Smoothing Riemannian metrics with bounded Ricci curvatures in dimension four

Ye Li

Department of Mathematics, Cornell University, Ithaca, NY 14853, United States

Received 12 February 2009; accepted 20 October 2009
Available online 30 October 2009
Communicated by Gang Tian

Abstract

We obtain a local smoothing result for Riemannian manifolds with bounded Ricci curvatures in dimension four. More precisely, given a Riemannian metric with bounded Ricci curvature and small $L^2$-norm of curvature on a metric ball, we can find a smooth metric with bounded curvature which is $C^{1,\alpha}$-close to the original metric on a smaller ball but still of definite size.

© 2009 Elsevier Inc. All rights reserved.

Keywords: Local Ricci flow

1. Introduction

It is of fundamental interest to understand topological and geometric structures of manifolds with suitable curvature bounds. To this end, a natural idea is to deform or smooth a given metric with bounded Ricci curvature to a metric with bounded curvature, since the local structure of the latter is well understood. On the other hand, by the results of Bemelmans, Min-Oo and Ruh [7], Bando [5], Abresch [1] and Shi [24], a metric with bounded curvature can always be approximated by smooth metrics with bounded covariant derivatives of curvature.

In general, one cannot always do the smoothing only under the assumption of bounded Ricci curvature since there is a sequence of Einstein metrics with fixed Einstein constant which degenerates to a singular space. Consequently, additional geometric conditions are needed for...
smoothing metrics with bounded Ricci curvature. In this paper, we consider Riemannian 4-
manifold with bounded Ricci curvature and provide a smoothing procedure locally in a ball 
of definite size with sufficiently small $L^2$-norm of curvature, where no Sobolev constant (or the 
volume lower bound equivalently) of the ball under consideration is assumed. The argument in 
this paper is inspired by the paper [14] in the Einstein 4-manifolds. In dimension 4, our result 
partially generalizes the one by Deane Yang [27], where he provided a way to deform the metric 
with some suitable integral curvature bound (Ricci curvature is $L^p$ for some $p > \frac{n}{2}$). His result 
needs the noncollapsing condition, that is, the $L^n$-norm of the curvature is small against the local 
Sobolev constant.

For the Einstein manifolds, we have the stronger result. It is well known that in dimensions 2 
and 3 any Einstein manifold has constant sectional curvature. Also, for Einstein $n$-manifolds the 
curvature tensor satisfies the following elliptic inequality

$$\triangle |Rm| + c(n)|Rm|^2 \geq 0.$$ 

Therefore, for $n \geq 4$, one can use Moser iteration argument to obtain the $\epsilon$-regularity theorem 
under the assumption that the $L^{n/2}$-norm of the curvature is sufficiently small against the Sobolev 
constant. For example, one can see [2,6,20] (in the Riemannian case) and [25] (in the Kähler 
case), where the Sobolev constant bound was assumed.

It is well known that under the condition of Ricci curvature lower bound, the local Sobolev 
constant bound can be replaced by the volume lower bound of a geodesic ball. In 1989, 
Varopoulos [26] proved the $L^1$-Sobolev inequality on the complete Riemannian manifold with 
lower bounded Ricci curvature. Later, Saloff-Coste in [21,22], Anderson in [4], independently 
showed the equivalence of the local Sobolev constant bound and the volume lower bound 
of a geodesic ball, when the Ricci curvature is bounded from below. Then, Anderson in [4] 
proved that the $\epsilon$-regularity theorem holds if the $L^{n/2}$-norm of the curvature is sufficiently 
small against the volume of a geodesic ball. This result can thus be applied to the collapsed 
Einstein manifold in the scale that there is a control on Sobolev constant. On the other hand, 
if the Ricci curvature is bounded from below, Buser [8] obtained a sharp Poincaré inequality 
apart from the dimensional constant, which is equivalent to the local Sobolev constant bound 
as shown in [23]. The author thanks Professor Laurent Saloff-Coste for pointing out to him 
Refs. [8,21,22,26].

One of the tools to study the existence of Einstein metrics is the Ricci flow. In general, the 
solutions may develop singularities in the limit. So we need to consider how Einstein metrics 
can degenerate. Although the noncollapsed situation has been studied thoroughly, for example, 
in [2,4,16,20,25], etc., the results for the collapsed case are limited, which initiates the Cheeger–
Tian’s program whose goal is to obtain a complete understanding of how Einstein metrics on 
4-manifolds can degenerate. Recently, there has been a breakthrough in the research of collapsed 
Einstein 4-manifolds. In [14], Cheeger and Tian proved an $\epsilon$-regularity theorem for such mani-

folds without the Sobolev constant bound assumption. Using this curvature estimate, they proved 
a compactness theorem for Einstein metrics in dimension 4. However, their argument in [14] may 
not be immediately generalized to higher dimensions, since the Gauss–Bonnet–Chern formula 
was heavily employed in the proof. This paper is also a part of Cheeger–Tian’s program. We 
provide a way to smooth the metrics of bounded Ricci curvatures locally (but of a definite size 
in the collapsed case) in dimension 4, when the curvature tensors are not concentrated.
For convenience, we will outline the arguments employed in [14] and discuss what have been already known to extend the result there to our Ricci bounded case and what have to be elaborated in this paper.

Suppose that $M$ is a complete Einstein 4-manifold with Einstein constant $|\lambda| \leq 3$. Fix $r \leq 1$.

To show the $\epsilon$-regularity theorem, Cheeger and Tian [14] obtained the following technical result:

If there exists some $\epsilon_0 > 0$ such that

$$\int_{B_1(x)} |\text{Rm}|^2 < \epsilon_0,$$

then for some constant $c > 0$,

$$\frac{\text{Vol}(B_{cr}(x))}{\text{Vol}(B_{cr}(x))} \int_{B_{cr}(x)} |Rm|^2 < \epsilon,$$

where $x$ is a point in the simply connected space of constant curvature $-1$. Hence, we can apply Moser’s iteration argument to get a curvature bound on the ball $B_{\frac{1}{2}cr}(x)$ (see [4] for details).

Notice that in the Einstein $n$-manifold the curvature can be bounded with respect to some local scale which is bounded from below by the maximal function of $|Rm|^2$ (see [14] for details), while in the Ricci bounded case, no such local scale is available. We have to smooth the metric in the local scale to obtain the curvature bound for some nearby metric (in the $C^0$ sense).

There are several steps to achieve this technical result.

First, one needs to obtain an equivariant version of good chopping in the case of locally bounded curvature. Using scaling, one can apply the argument in [13], which were originally used for spaces with bounded curvature, to constructing a submanifold $Z^n$ with smooth boundary which approximates a compact domain $K$ such that the boundary term in the Gauss–Bonnet–Chern formula satisfies

$$\left| \int_{\partial Z^n} T P_X \right| \leq c(n)r^{-1} \int_{A_{\frac{1}{2}r}(K)} \left( r^{-(n-1)} + (r_{\text{Rm}})^{(n-1)} \right),$$

where $A_{r_1,r_2}(K) = T_{r_2}(K) \setminus \overline{T_{r_1}(K)}$, $T_r(K) = \{ x \in M \mid \text{dist}(x,K) < r \}$, and $r_{\text{Rm}}(x)$ denotes the local scale at $x$ such that the curvature is locally bounded in the sense that $\sup_{B_r(x)} |Rm| \leq r^{-2}$, for any $r < r_{\text{Rm}}(x)$. If, in addition, we assume that $T_r(K)$ is sufficiently collapsed with locally bounded curvature, then one can further make $Z^n$ saturated with respect to the $N$-structure, and consequently, $\chi(Z) = 0$. This gives a bound on the Gauss–Bonnet–Chern integral $\int_Z P_X$. In particular, for Einstein 4-manifolds, one has a local bound on the $L^2$-norm of the curvature since $P_X$ is a multiple of the square norm of the curvature.

Next, by an iteration procedure, one can show the following key estimate: there exists $\delta > 0$, $c > 0$ such that if $E$ is a bounded open subset in a complete Einstein 4-manifold with $T_1(E)$ sufficiently collapsed and $\int_{B_1(x)} |Rm|^2 < \delta$, for any $x \in T_1(E)$, then $\int_E |Rm|^2 \leq c \cdot \text{Vol}(A_{0,1}(E))$. As a consequence, if the quantity

$$\frac{\text{Vol}(B_r(x))}{\text{Vol}(B_{cr}(x))} \int_{B_r(x)} |Rm|^2$$

is not sufficiently small, it has to be bounded.

Finally, in [14], Cheeger and Tian proved that the above quantity can be sufficiently small if one shrinks the ball to a smaller concentric one, whose radius is comparable to $r$. The proof used
the Gauss–Bonnet–Chern formula and a delicate estimate on the transgression form in terms of volume growth rate. Also in the proof, one needs a controlled and smooth approximation of the distance function which was provided by some previous works of Cheeger–Colding.

In [14], Cheeger and Tian have already made some remarks how to extend their results from Einstein 4-manifolds to bounded Ricci spaces. The purpose of this paper is two-fold. First, we want to fill out the details of the remarks mentioned in [14] and some of arguments have not been stated there. Second, we shall give a new result on the smoothing Riemannian metric in a local metric ball but of a definite size.

In the first step, it was mentioned in Remarks 2.7 and 4.8 in [14] that one can use local Ricci flow introduced by Deane Yang [27] to construct a suitable regularization of the metric, so that the theory of N-structure still holds. Using the heat flow method, in the Ricci bounded case, the curvature may not be bounded with respect to the local scale under consideration. Hence, to obtain the corresponding equivariant chopping result, we will use the fact that on $B_{\ell_a(x)}(x)$ the metric has $W^{2,p}$ covering geometry which can be checked if $\ell_a(x)$ is the local scale such that the local Ricci flow exists, then on the covering space of the ball $B_{\ell_a(x)}(x)$, the curvature tensor has a definite $L^p$ norm, which implies that we may assume

$$\frac{\text{Vol}(B_{\ell_a(x)}(x))}{\text{Vol}(B_{\ell_a(x)}(x)^{\text{Vol}(B_{\ell_a(x)}(x)})) \int_{B_{\ell_a(x)}(x)} |Rm|^2 \leq c,$$

for some constant only depending on the dimension of $M$.

In the second step, Remarks 1.4 and 5.11 state that for bounded Ricci spaces one cannot directly replace the Gauss–Bonnet–Chern form, $P_{\chi}$, by $\frac{1}{8\pi^2} |Rm|^2$ and the effect of changing the Einstein condition to the Ricci bounded case is only to add a definite constant in the corresponding estimates. However, to obtain those estimates, one remaining problem is still that the curvature may not be locally bounded, and hence we need the above mentioned covering argument to remedy this.

In the last step, Remark 8.22 states that we may show that under the Ricci bounded condition one can obtain a definite bound on the $L^p$-norm of the curvature for all $p < \infty$. By our smoothing procedure in this paper, we can recover this result. On the other hand, Remark 8.22 provides a way to construct a subset $U$ with $0 < \chi (U) < 1$ in the contradiction argument for the bounded Ricci case. As mentioned above, to find such subset $U$, Cheeger and Tian used the Cheeger–Colding approximation to find the smooth approximation of the distance function. In this paper, we will give details to show this fact. Also, our argument in constructing the subset $U$ does not involve the Cheeger–Colding approximation. Our proof here is more direct and elementary. The main difference is that we will use the exponential map of the regularized metric to lift the original metric and use this exponential map to regularize the distance function of the original metric. Our argument is based on the fact that when lifted to the covering space, the regularized distance function has a smooth convergence and the level surfaces of its limit have constant positive curvature.

Our main result is the following.

