetilde{A}) \geqslant q^{\operatorname{lev} A} \quad \text{and} \quad d(\tau(A), \complement \widetilde{\tau(A)}) \geqslant q^{\operatorname{lev} A}.$$

   (b) *There exists an injection* $\gamma_2 : \mathcal{J}_2 \cup \mathcal{J}_3 \to \mathcal{A}$ *such that the collection*

$$\{\tau(A), \gamma_2(\tau(A)), \tau(A) \cup \gamma_2(\tau(A)): A \in \mathcal{J}_2\}$$

   is nested and in addition we have for $A \in \mathcal{J}_2$

$$\gamma_2(\tau(A)) \subseteq \widetilde{\tau(A)}, \quad d(\gamma_2(\tau(A)), \complement \widetilde{\tau(A)}) \geqslant q^{\operatorname{lev} A} \quad \text{and} \quad d(A, \complement \widetilde{A}) \geqslant q^{\operatorname{lev} A}.$$

   (c) *For* $\mathcal{J}_3$ *and the injections* γ_1 *and* γ_2 *defined in* (a) *and* (b), *we have for* $A \in \mathcal{J}_3$

$$d(\gamma_1(A), \complement \widetilde{A}) \geqslant q^{\operatorname{lev} A} \quad \text{and} \quad d(\gamma_2(\tau(A)), \complement \widetilde{\tau(A)}) \geqslant q^{\operatorname{lev} A}.$$

   Additionally, the two collections

$$\{A, \gamma_1(A), A \cup \gamma_1(A): A \in \mathcal{J}_3\} \quad \text{and} \quad \{\tau(A), \gamma_2(\tau(A)), \tau(A) \cup \gamma_2(\tau(A)): A \in \mathcal{J}_3\}$$

   are nested.

The proof of this theorem is divided into four basic steps.

Step 1 *(Subsection 3.7.1).* We give the definition of the decomposition of $\mathcal{L}$ into $\mathcal{H}, \mathcal{I}, \mathcal{J}$ and we further define the decomposition of $\mathcal{I}$ into $\mathcal{I}_1, \mathcal{I}_2$ and also the decomposition of $\mathcal{J}$ into $\mathcal{J}_1$, $\mathcal{J}_2, \mathcal{J}_3$.

Step 2 *(Subsection 3.7.2).* We verify that $\mathcal{H}$ satisfies condition 1 of Theorem 3.17.

Step 3 *(Subsection 3.7.3).* We verify that $\mathcal{I}_1, \mathcal{I}_2$ satisfy condition 2 of Theorem 3.17. This involves a two-coloring of $\mathcal{I}$ and an application of the argument in Step 2.

Step 4 *(Subsection 3.7.4).* We first define the injections γ_1, γ_2 and verify condition 3 of the theorem. Here we use reduction to the arguments introduced in Steps 2 and 3.

3.7.1. Definition of the decomposition of $\mathcal{L}$

Fix $A \in \mathcal{L}$. We make the following case distinction:

1. If $\widetilde{A} = \widetilde{\tau(A)}$, we add A to $\mathcal{H}$.
2. If $\widetilde{A} \cap \widetilde{\tau(A)} = \emptyset$, we let
   (a) $A \in \mathcal{I}$, if the values of $d(A, \complement\widetilde{A})$ and $d(\tau(A), \complement\widetilde{\tau(A)})$ are both greater or equal $q^{\mathrm{lev}\, A}$,
   (b) $A \in \mathcal{J}$, if one of the values $d(A, \complement\widetilde{A})$ or $d(\tau(A), \complement\widetilde{\tau(A)})$ is less than $q^{\mathrm{lev}\, A}$.

For the case 2(a) we define the following collections: Take any $A \in \mathcal{I} \cup \tau(\mathcal{I})$, define

$$\mathcal{P}(A) := \big\{ B \in \mathcal{I}:\ \mathrm{lev}\, B < \mathrm{lev}\, A,\ \big[(B \cap A \neq \emptyset \wedge \tau(B) \cap \complement A \neq \emptyset) $$
$$\vee \big(B \cap \complement A \neq \emptyset \wedge \tau(B) \cap A \neq \emptyset \big) \big] \big\}$$

and set

$$\mathcal{R}(A) := \big\{ J, \tau(J):\ J \in \mathcal{P}(A) \big\}.$$

The purpose of the collection $\mathcal{P}(A)$ is that we get rid of overlappings that occur if we define a two-coloring on $\mathcal{I}$ (say with the colors black and white) and set

$$\mathcal{I}_1 := \{A \in \mathcal{I}:\ \mathrm{color}\, A = \mathrm{black}\}, \qquad \mathcal{I}_2 := \{A \in \mathcal{I}:\ \mathrm{color}\, A = \mathrm{white}\}.$$

This two-coloring will have the crucial property that if $A \in \mathcal{I}$ is white, then every element in $\mathcal{P}(A)$ is black. At last, we define a decomposition of $\mathcal{J}$ and let

$$\mathcal{J}_1 := \big\{ A \in \mathcal{J}:\ d(A, \complement\widetilde{A}) < q^{\mathrm{lev}\, A} \text{ and } d\big(\tau(A), \complement\widetilde{\tau(A)}\big) \geqslant q^{\mathrm{lev}\, A} \big\},$$
$$\mathcal{J}_2 := \big\{ A \in \mathcal{J}:\ d(A, \complement\widetilde{A}) \geqslant q^{\mathrm{lev}\, A} \text{ and } d\big(\tau(A), \complement\widetilde{\tau(A)}\big) < q^{\mathrm{lev}\, A} \big\},$$
$$\mathcal{J}_3 := \mathcal{J} \backslash (\mathcal{J}_1 \cup \mathcal{J}_2).$$

3.7.2. The collection $\mathcal{H}$

We first analyze the collection $\mathcal{H}$, which is simpler to handle than $\mathcal{I}$ and $\mathcal{J}$.

Lemma 3.18. *The collection* $\{A, \tau(A), A \cup \tau(A):\ A \in \mathcal{H}\}$ *is nested.*

Proof. Let $A, B \in \mathcal{H}$ with $A \neq B$. It suffices to look at the pairs $(A, B \cup \tau(B))$, $(\tau(A), B \cup \tau(B))$, $(A \cup \tau(A), B \cup \tau(B))$, since the other cases are trivial (this is the case if both elements in the pair are dyadic cubes themselves) or considered by symmetry (as for example the pair $(B, A \cup \tau(A))$). We begin with $(A, B \cup \tau(B))$:

We assume

$$A \cap \big(B \cup \tau(B) \big) \neq \emptyset. \tag{3.17}$$

Then we have to show that either $A \subseteq B \cup \tau(B)$ or $B \cup \tau(B) \subseteq A$. We have (3.17) if and only if

$$A \cap B \neq \emptyset \quad \text{or} \quad A \cap \tau(B) \neq \emptyset.$$

For $A \cap B \neq \emptyset$ we have the three possibilities

$$A = B \quad \text{or} \quad A \subset B \quad \text{or} \quad B \subset A,$$

where $\subset$ denotes a strict inclusion. Indeed these are the only cases that can happen, since A and B are dyadic cubes. But $A = B$ is impossible, since we assumed $A \neq B$. If $A \subset B$, we clearly have $A \subseteq B \cup \tau(B)$. If $B \subset A$, it holds also that

$$\widetilde{B} \subseteq A.$$

This yields $\tau(B) \subseteq A$, since $\widetilde{B} = \widetilde{\tau(B)}$. So $B \cup \tau(B) \subseteq A$. For the case $A \cap \tau(B) \neq \emptyset$, analogous arguments complete the analysis of the pair $(A, B \cup \tau(B))$.

The pair $(\tau(A), B \cup \tau(B))$ is then treated in the same manner.

