Topology and its Applications 156 (2009) 515–521

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

A nonaspherical cell-like 2-dimensional simply connected continuum and related constructions

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ARTICLE INFO

Article history:
Received 5 August 2005
Received in revised form 24 July 2008
Accepted 24 July 2008

MSC:
primary 54B15, 54G15, 54F15
secondary 55N10, 55Q52

Keywords:
Nonaspherical space
Simple connectivity
Peano continuum
Cell-like space
Trivial shape

ABSTRACT

We prove the existence of a 2-dimensional nonaspherical simply connected cell-like Peano continuum (the space itself was constructed in one of our earlier papers). We also indicate some relations between this space and the well-known Griffiths' space from the 1950s.

1. Introduction

It is well known (see [10,12]) that every n-dimensional compactum is weakly homotopy equivalent to an $(n+1)$-dimensional cell-like compactum (i.e. a compactum with the trivial shape). Therefore there exist nonaspherical cell-like simply connected compacta in all dimensions $\geqslant 3$.

It was heretofore unknown whether such compacta also exist in dimension 2. In this paper we give the affirmative answer to this question. We show that the space $SC(S^1)$ which we constructed in our earlier paper [9], is in fact, a *nonaspherical* cell-like 2-dimensional simply connected *Peano* continuum (i.e. locally connected continuum).

We also modify our original construction of the space $SC(S^1)$ and show that the modified construction gives a space which has the homotopy type of the classical well-known space [11] from the 1950s, which is a non-simply connected one-point union of two contractible spaces.

Our main result concerns $SC(X)$ for a non-simply connected path-connected space X. To analyze the singular homology $H_2(SC(X))$, we use infinitary words and a result from [5]. Although infinitary words have already been introduced in [1], they may not be a familiar notion. In the special case $X = S^1$, we can prove the result only by using finitary words—we present it at the end of Section 3. As a general reference for algebraic topology we refer the reader to [14].

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0166-8641/$ – see front matter
doi:10.1016/j.topol.2008.07.019

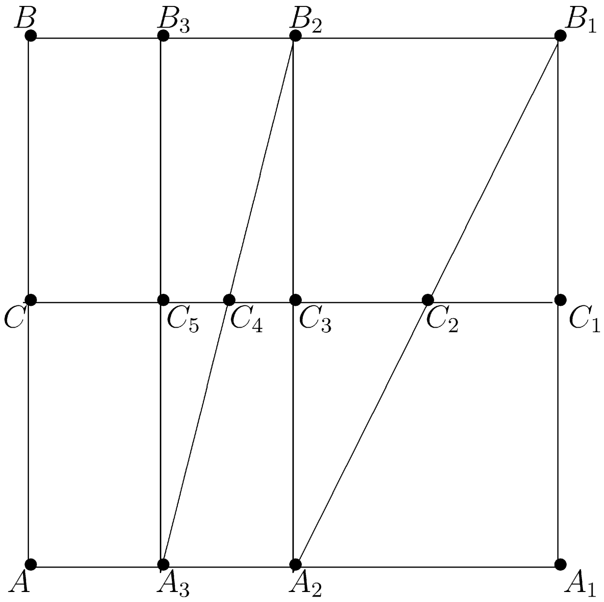

Fig. 1.

2. Preliminaries

We recall the construction of the space $SC(S^1)$ from [9]. Consider the so-called *Topologist sine curve* T and embed T into the square $\mathbb{I}^2 = \mathbb{I} \times \mathbb{I}$ as in Fig. 1, i.e. T is embedded as the union of $A_1B_1A_2B_2\ldots$ and AB. Let S^1 be the circle and s_0 any of its points which we consider as the base point. Consider the topological sum of $\mathbb{I}^2$ and $T \times S^1$. The space $SC(S^1)$ is now defined as the quotient space of this sum, obtained by identification of the points (t, s_0) with $t \in T \subset \mathbb{I}^2$, and by identification of each set $\{t\} \times S^1$ with t, when $t \in \{0\} \times \mathbb{I}$. For an arbitrary compactum X, one defines the space $SC(X)$ by replacing S^1 everywhere above by X. For the details of the definition of $SC(X)$ we refer the reader to [9].