**Theorem 1.1.** There exists $\varepsilon > 0$ such that the following holds: Let $(M, g)$ be a complete 4-dimensional Riemannian manifold satisfying

$$|\text{Ric}(g)| \leq 3,$$
and let \( r \leq 1 \). If
\[
\int_{B_r(x)} |Rm(g)|^2 \leq \varepsilon, \]
then for any \( \delta > 0 \), there exists a metric \( \bar{g} \) which is \( C^{1,\alpha} \)-close to \( g \) on \( B_{cr}(x) \) for some constant \( c < 1 \) (for example, \( c = \frac{3}{4} \)), and has bounds on all covariant derivatives of the curvature tensor, namely,
\[
|\bar{g} - g| < \delta,
\]
and for any \( k \geq 0 \),
\[
\sup_{B_{cr}(x)} |\nabla^k Rm(\bar{g})| \leq \frac{C}{r^{2+k}},
\]
where \( C > 0 \) is a constant depending only on \( c, \delta \) and \( k \).

As a corollary of Theorem 1.1, we can know the local topological structure of the metric ball \( B_{r}(x) \) if \( g \) has bounded Ricci curvature and small \( L^2 \)-norm of the curvature tensor on \( B_r(x) \). Roughly speaking, the metric ball is diffeomorphic in the \( C^{1,\alpha} \)-topology to the tubular neighborhood of the zero section of a vector bundle over some (tiny) infranilmanifold.

2. Smoothing Riemannian metric with respect to the collapsing

The result in this section can be generalized to any dimension. For our purpose we only consider the 4-dimension. We study smoothing procedure under the assumption that \( L^2 \)-norm of the curvature is sufficiently small against the lower bound of the volume. In the collapsed space, our result still holds in the scale that the above mentioned assumption is valid. The arguments here are a modified version of those in [27].

Fix an open set \( B_0 \subset M \) and a smooth compactly supported function \( \phi \in C_0^\infty(B_0) \).

Let \( g(t), 0 \leq t \leq T \), be a 1-parameter family of smooth Riemannian metrics. Let \( \nabla \) denote the covariant differentiation with respect to the metric \( g(t) \) and \( -\Delta \) be the corresponding Laplace–Beltrami operator. Let \( A > 0 \) be a constant that satisfies the standard Sobolev inequality
\[
\left( \int_{B_0} f^4 \, dV_g \right)^{\frac{1}{2}} \leq A \int_{B_0} |\nabla f|^2 \, dV_g, \quad f \in C_0^\infty(B_0), \tag{2.1}
\]
with respect to each metric \( g(t), 0 \leq t \leq T \).

Assume that for each \( t \in [0, T] \),
\[
\frac{1}{2} g_{ij}(0) \leq g_{ij}(t) \leq 2 g_{ij}(0) \quad \text{on } B_0.
\]

All geodesic balls in this section are defined with respect to the metric \( g(0) \), and therefore, are fixed open subsets of \( M \), and independent of \( t \).
We want to study the heat equation:

\[
\frac{\partial f}{\partial t} \leq \phi^2 (\Delta f + uf) + 2a\phi |\nabla \phi| |\nabla f| + b(|\nabla \phi|^2 - \phi \Delta \phi) f, \quad 0 \leq t \leq T, \tag{2.2}
\]

where \( f \) and \( u \) are nonnegative functions on \( B_0 \times [0, T] \), such that

\[
\frac{\partial}{\partial t} dV_g \leq c\phi^2 u dV_g \tag{2.3}
\]

and

\[
\left( \int_{B_0} \phi^2 u^3 \right)^{\frac{1}{3}} \leq \mu t^{-\frac{1}{3}}. \tag{2.4}
\]

**Lemma 2.1.** Suppose that \( f \) and \( u \) are nonnegative functions on \( B_0 \times [0, T] \) which satisfy (2.2), (2.3) and (2.4). For \( p \geq p' \geq p_0 > 1 \), we have

\[
\begin{align*}
\frac{\partial}{\partial t} \int \phi^{2p'} f^p + & \frac{p-1}{4p} \int |\nabla (\phi^{p'+1} f^\frac{p}{p'})|^2 \\
\leq & C((p'+1)^2 \|
abla \phi\|_\infty^2 + (p\mu)^3 A^2 t^{-1}) \int \phi^{2p'} f^p, \tag{2.5}
\end{align*}
\]

where \( A \) is a constant satisfying (2.1).

**Proof.** By direct computation, we obtain

\[
\begin{align*}
\frac{\partial}{\partial t} \int \phi^{2p'} f^p + & 2 \left( 1 - \frac{1}{p} \right) ^2 \int |\nabla (\phi^{p'+1} f^\frac{p}{p'})|^2 \\
\leq & (p'+1)^2 C \int |\nabla \phi|^2 \phi^{2p'} f^p + p \int u\phi^{2(p'+1)} f^p \\
\leq & (p'+1)^2 C \int |\nabla \phi|^2 \phi^{2p'} f^p + \varepsilon \frac{1}{2} A^2 \int |\nabla (\phi^{p'+1} f^\frac{p}{p'})|^2 \\
& + \varepsilon^{-\frac{1}{2}} p^3 \mu^3 t^{-1} \int \phi^{2p'} f^p.
\end{align*}
\]

Choosing \( \varepsilon \) so that \( \varepsilon^{\frac{1}{2}} A^2 \) is sufficient small, we have

\[
\begin{align*}
\frac{\partial}{\partial t} \int \phi^{2p'} f^p + & \frac{p-1}{4p} \int |\nabla (\phi^{p'+1} f^\frac{p}{p'})|^2 \\
\leq & C((p'+1)^2 \|
abla \phi\|_\infty^2 + (p\mu)^3 A^2 t^{-1}) \int \phi^{2p'} f^p.
\end{align*}
\]

This proves the lemma. \( \Box \)
The following theorem is the consequence of Moser’s iteration. The proof can be found in [27].

**Theorem 2.2.** Let \( f \) and \( u \) be nonnegative functions on \( B_0 \times [0, T] \) satisfying (2.2), (2.3) and (2.4). Then given \((x,t) \in B_0 \times [0, T]\), \( p_0 > 2 \),

\[
|\phi(x)^2 f(x,t)| \leq CA \frac{2}{p_0} \left( \|\nabla \phi\|_\infty^2 + t^{-1} \left( 1 + A^2 \mu_3 \right) \right) \frac{1}{p_0} \left( \int_0^t \int_{B_0} \phi^{2p_0-4} f \right) \frac{1}{p_0},
\]

where \( C \) depends on the dimension of \( M \), \( p_0 \), \( a \) and \( b \).

With help of Theorem 2.2, we consider a similar estimate for the nonlinear equation.

**Theorem 2.3.** Let \( f \geq 0 \) solve

\[
\frac{\partial f}{\partial t} \leq 2\phi^2 (\Delta f + C_0 f^2) + 2a\phi|\nabla \phi||\nabla f| + b(|\nabla \phi|^2 - 2\phi \Delta \phi) f, \quad 0 \leq t \leq T,
\]

(2.6) on \( B_0 \times [0, T] \). Assume that

\[
\frac{\partial}{\partial t} dV_g \leq C \phi^2 f \, dV_g
\]

and that

\[
\left( \int_{B_0} f_0^2 \right)^{\frac{1}{2}} \leq (3eC_0A)^{-1},
\]

where \( f_0(x) = f(x, 0) \). Then

\[
|\phi^2(x) f(x,t)| \leq C \left( t \|\nabla \phi\|_\infty^2 + 1 \right)^2 t^{-1},
\]

where \( 0 < t < \min(T, \|\nabla \phi\|_\infty^2) \), and \( C \) depends on the dimension of \( M \), \( C_0 \), \( a \), \( b \).

**Proof.** Let \([0, T'] \subset [0, T]\) be the maximal interval such that

\[
e_0 = \sup_{0 \leq t \leq T'} \left( \int_{B_0} f^2 \right)^{\frac{1}{2}} \leq (3C_0A)^{-1}.
\]

By a direct calculation, we have, for \( 0 \leq t \leq T' \),

\[
\frac{\partial}{\partial t} \int_{B_0} f^p + 2 \left( 1 - \frac{1}{p} \right)^2 \int |\nabla (\phi f^p)|^2 \\
\leq p \int |\nabla \phi|^2 f^p + pC_0A \left( \int_{B_0} f^2 \right)^{\frac{1}{2}} \int |\nabla (\phi f^p)|^2.
\]
Therefore, for $p = 2$, the bound on the $L^2$ norm of $f$ implies that for $0 \leq t \leq T'$,

$$\frac{\partial}{\partial t} \int f^2 \leq 2\|\nabla \phi\|_\infty^2 \int f^2,$$

which implies that

$$\int f^2 \leq e^{2\|\nabla \phi\|^2} \int f_0^2.$$

In particular, if $T' < \|\nabla \phi\|^{-2}$, then

$$e_0 \leq e \left( \int f_0^2 \right)^{\frac{1}{2}} \leq (3C_0A)^{-1}.$$

This contradicts the assumed maximality of $[0, T']$. We can therefore assume that $T' \geq \min(\|\nabla \phi\|^{-2}, T)$.

By the same argument of Lemma 2.1, we have an estimate of the form

$$\int f^p \int \int \left| \nabla (\phi f^\frac{p}{2}) \right|^2 \leq C(t^{-1} + \|\nabla \phi\|_\infty^2) \int \int f^p.$$

Therefore, by setting $\mu^3 = CA(1 + t\|\nabla \phi\|_\infty^2)^2 e_0^3$, we have

$$\int \phi^2 f^3 \leq C(t^{-1} + \|\nabla \phi\|_\infty^2) \int \int \phi^2 f^3 \leq C(t^{-1} + \|\nabla \phi\|_\infty^2) \int \int f^2 \left( \int (\phi f)^4 \right)^{\frac{1}{2}} dt$$

$$\leq C e_0 A(t^{-1} + \|\nabla \phi\|_\infty^2) \int \int |\nabla (\phi f)|^2$$

$$\leq C e_0 A(t^{-1} + \|\nabla \phi\|_\infty^2)^2 \int \int f^2$$

$$\leq CAt(t^{-1} + \|\nabla \phi\|_\infty^2)^2 e_0^3.$$
On the other hand, for $t \in [0, T']$

$$A^2 \mu^3 \leq CA^3 (1 + t\|\nabla \phi\|_\infty^2)^2 C_0^3$$

$$\leq CA^3 (1 + t\|\nabla \phi\|_\infty^2) (3C_0A)^{-3}$$

$$\leq CC_0^{-3} (1 + t\|\nabla \phi\|_\infty^2)^2.$$ 

Notice that Theorem 2.2 still holds, when $p_0 \to 2$. We then obtain the desired estimate. □

The argument also implies the following

**Corollary 2.4.** Let $f$ satisfy the assumptions of Theorem 2.3. Then given $u \geq 0$ such that

$$\frac{\partial u}{\partial t} \leq \phi^2 (\Delta u + c_0 fu) + a
\nabla \phi \cdot \nabla u + b\phi u,$$

the following estimate holds for $0 \leq t < \min(T, \|\nabla \phi\|_\infty^{-2})$,

$$|\phi(x)^2 u(x, t)| \leq CA^2 (1 + t\|\nabla \phi\|_\infty^2)^2 t^{-\frac{4}{3}} \left( \int_{\mathcal{B}_0} u_0^3 \right)^{\frac{1}{3}},$$

where $u_0(x) = u(x, 0)$, and $C$ depends on $a$ and $b$.

