Equilibrium of abstract economy and generalized quasi-variational inequality in $H$-spaces

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Received 28 November 2002; accepted 1 August 2003

Abstract

By using a neighborhood approximate method we give a new equilibrium existence theorem for abstract economies in $H$-spaces, and as applications, the existence problem for solution of generalized quasi-variational inequality in $H$-spaces is researched.

MSC: 47H04; 47H10; 52A99; 58E35; 90D13

Keywords: l.c.-space; Locally convex $H$-space; Lower semicontinuous multivalued mapping; Acyclic set

1. Introduction and preliminaries

Since Arrow and Debreu [2] established the existence theorem of Walrasian equilibria, the result has been generalized in many directions. In particular, by using Kakutani’s fixed point theorem, many mathematicians proved the powerful results on equilibrium existence of abstract economies (for example, see [8,11,15–18]).

In the present paper, by using a neighborhood approximate method we give a new equilibrium existence theorem for abstract economies in $H$-spaces, and as applications, the
existence problem for solution of generalized quasi-variational inequality in $H$-spaces is researched.

In order to establish our main results, we give some concepts and notations.

Let $X$ be a topological space and $\mathcal{F}(X)$ the family of all nonempty finite subset of $X$. Let $\{\Gamma_A\}$ be a family of some nonempty contractible subsets of $X$ indexed by $A \in \mathcal{F}(X)$ such that $\Gamma_A \subset \Gamma_{A'}$ whenever $A \subset A'$. The pair $(X, \{\Gamma_A\})$ is called an $H$-space. Given an $H$-space $(X, \{\Gamma_A\})$, a nonempty subset $D$ of $X$ is called to be

1. $H$-convex if $\Gamma_A \subset D$ for all $A \in \mathcal{F}(D)$;
2. weakly $H$-convex if $\Gamma_A \cap D$ is nonempty contractible for each $A \in \mathcal{F}(D)$.

For a nonempty subset $K$ of $X$, we define the $H$-convex hull of $K$, denoted by $H$-co $K$, as

$$H$$-co $K = \bigcap \{D \subset X : D$ is $H$-convex and $K \subset D\}.$

If $K = \emptyset$, we always consider $H$-co $K = \emptyset$ (see also, [3,9,10]).

An $H$-space $(X, \{\Gamma_A\})$ is called to be

1. a locally convex $H$-space if $X$ is an uniform space and if there exists a base $\{V_i : i \in I\}$ for the uniform structure $\mathcal{U}$ such that for each $i \in I$, $V_i(x) = \{y \in X : (y, x) \in V_i\}$ is $H$-convex for each $x \in X$ [17];
2. an l.c.-space (see [9]) if $X$ is an uniform space and if there exists a base $\{V_i : i \in I\}$ for the uniform structure such that for each $i \in I$, the set $\{x \in X : E \cap V_i[x] \neq \emptyset\}$ is $H$-convex whenever $E$ is $H$-convex, where $V_i[x] = \{y \in X : (x, y) \in V_i\}$.

**Remark.** The concept of an l.c.-space is different from a locally convex $H$-space. But an l.c.-space $(X, \{\Gamma_A\})$ with $\Gamma_\{x\} = \{x\}$ for all $x \in X$ must be a locally convex $H$-space. Otherwise, a nonempty convex subset $X$ of a locally convex topological vector space must be an l.c.-space with $\Gamma_A = \text{co} \ A$ for all $A \in \mathcal{F}(X)$, and hence $(X, \{\text{co} \ A\})$ must be a locally convex $H$-space.

Let $X$ be a topological space. We denote by $2^X$ the family of all subsets of $X$. If $A \subset X$ we shall denote by $\text{cl}(A)$ the closure of $A$. A topological space is called to be acyclic if all of its reduced Čech homology groups over rationals vanish. In particular, any contractible space is acyclic, and thus any convex or star-shaped set is acyclic.

Let $X, Y$ be two topological spaces and $S, T : X \rightarrow 2^Y$ two multivalued mappings. $T$ is called to be upper semicontinuous (respectively lower semicontinuous) if for each $x \in X$ and each open set $V \subset Y$ with $T(x) \subset V$ (respectively $T(x) \cap V \neq \emptyset$), there exists an open neighborhood $U$ of $x$ such that $T(z) \subset V$ (respectively $T(z) \cap V \neq \emptyset$) for each $z \in U$. $T$ is called to be almost upper semicontinuous if for each $x \in X$ and each open set $V \subset Y$ with $T(x) \subset V$ there exists an open neighborhood $U$ of $x$ such that $T(z) \subset \text{cl} \ V$ for all $z \in U$.