The subspace $\mathbb{H} = \bigcup_{m=1}^{\infty}\{(x, y)\colon\ (x-1/m)^2 + y^2 = 1/m^2\}$ of the Euclidean plane $\mathbb{R}^2$ is called the *Hawaiian earring*. Denote $\theta = (0, 0) \in \mathbb{H}$ and let $C(\mathbb{H})$ be the cone over $\mathbb{H}$. We consider $\mathbb{H}$ as the subspace of $C(\mathbb{H})$. A space $\mathcal{G}$ is then defined as the one-point union of two copies of $C(\mathbb{H})$, obtained by identifying two copies of θ at the point θ. This space is a well-known example of a non-contractible space which is a one-point union of contractible spaces—Griffiths was the first to investigate this kind of spaces [11, p. 190], where he also acknowledges ideas by James. The fact that $\mathcal{G}$ is aspherical was proved in [8]. For further information of this space and its generalizations we refer the reader to [4,6,7].

Throughout the paper, we shall denote the singular homology with integer coefficients by $H_*()$.

3. On nonasphericity of $SC(S^1)$ and $SC(X)$

Obviously, $SC(S^1)$ is a cell-like Peano continuum. It was shown in [9] that this space is simply connected. Therefore it suffices to show that $SC(S^1)$ is nonaspherical. In order to prove this it certainly suffices to verify that there exists a non-trivial 2-dimensional singular cycle in $SC(S^1)$. We shall prove this as a corollary of the following general result—Theorem 3.1 below—in the sense of [9]. Our notation for $SC(X)$ is the same as in [9].

Consider Fig. 1: the piecewise linear line $A_1B_1A_2B_2\ldots$ with the segment AB in this figure is the PL Topologist sine curve which was used to build $SC(X)$, i.e. along which we attached the "infinite tube".

Theorem 3.1. *Let X be any path-connected space. Then the following assertions hold*:

(1) *if X is not simply connected, then $H_2(SC(X))$ is not trivial*; *and*
(2) *if $\pi_1(X)$ and $\pi_2(X)$ are trivial, then $H_2(SC(X))$ is also trivial.*

Corollary 3.2. *The space $SC(S^1)$ is a nonaspherical cell-like* 2*-dimensional simply connected Peano continuum.*

For the proof of Theorem 3.1, we recall a notion of the free σ-product of groups and a lemma from [5]. Let (X_i, x_i) be any family of pointed spaces such that $X_i \cap X_j = \emptyset$, for $i \neq j$. The underlying set of a pointed space $(\widetilde{\bigvee}_{i\in I}(X_i, x_i), x^*)$ is the union of all X_i's obtained by identifying all x_i to a point x^* and the topology is defined by specifying the neighborhood bases as follows:

(1) If $x \in X_i \setminus \{x_i\}$, then the neighborhood base of x in $\widetilde{\bigvee}_{i\in I}(X_i, x_i)$ is the one of X_i;
(2) The point x^* has a neighborhood base, each element of which is of the form: $\widetilde{\bigvee}_{i\in I\setminus F}(X_i, x_i) \vee \bigvee_{j\in F} U_j$, where F is a finite subset of I and each U_j is an open neighborhood of x_j in X_j for $j \in F$.

Lemma 3.3. *(See [5, Theorem A.1].) Let X_i be locally simply-connected and first countable at x_n for each i. Then*

$$\pi_1\Big(\widetilde{\bigvee_{i\in I}}(X_i, x_i), x^*\Big) \simeq \times^{\sigma}_{i\in I}\pi_1(X_i, x_i).$$

In particular $I = \mathbb{N}$,

$$\pi_1\Big(\widetilde{\bigvee_{n\in\mathbb{N}}}(X_n, x_n), x^*\Big) \simeq \times_{n\in\mathbb{N}}\pi_1(X_n, x_n).$$

We also need basic descriptions of paths and loops. A loop $f : \mathbb{I} \to X$ is a continuous map with $f(0) = f(1)$. For a loop f, f^- denotes the loop defined by: $f^-(t) = f(1-t)$. For loops f, g with the same base point, the concatenation fg is a loop defined by: $fg(t) = f(2t)$ for $0 \leqslant t \leqslant 1/2$ and $fg(t) = g(2t-1)$ for $1/2 \leqslant t \leqslant 1$. We denote the homotopy class relative to end points of a loop f by $[f]$ and the homology class of f by $[f]_s$.

Proof of Theorem 3.1. Let p be the natural projection of $SC(X)$ onto $\mathbb{I}^2$ which we consider as a subspace of the plane $\mathbb{R}^2$.

Let $Y_0 = p^{-1}(\mathbb{I} \times [0, 2/3))$ and $Y_1 = p^{-1}(\mathbb{I} \times (1/3, 1])$. Then $SC(X) = Y_0 \cup Y_1$ and $Y_0 \cap Y_1$ is open in $SC(X)$.