**Proof.** Direct computation yields

$$\frac{\partial}{\partial t} \int u^3 + \frac{8}{9} \int |\nabla (\phi u^\frac{3}{2})|^2 \leq C_1 \int (|\nabla \phi|^2 + \phi^2) u^3 + \left( \int f^2 \right)^{\frac{1}{2}} \left( \int (\phi u^\frac{3}{2})^4 \right)^{\frac{1}{2}}$$

$$\leq C_1 \int (|\nabla \phi|^2 + \phi^2) u^3 + C_2 A \left( \int f^2 \right)^{\frac{1}{2}} \int |\nabla (\phi u^\frac{3}{2})|^2.$$ 

By the proof of the previous theorem, we obtain

$$\int u^3 \leq C \int u_0^3.$$

Also we have

$$\int \phi^2 f^3 \leq \mu^3 t^{-1}.$$ 

Thus the result follows from Theorem 2.2 with $p_0 = 3$. □

Let $M$ be a smooth manifold with Riemannian metric $g_0$ and $\Omega$ an open subset of $M$. Let $\phi$ be a nonnegative smooth compactly supported function on $\Omega$. Consider the following evolution equation
\[
\begin{aligned}
\frac{\partial g}{\partial t} &= -2\phi^2\text{Ric}(g), \\
g(0) &= g_0.
\end{aligned}
\] (2.7)

It is easy to check that the curvature tensor $R_m$ and Ricci tensor $\text{Ric}$ satisfy the following equations respectively,

\[
\begin{aligned}
\frac{\partial R_m}{\partial t} &= \phi^2(\triangle R_m + Q_1(R_m, R_m)) + 2\phi a_1(\nabla\phi, \nabla R_m) \\
&\quad + b_1(\nabla\phi, \nabla\phi, R_m) + \phi c_1(\nabla^2\phi, R_m)
\end{aligned}
\] (2.8)

and

\[
\begin{aligned}
\frac{\partial \text{Ric}}{\partial t} &= \phi^2(\triangle \text{Ric} + Q_2(R_m, \text{Ric})) + 2\phi a_2(\nabla\phi, \nabla \text{Ric}) \\
&\quad + b_2(\nabla\phi, \nabla\phi, \text{Ric}) + \phi c_2(\nabla^2\phi, \text{Ric}),
\end{aligned}
\] (2.9)

where $Q_i, a_i, b_i$ and $c_i$ are multi-linear functions of their arguments, $i = 1, 2$. Their definitions depend only on the dimension of $M$.

**Theorem 2.5.** There exist constants $C_1$ and $C_2$ such that if

\[
\left( \int_{\Omega} |R_m(g_0)|^2 \, dV_{g_0} \right)^{\frac{1}{2}} \leq \left[ C_1 C_4(\Omega) \right]^{-1}
\]

and

\[
|\text{Ric}(g_0)| \leq K,
\]

then Eq. (2.7) has a smooth solution for $t \in [0, T)$, where

\[T \geq \min(\|\nabla\phi\|_\infty^{-2}, C_2 K^{-1}).\]

Moreover, for $t \in (0, T)$, the Riemannian curvature tensor satisfies the following bound,

\[
\|\phi^2 R_m\|_\infty \leq C_3(t\|\nabla\phi\|_\infty^2 + 1)^2 t^{-1}.
\] (2.10)

Here $C_1$, $C_2$ and $C_3$ only depend on the dimension of $M$.

**Proof.** By the short time existence theorem of local Ricci flow (for example, see Theorem 8.2 in [27]), Eq. (2.7) has a smooth solution on a sufficiently small time interval starting at $t = 0$. Let $[0, T_{\text{max}})$ be a maximal time interval on which (2.7) has a smooth solution and such that the following hold for each metric $g(t)$,
\[ \| f \|_2^2 \leq 4A_0 \| \nabla f \|_2^2, \quad f \in C_0^\infty(\Omega), \]  
(2.11)

\[ \frac{1}{2}g_0 \leq g(t) \leq 2g_0, \]  
(2.12)

\[ \| \text{Rm}(g(t)) \|_2 \leq 2(C_1 A_0)^{-1}. \]  
(2.13)

Suppose that \( T_{\text{max}} < T_0 = \min(\| \nabla \phi \|_\infty^2, C_2 K^{-1}) \). We will show that this leads to a contradiction.

First, notice that the curvature tensor \( \text{Rm} \) satisfies (2.8), then we have

\[
\frac{\partial}{\partial t} |\text{Rm}|^2 = \phi^2 \Delta |\text{Rm}|^2 - 2\phi^2 |\nabla \text{Rm}|^2 + \phi^2 \text{Rm} \ast \text{Rm} \ast \text{Rm} + \phi \nabla \phi \ast \nabla \text{Rm} \ast \text{Rm} 
+ \nabla \phi \ast \nabla \phi \ast \text{Rm} \ast \text{Rm} + \phi \nabla^2 \phi \ast \text{Rm} \ast \text{Rm}.
\]

Here the last term, \( \phi \nabla^2 \phi \ast \text{Rm} \ast \text{Rm} \), on the right-hand side can be ignored in the energy estimate due to the negative term \(-2\phi^2 |\nabla \text{Rm}|^2\).

According to the proof of Theorem 2.3, we obtain

\[ \| \text{Rm}(g(t)) \|_2 < 2 \| \text{Rm}(g_0) \|_2 \leq 2[C_1 A_0]^{-1}, \]

which implies a strict inequality for (2.13).

Next, since the Ricci curvature satisfies (2.9), then Corollary 2.4 implies that

\[
|\phi^2 \text{Ric}(g(t))| \leq C\left[A_0^2 \left(1 + t \| \nabla \phi \|_\infty^2 \right)^2 t^{-\frac{5}{3}} \left( \int_\Omega |\text{Ric}(g_0)|^2 \right)^{\frac{1}{3}} \right]
\]

\[
\leq C \left(1 + t \| \nabla \phi \|_\infty^2 \right)^{\frac{2}{3}} t^{-\frac{5}{3}} \left(K A_0^2 \int_\Omega |\text{Ric}(g_0)|^2 \right)^{\frac{1}{3}}
\]

\[
\leq C_1^{-\frac{2}{3}} \left(1 + t \| \nabla \phi \|_\infty^2 \right)^{\frac{2}{3}} t^{-\frac{5}{3}} K^\frac{1}{3}
\]

\[
\leq C_4 t^{-\frac{5}{3}},
\]

where \( \| \nabla \phi \|_\infty \) can be evaluated at \( g(0) \), since the metrics \( g(t) \) are equivalent within the maximal time \( T_{\text{max}} \).

Applying the bound on Ric to the following

\[ \left| \frac{d}{dt} \int f^p \, dV_g \right| \leq 2 \| \phi^2 \text{Ric} \|_\infty \int f^p \, dV_g, \]

we have

\[ -2C_4 t^{-\frac{5}{3}} \, dt \leq d \log \int f^p \, dV_g \leq 2C_4 t^{-\frac{5}{3}} \, dt, \]

which implies that for some suitably chosen constants,
The differential inequality
\[ \left| \frac{d}{dt} \int |\nabla f|^2 \, dV_g \right| \leq 2 \| \phi^2 \text{Ric} \|_\infty \int |\nabla f|^2 \, dV_g \]
leads to a similar estimate. Therefore, it follows that for any \( t \leq T_0 \),
\[ \| f \|^4(t) < 2 \| f \|^4(0) \leq 2 A_0 \| \nabla f \|^2(0) < 4 A_0 \| \nabla f \|^2_2(t), \]
that is to say (2.11) holds with strict inequality.

To show that (2.12) holds with strict inequality, we use Hamilton’s trick. Simply fix a tangent vector \( v \) with respect to \( g(t) \), then
\[ \frac{d}{dt} |v|^2_{g(t)} = \frac{d}{dt} (g_{ij}(t)v^iv^j) = g'_{ij}(t)v^iv^j \]
implies
\[ \left| \frac{d}{dt} \log |v|^2_{g(t)} \right| \leq |g'_{ij}(t)| \leq 2\phi^2 |\text{Ric}|. \]

So for \( 0 \leq t \leq T_2 < T_0 \),
\[ \log \frac{|v|^2_{g(t)}}{|v|^2_{g(0)}} \leq \int_0^{T_2} |g'_{ij}(t)| \, dt \leq 2 \| \phi^2 \text{Ric} \|_\infty T_2 < \log 2, \]
which implies
\[ \frac{1}{2} |v|^2_{g(0)} < |v|^2_{g(t)} < 2 |v|^2_{g(0)}, \]
for \( t < T_0 \).

Finally, by differentiating the evolution equation for \( \text{Rm} \), we see that the covariant derivatives of \( \text{Rm} \) satisfy evolution equations for which \( L^2 \) energy bounds can be obtained. More precisely, we can compute
\[
\frac{\partial}{\partial t} |\nabla \text{Rm}|^2 = 2 \left\langle \nabla \left( \frac{\partial}{\partial t} \text{Rm} \right), \nabla \text{Rm} \right\rangle + \phi^2 \text{Rm} \ast \nabla \text{Ric} \ast \nabla \text{Rm} + \nabla \phi^2 \ast \text{Ric} \ast \text{Rm} \ast \nabla \text{Rm}
\]
\[= \nabla \phi^2 \ast \Delta \text{Rm} \ast \nabla \text{Rm} + \nabla \phi^2 \ast \text{Rm} \ast \text{Rm} \ast \nabla \text{Rm}
\]
\[+ 2\phi^2 \left\langle \nabla(\Delta \text{Rm} + \text{Rm} \ast \text{Rm}), \nabla \text{Rm} \right\rangle + \nabla^3 \phi^2 \ast \text{Ric} \ast \nabla \text{Rm}
\]
\[+ \nabla^2 \phi^2 \ast \nabla \text{Ric} \ast \nabla \text{Rm} + \nabla \phi^2 \ast \nabla^2 \text{Ric} \ast \nabla \text{Rm}
\]
\[+ \phi^2 \text{Rm} \ast \nabla \text{Ric} \ast \nabla \text{Rm} + \nabla \phi^2 \ast \text{Ric} \ast \text{Rm} \ast \nabla \text{Rm} \]
\[
\phi^2 (\Delta |\nabla R_m|^2 - 2|\nabla^2 R_m|^2) + \nabla \phi^2 \ast \Delta R_m \ast \nabla R_m
+ \nabla^2 \phi^2 \ast \nabla R_m + \nabla \phi^2 \ast \nu R_m \ast \nabla R_m
+ \nabla \phi^2 \ast R_m \ast \nabla R_m + \nabla^2 \phi^2 \ast \nabla R_m \ast \nabla R_m
+ \phi^2 R_m \ast \nabla R_m \ast \nabla R_m + \phi^2 \ast \nabla R_m \ast \nabla R_m.
\]

Thanks to the negative term \(-2\phi^2|\nabla^2 R_m|^2\), we can absorb the terms involving the second covariant derivatives of curvature in the above equation. Thus the remaining uncontrolled term is \(\nabla^3 \phi^2 \ast \nabla R_m\). However, when we do the energy estimate \(\frac{d}{dt} \int \phi^p |\nabla R_m|^2\), it can be bounded by using integration by parts and the negative term \(-2\phi^2|\nabla^2 R_m|^2\) again. Hence, the \(L^2\) energy estimate for \(|\nabla R_m|\) still holds. Direct calculation yields

\[
\frac{d}{dt} \int \phi^p |\nabla R_m|^2 \leq \int \phi^p \frac{\partial}{\partial t} |\nabla R_m|^2 + \int \phi^{p+2} |\nabla R_m|^2 |R_m|.
\]

By Theorem 2.3, we have for \(t \in (0, \text{T}_{\text{max}})\)

\[
\|\phi^2 R_m\|_\infty \leq C (t \|\nabla \phi\|_\infty^2 + 1)^2 t^{-1},
\]

thus

\[
\int \phi^{p+2} |\nabla R_m|^2 |R_m| \leq C (t \|\nabla \phi\|_\infty^2 + 1)^2 t^{-1} \int \phi^p |\nabla R_m|^2.
\]

Now we estimate the energy \(\int \phi^p \frac{d}{dt} |\nabla R_m|^2\). By direct computations, we have

\[
\int \phi^{p+2} \Delta |\nabla R_m|^2 = -(p + 2) \int \phi^{p+1} \nabla \phi \ast \nabla^2 R_m \ast \nabla R_m
\leq \epsilon \int \phi^{p+2} |\nabla^2 R_m|^2 + C_{\epsilon, p} \|\nabla \phi\|_\infty^2 \int \phi^p |\nabla R_m|^2,
\]

\[
\int \phi^p \nabla^3 \phi^2 \ast R_m \ast \nabla R_m = \int \phi^p \nabla \phi \ast \nabla^2 \phi \ast R_m \ast \nabla R_m + \int \phi^{p+1} \nabla^3 \phi \ast R_m \ast \nabla R_m
\leq \|\nabla \phi\|_\infty^2 \int \phi^p |\nabla R_m|^2 + \int \phi^p |\nabla^2 \phi|^2 |R_m|^2
+ (p + 1) \int \phi^p \nabla \phi \ast \nabla^2 \phi \ast R_m \ast \nabla R_m
- \int \phi^{p+1} \nabla^2 \phi \ast R_m \ast \nabla R_m - \int \phi^{p+1} \nabla^2 \phi \ast R_m \ast \nabla^2 R_m
\leq C_p (\|\nabla \phi\|_\infty^2 + \|\nabla^2 \phi\|_\infty) \int \phi^p |\nabla R_m|^2
+ C_{\epsilon, p} \int \phi^p |\nabla^2 \phi|^2 |R_m|^2 + \epsilon \int \phi^{p+2} |\nabla^2 R_m|^2,
\]
where in local coordinates $\nabla^2 \phi = \partial^2 \phi + \Gamma \ast \partial \phi$ has a uniform bound within time $T_2 < T_{\text{max}}$, since $\partial$ only depends on the manifold itself and the metrics $g(t)$ are equivalent. Here $\Gamma$ denotes the Christoffel symbol.