For each $y \in Y$, we denote $T^{-1}(y) = \{x \in X : y \in T(x)\}$, which is called the lower section of $T$. The multivalued mappings $S \cap T, \text{cl}(T) : X \rightarrow 2^Y$ are defined by

$$S \cap T(x) = S(x) \cap T(x), \quad \text{cl}(T)(x) = \text{cl}(T(x)), \quad \forall x \in X.$$
Now, we recall the following general definitions of equilibrium theory in mathematical economics. Let \( I \) be a finite or an infinite set of agents. For each \( i \in I \), let \( X_i \) be a nonempty set of actions. An abstract economy (or generalized game) \( \Omega = (X_i, A_i, B_i, P_i)_{i \in I} \) is defined as a family of ordered quadruples \( (X_i, A_i, B_i, P_i) \), where \( X_i \) is a nonempty topological space (a choice set), \( A_i, B_i : X := \prod_{j \in I} X_j \to 2^{X_i} \) are constraint correspondences and \( P_i : \bar{X} := \prod_{j \in I} \bar{X}_j \to 2^{X_i} \) are preference correspondences. An equilibrium of \( \Omega \) is a point \( \bar{x} \in X \) such that for each \( i \in I \), \( \bar{x}_i \in cl(B_i(\bar{x})) \) and \( P_i(\bar{x}) \cap A_i(\bar{x}) = \emptyset \). When \( A_i = B_i \) and \( X_i \) is a topological vector space for each \( i \in I \), our definitions of an abstract economy and an equilibrium coincide with the standard definition of Shafer and Sonnenschein [15].

2. Equilibrium existence theorems for abstract economies

We begin with the following lemmas.

**Lemma 1.** Let \( X \) be a topological space, \((Y, U)\) an uniform space and \( B \) a base of \( U \). Let \( T : X \to 2^Y \) be a multivalued mapping. If \( \bar{x} \in X \) and \( \bar{y} \in Y \) such that
\[
\bar{y} \in \bigcap_{V \in B} V(\bar{T}(\bar{x}))
\]
(where, \( \bar{T}(\bar{x}) = \{ y \in Y : (\bar{x}, y) \in cl(graph \ T) \} \), \( V(\bar{T}(\bar{x})) = \bigcup_{y \in \bar{T}(\bar{x})} V(y) \)), then \( \bar{y} \in \bar{T}(\bar{x}) \).

**Proof.** If \( \bar{y} \notin \bar{T}(\bar{x}) \), then \((\bar{x}, \bar{y}) \notin cl(graph \ T) \). Consequently, there exist an open neighborhood \( U \) of \( \bar{x} \) and an open symmetric element \( V \in U \) such that
\[
(U \times V(\bar{y})) \cap cl(graph \ T) = \emptyset.
\]
Take an element \( W \in B \) such that \( W \subset V \). Since \( \bar{y} \in W(\bar{T}(\bar{x})) \subset V(\bar{T}(\bar{x})) \), there is a point \( \bar{z} \in \bar{T}(\bar{x}) \) such that \( \bar{y} \in V(\bar{z}) \), and hence \( \bar{z} \in V(\bar{y}) \) since \( V \) is symmetric. Hence
\[
(\bar{x}, \bar{z}) \in (U \times V(\bar{y})) \cap cl(graph \ T).
\]
It contradicts that \( (U \times V(\bar{y})) \cap cl(graph \ T) = \emptyset \). Therefore, \( \bar{y} \in \bar{T}(\bar{x}) \). \( \square \)

**Lemma 2** [17]. Let \( \{(X_i, \{\Gamma_i^{(j)}\} : i \in I) \} \) be a family of Hausdorff locally convex H-spaces and \( X = \prod_{i \in I} X_i \). If for each \( i \in I \), \( D_i \) is a H-compact subset of \( X_i \) and \( T_i : X \to 2^{D_i} \) is an upper semicontinuous multivalued mapping with closed acyclic values, then there exists a point \( \bar{x} = \prod_{i \in I} \bar{x}_i \in D := \prod_{i \in I} D_i \) such that \( \bar{x}_i \in T_i(\bar{x}) \) for all \( i \in I \).