We now come to $(A \cup \tau(A), B \cup \tau(B))$: Again, we have to consider a few cases. First we assume that

$$\big(A \cup \tau(A)\big) \cap \big(B \cup \tau(B)\big) \neq \emptyset.$$

This is the case if and only if

$$A \cap B \neq \emptyset \quad \text{or} \quad A \cap \tau(B) \neq \emptyset \quad \text{or} \quad \tau(A) \cap B \neq \emptyset \quad \text{or} \quad \tau(A) \cap \tau(B) \neq \emptyset.$$

These four cases are treated in the same way as above. □

3.7.3. The collection $\mathcal{I}$

Lemma 3.19. *For each $B \in \mathcal{I} \cup \tau(\mathcal{I})$ there exists at most one $A \in \mathcal{I} \cup \tau(\mathcal{I})$ such that*

$$B \in \mathcal{R}(A).$$

Proof. Let $A_1, A_2 \in \mathcal{I} \cup \tau(\mathcal{I})$ with $A_1 \neq A_2$ such that

$$B \in \mathcal{R}(A_1) \quad \text{and} \quad B \in \mathcal{R}(A_2).$$

We split the proof into two parts. Part (a) treats the case $\operatorname{lev} A_1 = \operatorname{lev} A_2$ and part (b) treats the case $\operatorname{lev} A_1 < \operatorname{lev} A_2$, which is the general case, since we can always exchange A_1 and A_2. We additionally assume $B \in \mathcal{I}$, since the argument is symmetric if we assume $B \in \tau(\mathcal{I})$.

(a) We first treat the case $\operatorname{lev} A_1 = \operatorname{lev} A_2$. Here we get from the definition of $\mathcal{I}$ and from Proposition 3.13 that $d(A_1, A_2) > q^{\operatorname{lev} A_1}$ and $\operatorname{lev} B \leqslant \operatorname{lev} A_1 - L(m+1)$. Again we distinguish two cases. In view of the fact that $B \in \mathcal{R}(A_1)$, we split to (i) $B \cap A_1 \neq \emptyset$ and (ii) $\tau(B) \cap A_1 \neq \emptyset$.

(i) With $B \cap A_1 \neq \emptyset$ it holds that $B \subset A_1$ and so $B \cap A_2 = \emptyset$. Thus we have

$$d\big(A_1, \tau(B)\big) \leqslant d\big(B, \tau(B)\big) \leqslant q^{\operatorname{lev} B + m} \leqslant q^{\operatorname{lev} A_1}.$$

From these facts we infer that $\tau(B) \not\subset A_2$ and that implies $\tau(B) \cap A_2 = \emptyset$, which contradicts the assumption $B \in \mathcal{R}(A_2)$.

(ii) If $\tau(B) \cap A_1 \neq \emptyset$, this leads to $\tau(B) \subset A_1$ and $\tau(B) \cap A_2 = \emptyset$. Analogously to the above we get

$$d(A_1, B) \leqslant d\big(\tau(B), B\big) \leqslant q^{\operatorname{lev} B + m} \leqslant q^{\operatorname{lev} A_1}.$$

This implies $B \cap A_2 = \emptyset$, which contradicts $B \in \mathcal{R}(A_2)$.

(b) Now we assume without loss of generality that $\operatorname{lev} A_1 < \operatorname{lev} A_2$. Here we consider the two cases (i) $A_1 \subset A_2$ and (ii) $A_1 \cap A_2 = \emptyset$.

(i) For $A_1 \subset A_2$ we have by definition of $\mathcal{I}$

$$d(A_1, \complement A_2) \geqslant d(A_1, \complement \widetilde{A_1}) > q^{\operatorname{lev} A_1}. \tag{3.18}$$

Like in case (a) we have to consider the two cases $B \cap A_1 \neq \emptyset$ and $\tau(B) \cap A_1 \neq \emptyset$. We proceed with $B \cap A_1 \neq \emptyset$. (The case $\tau(B) \cap A_1 \neq \emptyset$ works analogously.) So it follows that $B \subset A_1$ and so $B \subset A_2$. We have the estimate

$$d\big(A_1, \tau(B)\big) \leqslant d\big(B, \tau(B)\big) \leqslant q^{m + \operatorname{lev} B} \leqslant q^{\operatorname{lev} A_1}. \tag{3.19}$$

Now it follows from (3.18) and (3.19) that $\tau(B) \not\subset \complement A_2$, i.e. $\tau(B) \subset A_2$. This contradicts $B \in \mathcal{R}(A_2)$.

(ii) Let $A_1 \cap A_2 = \emptyset$. In that case we have

$$d(A_1, A_2) \geqslant d(A_1, \complement \widetilde{A_1}) > q^{\operatorname{lev} A_1}.$$

If $B \cap A_1 \neq \emptyset$ (the other case $\tau(B) \cap A_1 \neq \emptyset$ is treated analogously), it follows that $B \subset A_1$ and hence $B \subset \complement A_2$. We have

$$d\big(A_1, \tau(B)\big) \leqslant d\big(B, \tau(B)\big) \leqslant q^{m + \operatorname{lev} B} \leqslant q^{\operatorname{lev} A_1}.$$

Thus we get

$$\tau(B) \not\subset A_2$$

and thus $\tau(B) \cap A_2 = \emptyset$. This identity together with $B \subset \complement A_2$ contradicts $B \in \mathcal{R}(A_2)$. This finishes the proof. $\square$

Lemma 3.19 allows us to introduce the announced two-coloring on $\mathcal{I} \cup \tau(\mathcal{I})$ with the colors black and white that satisfies the following three conditions:

1. For each $A \in \mathcal{I} \cup \tau(\mathcal{I})$ the collection $\mathcal{R}(A)$ is monochromatic.
2. If the color of $A \in \mathcal{I} \cup \tau(\mathcal{I})$ is already determined, then each $B \in \mathcal{R}(A)$ satisfies

$$\operatorname{color}(B) \neq \operatorname{color}(A).$$

3. For each $A \in \mathcal{I}$,

$$\text{color}(A) = \text{color}(\tau(A)).$$

Define

$$\mathcal{I}_1 = \{A \in \mathcal{I}: \text{color}(A) = \text{white}\} \quad \text{and} \quad \mathcal{I}_2 = \{A \in \mathcal{I}: \text{color}(A) = \text{black}\}.$$

Lemma 3.20. *If $A \in \mathcal{I}$ and $B \notin \mathcal{P}(A)$ with* lev $B <$ lev A, *then*

$$B \cup \tau(B) \subseteq A \quad or \quad B \cup \tau(B) \subseteq \complement A.$$

Proof. This is nothing else but a logical manipulation of the definition of $\mathcal{P}(A)$. □

Lemma 3.21. *The two subcollections*

$$\{A, \tau(A), A \cup \tau(A): A \in \mathcal{I}_j\} \quad for \; j \in \{1, 2\}$$

are nested.

Proof. Let $A, B \in \mathcal{I}_j$ for $j \in \{1, 2\}$. We consider the three pairs (a) $(A, B \cup \tau(B))$, (b) $(\tau(A), B \cup \tau(B))$, (c) $(A \cup \tau(A), B \cup \tau(B))$.

(a) We have to show that either

$$A \cap (B \cup \tau(B)) = \emptyset \quad \text{or} \quad A \subseteq B \cup \tau(B) \quad \text{or} \quad B \cup \tau(B) \subseteq A. \tag{3.20}$$

We consider the three cases (i) lev $A =$ lev B, (ii) lev $A <$ lev B and (iii) lev $B <$ lev A:

(i) This is clear, since A and B are dyadic cubes.
(ii) If $A \in \mathcal{P}(B)$ then either A or B is not in $\mathcal{I}_j$; if $A \notin \mathcal{P}(B)$, then due to Lemma 3.20 we have

$$A \cup \tau(A) \subseteq B \quad \text{or} \quad A \cup \tau(A) \subseteq \complement B.$$

In the first case, clearly, $A \subseteq B \cup \tau(B)$. In the second case $A \subseteq \complement B$ and so

$$A \cap (B \cup \tau(B)) = A \cap \tau(B) = \begin{cases} A, & \text{if } A \cap \tau(B) \neq \emptyset, \\ \emptyset, & \text{else.} \end{cases}$$

Both branches lead to one of the alternatives in (3.20).
(iii) Analogous to (even simpler than) case (ii).