The remaining terms in $\int \phi^p \frac{\partial}{\partial t} |\nabla Rm|^2$ only involve with the curvature and its first covariant derivatives and then can be estimated obviously. Therefore we have the $L^2$ energy estimate of $\nabla Rm$.

To obtain the $L^2$ energy estimate of $\nabla^2 Rm$, we only need to deal with the term involving $\phi \nabla^4 \phi \ast \Ric \ast \nabla^2 Rm$, since other terms only involve with the covariant derivatives of $Rm$ of lower orders or can be absorbed by the negative term $-2 \phi^{p+2} |\nabla^3 Rm|^2$ as before. Direct calculation yields

$$\int \phi^p \nabla Rm \ast \nabla^2 Rm = -(p+1) \int \phi^p \nabla(Rm \ast \nabla^2 Rm)$$

$$- \int \phi^p \nabla \nabla^3 \phi \ast \nabla^2 Rm$$

$$- \int \phi^p \nabla \nabla^3 \phi \ast \nabla Rm \ast \nabla Rm \ast \nabla^2 Rm.$$ 

All we need to do is to bound $\nabla^3 \phi$. Direct computation yields, in local coordinates

$$\nabla^3 \phi = \partial \nabla^2 \phi + \Gamma \ast \nabla^2 \phi = \partial^3 \phi + Rm \ast \nabla \phi + \Gamma \ast \nabla^2 \phi.$$ 

Since curvature has a uniform bound on $(0, T_2]$, we have a bound on $\nabla^3 \phi$. Thus we obtain the $L^2$ energy estimate of $\nabla^2 Rm$. Using the interpolation inequalities in [17], we have the $L^p$ bound on $\nabla Rm$. Thus by Moser iteration, we can bound $\nabla Rm$ on $(0, T_2]$. With this bound we can obtain the $L^2$ energy estimate of $\nabla^3 Rm$, since

$$\nabla^4 \phi = \partial^4 \phi + \Gamma \ast \nabla^3 \phi + \nabla Rm \ast \nabla \phi + \nabla^2 \phi + \nabla Rm \ast \nabla \nabla^3 \phi$$

$$+ Rm \ast \nabla^3 \phi + \Gamma \ast \nabla^2 \phi.$$ 

Repeat this argument, we can obtain the $L^2$ energy estimate of $\nabla^k Rm$ for all $k$.

Therefore we can use Hamilton’s argument in §14 of [17] to show that $g(t)$ has a smooth limit as $t \to T_{\text{max}}$. If $T_{\text{max}} < T_0$, we would be able to extend the solution to (2.7) smoothly beyond $T_\text{max}$ with (2.11), (2.12) and (2.13) still holding. This contradicts the assumed maximality of $T_{\text{max}}$. Hence, we conclude that $T_{\text{max}} \geq T_0$. □

According to Theorem 5 of [21], Theorem 3.1 of [22] and Theorem 4.1 of [4], there exists a constant $C$ depending only on the bounds of Ricci curvature and the dimension of $M$ such that the Sobolev constant for the geodesic ball $B_r(x)$ can be controlled as follows: for $r \leq 1$,

$$C_s(B_r(x)) \leq C \left( \frac{r^4}{\text{Vol}(B_r(x))} \right)^{\frac{1}{2}}.$$ 

Therefore, the previous theorem can be restated as the following.
Theorem 2.6. There exist constants $\varepsilon$ and $C_1$ such that for $r \leq 1$, if
\[
\frac{r^4}{\text{Vol}(B_r(x))} \int_{B_r(x)} |\text{Rm}(g_0)|^2 \, dV_{g_0} \leq \varepsilon
\]
and
\[
|\text{Ric}(g_0)| \leq K,
\]
then Eq. (2.7) has a smooth solution for $t \in [0, T)$, where
\[
T \geq \min \left( \|\nabla \phi\|^{-2}_\infty, C_1 K^{-1} \right).
\]
Moreover, for $t \in (0, T)$, the Riemannian curvature tensor satisfies the following bound,
\[
\|\phi^2 \text{Rm}\|_\infty \leq C_2 \left(t \|\nabla \phi\|_\infty^2 + 1\right)^2 t^{-1}.
\]
Here $\varepsilon$, $C_1$ and $C_2$ only depend on the dimension of $M$.

Under the assumptions of 2.6, we may pull back the metric $g$ to the tangent space via the exponential map of $\bar{g}$, which then is reduced to the noncollapsed case. Denote the pullback metrics of $g$ and $\bar{g}$ by $g_0$ and $\bar{g}_0$ respectively. Then we can show the existence of $C^{1,\alpha}$ or $W^{2, p}$ harmonic coordinates for $g_0$. Notice that $\bar{g}$ has bounded curvature, so the volume of the geodesic ball of a small but definite size of the pullback metric $\bar{g}_0$ is close to that of the corresponding Euclidean ball, so is for the volume of geodesic ball of $g_0$, since $\bar{g}$ is $C^0$-close to $g$. In view of Theorem 10.25 in [9] (which is a version of the result in [3]), such harmonic coordinates do exist. This implies that we may assume that the curvature tensor of pullback metric of $g$ has an $L^p$ bound on a geodesic ball of a definite size.

Theorem 2.7. There exists $\delta = \delta(4) > 0$, and for all $\alpha < 1$, a constant, $\theta = \theta(4, \alpha) > 0$, such that if
\[
|\text{Ric}| \leq 3,
\] (2.14)
and for some $r \leq 1$,
\[
\text{Vol}(B_r(x)) \geq (1 - \delta) \text{Vol}(B_r(0)),
\]
where $B_r(0) \subset \mathbb{R}^4$, then for all $y \in B_{\frac{1}{2}r}(x)$, the ball $B_{\theta r}(y)$ is the domain of a harmonic coordinate system satisfying
\[
\begin{align*}
r^{1+\alpha} \|g_{ij}\|_{C^{1,\alpha}} &\leq 2, \\
r^{2-\frac{3}{2}} \|\partial^2 g_{ij}\|_{L^p} &\leq 2 \quad (\text{for all } p < \infty), \\
\frac{1}{2}(\delta_{ij}) &\leq (g_{ij}) < 2(\delta_{ij}) \quad (\text{as bilinear forms}).
\end{align*}
\]
**Theorem 2.8.** There exist $C_1 > 0$ and $r_0 > 0$ such that the following holds. Let $r \leq r_0$. If a geodesic ball $B_r(x)$ satisfies

$$
\text{Vol}(B_r(x)) \geq v \cdot r^4,
$$

$$
r^{2p-4} \int_{B_r(x)} |\operatorname{Rm}(g)|^p \leq C_1,
$$

$$
|\operatorname{Ric}| \leq 3,
$$

there exists a metric $\tilde{g}$ of bounded curvature which is $C^{1,\alpha}$-close to $g$ on $B_{\frac{r}{2}}(x)$.

**Proof.** We solve the local Ricci flow on $B_r(x)$ so that we obtain a family of metrics $g(t)$ with $g(0) = g$, $0 \leq t \leq T$. By the proof of Theorem 2.3, we have

$$
\frac{\partial}{\partial t} \int_{B_r(x)} |\operatorname{Rm}|^p \leq C \int_{B_r(x)} |\operatorname{Rm}|^p,
$$

where $C$ depends on $p$, $r$, $v$ and the dimension of $M$. This implies that

$$
\int_{B_r(x)} |\operatorname{Rm}(g(t))|^p \leq e^{Ct} \int_{B_r(x)} |\operatorname{Rm}(g)|^p.
$$

Letting $t \to 0$, we obtain

$$
\limsup_{t \to 0} \int_{B_r(x)} |\operatorname{Rm}(g(t))|^p \leq \int_{B_r(x)} |\operatorname{Rm}(g)|^p.
$$

On the other hand, $g(t)$ are uniformly bounded in $W^{2,p}(B_r(x))$. This implies that $g(t)$ converge to $g$ weakly in $W^{2,p}(B_r(x))$, and then we have

$$
\int_{B_r(x)} |\operatorname{Rm}(g)|^p \leq \liminf_{t \to 0} \int_{B_r(x)} |\operatorname{Rm}(g_t)|^p.
$$

Therefore,

$$
\lim_{t \to 0} \int_{B_r(x)} |\operatorname{Rm}(g(t))|^p = \int_{B_r(x)} |\operatorname{Rm}(g)|^p,
$$

which, together with the fact that $g(t)$ converge to $g$ weakly in $W^{2,p}(B_r(x))$, implies that they converge to $g$ strongly in $W^{2,p}(B_r(x))$. This proves the theorem. \(\square\)

In view of the previous theorem, we can deduce a definite bound on $L^p$-norm of curvature for all $p < \infty$. More precisely, we have
Theorem 2.9. There exists $\varepsilon > 0$ such that the following holds: Let $(M, g)$ be a complete 4-dimensional Riemannian manifold satisfying

$$|\text{Ric}| \leq 3,$$

and let $r \leq 1$. If

$$\frac{r^4}{\text{Vol}(B_r(x))} \int_{B_r(x)} |\text{Rm}|^2 \leq \varepsilon,$$

then there exists a constant $c$ depending on $p$ such that

$$\int_{B_{\frac{1}{2r}}(x)} |\text{Rm}(g)|^p \leq c \cdot r^{2p-4}.$$

Once we prove that under the assumptions of Theorem 1.1, (2.15) holds, then as mentioned in Remark 8.22 of [14] the curvature tensor has a definite $L^p$ bound. This can be regarded as a generalization of $\varepsilon$-regularity theorem in the Einstein manifolds developed in [14].

3. The key estimate

In this section, we generalize the key estimate, i.e., Theorem 1.26 in [14], to our $W^{2,p}$ case. However the original metric $g$ may not be locally bounded on a suitable local scale mentioned below. Therefore we need to use the result in the previous section to construct a regularized metric which is $C^{1,\alpha}$-close to the original metric in this local scale.