Now, we establish our main results.

**Theorem 3.** Let \( \Omega = (X_i, A_i, B_i, P_i)_{i \in I} \) be an abstract economy. For each \( i \in I \), let \( (X_i, \{\Gamma_i^{(j)}\}) \) be a Hausdorff l.c.-space with the uniformity \( U_i \) and \( \Gamma_i^{(j)} = \{ x_j \} \) for all \( x_j \in X_i \), \( D_i \) a nonempty compact weakly H-convex subset of \( X_i \), and let \( V_i \) be a base of \( U_i \), which consists of closed symmetric elements in \( U_i \). If for each \( i \in I \), the following conditions are fulfilled:
(i) for each \( x \in X := \prod_{i \in I} X_i \), \( H\)-co \( A_i(x) \subset B_i(x) \subset D_i \) and \( D_i \cap [V(\text{cl}(B_i(x)))] \) is acyclic for each \( V \in \mathcal{V}_i \),

(ii) \( A_i : X \to 2^{D_i} \) is lower semicontinuous and \( P_i : X \to 2^{D_i} \) has open lower sections,

(iii) for each \( x \in X \), \( x_i \notin H\)-co \( P_i(x) \),

(iv) \( B_i : X \to 2^{D_i} \) is almost upper semicontinuous.

If \( X \) is perfectly normal, then there exists a point \( \bar{x} = \prod_{i \in I} \bar{x}_i \in D := \prod_{i \in I} D_i \) such that

\[
\bar{x}_i \in \text{cl}(B_i(\bar{x})) \quad \text{and} \quad A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset
\]

to all \( i \in I \).

**Proof.** For any fixed \( V \in \mathcal{V} := \prod_{i \in I} \mathcal{V}_i \), let \( V = \prod_{i \in I} V_i \). For each \( i \in I \), let \( B_i \) be a base of \( U_i \) such that for each \( W \in B_i \), the set \( \{ y \in X_i : W[y] \cap M \neq \emptyset \} \) is \( H \)-convex whenever \( M \subset X_i \) is \( H \)-convex.

For each \( i \in I \), there exist a \( U \in B_i \) and an open symmetric element \( G \in U_i \) such that \( G \subset U \subset V_i \). Since \( (X_i, \{ \Gamma_i \}) \) is a Hausdorff l.c.-space with \( \Gamma_i = \{ x_i \} \) for all \( x_i \in X_i \), \( (X_i, \{ \Gamma_i \}) \) is a Hausdorff locally convex \( H \)-space, and hence \( (D_i, D_i \cap \{ \Gamma_i \}) \) is also a Hausdorff locally convex \( H \)-space. For each \( x \in D \), let

\[
(A_i)_G(x) = G[A_i(x)] \cap D_i.
\]

Then \( (A_i)_G : D \to 2^{D_i} \) is a multivalued mapping such that for each \( y \in D_i \),

\[
(A_i)_G^{-1}(y) = \left\{ x \in D : y \in (A_i)_G(x) \right\} = \left\{ x \in D : y \in G[A_i(x)] \right\} = \left\{ x \in D : A_i(x) \cap G[y] \neq \emptyset \right\}
\]

is open in \( D \) since \( A_i : X \to 2^{D_i} \) is lower semicontinuous and \( G \in U_i \) is open and symmetric. Hence \( P_i \cap (A_i)_G : D \to 2^{D_i} \) has open lower sections in \( D \) by (ii). Consequently, the set

\[
W_i = \{ x \in D : P_i(x) \cap (A_i)_G(x) \neq \emptyset \}
\]

is open in \( D \). Note that \( X \) is perfectly normal and \( D \) is compact, and hence \( D \) is perfectly normal and compact so that \( W_i \) is paracompact. By virtue of Theorem 2 of Horvath [9], there exists a continuous mapping \( f_i : W_i \to D_i \) such that

\[
f_i(x) \in H\text{-co}[ (A_i)_G(x) ] \cap P_i(x) \subset H\text{-co} P_i(x)
\]

for all \( x \in W_i \).