Consider the following Mayer–Vietoris homology exact sequence:

$$H_2\big(SC(X)\big) \xrightarrow{\partial} H_1(Y_0 \cap Y_1) \xrightarrow{h} H_1(Y_0) \oplus H_1(Y_1).$$

We let $i_0 : Y_0 \cap Y_1 \to Y_0$ and $i_1 : Y_0 \cap Y_1 \to Y_1$ be the inclusion maps. Then $h = i_{0*} - i_{1*}$.

We now present the proof of property (1) above. We first observe that non-injectivity of h implies that $H_2(SC(X))$ is non-trivial.

Since $p^{-1}(\mathbb{I} \times \{0\})$, $p^{-1}(\mathbb{I} \times \{1/2\})$, $p^{-1}(\mathbb{I} \times \{1\})$ are strong deformation retracts of Y_0, $Y_0 \cap Y_1$ and Y_1 respectively, the homotopy types of Y_0, Y_1 and $Y_0 \cap Y_1$ have the same homotopy type as $p^{-1}(\mathbb{I} \times \{0\})$. We denote the deformation retractions by $r_0 : Y_0 \to p^{-1}(\mathbb{I} \times \{0\})$ and $r_1 : Y_1 \to p^{-1}(\mathbb{I} \times \{1\})$.

Choose a point $x^\# \in X$ and form a one-point union $(X, x^\#) \vee (\mathbb{I}, 0)$ under the identification of $x^\#$ and 0. Let X_n's be copies of $(X, x^\#) \vee (\mathbb{I}, 0)$ and x_n's copies of $1 \in \mathbb{I}$. Then the space $p^{-1}(\mathbb{I} \times \{0\})$ has the same homotopy type $Y = \widetilde{\bigvee}_{n\in\mathbb{N}}(X_n, x_n)$. Hence (X_n, x_n) is locally simply-connected and first countable at x_n. Lemma 3.3 implies that $\pi_1(Y) \simeq \times_{n\in\mathbb{N}}\pi_1(X_n, x_n)$.

Since X is not simply connected, we can find an essential loop f in X whose base point is $x^\#$. Observe that $p^{-1}(\{P\})$ is a copy of X for each point P on $A_1B_1A_2B_2\ldots$. A point P on $A_1B_1A_2B_2\ldots$ is written as $P = (x, y)$ for $x, y \in \mathbb{I}$. Define

$$f_P(t) = \begin{cases} (3xt, y), & \text{for } 0 \leqslant t \leqslant 1/3, \\ (P, f(3(t-1/3))), & \text{for } 1/3 \leqslant t \leqslant 2/3, \\ (3(1-t)x, y), & \text{for } 2/3 \leqslant t \leqslant 1. \end{cases}$$

Then for $n \geqslant 1$, f_{A_n} is a loop in $p^{-1}(\mathbb{I} \times \{0\}) \subseteq Y_0$ with the base point A and f_{B_n} one in $p^{-1}(\mathbb{I} \times \{1\}) \subseteq Y_1$ with the base point B and f_{C_n} one in $p^{-1}(\mathbb{I} \times \{1/2\}) \subseteq Y_0 \cap Y_1$ with the base point C respectively. Since the images of f_{C_n}'s converge to C, we have two loops $g_0 = f_{C_1} f^-_{C_2} f_{C_3} f^-_{C_4} \ldots$ and $g_1 = f^-_{C_1} f_{C_2} f^-_{C_3} f_{C_4} \ldots$ in $Y_0 \cap Y_1$. (These infinite concatenations make sense, since the ranges of loops converge to C.)

Observe that $r_{0*} \circ i_{0*}([f_{C_1}]) = [f_{A_1}]$, $r_{1*} \circ i_{1*}([f_{C_1}]) = [f_{B_1}]$, $r_{0*} \circ i_{0*}([f_{C_{2n}}]) = [f_{A_{n+1}}] = r_{0*} \circ i_{0*}([f_{C_{2n+1}}])$ and $r_{1*} \circ i_{1*}([f_{C_{2n-1}}]) = [B_n] = r_{1*} \circ i_{1*}([f_{C_{2n}}])$ for each natural number n.

Since we have homotopies from $f^-_{A_{n+1}} f_{A_{n+1}}$ to the constant A and the images of the homotopies converge to A, it follows that $r_{0*} \circ i_{0*}([g_1]) = [f_{A_1}]$ and $r_{0*} \circ i_{0*}([g_2]) = [f^-_{A_1}]$. Hence $i_{0*}([g_0 g_1]) = e$. Similarly, $r_{1*} \circ i_{1*}([g_0]) = e$ and $r_{1*} \circ i_{1*}([g_1]) = e$ and hence $r_{1*} \circ i_{1*}([g_0 g_1]) = e$. Now we have $i_{0*}([g_0 g_1]_s) = 0$ and $i_{1*}([g_0 g_1]_s) = 0$, i.e. $h([g_0 g_1]_s) = 0$.