Let $(M, g)$ denote an arbitrary 4-dimensional Riemannian manifold with bounded Ricci curvature, without loss of generality, we may assume that $|\text{Ric}(g)| \leq 3$. Let $r^r_{B_r(x)} > 0$ denote the supremum of those $r$ such that $B_r(x)$ is compact for $s \leq r$ and the lifted metric on $B_r(O_x) \subset T_x M$ has uniform bounded $W^{2,p}$ geometry in harmonic coordinates, for all $p < \infty$. That is, on $B_r(O_x)$, there is a harmonic coordinate chart such that the pullback metric is $W^{2,p}$-controlled in these coordinates. The properties of harmonic coordinates can be found in [3,15,18,19].

Let $M_\eta$ denote the space form with constant curvature $\eta$.

If $x \in M$ satisfies

$$\frac{\text{Vol}(B_{1}(x))}{\text{Vol}(B_{1}(x))} \int_{B_{1}(x)} |\text{Rm}|^2 \leq \varepsilon,$$

put $\rho(x) = 1$, where $x \in M_{-1}$ and $B_{1}(x)$ is the unit geodesic ball in $M_{-1}$. Otherwise, define $\rho(x)$ to be the largest solution of

$$\frac{\text{Vol}(B_{\rho(x)}(x))}{\text{Vol}(B_{\rho(x)}(x))} \int_{B_{\rho(x)}(x)} |\text{Rm}|^2 = \varepsilon.$$
Then Theorem 2.6 implies that there exists a regularized metric \( \bar{g} \) which is \( C^0 \)-close to \( g \) on \( \text{Vol}(B_{\frac{1}{2}\rho(x)}(x)) \). We then pull back \( g \) by the exponential map of \( \bar{g} \) to the tangent space. By Proposition 2.5 in [3], there exists an absolute constant \( c \) such that

\[
\rho(x) \leq r_{|R|}^c(x).
\]

A subset, \( U \subseteq M \), such that for all \( x \in U \), \( \sup_{B_1(x)} \text{Ric}(g) \geq -3 \), is called \( v \)-collapsed if for all \( x \in U \),

\[
\text{Vol}(B_1(x)) \leq v.
\]

Here the lower bound of Ricci curvature is obtained by normalization.

We say that \( U \) is \( v \)-collapsed with locally bounded \( W^{2,p} \) covering geometry if for all \( x \in U \),

\[
\text{Vol}(B_{r_{|R|}^c(x)}(x)) \leq v \cdot (r_{|R|}^c(x))^4,
\]

and that \( U \) is \((v, a)\)-collapsed with locally bounded \( W^{2,p} \) covering geometry if, in addition, for all \( x \) with \( r_{|R|}^c(x) \geq a \),

\[
\text{Vol}(B_a(x)) \leq v \cdot a^4.
\]

For some \( 0 < a \leq 1 \), put

\[
\ell_a = \min(r_{|R|}^c, a).
\]

Fix a small constant, \( \zeta > 0 \) such that on each ball, and let \( \{x_\alpha\} \) denote a maximal set of points such that for \( \alpha_1 \neq \alpha_2 \),

\[
x_{\alpha_1}, x_{\alpha_2} \geq \zeta \cdot \min(\ell_a(x_{\alpha_1}), \ell_a(x_{\alpha_2})).
\]

Since \( M^4 \) has bounded Ricci curvatures, \( \{B_{2\zeta \ell_a(x_\alpha)}(x_\alpha)\} \) is a covering with multiplicity \( \leq N \), where \( N \) is a positive constant only depending on the dimension of \( M \). The covering, \( \{B_{2\zeta \ell_a(x_\alpha)}(x_\alpha)\} \), can be partitioned into at most \( N \) disjoint subcollections, \( S_i \), of mutually nonintersecting balls, \( \{B_{2\zeta \ell_a(x_{i,j})}(x_{i,j})\} \), such that a given member of any such subcollection intersects at most one member of any other such subcollection; see Lemma 2.2 of [12]. In addition, if

\[
B_{2\zeta \ell_a(x_{i_1,j_1})}(x_{i_1,j_1}) \cap B_{2\zeta \ell_a(x_{i_2,j_2})}(x_{i_2,j_2}) \neq \emptyset,
\]

then

\[
(1 - 2\zeta)\ell_a(x_{i_1,j_1}) \leq \ell_a(x_{i_2,j_2}) \leq (1 + 2\zeta)\ell_a(x_{i_1,j_1}).
\]

On each ball, \( B_{r_{|R|}^c(x_\alpha)}(x_\alpha) \), we can smooth the rescaled metric, \( (\ell_a(x_\alpha))^{-2}g \), to obtain a metric, \( (\ell_a(x_\alpha))^{-2}\hat{g}_\alpha \), with bounded curvature. In view of Theorem 2.8, we choose \( \zeta \) such that \( (\ell_a(x_\alpha))^{-2}\hat{g}_\alpha \) is \( C^{1,\alpha} \)-close to \( (\ell_a(x_\alpha))^{-2}g \) on \( B_{2\zeta \ell_a(x_\alpha)}(x_\alpha) \). Then we can apply the standard
regularized procedure of [1] to \((\ell_a(x_\alpha))^{-2}\tilde{g}_\alpha\). We denote the resulting regularized metric of bounded curvature by \((\ell_a(x_\alpha))^{-2}\tilde{g}_\alpha\).

Next, we regularize the distance function of the metric, \((\ell_a(x_\alpha))^{-2}\tilde{g}_\alpha\), to obtain a smooth function with definite bounds on all covariant derivatives with respect to the metric \((\ell_a(x_\alpha))^{-2}\tilde{g}_\alpha\).

By composing the regularized distance functions with standard cut-off functions, we obtain a partition of unity, \(\{\phi_\alpha\}\), subordinate to the cover \(\{B_{2\xi}\ell_a(x_\alpha)(x_\alpha)\}\). Then for some \(\eta > 0\), the metric, \(\tilde{g}\), defined by

\[
\tilde{g} = \sum_\alpha \phi_\alpha \tilde{g}_\alpha,
\]

satisfies

\[
(1 + \eta)^{-2}g \leq \tilde{g} \leq (1 + \eta)^2g, \tag{3.1}
\]

\[
|\nabla - \tilde{\nabla}|_{\tilde{g}} \leq c\eta\ell_a^{-1}, \tag{3.2}
\]

and

\[
|\tilde{\nabla}^k Rm|_{\tilde{g}} \leq c(k, \eta)(\ell_a)^{-(k+2)},
\]

where \(\nabla\) and \(\tilde{\nabla}\) are the Riemannian connections of \(g\) and \(\tilde{g}\) respectively, \(c\) is a constant depending only on dimension of the manifold, and \(c(k, \eta)\) only depends on \(K, \eta\) and the dimension. Notice that (3.2) is important in the construction of standard \(N\)-structure, which guarantees that the local fibrations constructed as in Section 6 of [11] are also compatible with the affine structures. For details, one can see [11,14].

Similar in [14], we need to develop the following key estimate.

**Theorem 3.1.** Let \((M, g)\) denote a complete 4-dimensional Riemannian manifold with bounded Ricci curvature. Let \(E \subset M^4\) denote a bounded open subset so that there exists \(\varepsilon > 0\) and \(T_1(E)\) is \(t\)-collapsed with

\[
\int_{B_1(x)} |Rm|^2 \leq \varepsilon,
\]

for all \(x \in T_1(E)\). Then there exists \(c > 0\) such that

\[
\int_E |Rm|^2 \leq c \cdot \text{Vol}(T_1(E)).
\]

Before proving the theorem, we remark that this implies the following important fact, which says that for the metric \(g\), either the hypothesis of Theorem 2.6 holds or there exists a definite constant \(c\) such that

\[
\frac{\text{Vol}(B_r(x))}{\text{Vol}(B_r(x))} \int_{B_r(x)} |Rm|^2 \leq c,
\]
where \( B_r(x) \) is the geodesic ball in the space form with respect to \( g \), i.e., \( B_r(x) \subset M_{-1} \) if \(|\text{Ric}(g)| \leq 3\).

**Corollary 3.2.** There exist \( \varepsilon > 0 \) and \( \tau > 0 \), such that the following holds. Let \((M, g)\) be a complete 4-dimensional Riemannian manifold satisfying

\[ |\text{Ric}| \leq 3, \]

and

\[ \int_{B_r(x)} |\text{Rm}|^2 \leq \varepsilon. \]

Let \( r \leq 1 \) such that \( B_s(x) \) has compact closure for all \( s \leq r \). If

\[ \frac{\text{Vol}(B_r(x))}{\text{Vol}(B_r(x))} \int_{B_r(x)} |\text{Rm}|^2 > \tau, \]

then there exists a definite constant \( c > 0 \), such that

\[ \frac{\text{Vol}(B_r(x))}{\text{Vol}(B_r(x))} \int_{B_r(x)} |\text{Rm}|^2 \leq c. \]

**Proof.** After rescaling the metric, the assumptions of Theorem 3.1 are satisfied with \( E = B_1(x) \) and \( T_1(E) = B_2(x) \). Then

\[ \frac{\text{Vol}(B_1(x))}{\text{Vol}(B_1(x))} \int_{B_1(x)} |\text{Rm}|^2 \leq c \cdot \frac{\text{Vol}(B_1(x))}{\text{Vol}(B_1(x))} \cdot \text{Vol}(B_2(x)) \]

\[ \leq c \cdot \text{Vol}(B_2(x)). \]

This finishes the proof of the corollary. \( \Box \)

The proof of Theorem 3.1 is similar as that of Theorem 1.26 in [14]. However, in the Ricci bounded case, we cannot directly replace the Gauss–Bonnet–Chern form, \( P_x \), by \( \frac{1}{8\pi^2} |\text{Rm}|^2 \). According to Remark 5.11 in [14], the effect of changing the Einstein condition to our Ricci curvature bounded case is only to add a definite constant in the corresponding estimates. On the other hand, the metric \( g \) may not be locally bounded on the scale \( \ell_a(x_\alpha) \). To overcome this, we use the fact that on \( B_{\ell_a(x_\alpha)}(x_\alpha) \) the metric has \( W^{2,p} \) covering geometry, then on the covering space of the ball \( B_{\ell_a(x_\alpha)}(x_\alpha) \), the curvature tensor has a definite \( L^p \) norm, which implies that we may assume

\[ \frac{\text{Vol}(B_{\ell_a(x_\alpha)}(x_\alpha))}{\text{Vol}(B_{\ell_a(x_\alpha)}(x_\alpha))} \int_{B_{\ell_a(x_\alpha)}(x_\alpha)} |\text{Rm}|^2 \leq c, \]
for some constant only depending on the dimension of $M$. This yields

$$\int_{B_{\ell a(x_\alpha)}(x_\alpha)} |\text{Rm}|^2 \leq c \cdot \ell a(x_\alpha)^{-4} \text{Vol}(B_{\ell a(x_\alpha)}(x_\alpha)). \quad (3.3)$$

The proof follows from an iteration argument. As in [14], we need the following results which generalize the equivariant chopping theorem of [13]. Unlike the Einstein case, again, the metric may not be locally bounded on the local scale. Thus, to obtain (3.6) below, we will still use (3.3) to remedy the difference between Einstein and bounded Ricci cases.

For $K \subset M$, $r > 0$, put

$$T_r(K) = \{ x \in M \mid x, K < r \},$$

and for $0 \leq r_1 < r_2$,

$$A_{r_1, r_2}(K) = T_{r_2}(K) \setminus T_{r_1}(K).$$

Let $K \subset M^4$ be a closed subset. For $N^k \subset M^4$ a smooth submanifold without boundary, we denote by $II_{N^k}$ the second fundamental form of $N^k$.