(b) Analogous to (a).
(c) We have to show that either

$$(A \cup \tau(A)) \cap (B \cup \tau(B)) = \emptyset \quad \text{or} \quad A \cup \tau(A) \subseteq B \cup \tau(B)$$

$$\text{or} \quad B \cup \tau(B) \subseteq A \cup \tau(A). \tag{3.21}$$

We consider the two cases (i) lev $A = $ lev B, (ii) lev $A < $ lev B:

(i) Since A and B are in a collection $\mathcal{G}$, we have that $d(\tau(A), B)$ and $d(A, \tau(B))$ are greater than $q^{\text{lev } A}$, and so

$$\big(A \cup \tau(A)\big) \cap \big(B \cup \tau(B)\big) = \emptyset.$$

(ii) If $A \in \mathcal{P}(B)$ then either A or B is not in $\mathcal{I}_j$. If $A \notin \mathcal{P}(B)$ we get with Lemma 3.20 that either

$$A \cup \tau(A) \subseteq B \quad \text{or} \quad A \cup \tau(A) \subseteq \complement B.$$

In the first case, clearly, $A \cup \tau(A) \subseteq B \cup \tau(B)$. In the second case we get from part (b) of the lemma that for

$$\big(A \cup \tau(A)\big) \cap \tau(B)$$

we only have the three possibilities $\emptyset$, $A \cup \tau(A)$ or $\tau(B)$. The former two lead to $(A \cup \tau(A)) \cap (B \cup \tau(B)) = \emptyset$ and $A \cup \tau(A) \subseteq B \cup \tau(B)$ respectively. The third one gives

$$\tau(B) \subseteq A \cup \tau(A),$$

which is not possible (cf. Lemma 3.14). $\quad\square$

Remark. We remark that this decomposition of $\mathcal{I}$ into $\mathcal{I}_1$ and $\mathcal{I}_2$, in particular the proof of Lemma 3.19, does not depend on the explicit form of $\mathcal{I}$ and, what is even more important, the corresponding injection τ. In fact, the same proof works if there exists a constant C_R such that the following conditions are satisfied:

1. τ is an injection on $\mathcal{I}$ that lev $A = $ lev $\tau(A)$ whenever $A \in \mathcal{I}$,
2. for every $A \in \mathcal{I}$, $\widetilde{A} \cap \widetilde{\tau(A)} = \emptyset$,
3. for $A \in \mathcal{I}$, we have that $\min(d(A, \complement\widetilde{A}), d(\tau(A), \complement\widetilde{\tau(A)})) > C_R q^{\text{lev } A}$,
4. for $A, B \in \mathcal{I}$ with lev $A = $ lev B, it holds that $d(\tau^j(A), \tau^i(B)) > C_R q^{\text{lev } A}$ for $i, j \in \{0, 1\}$,
5. for $A \in \mathcal{I}$, we have that $\max(d(\widetilde{A}, \tau(A)), d(\widetilde{\tau(A)}, A)) \leqslant C_R q^{\text{lev } \widetilde{A}}$,
6. for two distinct sets A, B in $\mathcal{I}$ such that lev $A > $ lev B, we have that lev $A \geqslant $ lev $B + L(m+1)$.

3.7.4. The collection $\mathcal{J}$

Lemma 3.22. *There exists a constant C_2 such that for all $A \in \mathcal{A}_n$ and every $l \in \mathbb{N}$, the number Y_l^A of sets B in $\mathcal{A}_{n-l}$, for which we have $B \subseteq A$, is bounded from below by*

$$C_2 q^l.$$

Proof. If we use the normality of X and point 4 of Theorem 3.3 the conclusion of the lemma follows from the subsequent chain of inequalities:

$$\frac{b_1 c_1}{b_2 c_2} q^n \leqslant \frac{1}{b_2 c_2} \mu(A) = \frac{1}{b_2 c_2} \sum_{B \subset A, \, B \in \mathcal{A}_{n-l}} \mu(B)$$

$$\leqslant \sum_{B \subset A, \, B \in \mathcal{A}_{n-l}} q^{n-l} = Y_l^A q^{n-l}. \qquad \square$$

Now, recall the definition of the boundary layer $\partial_t A$ of a cube A with level n, which we defined as

$$\partial_t A = \{ x \in A : d(x, X \backslash A) \leqslant t q^n \}.$$

Additionally, due to Theorem 3.3, the measure of $\partial_t A$ admits the following upper bound

$$\mu(\partial_t A) < c_3 t^\eta \mu(A)$$

for some universal constants $c_3, \eta > 0$.

Lemma 3.23. *There exists a constant C_3 such that for all $A \in \mathcal{A}_n$ and every $l \in \mathbb{N}$, the number X_l^A of sets $B \in \mathcal{A}_{n-l}$ for which we have*

$$B \cap \partial_{q^{-l}} A \neq \emptyset$$

is bounded from above by

$$C_3 q^{l(1-\eta)}.$$

Proof. It is a simple consequence of the quasi-triangle inequality that there exists $d \geqslant 1$ depending only on X such that if $B \in \mathcal{A}_{n-l}$ we have

$$B \cap \partial_{q^{-l}} \neq \emptyset \quad \Rightarrow \quad B \subset \partial_{d q^{-l}} A.$$

With this fact, the normality of X and Theorem 3.3, points 4 and 5, the conclusion of the lemma follows from the subsequent chain of inequalities:

$$X_l^A q^{n-l} \leqslant b_2 c_2 \sum_{B \subset \partial_{d q^{-l}} A, \, B \in \mathcal{A}_{n-l}} q^{n-l} \leqslant \frac{1}{b_1 c_1} \sum_{B \subset \partial_{d q^{-l}} A, \, B \in \mathcal{A}_{n-l}} \mu(B)$$

$$\leqslant \frac{1}{b_1 c_1} \mu(\partial_{d q^{-l}} A) \leqslant c_3 \frac{1}{b_1 c_1} d^\eta q^{-l\eta} \mu(A) \leqslant c_3 \frac{b_2 c_2}{b_1 c_1} d^\eta q^{-l\eta} q^n. \qquad \square$$

In view of the above two lemmas and Lemma 3.14, we can choose L in (3.15) large enough that for all $l \geqslant L$, $A \in \mathcal{A}_n$ and $B \in \mathcal{A}_{n-l}$ we don't have

$$A = B,$$

and in addition that the quotient $\frac{Y_L^A}{X_L^A}$ admits the bound

$$\frac{Y_L^A}{X_L^A} > 2.$$

This property is crucial, since it enables us to define an injection $\gamma_1 : \mathcal{J}_1 \cup \mathcal{J}_3 \to \mathcal{A}$, such that we have $\gamma_1(A) \subseteq \widetilde{A}$ and we have moved away from the boundary of $\widetilde{A}$:

$$d\big(\gamma_1(A), \complement\widetilde{A}\big) \geqslant q^{\mathrm{lev}\,A}.$$

We extend γ_1 to $\mathcal{J}_1 \cup \mathcal{J}_3 \cup \gamma_1(\mathcal{J}_1) \cup \gamma_1(\mathcal{J}_3)$ and define for $A \in \gamma_1(\mathcal{J}_1) \cup \gamma_1(\mathcal{J}_3)$ that

$$\gamma_1(A) := \gamma_1^{-1}(A).$$

It as now a straightforward consequence of the definitions that the following holds:

Lemma 3.24. *The collections*

$$\big\{A, \gamma_1(A), A \cup \gamma_1(A) \colon A \in \mathcal{J}_1\big\} \quad \textit{and} \quad \big\{A, \gamma_1(A), A \cup \gamma_1(A) \colon A \in \mathcal{J}_3\big\}$$

are nested.

Proof. The proof is in fact nothing else than the proof of Lemma 3.18 with τ replaced by γ_1. $\quad\square$

Analogously, we define an injection $\gamma_2 : \tau(\mathcal{J}_2) \cup \tau(\mathcal{J}_3) \to \mathcal{A}$, such that we have $\gamma_2(\tau(A)) \subseteq \widetilde{\tau(A)}$ and

$$d\big(\gamma_2\big(\tau(A)\big), \complement\widetilde{\tau(A)}\big) \geqslant q^{\mathrm{lev}\,A}$$

and extend it to $\tau(\mathcal{J}_2) \cup \tau(\mathcal{J}_3) \cup \gamma_2(\tau(\mathcal{J}_2)) \cup \gamma_2(\tau(\mathcal{J}_3))$ by defining for $A \in \gamma_2(\tau(\mathcal{J}_2)) \cup \gamma_2(\tau(\mathcal{J}_3))$:

$$\gamma_2(A) := \gamma_2^{-1}(A).$$

Again it follows as in Lemma 3.24 that

Lemma 3.25. *The collections*

$$\big\{\tau(A), \gamma_2\big(\tau(A)\big), \tau(A) \cup \gamma_2\big(\tau(A)\big) \colon A \in \mathcal{J}_2\big\}$$

and

$$\big\{\tau(A), \gamma_2\big(\tau(A)\big), \tau(A) \cup \gamma_2\big(\tau(A)\big) \colon A \in \mathcal{J}_3\big\}$$

are nested.