Define a multivalued mapping \( F_i : D \to 2^{D_i} \) by

\[
F_i(x) = \begin{cases} 
\{ f_i(x) \}, & \text{if } x \in W_i, \\
D_i \cap [ V_i(\text{cl}(B_i(x))) ] & \text{if } x \in D \setminus W_i.
\end{cases}
\]

Then \( F_i : D \to 2^{D_i} \) is a multivalued mapping with closed acyclic values by the second part of (i).
By (iv) and the compactness of $D_i$ we know that the mapping $\text{cl} B_i : D \to 2^{D_i}$ is upper semicontinuous and hence the mapping $K_i : D \to 2^{D_i}$ defined by

$$K_i(x) = D_i \cap V_i(\text{cl}(B_i(x))), \quad \forall x \in D,$$

is also upper semicontinuous by the closeness of $V_i$. Note that

$$H-\text{co}\left[(A_i)G(x) \cap P_i(x)\right] \subset H-\text{co}\left[A_i(x)\right] \subset H-\text{co}(U[A_i(x)]) \subset U[H-\text{co}A_i(x)] \subset V_i[\text{cl}(B_i(x))].$$

Hence, for each open set $N$ in $D_i$, the set

$$\{x \in D : F_i(x) \subset N\} = \{x \in W_i : f_i(x) \in N\} \cup \{x \in D \setminus W_i : D_i \cap V_i(\text{cl}(B_i(x))) \subset N\}$$

is open in $D$. It shows that $F_i : D \to 2^{D_i}$ is upper semicontinuous.

By virtue of Theorem 3 of Wu [17] i.e., Lemma 2, there exists a point $\bar{z} \in D$ such that $\bar{z}_i \in F_i(\bar{z})$ for all $i \in I$. Note that $\bar{z}_i \notin H-\text{co}P_i(\bar{z})$ (by (iv)) and $f_i(\bar{z}) \in H-\text{co}P_i(\bar{z})$, we have

$$\bar{z}_i \in V_i(\text{cl}(B_i(\bar{z}))) \quad \text{and} \quad (A_i)G(\bar{z}) \cap P_i(\bar{z}) = \emptyset$$

for all $i \in I$. Let

$$K_V = \{x \in D : x_i \in K_i(x) \text{ and } (A_i)G(x) \cap P_i(x) = \emptyset, \forall i \in I\}.$$

Then the set $K_V$ is nonempty closed and the family $\{K_V : V \in V\}$ has finite intersection property, obviously. Hence

$$\bigcap_{V \in V} K_V \neq \emptyset$$

since $D$ is compact. Take a point

$$\bar{x} \in \bigcap_{V \in V} K_V.$$

Then for each $i \in I$, we have $\bar{x} \in D_i$ and

$$\bar{x}_i \in V_i(\text{cl}(B_i(\bar{x}))) \quad \text{and} \quad A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset,$$

for all $V_i \in V_i$, and hence

$$\bar{x}_i \in \text{cl}(B_i(\bar{x})) \quad \text{and} \quad A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$$

for all $i \in I$ by Lemma 1. This completes the proof. \qed

**Corollary 4.** Let $\Omega = (X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy. Suppose that for each $i \in I$, the following conditions are fulfilled:
(i) \( X_i \) is a nonempty convex subset of a Hausdorff locally convex topological vector space \( E_i \) and \( D_i \) is a nonempty compact convex subset of \( X_i \),
(ii) for each \( x \in X := \prod_{i \in I} X_i \), \( A_i(x) \subset B_i(x) \subset D_i \) and \( B_i(x) \) is nonempty convex,
(iii) \( A_i : X \to 2^{D_i} \) is lower semicontinuous and \( P_i : X \to 2^{D_i} \) has open lower sections,
(iv) for each \( x \in X \), \( x_i \notin \text{co} \ P_i(x) \),
(v) \( B_i : X \to 2^{D_i} \) is almost upper semicontinuous.

If \( X \) is perfectly normal, then there exists a point \( \bar{x} = \prod_{i \in I} \bar{x}_i \in D := \prod_{i \in I} D_i \) such that
\[
\bar{x}_i \in \text{cl}(B_i(\bar{x})) \quad \text{and} \quad A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset
\]
for all \( i \in I \).