Put

$$S_a(K) = \bigcup_{x \in \partial K} B_{2\xi \cdot \ell a(x)}(x).$$

**Theorem 3.3.** There exists a smooth manifold with boundary, $Z^4$, so that we have

$$K \subset Z^4 \subset K \cup S_a(K), \quad (3.4)$$

$$|II_{\partial Z}| \leq c \cdot \ell_{a}^{-1}, \quad (3.5)$$

where $c$ is a constant only depending on the dimension of $M$. Also for all nonnegative integers $k_1, k_2$ satisfying $k_1 + 2k_2 = 3$,

$$\int_{\partial Z} |II_{\partial Z}|^{k_1} |\text{Rm}|^{k_2} \leq c \cdot \int_{S_a(K)} (a^{-4} + (r_{[R]}^{-4})^{-4}). \quad (3.6)$$

**Proof.** We only indicate the proof of (3.4) and (3.5). The details can be found in [14]. Given $r \leq 1$, we can construct a function $F$ satisfying

$$F \leq (2\xi \cdot \ell a)^{-1} \rho_K \leq F + (1 - \delta)r,$$

$$|\nabla F| \leq (\xi \cdot \ell a)^{-1},$$

$$|\text{Hess} F| \leq cr^{-1}(2\xi \cdot \ell a)^{-2},$$

and
\[ |\nabla F| \geq \varepsilon (2\zeta \cdot \ell_a)^{-1} \quad \text{on } F^{-1}([0, \delta r]), \]

where \( \rho_K \) denotes the distance function from \( K \), \( \delta \) and \( \varepsilon \) are positive constants depending only on the dimension of \( M \). By (3.1) and (3.2) we can find such function \( F \) for the locally regularized metric \( \tilde{g} \). This is necessary because a key step in the construction depends on Yomdin’s quantitative version of the A.P. Morse lemma, which requires that the metric under consideration has certain regularity.

In view of coarea formula there exists some \( b \in [0, \delta r] \) such that

\[ Z = F^{-1}((-\infty, b]), \]

and

\[ \int_{F=b} |II_{\partial Z}|^{k_1} |Rm|^{k_2} \leq \frac{2}{\delta r \varepsilon} \int_{F^{-1}([0, \delta r])} |II_{\partial Z}|^{k_1} |Rm|^{k_2} |\nabla F|. \]

Also it is clear that for some \( r \leq 1 \),

\[ F^{-1}([0, \delta r]) \subset S_a(K). \]

This proves (3.4) and (3.5) immediately.

Now it remains to prove (3.6). For our purposes it suffices to prove it in the case that \( k_1 = k_2 = 1 \). Recall that when restricted to any ball \( B_{\ell_a(x)}(x) \) the function \( \ell_a \) satisfies a Harnack inequality with constant \( \frac{1+2\xi}{2-2\zeta} \). So the proof is a direct consequence of (3.3) and the covering argument. More precisely,

\[ \int_{\partial Z} |II_{\partial Z}| |Rm| \leq c_1 \int_{S_a(K)} |II_{\partial Z}| |Rm| \ell_a^{-1} \]

\[ \leq c_2 \sum_{x \in \partial K} \int_{B_{2\zeta \cdot \ell_a(x)}(x)} |II_{\partial Z}| |Rm| \ell_a^{-1} \]

\[ \leq c_3 \sum_{x \in \partial K} \ell_a(x)^{-2} \int_{B_{2\zeta \cdot \ell_a(x)}(x)} |Rm| \]

\[ \leq c_3 \sum_{x \in \partial K} \ell_a(x)^{-2} (\text{Vol}(B_{2\zeta \cdot \ell_a(x)}(x)))^{\frac{1}{2}} \left( \int_{B_{2\zeta \cdot \ell_a(x)}(x)} |Rm|^2 \right)^{\frac{1}{2}} \]

\[ \leq c_4 \sum_{x \in \partial K} \ell_a(x)^{-4} \text{Vol}(B_{2\zeta \cdot \ell_a(x)}(x)) \]

\[ \leq c_5 \int_{S_a(K)} \ell_a^{-4} \]
\[ \leq c_5 \int_{S_a(K)} \left( a^{-4} + (r_{|R|^4})^{-4} \right). \]

This proves the theorem. \( \square \)

By [14], Theorem 3.3 implies a better estimate which is important for the iteration procedure.

**Theorem 3.4.** There exists a smooth manifold with boundary, \( Z^4 \), so that we have

\[ T_{\frac{1}{2}r}(K) \subset Z^4 \subset T_{\frac{3}{2}r}(K), \quad \text{(3.7)} \]

\[ |\Pi_{\partial Z}| \leq c \cdot \left( r^{-1} + (r_{|R|^4})^{-1} \right), \quad \text{(3.8)} \]

where \( c \) is a constant only depending on the dimension of \( M \). Also for all \( k_1, k_2 \geq 0 \) satisfying \( k_1 + 2k_2 = 3 \),

\[ \int_{\partial Z} |\Pi_{\partial Z}|^{k_1} |Rm|^{k_2} \leq \frac{c}{r} \cdot \int_{T_{\frac{1}{2}r}(K)} \left( r^{-3} + (r_{|R|^4})^{-3} \right). \quad \text{(3.9)} \]

To obtain an estimate on the boundary term of the Gauss–Bonnet–Chern formula applied to a good chopping, we need the notion of maximal function.

For \( (X, \mu) \) a metric measure space, with \( \mu \) a finite Radon measure, and \( f \in L^1 \), define the maximal function for balls of radius at most \( r \) by

\[ M_f(x, r) = \sup_{s \leq r} \frac{1}{\mu(B_s(x))} \int_{B_s(x)} |f|. \]

Let \( W \subset X \) denote a measurable subset. We have the following lemma; see [14] for the proof.

**Lemma 3.5.** Suppose \( x \in W \) and \( s \leq 4r \). If every ball \( B_s(x) \) satisfies

\[ \mu(B_{2s}(x)) \leq 2^k \mu(B_s(x)), \]

then for all \( \Omega \geq \mu(W) \), \( \alpha < 1 \),

\[ \left( \frac{1}{\Omega} \int_W (M_f(x, r))^\alpha \, d\mu \right)^\frac{1}{\alpha} \leq \frac{c(\kappa, \alpha)}{\Omega} \int_{T_{\frac{1}{2}r}(W)} |f| \, d\mu. \]

For \( s \leq 1 \), we have \( \text{Vol}(B_s(x)) \leq c \cdot s^4 \). Thus, by the definition of \( \rho(x) \), we get for any \( 0 < s \leq 1 \),

\[ \rho(x)^{-1} \leq c \cdot \max((M_{|Rm|^2}(x, s))^\frac{1}{2}, s^{-1}). \]
which implies
\[
(r^c_{|R_1}(x))^{-3} \leq c \cdot (s^{-3} + (M_{|Rm|^2}(x,s))^\frac{3}{4}).
\]

Let \( s \leq r \leq 1 \). Following Remark 4.8 in [14], by Theorem 3.4, we can approximate a compact subset, \( K \), from the outside, by a submanifold with boundary, \( Z \), with \( K \subset Z \subset T_r(K) \), where the boundary term in the Gauss–Bonnet–Chern formula for \( Z \) satisfies the estimate (3.9).

With the above estimate for \( r^c_{|R_1}(x) \), this gives, for some constant \( c \),
\[
\left| \int_{\partial Z^4} TP_X \right| \leq c \cdot r^{-1} \int_{A_{\frac{1}{4}r, \frac{3}{4}r}(K)} (s^{-3} + (M_{|Rm|^2}(\cdot,s))^\frac{3}{4}).
\]

Choosing \( s = \frac{1}{312} r \) and employing Lemma 3.5, we get,
\[
\text{Vol}(A_{0,r}(K))^{-1} \left| \int_{\partial Z^4} TP_X \right| \leq c \cdot \left( r^{-4} + r^{-1} \left( \frac{1}{\text{Vol}(A_{0,r}(K))} \int_{A_{\frac{1}{4}r, \frac{3}{4}r}(K)} |Rm|^2 \right)^\frac{3}{4} \right).
\]

Note that \( \tilde{g} \) is \( C^0 \)-close to the original metric \( g \) and has locally bounded curvature. If \((M, g)\) is \((t, a)\)-collapsed with locally bounded \( W^{2,p} \) covering geometry, then we can choose \( \tilde{g} \) such that \((M, \tilde{g})\) is \((2t, a)\)-collapsed with locally bounded curvature. Therefore by Remark 2.7 in [14], Theorem 2.3 of [14] implies the existence of standard \( N \)-structure with respect to \( \tilde{g} \) on \( M \). That is, there exists \( t > 0 \) such that if \( M^4 \) is complete and \( W \subset M^4 \) is \((t, a)\)-collapsed with locally bounded \( W^{2,p} \) covering geometry, then there exists an \( a \)-standard \( N \)-structure on a subset containing \( W \). For the definition of \( a \)-standard \( N \)-structure, see [14] for details. Like \( \tilde{g} \), the invariant metric constructed with respect to this structure is also close to the original metric in the sense of (3.1) and (3.2). This is crucial in the equivariant chopping theorem above. In fact, if \( T_r(K) \) is \((t, r)\)-collapsed with locally bounded \( W^{2,p} \) covering geometry, then \( Z^4 \) can be chosen to be saturated for some standard \( N \)-structure and hence the Euler characteristic of \( Z^4 \) vanishes. Thus for any \( \Omega \geq \text{Vol}(A_{\frac{1}{4}r, \frac{3}{4}r}(K)) \), we have
\[
\left| \int_{Z^4} P_X \right| \leq c \cdot \Omega \cdot r^{-1} \left( r^{-3} + \left( \int_{A_{\frac{1}{4}r, \frac{3}{4}r}(K)} |Rm|^2 \right)^\frac{3}{4} \right).
\]

Assume \( r = 1 \). Let \( E \in M^4 \) denote a bounded open subset such that \( T_1(E) \) is \( t \)-collapsed with
\[
\int_{B_1(x)} |Rm|^2 \leq \varepsilon,
\]
for all \( x \in T_1(E) \). Recall that
\[
P_X = \frac{1}{8\pi^2} \cdot (|Rm|^2 - 4|\text{Ric}|^2) \, d\text{Vol},
\]
where $\text{Ric}$ denotes the trace free part of the Ricci curvature. Then we have

$$\int_E |\text{Rm}|^2 \leq c \cdot \text{Vol}(A_{0,1}(E)) \left( 1 + \left( \frac{1}{\text{Vol}(A_{0,1}(E))} \int_{A_{\frac{1}{2}, \frac{3}{4}}(E)} |\text{Rm}|^2 \right)^{\frac{3}{4}} \right) + c \cdot \text{Vol}(E).$$

For $i = 2, 3, \ldots$, put

$$D_i = \{ x \in A_{2^{-i}, 1 - 2^{-i}}(E) \mid r^c_{|R|}(x) \leq 2^{-(i+1)} \} ,$$

$$F_i = A_{2^{-i}, 1 - 2^{-i}}(E) \setminus D_i.$$

We have

$$T_{2^{-(i+1)}}(D_i) \subset A_{2^{-(i+1)}, 1 - 2^{-(i+1)}}(E).$$

Moreover, $\text{Lip} r^c_{|R|} \leq 1$ implies

$$\sup_{T_{2^{-(i+1)}}(D_i)} r^c_{|R|} \leq 2^{-i}.$$

Since $A_{0,1}(E)$ is $t$-collapsed with locally bounded $W^{2,p}$ covering geometry, it follows that $T_{2^{-(i+1)}}(D_i)$ is $(16t, 2^{-(i+1)})$-collapsed with locally bounded $W^{2,p}$ covering geometry. Hence, we have

$$\int_{A_{2^{-i}, 1 - 2^{-i}}(E)} |\text{Rm}|^2$$

$$= \int_{D_i} |\text{Rm}|^2 + \int_{F_i} |\text{Rm}|^2$$

$$\leq c \cdot 2^{4i} \text{Vol}(A_{0,1}(E)) \left( 1 + \left( \frac{1}{\text{Vol} A_{0,1}(E)} \int_{A_{2^{-(i+1)}, 1 - 2^{-(i+1)}}(E)} |\text{Rm}|^2 \right)^{\frac{3}{4}} \right)$$

$$+ c \cdot \text{Vol}(A_{2^{-i}, 1 - 2^{-i}}(E)) + c \cdot 2^{4i} \text{Vol}(A_{0,1}(E))$$

$$\leq c_1 \cdot 2^{4i} \text{Vol}(A_{0,1}(E)) \left( 1 + \left( \frac{1}{\text{Vol} A_{0,1}(E)} \int_{A_{2^{-(i+1)}, 1 - 2^{-(i+1)}}(E)} |\text{Rm}|^2 \right)^{\frac{3}{4}} \right),$$

where the estimate of $\int_{F_i} |\text{Rm}|^2$ follows from (3.3) and the covering argument mentioned above. Therefore, by applying the same iteration argument as in Lemma 5.1 of [14], we can obtain the estimate for the term $\frac{1}{\text{Vol}(A_{0,1}(E))} \int_{A_{\frac{1}{2}, \frac{3}{4}}(E)} |\text{Rm}|^2$ and hence a similar key estimate as in Theorem 3.1.
4. Smoothing the Riemannian metric

In this section, we prove Theorem 1.1. Notice that we shall give a more self-contained proof. The argument in [14] relied on some approximation used by Cheeger–Colding as well as Cheeger–Colding–Tian. Here we give a direct argument. We will mainly use the exponential map of the regularized metric to lift the original metric and use this exponential map to regularize the distance function of the original metric. Then we will analyze the behavior of the limiting distance function appeared in the contradiction argument. With this argument we don’t need to quote the results in Section 3 of [10].