It suffices to show that $[g_0 g_1]_s$ is non-zero, i.e. that $[g_0 g_1]$ does not belong to the commutator subgroup of $\pi_1(Y_0 \cap Y_1)$. The isomorphism from $\pi_1(Y_0 \cap Y_1)$ to $\widetilde{\bigvee}_{n\in\mathbb{N}}(X_n, x_n)$ maps $[g_0 g_1]$ to $c_1 c_2^{-1} c_3 c_4^{-1} \ldots c_1^{-1} c_2 c_3^{-1} c_4 \ldots$, where c_n is the letter corresponding to $[f_{C_n}]$. To show the conclusion by contradiction, suppose that $c_1 c_2^{-1} c_3 c_4^{-1} \ldots c_1^{-1} c_2 c_3^{-1} c_4 \ldots$ belongs to the commutator subgroup. Then, by [5, Lemma 4.11] there exist non-empty reduced words $U_1, \ldots, U_{2m}$ such that $c_1 c_2^{-1} c_3 c_4^{-1} \ldots c_1^{-1} c_2 c_3^{-1} c_4 \ldots = U_1 \ldots U_{2m}$, where $U_1, \ldots, U_{2m}$ is of the canonical commutator form, i.e. there are j_l, k_l such that $\{j_1, \ldots, j_m\} \cup \{k_1, \ldots, k_m\} = \{1, \ldots, 2m\}$, $U_{j_l} = U^{-1}_{k_l}$ and the reduced word $c_1 c_2^{-1} c_3 c_4^{-1} \ldots c_1^{-1} c_2 c_3^{-1} c_4 \ldots$ is obtained by multiplying the rightmost elements U_i and the leftmost elements of U_{i+1} at most $(2m-1)$-times. Therefore, W_{2m} is of infinite length and is well ordered from the left to the right, and hence there exists U_i which is of infinite length and is well ordered from the right to the left. But this is impossible, because $c_1 c_2^{-1} c_3 c_4^{-1} \ldots c_1^{-1} c_2 c_3^{-1} c_4 \ldots$ is well ordered from the left to the right.

Next we show the second statement (2). Suppose that $\pi_1(X)$ and $\pi_2(X)$ are trivial. Consider another part of the Mayer–Vietoris sequence:

$$H_2(Y_0)\oplus H_2(Y_1)\longrightarrow H_2\big(SC(X)\big)\xrightarrow{\partial} H_1(Y_0\cap Y_1).$$

By $\pi_1(Y_0\cap Y_1)\simeq \times_{n\in\mathbb{N}}\pi_1(X_n,x_n)$, we conclude that $\pi_1(Y_0\cap Y_1)$ is trivial. Hence $H_1(Y_0\cap Y_1)$ is trivial. Since $\pi_1(Y_0)$ is trivial, it follows that $H_2(Y_0)$ is isomorphic to $\pi_2(Y_0)$. Now we have $H_2(Y_0)=\pi_2(Y_0)\simeq\prod_{n\in\mathbb{N}}\pi_2(X_n,x_n)=\{0\}$ by [7, Theorem 1.1]. Similarly, $H_2(Y_1)=0$ and $H_2(Y_0)\oplus H_2(Y_1)=\{0\}$. Now the above exact sequence implies that $H_2(SC(X))$ is trivial. □

We denote the commutator $aba^{-1}b^{-1}$ by $[a,b]$.

Alternative proof of Corollary 3.2. For the case $X=S^1$ we take c_n as the generator of the fundamental group of X_{C_n}, which is isomorphic to $\mathbb{Z}$. As in the preceding proof of Theorem 3.1, it suffices to show that the element $c=c_1c_2^{-1}c_3c_4^{-1}\ldots c_1^{-1}c_2c_3^{-1}c_4\ldots$ does not belong to the commutator subgroup of the group $\pi_1(Y_0\cap Y_1)$. To prove this by contradiction, suppose that c belongs to the commutator subgroup, i.e. c is a product of m commutators for some m.

Consider natural homomorphism $f:\pi_1(Y_0\cap Y_1)\to\pi_1(\bigvee_{1\leqslant i\leqslant 2m+2}(X_{C_i},C_i))$, where $X_{C_i}=S^1$. The group $\pi_1(\bigvee_{1\leqslant i\leqslant 2m+2}(X_{C_i},C_i))$ is a free group with $(2m+2)$-generators $\langle c_1,c_2,\ldots,c_{2m+1},c_{2m+2}\rangle$. We have

$$f(c)=c_1c_2^{-1}\ldots c_{2m+1}c_{2m+2}^{-1}c_1^{-1}c_2\ldots c_{2m+1}^{-1}c_{2m+2}.$$

Let $d_1=c_1, d_2=c_2^{-1}$, $d_{2k-1}=c_{2k-2}^{-1}c_{2k-3}\ldots c_2^{-1}c_1c_{2k-1}$ and $d_{2k}=c_{2k}^{-1}c_{2k-1}$.