**Proof.** Let \( U_i \) be the family of all neighborhoods of zero in \( E_i \). For each \( U_i \in U_i \), let
\[
\tilde{U}_i = \{(x, y) \in X_i \times X_i : x - y \in U_i \}\.
\]
Then \( (X_i, \text{co} A) (A \in \mathcal{F}(X_i)) \) is an l.c.-space with uniformity
\[
\tilde{U}_i = \{\tilde{U}_i : U_i \in U_i \}.
\]
Consequently, the conclusion follows from Theorem 3. \( \square \)

**Corollary 5.** Let \( \Omega = (X_i, A_i, B_i, P_i)_{i \in I} \) be an abstract economy. Suppose that for each \( i \in I \), the following conditions are fulfilled:

(i) \( X_i \) is a nonempty closed convex subset of a Hausdorff quasi-complete locally convex topological vector space \( E_i \) and \( D_i \) is nonempty compact subset of \( X_i \),
(ii) for each \( x \in X := \prod_{i \in I} X_i \), \( A_i(x) \subset B_i(x) \subset D_i \) and \( B_i(x) \) is nonempty convex,
(iii) \( A_i : X \to 2^{D_i} \) is lower semicontinuous and \( P_i : X \to 2^{D_i} \) has open lower sections,
(iv) for each \( x \in X \), \( x_i \notin \text{co} \ P_i(x) \),
(v) \( B_i : X \to 2^{D_i} \) is almost upper semicontinuous.

Then there exists a point \( \bar{x} = \prod_{i \in I} \bar{x}_i \in D := \prod_{i \in I} D_i \) such that
\[
\bar{x}_i \in \text{cl}(B_i(\bar{x})) \quad \text{and} \quad A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset
\]
for all \( i \in I \).

**Proof.** Since \( E_i \) is a Hausdorff quasi-complete locally convex topological vector space and \( D_i \) is nonempty compact subset of \( E_i \), \( \overline{\text{co}} \ D_i \) is compact convex and hence \( \prod_{i \in I} \overline{\text{co}} \ D_i \) is compact convex. By Corollary 4 there exists a point \( \bar{x} = \prod_{i \in I} \bar{x}_i \in \prod_{i \in I} \overline{\text{co}} \ D_i \) such that
\[
\bar{x}_i \in \text{cl}(B_i(\bar{x})) \subset D_i \quad \text{and} \quad A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset
\]
for all \( i \in I \). \( \square \)

**Remark.** Corollaries 4 and 5 (consequently, Theorem 3) are new equilibrium existence theorems for abstract economies. (For example, see [8,11,15–18].)
3. Generalized quasi-variational inequalities

In the following, we research the existence problem for solutions of generalized quasi-variational inequalities by using the above Theorem 3.

**Theorem 6.** Let \((X, \Gamma^x)\) be a perfectly normal Hausdorff l.c.-space with uniformity \(U\) and \(\Gamma^x = \{x\}\) for all \(x \in X\), \(D \subset X\) a nonempty compact weakly \(H\)-convex subset, \(V\) a base of \(U\), which consists of closed symmetric elements in \(U\), and \(Y\) a Hausdorff topological space. Let \(T : X \to 2^Y\) be an upper semicontinuous multivalued mapping with nonempty compact values, \(A : X \to 2^D\) be a lower semicontinuous and almost upper semi-continuous multivalued mapping with nonempty \(H\)-convex values. If \(\phi : X \times Y \times X \to \mathbb{R} \cup \{\pm \infty\}\) such that

1. \(\phi(x, y, z)\) is upper semicontinuous in \((x, y)\) and \(H\)-quasiconvex in \(z\),
2. for each \(x \in X\), there exists \(y \in T(x)\) such that \(\phi(x, y, x) \geq 0\),
3. for each \(x \in X\) and each \(V \in V\), the set \(D \cap [V(\text{cl}(A(x)))]\) is acyclic,

then there exists a point \(\bar{x} \in D\) such that \(\bar{x} \in \text{cl}(A(\bar{x}))\) and

\[
\sup_{y \in T(\bar{x})} \phi(\bar{x}, y, x) \geq 0
\]

for all \(x \in A(\bar{x})\).