We can now summarize our considerations and thus prove our main theorem in this section:

Proof of Theorem 3.17. The collections $\{A, \tau(A), A \cup \tau(A): A \in \mathcal{H}\}$ and $\{A, \tau(A), A \cup \tau(A): A \in \mathcal{I}_i\}$ for $i \in \{1, 2\}$ are nested by Lemmas 3.18 and 3.21 respectively. Lemmas 3.24 and 3.25 yield that the collections $\{A, \gamma_1(A), A \cup \gamma_1(A): A \in \mathcal{J}_1\}$, $\{A, \gamma_1(A), A \cup \gamma_1(A): A \in \mathcal{J}_3\}$, $\{\tau(A), \gamma_2(\tau(A)), \tau(A) \cup \gamma_2(\tau(A)): A \in \mathcal{J}_2\}$ and $\{\tau(A), \gamma_2(\tau(A)), \tau(A) \cup \gamma_2(\tau(A)): A \in \mathcal{J}_3\}$ are nested. The additional properties of the mappings γ_1 and γ_2 follow from the definition. We have thus completely proved the theorem. $\quad\square$

4. Decomposing singular integral operators

In this section we decompose singular integral operators as absolutely convergent series of simple rearrangements, shifts and two paraproducts.

4.1. Integral operators

We now define the integral operator K with the kernel $k: X \times X \to \mathbb{R}$, $k \in L^2(X \times X)$ by

$$K(f)(x) := \int_X k(x, y) f(y) \, d\mu(y)$$

for $f \in L^2_E(X)$ and E is a UMD Banach space. We assume structural estimates on k, in particular a strong off-diagonal decay and also a weak boundedness estimate on the diagonal. This is formalized with the following definition using the orthonormal basis from Lemma 3.9. First recall that q was the number with that $q^{\text{lev} A}$ represents roughly the "size" of A.

Definition 4.1. Let $k \in L^2(X \times X)$. We say that k is an *admissible kernel* if there exist $C_S > 0$ and $\delta > 0$ such that $|\langle k, 1_X \otimes 1_X \rangle| \leqslant C_S$ and for all $Q, R \in \mathcal{E}(\mathcal{A})$ with $\text{lev } Q = \text{lev } R$ we have

$$\left| \langle k, d_Q^{(\varepsilon_1)} \otimes d_R^{(\varepsilon_2)} \rangle \right| \leqslant C_S \left(1 + \frac{d(\text{pre } Q, \text{pre } R)}{q^{\text{lev } Q+1}} \right)^{-1-\delta},$$

$$\varepsilon = (\varepsilon_1, \varepsilon_2) \in \{0, 1\}^2 \backslash \{(0, 0)\}. \tag{4.1}$$

Remark. The condition on the Haar coefficients of the kernel k in the previous definition corresponds to Calderón–Zygmund kernels. We remark that in the following (see especially (4.27)) we only need in (4.1) the strictly weaker condition that

$$\left| \langle k, d_Q^{(\varepsilon_1)} \otimes d_R^{(\varepsilon_2)} \rangle \right| \leqslant \Phi \left(1 + \frac{d(\text{pre } Q, \text{pre } R)}{q^{\text{lev } Q+1}} \right)$$

for some positive decreasing function Φ on $\mathbb{R}^+$ satisfying

$$\sum_{n \in \mathbb{N}} \Phi(n) \log n < \infty.$$

The Calderón–Zygmund class corresponds to the special choice $\Phi(t) = t^{-1-\delta}$.

In this section we provide vector valued norm estimates for integral operators defined by admissible kernels. We point out that the L^p-norm of the integral operators depends just on the structural constants C_S and δ, the value of p and the BMO-norms of $K(1)$, $K^*(1)$. In particular, the L^2-norm of k is not present in the estimates. From now on, we work with admissible kernels k. We expand the kernel k in the isotropic orthonormal basis introduced in Section 3.4. The division of Z into three groups (see (3.6)) gives rise to the following decomposition of the kernel k. We let

$$k_1 := \sum_{n=0}^{\infty} \sum_{A,B \in \mathcal{A}_{-n}} \sum_{Q \in \mathcal{E}(A)} \sum_{R \in \mathcal{E}(B)} \langle k, d_Q \otimes d_R \rangle d_Q \otimes d_R,$$

$$k_2 := \sum_{n=0}^{\infty} \sum_{A,B \in \mathcal{A}_{-n}} \sum_{Q \in \mathcal{E}(A)} \frac{\langle k, d_Q \otimes 1_B \rangle}{\mu(B)} d_Q \otimes 1_B,$$

$$k_3 := \sum_{n=0}^{\infty} \sum_{A,B \in \mathcal{A}_{-n}} \sum_{R \in \mathcal{E}(B)} \frac{\langle k, 1_A \otimes d_R \rangle}{\mu(A)} 1_A \otimes d_R.$$

If we decompose k into the isotropic orthonormal basis we see that

$$k = \int\limits_X \int\limits_X k(s,t) \, d\mu(s) \, d\mu(t) + k_1 + k_2 + k_3.$$

We note the following identities (which follow from $X = \bigcup_{A \in \mathcal{A}_{-n}} A$ for every $n \in \mathbb{N}_0$)

$$\sum_{B \in \mathcal{A}_{-n}} \langle k, d_Q \otimes 1_B \rangle = \langle K(1), d_Q \rangle, \qquad \sum_{A \in \mathcal{A}_{-n}} \langle k, 1_A \otimes d_R \rangle = \langle K^*(1), d_R \rangle.$$

Now we let $f \in L^p_E(X)$ and $g \in L^q_{E'}(X)$ be finite linear combinations of Haar functions and E be a UMD-space. Then we see that k_2 has the further decomposition

$$\langle k_2, g \otimes f \rangle = B_2(f,g) + \widetilde{B}_2(f,g), \tag{4.2}$$

where

$$B_2(f,g) := \sum_{n=0}^{\infty} \sum_{A,B \in \mathcal{A}_{-n}} \sum_{Q \in \mathcal{E}(A)} \frac{\langle k, d_Q \otimes 1_B \rangle}{\mu(B)} \left\langle d_Q \otimes \left(1_B - \frac{\mu(B)}{\mu(A)} 1_A \right), g \otimes f \right\rangle \tag{4.3}$$

and

$$\widetilde{B}_2(f,g) := \sum_{n=0}^{\infty} \sum_{A \in \mathcal{A}_{-n}} \sum_{Q \in \mathcal{E}(A)} \frac{\langle K(1), d_Q \rangle}{\mu(A)} \langle d_Q \otimes 1_A, g \otimes f \rangle. \tag{4.4}$$

We also decompose k_3 further and get the following identity, which is valid in $L^2_E(X)$

$$\int\limits_X k_3(x, y) f(y) \, d\mu(y) = K_3 f(x) + \widetilde{K}_3 f(x), \tag{4.5}$$

where

$$K_3 f(x) := \sum_{n=0}^{\infty} \sum_{A, B \in \mathcal{A}_{-n}} \sum_{R \in \mathcal{E}(B)} \frac{\langle k, 1_A \otimes d_R \rangle}{\mu(A)} \langle d_R, f \rangle \left(1_A(x) - \frac{\mu(A)}{\mu(B)} 1_B(x) \right) \tag{4.6}$$

and

$$\widetilde{K}_3 f(x) := \sum_{n=0}^{\infty} \sum_{B \in \mathcal{A}_{-n}} \sum_{R \in \mathcal{E}(B)} \frac{\langle K^*(1), d_R \rangle}{\mu(B)} \langle d_R, f \rangle 1_B(x). \tag{4.7}$$

Furthermore we set

$$K_1 f(x) := \int\limits_X k_1(x, y) f(y) \, d\mu(y).$$

4.2. Statement of the main theorems

Recall the definition of the σ-algebras $\mathcal{F}_k^{\text{lev}}$, which were defined to be the σ-algebras generated by the dyadic cubes of level $-k$. In this section (Section 4), each occurrence of BMO means the space $\text{BMO}(X, \mathcal{F}_k^{\text{lev}})$ with these σ-algebras. Further, we let E be a UMD-space (see Section 2.2). We now state the main result in this article.