It is easy to prove by induction the equality $c_1c_2^{-1}\ldots c_{2k-1}c_{2k}^{-1}c_1^{-1}c_2\ldots c_{2k-1}^{-1}c_{2k}=[d_1,d_2]\ldots[d_{2k-1},d_{2k}]$.

Since $(d_1,d_2,\ldots,d_{2m+1},d_{2m+2})$ is obtained by a Nielsen transformation [13, p. 5] from $(c_1,c_2,\ldots,c_{2m+1},c_{2m+2})$, the set $\{d_0,d_1,\ldots,d_{2m},d_{2m+2}\}$ generates the free group $\langle c_1,c_2,\ldots,c_{2m+1},c_{2m+2}\rangle$. It follows from this and by [13, Proposition 6.8, p. 55] (see also [2, p. 137]) that $f(c)$ cannot be presented as a product of less than $m+1$ commutators. This contradicts our assumption. □

4. A PL model for $SC(S^1)$ and some related constructions

In this section we demonstrate piecewise linear constructions which are similar to $SC(S^1)$, using parameters for oscillations of a tube. Actually we prove in Theorem 4.3 that they are homotopy equivalent to the point, $SC(S^1)$, or $\mathcal{G}$ depending on their parameters.

For $0\leqslant y\leqslant 1$ and $\varepsilon\geqslant 0$ with $0<y+\varepsilon\leqslant 1$, we construct a space $S(y,\varepsilon)\subseteq\mathbb{R}^3$ as follows. Consider the following points on $\mathbb{I}^2$ for $n\in\mathbb{N}$ (see Fig. 2), where we regard $\mathbb{I}^2\subseteq\mathbb{R}^2$ as $\mathbb{I}^2\times\{0\}$:

$$A_n=\left(\frac{1}{2n-1},0\right),\qquad B_n=\left(\frac{1}{2n},1\right),\qquad C_n=\left(\frac{1}{2n-1},y+\frac{\varepsilon}{2n-1}\right),$$
$$D_n=\left(\frac{1}{2n},1-y-\frac{\varepsilon}{2n}\right),\qquad E_n=\left(\frac{1}{2n-1},\frac{1}{2}\left(y+\frac{\varepsilon}{2n-1}\right)\right),\qquad F_n=\left(\frac{1}{2n},\frac{1}{2}\left(2-y-\frac{\varepsilon}{2n}\right)\right).$$

Let $\overline{E}_n$ and $\overline{F}_n$ be points on the plane $\{(z,x,y)\in\mathbb{R}^3\mid z=\frac{1}{2}x\}$ the projections of which to the plane $\mathbb{R}^2$ are points E_n and F_n respectively, i.e.,

$$\overline{E}_n=\left(\frac{1}{2n-1},\frac{1}{2}\left(y+\frac{\varepsilon}{2n-1}\right),\frac{1}{2(2n-1)}\right),\qquad \overline{F}_n=\left(\frac{1}{2n},\frac{1}{2}\left(2-y-\frac{\varepsilon}{2n}\right),\frac{1}{4n}\right).$$

Let H_{2n-1} be the convex hull of the points A_n, B_n, C_n, D_n, $\overline{E}_n$ and $\overline{F}_n$ and H_{2n} the convex hull of the points A_{n+1}, B_n, C_{n+1}, D_n, $\overline{F}_n$ and $\overline{E}_{n+1}$.

Let H_∞ be the set $\bigcup_{n=1}^{\infty}H_n$ and ∂H_∞ its boundary. Let $\triangle A_1C_1\overline{E}_1$ be an open triangle in ∂H_∞. Finally, define $S(y,\varepsilon)$ to be the subspace $(\mathbb{I}^2\times\{0\})\cup\partial H_\infty\setminus\triangle A_1C_1\overline{E}_1$ of $\mathbb{R}^3$.

The first lemma is easy to prove and we therefore omit its proof.