**Proof.** Define a multivalued mapping \(P : X \to 2^D\) by

\[
P(x) = \left\{ z \in D : \sup_{y \in T(x)} \phi(x, y, z) < 0 \right\}, \quad \forall x \in X.
\]

Then \(P(x)\) is \(H\)-convex by the second part of (i). Since \(\phi(x, y, z)\) is upper semicontinuous in \((x, y)\) and \(T : X \to 2^Y\) is an upper semicontinuous multivalued mapping with nonempty compact values, \(\sup_{y \in T(x)} \phi(x, y, z)\) is upper semicontinuous in \(x\) by Proposition 21 of [1, p. 119], and hence for each \(z \in D\),

\[
P^{-1}(z) = \left\{ x \in X : z \in P(x) \right\}
\]

is open in \(X\). Moreover, by (ii) we know that \(x \notin P(x) = H\text{-co } P(x)\) for all \(x \in X\). Consequently, by virtue of Theorem 3, there exists a point \(\bar{x} \in D\) such that \(\bar{x} \in \text{cl}(A(\bar{x}))\) and \(A(\bar{x}) \cap P(\bar{x}) = \emptyset\), i.e.

\[
\sup_{y \in T(\bar{x})} \phi(\bar{x}, y, x) \geq 0
\]

for all \(x \in A(\bar{x})\). This completes the proof. \(\square\)

Corollary 7 (Kim [12]). Let $X$ be a nonempty compact convex subset of a Hausdorff locally convex topological vector space $E$ and $E^*$ the conjugate space. Let $T : X \to 2^{E^*}$ be an upper semicontinuous multivalued mapping with nonempty compact values, $A : X \to 2^X$ be a lower semicontinuous and almost upper semicontinuous multivalued mapping with nonempty convex values. Then there exists a point $\tilde{x} \in X$ such that $\tilde{x} \in cl(A(\tilde{x}))$ and
\[
\sup_{y \in T(\tilde{x})} \text{Re} \langle y, x - \tilde{x} \rangle \geq 0
\]
for all $x \in A(\tilde{x})$.

Proof. Let $\mathcal{U}$ be the family of all neighborhoods of zero in $E$. For each $U \in \mathcal{U}$, let $\tilde{U} = \{ (x, y) \in X \times X : x - y \in U \}$. Then $(X, \{co A\}) (A \in \mathcal{F}(X))$ is an l.c.-space with the uniformity $\tilde{U} = \{ U : U \in \mathcal{U} \}$. For each $(x, y, z) \in X \times E^* \times X$, let $\phi(x, y, z) = \text{Re} \langle y, z - x \rangle$. Then $\phi$ is continuous by Lemma B of Kum [13]. Consequently, the conclusion follows from Theorem 6. □

Theorem 8. Let $(X, \{\Gamma_A\})$ be a perfectly normal Hausdorff l.c.-space with uniformity $\mathcal{U}$ and $\Gamma_{[x]} = \{ x \}$ for all $x \in X$, $D \subseteq X$ a nonempty compact weakly $H$-convex subset of $X$, $V$ a base of $\mathcal{U}$, which consists of closed symmetric elements in $\mathcal{U}$, and $Y$ a Hausdorff topological space. Let $T : X \to 2^Y$ be an upper semicontinuous multivalued mapping with nonempty compact values, $A : X \to 2^D$ be a lower semicontinuous and almost upper semicontinuous multivalued mapping with nonempty $H$-convex values. If an upper semicontinuous function $\phi : X \times Y \times X \to R \cup \{ \pm \infty \}$ is such that

(i) $\phi(x, y, z)$ is $H$-quasiconvex in $z$,
(ii) for each $x \in X$, there exists $y \in T(x)$ such that $\phi(x, y, x) \geq 0$,
(iii) for each $x \in X$ and each $V \in \mathcal{V}$, the set $D \cap [V(cl(A(x)))]$ is acyclic,
(iv) for each $(x, z) \in D \times D$, the set $\{ y \in T(x) : \phi(x, y, z) \geq 0 \}$ is acyclic,

then there exists a point $\tilde{x} \in cl(A(\tilde{x}))$ and a point $\tilde{y} \in T(\tilde{x})$ such that
\[
\phi(\tilde{x}, \tilde{y}, x) \geq 0, \quad \forall x \in A(\tilde{x}).
\]