Theorem 4.2. *Let K be the integral operator defined in the last section satisfying* (4.1). *Then K, initially defined on finite linear combinations of Haar functions, extends linearly to a unique bounded operator on L^p for $1 < p < \infty$, i.e. we have a constant C_K such that*

$$\left\| K : L_E^p(X) \to L_E^p(X) \right\| \leqslant C_K$$

and C_K depends only on p, the BMO*-norms of $K(1)$ and $K^*(1)$, the constants C_S and δ coming from the structural estimate* (4.1) *and the* UMD *constant of E.*

The starting point and basic idea of the proof is the following decomposition of the bilinear form $\langle Kf, g \rangle$:

$$\langle Kf, g \rangle = \widetilde{B}_2(f, g) + \langle \widetilde{K}_3 f, g \rangle + \sum_{m=0}^{\infty} \langle k_{1,m}, g \otimes f \rangle + B_{2,m}(f, g) + \langle K_{3,m} f, g \rangle, \tag{4.8}$$

where these operators are defined in (4.4), (4.7), (4.10), (4.11) and (4.12). Clearly, we assumed here that k has mean zero with respect to the product measure $\mu \otimes \mu$. In fact, as we will see in the proof of Theorem 4.3 and the proof of Theorem 4.4, this decomposition can be further split as

$$K = P_{K(1)}^* + P_{K^*(1)}$$

$$+ \sum_{m=0}^{\infty} \sum_{i=1}^{M_1 q^m} \left(\sum_{j,k=1}^{N-1} T_{m,i}^{(j,k)} \circ \mathcal{M}_{m,i}^{(j,k)} + \sum_{j=1}^{N-1} W_{m,i}^{(j)} \circ \widetilde{\mathcal{M}}_{m,i}^{(j)} + \sum_{k=1}^{N-1} U_{m,i}^{(k)} \circ \mathcal{M}_{m,i}^{(k)} \right),$$

$$(4.9)$$

where $P_{K(1)}$ and $P_{K^*(1)}$ are paraproducts defined in the proof of Theorem 4.4, $T_{m,i}^{(j,k)}$, $W_{m,i}^{(j)}$, $U_{m,i}^{(k)}$ are shift and rearrangement operators defined in Section 4.3 and the operators $\mathcal{M}$ are suitable Haar multipliers. The five summands in (4.8) correspond to the summands in (4.9) in the same order.

Remark. We note explicitly that the constant C_K in the last theorem does *not* depend on the L^2-norm of $k(x, y)$, which is the crucial fact about the statement. It thus can be shown that

1. Theorem 4.2 yields a direct generalization of T. Figiel's $T(1)$ theorem [13] to spaces of homogeneous type, and
2. Theorem 4.2 yields a direct generalization of Coifmans $T(1)$ theorem (as presented in Christ [4], for the origin of the method see also Coifman, Jones, Semmes [6]) to vector valued singular integral operators given by standard kernels.

According to the decomposition of $\mathcal{C}$ we split k_1, B_1 and K_2 further and define

$$k_{1,m} := \sum_{(A,B) \in \mathcal{C}_m} \sum_{Q \in \mathcal{E}(A)} \sum_{R \in \mathcal{E}(B)} \langle k, d_Q \otimes d_R \rangle d_Q \otimes d_R, \quad \text{in } L^2(X \times X), \qquad (4.10)$$

$$B_{2,m}(f, g) := \sum_{(A,B) \in \mathcal{C}_m} \sum_{Q \in \mathcal{E}(A)} \frac{\langle k, d_Q \otimes 1_B \rangle}{\mu(B)} \left\langle d_Q \otimes \left(1_B - \frac{\mu(B)}{\mu(A)} 1_A \right), g \otimes f \right\rangle, \qquad (4.11)$$

$$K_{3,m} f := \sum_{(A,B) \in \mathcal{C}_m} \sum_{R \in \mathcal{E}(B)} \frac{\langle k, 1_A \otimes d_R \rangle}{\mu(A)} \langle d_R, f \rangle \left(1_A - \frac{\mu(A)}{\mu(B)} 1_B \right), \quad \text{in } L_E^2(X). \quad (4.12)$$

Associated to the kernel $k_{1,m}$ we define the integral operator

$$K_{1,m}(f)(x) := \int_X k_{1,m}(x, y) f(y) \, d\mu(y).$$

In later sections we prove the following theorems, from which our main result (Theorem 4.2) follows. In the subsequent theorem, δ is the positive number coming from the structural estimate (4.1) and q is the constant appearing in Theorem 3.3.

Theorem 4.3. *For all $1 < p < \infty$ there exists a constant C_p depending only on p, X, the* UMD *constant of E and C_S from (4.1), such that for all $f \in L_E^p(X)$, $g \in L_{E'}^{p'}(X)$, which are finite linear combinations of Haar functions, the operators $K_{1,m}$, $K_{3,m}$ and the bilinear form $B_{2,m}$ satisfy the following estimates*

$$\left\| K_{1,m}(f) \right\|_{L_E^p(X)} \leqslant C_p(m+1)q^{-m\delta} \| f \|_{L_E^p(X)}, \tag{4.13}$$

$$\left\| K_{3,m}(f) \right\|_{L_E^p(X)} \leqslant C_p(m+1)q^{-m\delta} \| f \|_{L_E^p(X)}, \tag{4.14}$$

$$\left| B_{2,m}(f,g) \right| \leqslant C_p(m+1)q^{-m\delta} \| f \|_{L_E^p(X)} \| g \|_{L_{E'}^{p'}(X)}. \tag{4.15}$$

Here, $p' = p/(p-1)$ denotes the conjugate exponent to p.

Remark. For this theorem, we need the L^p-boundedness of rearrangement and shift operators, which will be introduced in Section 4.3 and the boundedness of these operators will be proved in Sections 4.4 and 4.5.

Theorem 4.4. *For all* $1 < p < \infty$ *there exists a constant* C_p, *which depends only on* p, X *and the* UMD *constant of* E *such that for all* $f \in L_E^p(X)$, $g \in L_{E'}^{p'}(X)$ *which are finite linear combinations of Haar functions, the operator* $\widetilde{K}_3$ *and the bilinear form* $\widetilde{B}_2$ *satisfy the estimates*

$$\left| \widetilde{B}_2(f,g) \right| \leqslant C_p \| K(1) \|_{\mathrm{BMO}} \| f \|_{L_E^p(X)} \| g \|_{L_{E'}^{p'}(X)}, \tag{4.16}$$

$$\left| \langle \widetilde{K}_3 f, g \rangle \right| \leqslant C_p \| K^*(1) \|_{\mathrm{BMO}} \| f \|_{L_E^p(X)} \| g \|_{L_{E'}^{p'}(X)}. \tag{4.17}$$

Again, $p' = p/(p-1)$ *is the conjugate exponent to* p.