Lemma 4.1. *Let $\varepsilon,\varepsilon'\in(0,1)$. Then the spaces $S(0,\varepsilon)$ and $S(0,\varepsilon')$ are homeomorphic and $S(0,1)$ is homotopy equivalent to $S(0,\varepsilon)$.*

Lemma 4.2. *If $0<y\leqslant 1/2$ and $0<y+\varepsilon\leqslant 1$, the space $S(y,\varepsilon)$ is homotopy equivalent to $S(1/2,0)$.*

Proof. It is easy to see that $S(y,\varepsilon)$ and $S(y,0)$ are homeomorphic and so we only need to prove that $S(y,0)$ for $0<y<1/2$ and $S(1/2,0)$ are homotopy equivalent. (Without any loss of generality we may assume that $y=1/3$.)

Since there might be some confusion regarding the homotopy equivalence, we explain this first. Let $A_n,B_n,C_n,D_n,\ldots$ be the notation for $S(1/2,0)$ and $C'_n,D'_n,\ldots$ be the corresponding notation for $S(1/3,0)$.

If we remove $\{0\}\times\mathbb{I}$ from $S(1/2,0)$ and $S(1/3,0)$, then the resulting spaces are homeomorphic, that is, $S(1/2,0)\setminus\{0\}\times\mathbb{I}$ and $S(1/3,0)\setminus\{0\}\times\mathbb{I}$ are homeomorphic. However, this homeomorphism cannot be extended over to $S(1/2,0)$, since the

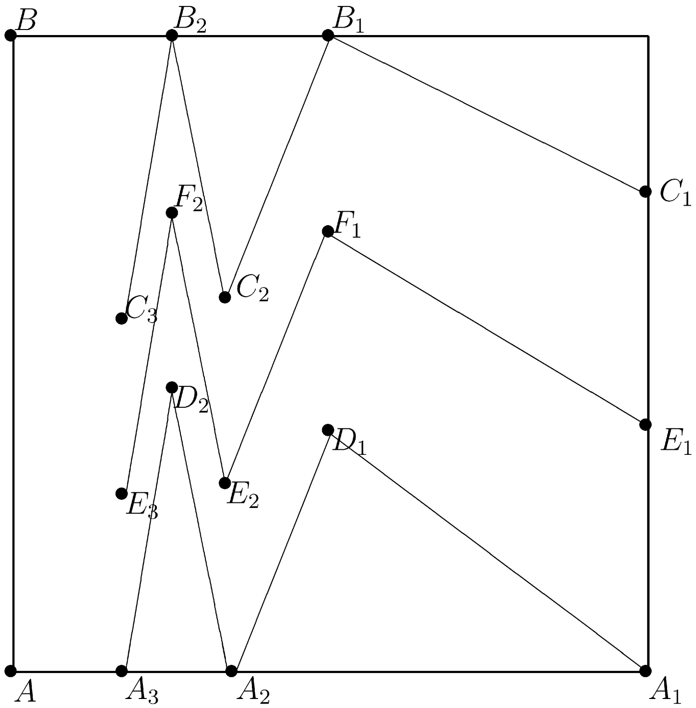

Fig. 2. $S(1/2, 1/4)$.

homeomorphism maps C_n to C'_n and D_n to D'_n, that is, upwards for C_n and downwards for D_n, with respect to the y-coordinate. Conversely, if we construct a homotopy on $S(1/2,0) \setminus \{0\} \times \mathbb{I}$ or $S(1/3,0) \setminus \{0\} \times \mathbb{I}$, whose projection to the y-coordinate only depends on the y-coordinate on the domain, it extends on $SC(1/2,0)$ or $SC(1/3,0)$.

We define $\varphi : S(1/2,0) \to S(1/3,0)$ and $\psi : S(1/2,0) \to S(1/3,0)$ piecewise linearly as follows:

Let $\varphi(x,y,0) = (x,y,0)$ and $\varphi(x,y,z) = (x,y,\varphi_2(x,y,z))$, for $z > 0$, where $\varphi_2(x,y,z) > 0$ if and only if $z > 0$ and there exists $z' > 0$ such that $(x,y,z') \in S(1/3,0)$. Let

$$\psi_1(y) = \begin{cases} 3y/2, & \text{for } 0 \leqslant y \leqslant 1/3, \\ 1/2, & \text{for } 1/3 \leqslant y \leqslant 2/3, \\ 3y/2 - 1/2, & \text{for } 2/3 \leqslant y \leqslant 1, \end{cases}$$

and $\psi(x,y,z) = (\psi_0(x,y,z), \psi_1(y), \psi_2(x,y,z))$, where $\psi_2(x,y,0) = 0$ and $\psi_2(x,y,z) > 0$, for $z > 0$ and $\psi_0(x,y,z)$ is defined as we explain using Fig. 3 in the sequel.

Fig. 3 demonstrates how $[\frac{1}{2n+1}, \frac{1}{2n}] \times \mathbb{I}$ of $S(1/2,0)$ and $S(1/3,0)$ are mapped by φ and ψ.