Proof. By Theorem 6 there exists a point $\tilde{x} \in D$ such that $\tilde{x} \in cl(A(\tilde{x}))$ and
\[
\sup_{y \in T(\tilde{x})} \phi(\tilde{x}, y, x) \geq 0
\]
for all $x \in A(\tilde{x})$. Since $\phi$ is upper semicontinuous and $T(\tilde{x})$ is compact, for each $x \in A(\tilde{x})$ there exists a point $y(x) \in T(\tilde{x})$ such that $\phi(\tilde{x}, y(x), x) \geq 0$. Define a multivalued mapping $G : A(\tilde{x}) \to 2^{T(\tilde{x})}$ by
\[
G(x) = \{ y \in T(\tilde{x}) : \phi(\tilde{x}, y, x) \geq 0 \}, \quad \forall x \in A(\tilde{x}).
\]
Then $G : A(\tilde{x}) \to 2^{T(\tilde{x})}$ is a multivalued mapping with nonempty acyclic values and closed graph (in $A(\tilde{x}) \times T(\tilde{x})$) by (iv) and the upper semicontinuity of $\phi$. Consequently, $G : A(\tilde{x}) \to 2^{T(\tilde{x})}$ is upper semicontinuous since $T(\tilde{x})$ is compact.
If the conclusion of Theorem 8 is false, then for each \( y \in T(\bar{x}) \), there exists a point \( x \in A(\bar{x}) \) such that \( \phi(\bar{x}, y, x) < 0 \). Define a multivalued mapping \( F : T(\bar{x}) \to 2^{A(\bar{x})} \) by

\[
F(y) = \left\{ x \in A(\bar{x}) : \phi(\bar{x}, y, x) < 0 \right\}, \quad \forall y \in T(\bar{x}).
\]

Then \( F : T(\bar{x}) \to 2^{A(\bar{x})} \) is a multivalued mapping with nonempty \( H \)-convex values by (i) and the \( H \)-convexity of \( A(\bar{x}) \). For each \( x \in A(\bar{x}) \), the set

\[
F^{-1}(x) = \left\{ y \in T(\bar{x}) : x \in F(y) \right\} = \left\{ y \in T(\bar{x}) : \phi(\bar{x}, y, x) < 0 \right\}
\]

is open in \( T(\bar{x}) \). By virtue of Theorem 3.1 of Ding and Tarafdar [7], there exist \( x_0 \in A(\bar{x}) \) and \( y_0 \in T(\bar{x}) \) such that \( x_0 \in F(y_0) \) and \( y_0 \in G(x_0) \), i.e. \( \phi(\bar{x}, y_0, x_0) < 0 \) and \( \phi(\bar{x}, y_0, x_0) \geq 0 \). It is a contradiction. This completes the proof.

**Corollary 9.** Let \( X \) be a nonempty compact convex subset of a Hausdorff locally convex topological vector space \( E \) and \( E^* \) the conjugate space. Let \( T : X \to 2^{E^*} \) be an upper semicontinuous multivalued mapping with nonempty compact convex values, \( A : X \to 2^X \) be a lower semicontinuous and almost upper semicontinuous multivalued mapping with nonempty convex values. Then there exists a point \( \bar{x} \in \text{cl}(A(\bar{x})) \) and a point \( \bar{y} \in T(\bar{x}) \) such that

\[
\text{Re} \langle \bar{y}, \bar{x} - x \rangle \leq 0
\]

for all \( x \in A(\bar{x}) \).

**Proof.** Let \( \mathcal{U} \) be the family of all neighborhoods of zero in \( E \). For each \( U \in \mathcal{U} \), let \( \tilde{U} = \{(x, y) \in X \times X : x - y \in U\} \). Then (\( X, \{\text{co} A\} \) (\( A \in \mathcal{F}(X) \)) is a l.c.-space with uniformity \( \tilde{U} = \{\tilde{U} : U \in \mathcal{U}\} \). For each \( (x, y, z) \in X \times E^* \times X \), let \( \phi(x, y, z) = \text{Re} \langle y, z - x \rangle \). Then \( \phi \) is continuous by Lemma B of Kum [13]. Consequently, the conclusion follows from Theorem 7.

**Remark.** Corollary 9 (further, Theorem 8) generalizes and improves Theorem 4 of Shih and Tan [14], Theorem 6 of Browder [4] and corresponding results in Chan and Pang [5] and Kim [12].

**References**