Proof. For the proof we use paraproduct operators which are formally given by

$$(P_a f)(x) = \sum_{n=0}^{\infty} \sum_{B \in \mathcal{A}_{-n}} \sum_{R \in \mathcal{E}(B)} \frac{\langle a, d_R \rangle}{\mu(B)} \langle d_R, f \rangle 1_B(x), \tag{4.18}$$

where $a \in \mathrm{BMO}$. Observe that P_a is the linear extension of the mapping

$$d_R \mapsto \langle a, d_R \rangle \frac{1_B}{\mu(B)}, \quad R \in \mathcal{E}(B)$$

and

$$1_X \mapsto 0,$$

so that for finite linear combinations of Haar functions f, g we have

$$\left| \widetilde{B}_2(f,g) \right| = \left| \langle P_{K(1)} g, f \rangle \right| \quad \text{and} \quad \left| \langle \widetilde{K}_3 f, g \rangle \right| = \left| \langle P_{K^*(1)} f, g \rangle \right|. \tag{4.19}$$

Now let both $f : X \to E$ and $a : X \to \mathbb{R}$ be finite linear combinations of Haar functions. We then consider the bilinear operation

$$P(a, f) := \sum_k (\mathbb{E}_k f)(\Delta_{k+1} a),$$

which has an immediate connection to a paraproduct operator, since we can compute for $R \in \mathcal{E}(B)$

$$\langle P(a, f), d_R \rangle = \sum_k \langle (\mathbb{E}_k f)(\Delta_{k+1} a), d_R \rangle = \langle a, d_R \rangle \frac{1}{\mu(B)} \int_B f \, d\mu = \langle f, P_a d_R \rangle.$$

Additionally, $\langle P(a, f), 1 \rangle = \langle f, P_a 1 \rangle$, since

$$\langle P(a, f), 1 \rangle = \sum_k \mathbb{E}[(\mathbb{E}_k f)(\Delta_{k+1} a)] = 0 = \langle f, P_a 1 \rangle, \tag{4.20}$$

so $P(a, \cdot)$ is the adjoint of P_a. Now we use a result that can be found in Figiel [13, pp. 108–109], and Figiel, Wojtaszczyk [14, p. 593], which allows us to deduce the L_E^p-boundedness of the operator $P(a, \cdot)$ (note that we have a regular sequence of σ-algebras $\mathcal{F}_k^{\mathrm{lev}}$) and the estimate

$$\|P(a, f)\|_{L_E^p(X)} \leqslant C \|a\|_{\mathrm{BMO}} \|f\|_{L_E^p(X)} \tag{4.21}$$

for $f \in L_E^p(X)$ and $a \in \mathrm{BMO}$.

With the L_E^p-boundedness of $P(a, \cdot)$, (4.19) and the fact that P_a is the adjoint of $P(a, \cdot)$, we finally get that

$$|\widetilde{B_2}(f, g)| = |\langle P(K(1), f), g \rangle| \leqslant C \|K(1)\|_{\mathrm{BMO}} \|f\|_{L_E^p(X)} \|g\|_{L_{E'}^{p'}(X)}$$

and

$$|\langle \widetilde{K_3} f, g \rangle| = |\langle f, P(K^*(1), g) \rangle| \leqslant C \|K^*(1)\|_{\mathrm{BMO}} \|f\|_{L_E^p(X)} \|g\|_{L_{E'}^{p'}(X)},$$

since E is a UMD-space and thus reflexive. $\square$

Proof of Theorem 4.2. For $1/p + 1/q = 1$, let $f \in L_E^p(X)$ and $g \in L_{E'}^{p'}(X)$ be finite linear combinations of Haar functions, then we have

$$\langle Kf, g \rangle = \widetilde{B_2}(f, g) + \langle \widetilde{K_3} f, g \rangle + \sum_{m=0}^{\infty} \langle k_{1,m}, g \otimes f \rangle + B_{2,m}(f, g) + \langle K_{3,m} f, g \rangle,$$

where these operators are defined in (4.4), (4.7), (4.10), (4.11) and (4.12). Thus we obtain from Theorems 4.3 and 4.4 that there exists a constant C_K which has only the stated dependences and we have

$$|\langle Kf, g \rangle| \leqslant C_K \|f\|_{L_E^p(X)} \|g\|_{L_{E'}^{p'}(X)}.$$

Hence for fixed $f \in L_E^p(X)$ which is a finite linear combination of Haar functions, the functional S_f defined by

$$S_f : g \mapsto \langle Kf, g \rangle$$

is bounded on the subspace U consisting of finite linear combinations of Haar functions of $L_{E'}^{p'}(X)$. Since U is dense in $L_{E'}^{p'}(X)$, it has a unique continuous extension to the whole space $L_{E'}^{p'}(X)$. Recall now that UMD-spaces are reflexive, and so $L_{E'}^{p'}(X)$ is canonically identified with $L_E^p(X)$. Hence there exists $z \in L_E^p(X)$ such that

$$\langle z, g \rangle = \langle Kf, g \rangle \quad \text{for all } g \in U \quad \text{and} \quad \|z\|_{L_E^p(X)} \leqslant C_K \|f\|_{L_E^p(X)}.$$

We get that $z = Kf$ since they have the same Haar coefficients, and so

$$\|Kf\|_{L_E^p(X)} \leqslant C_K \|f\|_{L_E^p(X)}$$

for all finite linear combinations of Haar functions f. Since again these functions are dense in $L_E^p(X)$, K has a unique bounded linear extension to all of $L_E^p(X)$ and the theorem is proved. □

The rest of Section 4 is now devoted to proving Theorem 4.3.

4.3. Rearrangement and shift operators

Definition 3.12 (the definition of the injection τ on $\mathcal{A}_{m,i}$) gives rise to rearrangement and shift operators, which are closely related to the integral operators K_1, K_3 and the bilinear form B_2. For $m \in \mathbb{N}_0$, $1 \leqslant i \leqslant M_1 q^m$ (see Proposition 3.11) we define

$$U_{m,i}^{(k)}(f) := \sum_{A \in \mathcal{A}_{m,i}} \frac{\langle d_{Q_k(\tau(A))}, f \rangle}{\sqrt{\mu(A)}} \left(1_A - \frac{\mu(A)}{\mu(\tau(A))} 1_{\tau(A)} \right), \tag{4.22}$$

$$T_{m,i}^{(j,k)}(f) := \sum_{A \in \mathcal{A}_{m,i}} \langle d_{Q_k(\tau(A))}, f \rangle d_{Q_j(A)}, \tag{4.23}$$

where f is a finite linear combination of Haar functions and $Q_j(A)$ is any enumeration of the elements in $\mathcal{E}(A)$. If the parameter k is greater than the number $N(A)$ of Haar functions corresponding to children of A, we simply set $d_{Q_k(A)} \equiv 0$.

Remark. We see that $U_{m,i}^{(k)}$ is the linear extension of the map

$$d_{Q_k(\tau(A))} \mapsto \frac{1}{\sqrt{\mu(A)}} \left(1_A - \frac{\mu(A)}{\mu(\tau(A))} 1_{\tau(A)} \right), \quad \text{for } 1 \leqslant k \leqslant N(\tau(A)),$$

with $A \in \mathcal{A}_{m,i}$. Analogously the mapping $T_{m,i}^{(j,k)}(f)$ is the linear extension of

$$d_{Q_k(\tau(A))} \mapsto d_{Q_j(A)}, \quad \text{for } 1 \leqslant k \leqslant N(\tau(A)),$$

where $A \in \mathcal{A}_{m,i}$.

In order to show Theorem 4.3, we prove the following L^p-bounds of the operators $U_{m,i}^{(k)}$ and $T_{m,i}^{(j,k)}$:

Proposition 4.5. *The operators $U_{m,i}^{(k)}$ and $T_{m,i}^{(j,k)}$ satisfy the $L_E^p(X)$-estimate ($1 < p < \infty$)*

$$\left\| U_{m,i}^{(k)} : L_E^p(X) \to L_E^p(X) \right\| \leqslant C_p(m+1), \tag{4.24}$$

$$\left\| T_{m,i}^{(j,k)} : L_E^p(X) \to L_E^p(X) \right\| \leqslant C_p(m+1) \tag{4.25}$$

for all $1 \leqslant j, k \leqslant N - 1$, where C_p depends only on p, X and the UMD-*constant of E. Here, as in Section* 3, *N is the maximal number of children a dyadic cube can have.*

The rough idea of the proof of these bounds is the following: We prove a version of Proposition 4.5 under the constraint that we restrict the sum in (4.22) and (4.23) from $\mathcal{A}_{m,i}$ to a collection that satisfies the so called Figiel's compatibility condition. In this case we get a bound, which is independent of m. Thereafter we invoke the decomposition of $\mathcal{A}_{m,i}$ into such subcollections introduced in Section 3.7.