First we explain the map ψ. The two shadowed triangles are mapped to C_{n+1} or D_n, respectively. Accordingly, the segments $B'_nC'_{n+1}$ and $D'_nA'_{n+1}$ are mapped onto B_nC_{n+1} and D_nA_{n+1} respectively. The segments N'_nD_n and $C'_{n+1}M'_n$ are mapped bijectively to $C_{n+1}D_n$.

Next we explain the map $\varphi\psi$. The two shadowed triangles are mapped to $\varphi(C_{n+1})$ or $\varphi(D_n)$, which are the dotted point. The two bending segments are mapped onto $C'_{n+1}B'_n$ or $A'_{n+1}D'_n$.

Last we explain the map $\psi\varphi$. The two shadowed triangles are mapped to C_{n+1} or D_n. The two segments having slope greater than 1 are mapped to $C_{n+1}B_n$ or $A_{n+1}D_n$.

We have a homotopy $H(x,y,z,t)$ on $S(1/2,0) \setminus (\{0\} \times \mathbb{I})$ such that:

(1) $H(x,y,z,0) = (x,y,z)$ and $H(x,y,z,1) = \psi\varphi(x,y,z)$;
(2) for the y-coordinate $H_1(x,y,z,t)$ of $H(x,y,z,t)$,

$$H_1(x,y,z,t) = \begin{cases} y + yt/2, & \text{for } 0 \leqslant y \leqslant 1/3, \\ y + t/2 - yt, & \text{for } 1/3 \leqslant y \leqslant 2/3, \\ y - t/2 + yt/2, & \text{for } 2/3 \leqslant y \leqslant 1; \end{cases}$$

(3) $H(*,*,*,t)$ maps $p^{-1}([\frac{1}{n+1}, \frac{1}{n}] \times \mathbb{I})$ onto itself for each n.

Then we can extend $H(*,*,*,t)$ to $S(1/2,0)$ uniquely and continuously.

Concerning $S(1/3,0)$ with $\varphi\psi$, we have a homotopy with the same properties as above and we now see that $S(1/2,0)$ and $S(1/3,0)$ are homotopy equivalent. $\square$

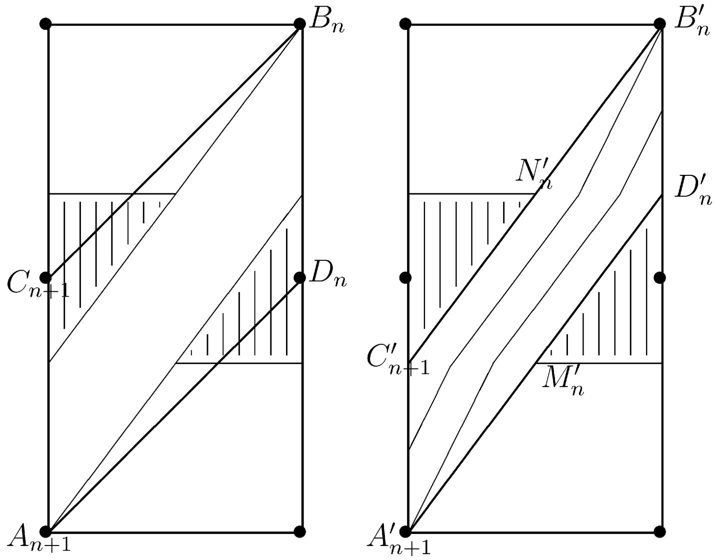

Fig. 3. Parts of $S(1/2, 0)$ and $S(1/3, 0)$.

Theorem 4.3. *Suppose that $0 \leqslant y \leqslant 1$, $\varepsilon \geqslant 0$ and $0 < y + \varepsilon \leqslant 1$. Then the following assertions hold*:

(1) *for every $1/2 < y \leqslant 1$, the spaces $S(1, 0)$ and $S(y, \varepsilon)$ are contractible*;
(2) *for $y = 0$, the space $S(y, \varepsilon)$ is homotopy equivalent to $SC(S^1)$*; *and*
(3) *for every $0 < y \leqslant 1/2$, the space $S(y, \varepsilon)$ is homotopy equivalent to the space $\mathcal{G}$.*

Proof. The statements (1) and (2) are easy to verify. Therefore we shall only prove (3).

By Lemma 4.2, it suffices to show that $S(1/2, 1/4)$ is homotopy equivalent to the space $\mathcal{G}$. Let $\triangle$ be the half-open triangle, defined as $\triangle = \{(x, y) \mid x \in (0, 1],\ y \in (-x/4 + 1/2, x/4 + 1/2)\}$. Then $p^{-1}(\mathbb{I}^2 \setminus \triangle)$ is a strong deformation retract of $S(1/2, 1/4)$.