As in [14], it suffices to prove the following result.

Proposition 4.1. For all $C_1 > 0$, there exists $\eta = \eta(C_1) > 0$, such that if

$$0 < r \leq \eta, \quad \int_{B_r(x)} |Rm|^2 \leq 4\pi^2,$$  \hfill (4.1)

$$\frac{Vol(B_r(x))}{Vol(B_r(\bar{x}))} \int_{B_r(x)} |Rm|^2 \leq C_1,$$  \hfill (4.2)

and

$$\frac{Vol(B_{\eta r}(x))}{Vol(B_{\eta r}(\bar{x}))} \leq \frac{1}{4},$$  \hfill (4.3)

then

$$\frac{Vol(B_{\eta r}(x))}{Vol(B_{\eta r}(\bar{x}))} \int_{B_{\eta r}(x)} |Rm|^2 \leq (1 - \eta) \frac{Vol(B_r(x))}{Vol(B_r(\bar{x}))} \int_{B_r(x)} |Rm|^2.$$  \hfill (4.4)

Proof. By scaling, we can suppose $r = 1$, $|\text{Ric}_{M^4}| \leq 3\eta^2$, $\bar{x} \in M^4_{-\eta^2}$.

Assume that for some $\eta > 0$, (4.2)–(4.4) hold but (4.5) fails. Then

$$\frac{Vol(B_{\frac{1}{2}}(x))}{Vol(B_{\frac{1}{2}}(\bar{x}))} \geq (1 - \eta) \frac{Vol(B_1(x))}{Vol(B_1(\bar{x}))},$$  \hfill (4.6)

$$\frac{1}{Vol(B_1(x) \setminus B_{\frac{1}{2}}(x))} \int_{B_1(x) \setminus B_{\frac{1}{2}}(x)} |Rm|^2 \leq c \cdot \eta,$$  \hfill (4.7)

for some absolute constant $c$ (arising from a relative volume comparison in dimension 4).

We may choose $\eta$ sufficiently small such that the hypothesis of Theorem 2.6 is valid for balls $B_\delta(y) \subset (B_1(x) \setminus B_{\frac{1}{2}}(x))$. Thus we can smooth the metric on $B_\delta(x) \setminus B_{\frac{1}{2}}(x)$ and the regularized metric has small curvature with respect to $\eta$. In particular, $B_\delta(x) \setminus B_{\frac{1}{2}}(x)$ has a $W^{2,p}$ bounded covering geometry by Proposition 2.5 in [3], since we have the smallness of the $L^2$-norm of the
curvature and the volume lower bound on the covering space. Then, there exists \( s > 0 \) such that for all \( y \in B_{\frac{3}{4}}(x) \setminus \overline{B}_{\frac{1}{2}}(x) \), the lifted metric with respect to the exponential map of regularized metric on \( B_{s}(O_y) \subset T_{y}M \) has \( C^{1,\alpha} \) bounded covering geometry.

Denote \( r(\cdot) = \overline{r}(\cdot, x) \) the distance function with respect to \( x \). Then we can smooth \( r \) as follows. Let \( \psi(s) \) be a nonnegative \( C_{\infty} \)-function on \( [0, \frac{1}{2}] \) such that \( \psi \equiv 1 \) near \( s = 0 \) and \( \psi \equiv 0 \) near \( s = \frac{1}{2} \). Put \( \psi_{\varepsilon}(s) = \psi(\varepsilon s) \). Let \( B_{\varepsilon}(O_y) \) denote the ball of radius \( \varepsilon \) about the origin in the tangent space at \( y \) equipped with the pull-back metric. Set

\[
\varepsilon(y) = \frac{\int_{B_{\varepsilon}(O_y)} r(\text{exp}_{y}(v)) \psi_{\varepsilon}(|v|) \, dv}{\int_{B_{\varepsilon}(O_y)} \psi_{\varepsilon}(|v|) \, dv},
\]

where \( \text{exp} \) denotes the exponential map of regularized metric, and \( dv \) is the volume form on the tangent space with respect to the pullback metric of regularized metric. It is clear that

\[
\lim_{\varepsilon \to 0^{+}} \varepsilon(y) = r(y),
\]

on any compact subset in \( B_{\frac{3}{4}}(x) \setminus \overline{B}_{\frac{1}{2}}(x) \). In fact, we have

\[
|\varepsilon(y) - r(y)| \leq \frac{\int_{B_{\varepsilon}(O_y)} |r(\text{exp}_{y}(v)) - r(y)| \psi_{\varepsilon}(|v|) \, dv}{\int_{B_{\varepsilon}(O_y)} \psi_{\varepsilon}(|v|) \, dv} \leq \frac{\int_{B_{\varepsilon}(O_y)} \text{exp}_{y}(v), y \cdot \psi_{\varepsilon}(|v|) \, dv}{\int_{B_{\varepsilon}(O_y)} \psi_{\varepsilon}(|v|) \, dv} \leq c\varepsilon,
\]

where \( c \) is a definite constant depending only on the difference between \( g \) and its regularized metric.

On the other hand, fix \( \frac{1}{2} \leq b < a \leq \frac{3}{4} \) and set

\[
\lambda_{\varepsilon}(s) = \begin{cases} 
1 & \text{if } b + \varepsilon \leq s \leq a - \varepsilon, \\
0 & \text{if } s \leq b, \text{ or } s \geq a \end{cases}
\]

with \( |\nabla \lambda_{\varepsilon}| = \frac{1}{\varepsilon} \) for \( b \leq s \leq b + \varepsilon \) and \( a - \varepsilon \leq s \leq a \). Then

\[
- \int_{A_{b,a}(x)} \lambda_{\varepsilon}(r(y)) \Delta r(y) = \int_{A_{b,a}(x)} \nabla \lambda_{\varepsilon} \nabla r = \int_{A_{b,a}(x)} \frac{1}{\varepsilon} - \int_{A_{a-\varepsilon,a}(x)} \frac{1}{\varepsilon}.
\]

Letting \( \varepsilon \to 0 \), we obtain

\[
- \int_{A_{b,a}(x)} \Delta r = \text{Vol}(\partial B_{b}(x)) - \text{Vol}(\partial B_{a}(x)).
\]
A similar argument then implies that
\[- \int_{A_{b,a}(x)} \Delta (r^2) = 2b \text{Vol}(\partial B_b(x)) - 2a \text{Vol}(\partial B_a(x)).\]

By the relative volume comparison theorem, if
\[(1 - \eta) \frac{\text{Vol}(B_1(x))}{\text{Vol}(B_\frac{1}{2}(x))} \leq \frac{\text{Vol}(B_1(x))}{\text{Vol}(B_1(x))},\]
then for \(\theta \geq \frac{1}{2},
\[(1 - \eta') \frac{\text{Vol}(B_\theta(x))}{\text{Vol}(B_{\frac{1}{2}}(x))} \leq \frac{\text{Vol}(B_1(x) \setminus B_\theta(x))}{\text{Vol}(B_1(x) \setminus B_{\frac{1}{2}}(x))} \leq \frac{\text{Vol}(\partial B_\theta(x))}{\text{Vol}(\partial B_{\frac{1}{2}}(x))},\]
where
\[\eta' = \eta \frac{\text{Vol}(B_1(x))}{\text{Vol}(B_1(x) \setminus B_{\frac{1}{2}}(x))}.\]

So we have, for some \(\Psi_1(\eta)\) with \(\Psi_1(\eta) \to 0\) as \(\eta \to 0,
\[(1 - \Psi_1(\eta)) \frac{\text{Vol}(\partial B_b(x))}{\text{Vol}(\partial B_b(x))} \leq (1 - \Psi_1(\eta)) \frac{\text{Vol}(B_b(x))}{\text{Vol}(B_b(x))} \leq \frac{\text{Vol}(\partial B_{\frac{1}{2}}(x))}{\text{Vol}(\partial B_{\frac{1}{2}}(x))}. \quad (4.8)\]

Noticing that by Laplacian comparison theorem, at any point where the distance function is smooth, we have
\[\Delta r \leq \frac{3}{r} + 3\eta.\]

Then
\[\Delta (r^2) = 2r \Delta r + 2|\nabla r|^2 \leq 8 + \Psi_2(\eta),\]
at any smooth point of \(r\).

We want to show that
\[\frac{1}{\text{Vol}(A_{b,a}(x))} \int_{A_{b,a}(x)} \left| \Delta (r^2) - 8 \right| \leq \Psi_3(\eta). \quad (4.9)\]

In fact,
\[
\frac{1}{\text{Vol}(A_{b,a}(x))} \int_{A_{b,a}(x)} |\triangle(r^2) - 8| \\
= \frac{1}{\text{Vol}(A_{b,a}(x))} \int_{A_{b,a}(x)} |\triangle(r^2) - (8 + \Psi_2(\eta)) + \Psi_2(\eta)| \\
\leq - \frac{1}{\text{Vol}(A_{b,a}(x))} \int_{A_{b,a}(x)} \triangle(r^2) + 8 + 2\Psi_2(\eta) \\
= \frac{2}{\text{Vol}(A_{b,a}(x))} (b \text{Vol}(\partial B_b(x)) - a \text{Vol}(\partial B_a(x))) + 8 + 2\Psi_2(\eta).
\]

If there exists some \( C > 0 \) such that
\[
\frac{1}{\text{Vol}(A_{b,a}(x))} \int_{A_{b,a}(x)} |\triangle(r^2) - 8| \geq C > 0,
\]
then
\[
C \leq \frac{2}{\text{Vol}(A_{b,a}(x))} (b \text{Vol}(\partial B_b(x)) - a \text{Vol}(\partial B_a(x))) + 8 + 2\Psi_2(\eta),
\]
which implies that, together with (4.8),
\[
\frac{2}{\text{Vol}(A_{b,a}(x))} \text{Vol}(\partial B_b(x)) \left( (1 - \Psi_1(\eta))a \frac{\text{Vol}(\partial B_a(x))}{\text{Vol}(\partial B_b(x))} - b \right) \leq -C + 8 + 2\Psi_2(\eta).
\]

Since relative volume comparison theorem implies that
\[
\text{Vol}(A_{b,a}(x)) = \int_{b}^{a} \text{Vol}(\partial B_s(x)) \, ds \\
\leq \int_{b}^{a} \text{Vol}(\partial B_b(x)) \left( \frac{s}{b} \right)^3 \left( 1 + \Psi_4(\eta) \right) \, ds \\
= \frac{1}{4} (1 + \Psi_4(\eta)) \left( a \left( \frac{a}{b} \right)^3 - b \right) \text{Vol}(\partial B_b(x)),
\]
we have
\[
\frac{8}{1 + \Psi_4(\eta)} \cdot \frac{(1 - \Psi_1(\eta))a \frac{\text{Vol}(\partial B_a(x))}{\text{Vol}(\partial B_b(x))} - b}{a \left( \frac{a}{b} \right)^3 - b} - 8 - 2\Psi_2(\eta) \leq -C < 0.
\]

But the left-hand side tends to 0 as \( \eta \to 0 \), which is a contradiction. This proves (4.9).
Now we claim that
\[
\left| \int_{r^{-1}_e(\alpha)} T P \chi - \frac{\text{Vol}(r^{-1}_e(\alpha))}{\text{Vol}(\partial B_\alpha(x))} \right| \leq \Psi(\eta, \varepsilon) \cdot \text{Vol}(\partial B_\alpha(x)), \tag{4.10}
\]
where \( \Psi(\eta, \varepsilon) \to 0 \) as \( \eta \to 0 \) and \( \varepsilon \to 0 \). Here \( T P \chi \) denotes the boundary term of Gauss–Bonnet–Chern formula.