4.4. Figiel's compatibility condition

Here we review the martingale estimates of rearrangement operators that satisfy Figiel's compatibility condition. We follow Figiel [12] and the expositions Figiel, Wojtaszczyk [14] and Müller [25].

Definition 4.6. Let $\mathcal{D} \subseteq \mathcal{A}_{m,i}$ be a subset of $\mathcal{A}_{m,i}$ and $\tau : \mathcal{D} \to \mathcal{A}$ be an injective map. We say that the pair $(\tau, \mathcal{D})$ satisfies *Figiel's compatibility condition* if the collection

$$\mathcal{Z} := \left\{ A, \tau(A), A \cup \tau(A) \colon A \in \mathcal{D} \right\}$$

is nested, $\operatorname{lev} A = \operatorname{lev} \tau(A)$ and $\tau(A) \notin \mathcal{D}$ for all $A \in \mathcal{D}$.

Recall that a collection of sets $\mathcal{Z}$ is said to be *nested*, if for every choice $A, B \in \mathcal{Z}$ it holds that either

$$A \subseteq B \quad \text{or} \quad B \subseteq A \quad \text{or} \quad A \cap B = \emptyset.$$

We remark that if $(\tau, \mathcal{D})$ satisfies Figiel's compatibility condition, the pair $(\tau^{-1}, \tau(\mathcal{D}))$ also satisfies Figiel's compatibility condition. Then the following theorems concerning the boundedness of the operators T and U hold.

Theorem 4.7. *Let $(\tau, \mathcal{D})$ satisfy the compatibility condition. Then the operator*

$$T_{m,i}^{(j,k),\mathcal{D}} f := \sum_{A \in \mathcal{D}} \langle d_{Q_k(\tau(A))}, f \rangle d_{Q_j(A)}$$

is bounded in L^p for all $1 \leqslant j, k \leqslant N - 1$ and satisfies the estimate

$$\left\| T_{m,i}^{(j,k),\mathcal{D}} f \right\|_{L_E^p(X)} \leqslant C_p \| f \|_{L_E^p(X)}$$

where C_p depends only on p, X and the UMD-*constant of E.*

Theorem 4.8. *Let $(\tau, \mathcal{D})$ satisfy the compatibility condition. Then the operator*

$$U_{m,i}^{(k),\mathcal{D}} f := \sum_{A \in \mathcal{D}} \frac{\langle d_{Q_k(\tau(A))}, f \rangle}{\sqrt{\mu(A)}} \left(1_A - \frac{\mu(A)}{\mu(\tau(A))} 1_{\tau(A)} \right)$$

is bounded in L^p and satisfies the estimate

$$\left\| U_{m,i}^{(k),\mathcal{D}} f \right\|_{L_E^p(X)} \leqslant C_p \| f \|_{L_E^p(X)},$$

where C_p depends only on p, X and the UMD-*constant of E.*

The proofs of the foregoing two theorems are slight modifications of the analogous results for the Haar system in $\mathbb{R}$ (see for instance the exposition Müller [24]).

Remark. If we apply this theorem to the collection $\tau(\mathcal{D})$ and the map τ^{-1}, we get that the operator

$$f \mapsto W_{m,i}^{(k),\mathcal{D}}(f) := \sum_{A \in \mathcal{D}} \frac{\langle d_{Q_k(A)}, f \rangle}{\sqrt{\mu(\tau(A))}} \left(1_{\tau(A)} - \frac{\mu(\tau(A))}{\mu(A)} 1_A \right)$$

is bounded on L_E^p. $W_{m,i}^{\mathcal{D}}$ is the linear extension of the mapping

$$d_{Q_k(A)} \mapsto \frac{1}{\sqrt{\mu(\tau(A))}} \left(1_{\tau(A)} - \frac{\mu(\tau(A))}{\mu(A)} 1_A \right),$$

for $A \in \mathcal{D}$.

4.5. The boundedness of the operators $W_{m,i}^{(k)}$, $U_{m,i}^{(k)}$, $T_{m,i}^{(j,k)}$

Using the decomposition theorems proved in Section 3, we are now able to reduce the general case of Proposition 4.5 to the special case of nested collections proved in the preceding Section 4.4.

Proof of Proposition 4.5. If we invoke the decomposition results of Chapter 3, we see that $\mathcal{C}_{m,i}$ splits into M collections $\mathcal{G}$, where M is constant. Further, every $\mathcal{A}_{\mathcal{G}}$ splits into $L(m+1)$ collections $\mathcal{L}$. $\mathcal{L}$ decomposes in $\mathcal{H}$, $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$, where on $\mathcal{H}$, $\mathcal{I}_1$ and $\mathcal{I}_2$, the operators $W_{m,i}^{(k)}$, $U_{m,i}^{(k)}$ and $T_{m,i}^{(j,k)}$ are bounded by a constant which is independent of m. Since the collections $\mathcal{H}$, $\mathcal{I}_1$ and $\mathcal{I}_2$ satisfy Figiel's compatibility condition (with the injection τ) by Theorem 3.17, this follows directly from Theorems 4.7, 4.8 and the remark after them. The collections $\mathcal{J}_i$, $i \in \{1, 2, 3\}$ need further arguments. For the following we fix an index $1 \leqslant j \leqslant N - 1$ and define the following map on $\gamma_1(\mathcal{J}_1)$

$$\rho : \gamma_1(\mathcal{J}_1) \to \mathcal{A},$$

$$A \mapsto \tau \circ \gamma_1(A).$$

Since the mapping $(\gamma_1, \mathcal{J}_1)$ (and hence also $(\gamma_1^{-1}, \gamma_1(\mathcal{J}_1))$) satisfies Figiel's compatibility condition (note Lemma 3.24), we see from Theorem 4.7 and Theorem 4.8 that the linear extensions of the mappings

$$T_{\gamma_1}^{(j,k)} : d_{Q_k(A)} \mapsto d_{Q_j(\gamma_1(A))}, \qquad W_{\gamma_1}^{(k)} : d_{Q_k(A)} \mapsto \frac{1}{\sqrt{\mu(\gamma_1(A))}}\left(1_{\gamma_1(A)} - \frac{\mu(\gamma_1(A))}{\mu(A)}1_A\right)$$

where $A \in \mathcal{J}_1$, are bounded on L^p. In Theorem 3.17 we constructed a decomposition of $\mathcal{I}$ into $\mathcal{I}_1$ and $\mathcal{I}_2$ which both satisfied Figiel's compatibility condition with the injection τ. Since with γ_1 we moved sets in $\mathcal{J}_1$ away from the boundary of their arithmetic predecessors, we are in the same position for the collection $\gamma_1(\mathcal{J}_1)$ and the injection ρ, since if we again perform a decomposition like in Proposition 3.13 we are able to use the remark after the proof of Lemma 3.21. We thus obtain from the $\mathcal{I}$-part of Theorem 3.17 and again from Theorems 4.7, 4.8 and the remark after them that the linear extension of the mappings

$$T_{\rho}^{(j,k)} : d_{Q_k(A)} \mapsto d_{Q_j(\rho(A))}, \qquad W_{\rho}^{(k)} : d_{Q_k(A)} \mapsto \frac{1}{\sqrt{\mu(\rho(A))}}\left(1_{\rho(A)} - \frac{\mu(\rho(A))}{\mu(A)}1_A\right)$$

where $A \in \gamma_1(\mathcal{J}_1)$, are bounded on L_E^p by a constant which depends only on X. For the same reason, we may even replace ρ by ρ^{-1} and the assertions stay valid. We conclude that the composition

$$T_{\gamma_1}^{(j,1)} \circ T_{\rho^{-1}}^{(1,k)} =: T_{\tau^{-1}}^{(j,k)}$$

is bounded on L_E^p and it is the linear extension of the map

$$d_{Q_k(\tau(A))} \mapsto d_{Q_j(A)},$$

where $A \in \mathcal{J}_1$. We remark that $T_{\tau^{-1}}^{(j,k)}$ is the shift operator $T_{m,i}^{(j,k),\, \mathcal{J}_1}$, which is thus shown to be bounded. Now we come to the linear extension of the map

$$d_{Q_k(A)} \mapsto \frac{1}{\sqrt{\mu(\tau(A))}}\left(1_{\tau(A)} - \frac{\mu(\tau(A))}{\mu(A)}1_A\right), \quad \text{for } A \in \mathcal{J}_1$$

which is the mapping $W_{m,i}^{(k),\mathcal{J}_1}$. For finite linear combinations of Haar functions $f = \sum_l a_l d_{Q_l}$, where $Q_l = Q_k(A_l)$, $A_l \in \mathcal{J}_1$ and $a_l \in E$, $W_{m,i}^{(k),\mathcal{J}_1}$ has the representation

$$W_{m,i}^{(k),\mathcal{J}_1} f = \left(W_{\rho}^{(1)} \circ T_{\gamma_1}^{(1,k)}\right)(f) + \sum a_{Q_l}\sqrt{\frac{\mu(\tau(A_l))}{\mu(\gamma_1(A_l))}}\, W_{\gamma_1}^{(k)}(d_{Q_l}).$$