Identifying $\{(x, y) \mid y = a + (1 - a)x/4,\ x \in \mathbb{I}\}$ as one point for $a \in [1/2, 1]$ and $\{(x, y) \mid y = a - ax/2,\ x \in \mathbb{I}\}$ as one point for $a \in [0, 1/2]$, we get the quotient space of $p^{-1}(\mathbb{I}^2 \setminus \triangle)$, which is homeomorphic to $\mathcal{G}$. Now the homotopy equivalence between $p^{-1}(\mathbb{I}^2 \setminus \triangle)$ and $\mathcal{G}$ is evident and so $S(1/2, 1/4)$ is indeed homotopy equivalent to $\mathcal{G}$. □

Remark 4.4. The space $SC(S^1)$ is simply connected (see [9]), whereas the space $\mathcal{G}$ is not simply connected (see [11]). We remark that $H_2(\mathcal{G}) = \{0\}$, which contrasts with Theorem 3.2.

To show this, we introduce some notation. Since the cone $C(X)$ over the space X is the quotient space of $X \times \mathbb{I}$, obtained by identifying $X \times \{1\}$ to a point, we let $p : X \times \mathbb{I} \to C(X)$ be the canonical projection.

For a subset A of $\mathbb{I}$, let $C_A(X) = p(X \times A) \subset C(X)$. Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be copies of the Hawaiian earring $\mathbb{H}$ and $\mathcal{G} = C(\mathbb{H}_1) \vee C(\mathbb{H}_2)$ be the one point union of $C(\mathbb{H}_1)$ and $C(\mathbb{H}_2)$ defined in Section 2. Let X_1 be the disjoint union of $C_{(1/3,1]}(\mathbb{H}_1)$ and $C_{(1/3,1]}(\mathbb{H}_2)$ and X_2 be $C_{[0,2/3)}(\mathbb{H}_1) \vee C_{[0,2/3)}(\mathbb{H}_2)$.

Then $\mathcal{G} = X_1 \cup X_2$ and we have the following part of the Mayer–Vietoris sequence:

$$H_2(X_1) \oplus H_2(X_2) \longrightarrow H_2(\mathcal{G}) \xrightarrow{\partial} H_1(X_1 \cap X_2) \xrightarrow{h} H_1(X_1) \oplus H_1(X_2).$$

Obviously, $H_2(X_1) = \{0\}$. Since X_2 is homotopy equivalent to $\mathbb{H}_1 \vee \mathbb{H}_2$ which is a 1-dimensional compact metric space, $H_2(X_2)$ is trivial [3]. Hence ∂ is injective. We observe that $X_1 \cap X_2$ is the disjoint union of $C_{(1/3,2/3)}(\mathbb{H}_1)$ and $C_{(1/3,2/3)}(\mathbb{H}_2)$.

Since $C_{[1/3,2/3)}(\mathbb{H}_1)$ and $C_{[1/3,2/3)}(\mathbb{H}_2)$ are retracts of $C_{[0,2/3)}(\mathbb{H}_1) \vee C_{[0,2/3)}(\mathbb{H}_2)$ and are homotopy equivalent to $C_{(1/3,2/3)}(\mathbb{H}_1)$ and $C_{(1/3,2/3)}(\mathbb{H}_2)$ respectively, it follows that h is injective. Therefore we obtain that $H_2(\mathcal{G}) = \{0\}$.

Problem 4.5. Does there exist a finite-dimensional non-contractible Peano continuum all homotopy groups of which are trivial?

Remark 4.6. Recently we have strengthened Theorem 3.1(2) by proving the following: If X is simply connected, then $\pi_2(SC(X))$ is trivial. We have proved earlier that $SC(X)$ is also simply connected [9]. Therefore by Theorem 3.1(1) the following statements are equivalent for any path-connected space X:

(1) X is simply connected;
(2) $\pi_2(SC(X))$ is trivial; and

(3) $H_2(SC(X))$ is trivial.

Acknowledgements

This paper was presented (by the third author) at the special session *Topology of Continua* of the AMS Spring Central Section Meeting in Lubbock, Texas (April 8–10, 2005). In June 2005, during his visit to Ljubljana, J. Dydak informed the third author that since then together with A. Mitra they have obtained an independent proof of Corollary 3.2 (their manuscript is not yet available). We were supported in part by the Japanese–Slovenian research grant BI-JP/03-04/2 and the Slovenian Research Agency research project No. J1-6128-0101-04 and program P1-0292-0101-04.

We thank the referee for several useful comments and suggestions.

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