By pulling back the distance functions \( r \) to the tangent space \( B_y(O_y) \) via the exponential map of the regularized metric, of \( y \in B_{1/2}(x) \setminus B_1(x) \), we reduce to the noncollapsed case. We denote the resulting pull-back functions by \( \bar{r}_i \).

Suppose the claim fails to hold. Then for some fixed \( \varepsilon > 0 \), which will be determined later, there are sequences \( \eta_i \to 0 \) and counterexamples, \( \tilde{K}_i \), in manifolds \( (\tilde{M}_i^4, \tilde{g}_i) \) converging in the \( C^{1,\alpha} \)-topology to \( (\tilde{K}_\infty, \tilde{g}_\infty) \), a portion of an annulus in a flat cone so that the corresponding distances functions, \( \tilde{r}_i \), converge in the \( C^\alpha \)-topology to some function, \( \tilde{r}_\infty \).

Using (4.9), we can show that, on \( \tilde{K}_\infty \), \( \tilde{r}_\infty \) is a weak solution of the equation
\[
\Delta(\tilde{r}_\infty)^2 = 8,
\]
and then from elliptic regularity theory, we conclude that \( \tilde{r}_\infty \) is a smooth solution. We also need to show that, on \( \tilde{K}_\infty \), \( \tilde{r}_\infty \) is a strong solution of the equation
\[
\Delta \tilde{r}_\infty = 3 \tilde{r}_\infty. \tag{4.11}
\]

In fact, first notice that \( \tilde{r}_i \geq \frac{1}{2} \) on \( \tilde{K}_i \), then
\[
\int_{\tilde{K}_i} \left| \Delta \tilde{g}_i \tilde{r}_i - \frac{3}{\tilde{r}_i} \right| = \int_{\tilde{K}_i} \frac{1}{2\tilde{r}_i} \left| \Delta \tilde{g}_i (\tilde{r}_i)^2 - 8 \right| \leq \int_{\tilde{K}_i} \left| \Delta \tilde{g}_i (\tilde{r}_i)^2 - 8 \right| \to 0,
\]
as \( i \to \infty \), so for any \( \phi \in C^\infty_0(\tilde{K}_\infty) \), we have
\[
\int_{\tilde{K}_\infty} \tilde{r}_\infty \Delta \phi = \int_{\tilde{K}_\infty} \left( \lim_{i \to \infty} \tilde{r}_i \right) \Delta \phi
= \lim_{i \to \infty} \int_{\tilde{K}_\infty} \tilde{r}_i \Delta \tilde{g}_i \phi
= \lim_{i \to \infty} \int_{\tilde{K}_\infty} (\Delta \tilde{g}_i \tilde{r}_i) \phi
= \lim_{i \to \infty} \int_{\tilde{K}_\infty} \left( \Delta \tilde{g}_i \tilde{r}_i - \frac{3}{\tilde{r}_i} \right) + \lim_{i \to \infty} \int_{\tilde{K}_\infty} \frac{3}{\tilde{r}_i} \phi.
\[ \int_{\tilde{K}_\infty} \frac{3}{r_\infty^3} \phi, \]

which shows that \( r_\infty \) is a weak solution of Eq. (4.11), and hence, a strong solution as claimed. Then, a direct computation yields

\[
8 = \Delta (r_\infty)^2 \\
= 2(r_\infty \Delta r_\infty + |\nabla r_\infty|^2) \\
= 2(3 + |\nabla r_\infty|^2),
\]

which gives that

\[ |\nabla r_\infty| = 1, \]

and thus, \( \nabla^2 r_\infty(\nabla r_\infty, \cdot) \equiv 0. \) From the Bochner formula, since \( \tilde{K}_\infty \) is flat, we have

\[
0 = \Delta |\nabla r_\infty|^2 = \tilde{g}_\infty(\nabla r_\infty, \nabla \Delta r_\infty) + |\nabla^2 r_\infty|^2 \\
= \tilde{g}_\infty(\nabla r_\infty, \nabla \frac{3}{r_\infty}) + |\nabla^2 r_\infty|^2 \\
= -\frac{3}{r_\infty^2} + |\nabla^2 r_\infty|^2 \\
= |\nabla^2 r_\infty| - \frac{1}{r_\infty}(\tilde{g}_\infty - d\tilde{r}_\infty \otimes d\tilde{r}_\infty).
\]

This implies that, on \( \tilde{K}_\infty, \) \( r_\infty \) is a strong solution of the equation

\[ \nabla^2 r_\infty = \frac{1}{r_\infty}(\tilde{g}_\infty - d\tilde{r}_\infty \otimes d\tilde{r}_\infty), \]

and in particular, that level sets of \( r_\infty \) have constant positive sectional curvature, and therefore are locally isometric to portions of Euclidean spheres. Therefore, we have

\[ \left| \int_{\tilde{r}_\infty^{-1}(a)} TP_x - \frac{\text{Vol}(\tilde{r}_\infty^{-1}(a))}{2\pi^2 a^3} \right| = 0. \]

Notice that the regularized distance function \( r_\varepsilon(y) \) is defined with respect to the exponential map of \( y \) and correspondingly a small ball on the tangent space of \( y. \) So it is unclear at this stage how the behavior of this regularized function changes when the points under consideration are close to each other. Let \( \exp_y : B_\varepsilon(O_y) \to M \) denote the exponential map of the regularized metric at \( y. \) For \( z \in B_\frac{1}{2\varepsilon}(y), \) there exists some \( \tilde{z} \in B_\frac{1}{2\varepsilon}(O_y) \) such that \( \exp_y \tilde{z} = z. \) Then we claim that the map

\[ \exp_{\tilde{z}}^{-1} \cdot \exp_y : B_\frac{1}{2\varepsilon}(\tilde{z}) \subset T_{\tilde{z}}M \to B_\frac{1}{2\varepsilon}(O_z) \subset T_{\tilde{z}}M \]
is one-to-one and an isometry. In fact, for any \( \tilde{w} \in B^{1/2}_{1/2}(\tilde{z}) \), connect it with the center \( \tilde{z} \) by the unique minimal geodesic, which can be projected down to a geodesic on \( M \) and hence can be mapped to a geodesic in \( B^{1/2}_{1/2}(O_y) \) by the uniqueness of lifting property. The end point of this geodesic on \( B^{1/2}_{1/2}(O_y) \) gives the image of \( \tilde{w} \). Therefore, we obtain

\[
r_\varepsilon(z) = \frac{\int_{B^{1/2}_{1/2}(O_z)} r(\exp_v(z)) \psi_\varepsilon(|v|) \, dv}{\int_{B^{1/2}_{1/2}(O_z)} \psi_\varepsilon(|v|) \, dv}
\]

which gives the smooth dependence of the regularized distance function as required.

Notice that our estimate is only valid on a portion of \( r^{-1}_i (a) \). To remedy this, we can cover \( r^{-1}_i (a) \) by balls \( B_{r_i}(y_i) \) such that the multiplicity of this covering is bounded by a definite constant. Then by choosing \( \varepsilon \) sufficiently small first and then \( i \) sufficiently large, we have

\[
\left| \int_{\tilde{r}^{-1}_{i,\varepsilon}(a)} TP_X - \frac{\text{Vol}(\tilde{r}^{-1}_{i,\varepsilon}(a))}{\text{Vol}(\partial B_a(x_i))} \right| \leq \Psi(\eta_i, \varepsilon) \cdot \text{Vol}(\tilde{r}^{-1}_{i,\varepsilon}(a)).
\]

(4.12)

Now (4.10) follows by descending the metric to the base space \( M \).

On the other hand, we have, for some subset, \( A \subset \left[ \frac{1}{2}, \frac{3}{4} \right] \), of regular values of \( r_\varepsilon \) if \( a \in A \),

\[
\left| 1 - \frac{\text{Vol}(r_\varepsilon^{-1}(a))}{\text{Vol}(\partial B_a(x))} \right| \leq \Psi(\varepsilon).
\]

(4.13)

Therefore, we obtain a contradiction, and hence, (4.10) holds.

Taking \( U = r_\varepsilon^{-1}((0, a]) \), we get from (4.4), (4.10) and (4.13),

\[
0 < \int_{\partial U} TP_X < \frac{1}{2}.
\]

Since

\[
P_X = \frac{1}{8\pi^2} \cdot (|\mathcal{R}m|^2 - 4|\mathring{\mathcal{Ric}}|^2) \, d\text{Vol},
\]

we then get
\[
\int_U P_\chi \leq \frac{1}{8\pi^2} \int_U |\text{Rm}|^2 < \frac{1}{2}.
\]

Unlike the Einstein case in [14], it is not obvious that there holds

\[
0 < \int_U P_\chi + \int_{\partial U} T P_\chi.
\]

(4.14)

However, by Remark 8.22 in [14], we can still prove (4.14). In fact, by relative volume comparison and almost volume cone condition as above, we have

\[
\int_{r_\varepsilon^{-1}(a)} T P_\chi \geq \frac{\text{Vol}(r_\varepsilon^{-1}(a))}{\text{Vol}(\partial B_a(\chi))} - \Psi(\eta, \varepsilon) \text{Vol}(\partial B_a(\chi)) \\
\geq (1 - \eta') (1 - \Psi(\varepsilon)) (1 - 2\Psi(\eta, \varepsilon)) \frac{\text{Vol}(B_a(\chi))}{\text{Vol}(B_a(\chi))},
\]

where \(\eta' = \eta \frac{\text{Vol}(B_1(\chi))}{\text{Vol}(B_1(\chi) \setminus B_a(\chi))}\). On the other hand,

\[
\int_U P_\chi \geq -\frac{1}{2} \eta^4 \text{Vol}(U) \\
\geq -\eta^4 \text{Vol}(B_a(\chi)).
\]

Choosing \(\varepsilon\) sufficiently small first and then \(\eta\) sufficiently small, we get (4.14) and thus,

\[0 < \chi(U) < 1,
\]

which leads to a contradiction. This proves the proposition as mentioned above. \(\square\)

Acknowledgments

The author would like to thank Professor Gang Tian for bringing this problem to his attention, and for many stimulating discussions and suggestions, especially for pointing out the argument in Section 4 to simplify the original proof by Cheeger and Tian using Cheeger–Colding theory. The author feels grateful to Professor Jeff Cheeger and benefits a lot from his constructive comments. This paper could not be finished without their ingenious ideas in the Einstein case and their outlines in the bounded Ricci curvature case. The author also thanks the referee for helpful suggestions.
References