With the unconditionality of the $\{d_Q\}$ and Kahane's contraction principle, we conclude that $W_{m,i}^{(k),\mathcal{J}_1}$ is bounded on L_E^p. Analogously, for $f = \sum_l a_l d_{Q_l}$, where $Q_l = Q_k(\tau(A_l))$ and $A_l \in \mathcal{J}_1$, we have the representation

$$U_{m,i}^{(k),\mathcal{J}_1} f = \sum_l a_l \sqrt{\frac{\mu(A_l)}{\mu(\gamma_1(A_l))}} W_{\rho^{-1}}^{(k)}(d_{Q_l}) + \big(W_{\gamma_1^{-1}}^{(1)} \circ T_{\rho^{-1}}^{(1,k)}\big)(f).$$

A similar reasoning applies to $\mathcal{J}_2$, where in this case we let

$$\gamma_2 : \tau(\mathcal{J}_2) \to \mathcal{A}$$

to move away from the boundary of the arithmetic predecessor. The mapping ρ is defined as

$$\rho : \mathcal{J}_2 \to \mathcal{A},$$
$$A \mapsto \gamma_2 \circ \tau(A).$$

In the case for $\mathcal{J}_3$ we define both injections γ_1 and γ_2 from above to act on $\mathcal{J}_3$ and $\tau(\mathcal{J}_3)$ respectively and set

$$\rho : \gamma_1(\mathcal{J}_3) \to \mathcal{A},$$
$$A \mapsto \gamma_2 \circ \tau \circ \gamma_1(A).$$

If we summarize these considerations, we get a decomposition of the operators $W_{m,i}^{(k)}$, $U_{m,i}^{(k)}$ and $T_{m,i}^{(j,k)}$ into a sum of $C(m+1)$ bounded operators on L^p, where their bound depends only on p, X and the UMD-constant of E. Since so does C, we get the assertions of Proposition 4.5 and that $W_{m,i}$ is bounded on L^p by $C_p(m+1)$. $\quad\square$

4.6. The proof of Theorem 4.3

Theorem 4.3. *For all $1 < p < \infty$ there exists a constant C_p depending only on p, X, the* UMD *constant of E and C_S from (4.1), such that for all $f \in L_E^p(X)$, $g \in L_{E'}^{p'}(X)$, which are finite linear combinations of Haar functions, the operators $K_{1,m}$, $K_{3,m}$ and the bilinear form $B_{2,m}$ satisfy the following estimates*

$$\big\|K_{1,m}(f)\big\|_{L_E^p(X)} \leqslant C_p(m+1)q^{-m\delta}\|f\|_{L_E^p(X)}, \tag{4.13}$$

$$\big\|K_{3,m}(f)\big\|_{L_E^p(X)} \leqslant C_p(m+1)q^{-m\delta}\|f\|_{L_E^p(X)}, \tag{4.14}$$

$$\big|B_{2,m}(f,g)\big| \leqslant C_p(m+1)q^{-m\delta}\|f\|_{L_E^p(X)}\|g\|_{L_{E'}^{p'}(X)}. \tag{4.15}$$

Here, $p' = p/(p-1)$ denotes the conjugate exponent to p.

Proof. It holds that

$$K_{1,m} = \sum_{i=1}^{M_1 q^m} \sum_{j,k=1}^{N-1} T_{m,i}^{(j,k)} \circ \mathcal{M}_{m,i}^{(j,k)},$$

where $T_{m,i}^{(j,k)}$ is the shift operator introduced in (4.23) and $\mathcal{M}_{m,i}^{(j,k)}$ is the Haar multiplication operator which maps

$$d_{Q_k(\tau(A))} \mapsto \langle k, d_{Q_j(A)} \otimes d_{Q_k(\tau(A))} \rangle d_{Q_k(\tau(A))} \quad \text{for } A \in \mathcal{A}_{m,i}.$$

Analogously we get

$$K_{3,m} = \sum_{i=1}^{M_1 q^m} \sum_{k=1}^{N-1} U_{m,i}^{(k)} \circ \mathcal{M}_{m,i}^{(k)},$$

$$B_{2,m}(f,g) = \left\langle \sum_{i=1}^{M_1 q^m} \sum_{j=1}^{N-1} W_{m,i}^{(j)} \circ \widetilde{\mathcal{M}}_{m,i}^{(j)} f, g \right\rangle, \tag{4.26}$$

where $\mathcal{M}_{m,i}^{(k)}$ and $\widetilde{\mathcal{M}}_{m,i}^{(j)}$ are Haar multiplication operators which map

$$d_{Q_k(\tau(A))} \mapsto \frac{\langle k, 1_A \otimes d_{Q_k(\tau(A))} \rangle}{\sqrt{\mu(A)}} d_{Q_k(\tau(A))} \quad \text{and} \quad d_{Q_j(A)} \mapsto \frac{\langle k, d_{Q_j(A)} \otimes 1_{\tau(A)} \rangle}{\sqrt{\mu(\tau(A))}} d_{Q_j(A)},$$

respectively. These decompositions follow from the definition of $K_{1,m}$, $K_{3,m}$, $B_{2,m}$ in (4.10)–(4.12), Proposition 3.11, the definition of the shifts $T_{m,i}^{(j,k)}$ and rearrangements $U_{m,i}^{(k)}$, $W_{m,i}^{(j)}$ in (4.22), (4.23) and the remark after Theorem 4.8. Since Haar multipliers are bounded on $L_E^p(X)$ by the supremum of their coefficients, we deduce by the structural estimate (4.1) and Proposition 4.5

$$\begin{aligned}
\|K_{1,m} f\|_{L_E^p(X)} &= \left\| \sum_{i=1}^{M_1 q^m} \sum_{j,k=1}^{N-1} T_{m,i}^{(j,k)} \circ \mathcal{M}_{m,i}^{(j,k)} f \right\|_{L_E^p(X)} \\
&\leqslant \sum_{i=1}^{M_1 q^m} \sum_{j,k=1}^{N-1} \|T_{m,i}^{(j,k)}\|_{L_E^p(X) \to L_E^p(X)} \|\mathcal{M}_{m,i}^{(j,k)}\|_{L_E^p(X) \to L_E^p(X)} \|f\|_{L_E^p(X)} \\
&\leqslant C q^m (m+1) \sup_{\substack{1 \leqslant j,k \leqslant N-1 \\ A \in \mathcal{A}_{m,i}}} \left| \langle k, d_{Q_j(A)} \otimes d_{Q_k(\tau(A))} \rangle \right| \|f\|_{L_E^p(X)} \\
&\leqslant C q^m (m+1) \sup_{A \in \mathcal{A}_{m,i}} \left(1 + \frac{d(A, \tau(A))}{q^{\text{lev } A}} \right)^{-1-\delta} \|f\|_{L_E^p(X)}, \tag{4.27}
\end{aligned}$$

where C is a constant only depending on p, X, the UMD-constant of E and C_S that possibly changes from line to line. Now we get from the definition of $\mathcal{C}_m$ in Section 3.5 (and hence from

the corresponding property for $\mathcal{A}_{m,i}$) that the last expression in the previous display is less or equal

$$Cq^{-\delta m}(m+1)\|f\|_{L^p_E(X)},$$

which is the required conclusion for $K_{1,m}$. The two remaining assertions follow from similar arguments using the decompositions in (4.26). □

Acknowledgments

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