A representation of solution of stochastic differential equations

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Abstract

We prove that the logarithm of the formal power series, obtained from a stochastic differential equation, is an element in the closure of the Lie algebra generated by vector fields being coefficients of equations. By using this result, we obtain a representation of the solution of stochastic differential equations in terms of Lie brackets and iterated Stratonovich integrals in the algebra of formal power series.

Keywords: Specht–Wever theorem; Iterated Stratonovich integrals; Lie brackets; Stochastic differential equation; Brownian motion

1. Introduction

Let us consider a SDE (Stochastic Differential Equation) on $\mathbb{R}^d$,

\[
\begin{aligned}
  d\xi_t &= \sum_{i=1}^r X_i(\xi_t) \circ dB^i_t + X_0(\xi_t) \, dt, \\
  \xi_0 &= x_0,
\end{aligned}
\]

where $X_0, \ldots, X_r$ are $C^\infty$ bounded vector fields on $\mathbb{R}^d$, $B_t = (B^1_t, \ldots, B^r_t)$ is an $r$-dimensional Brownian motion and the symbol $\circ d$ denotes the Stratonovich stochastic differential. In the past...
years, under some conditions on vector fields $X_0, \ldots, X_r$, a number of authors have expressed the solution as a functional of $B_t$ (see, e.g., [4,8,9]). Kunita [7] has obtained the explicit formula in the case when the Lie algebra generated by vector fields $X_0, \ldots, X_r$ is nilpotent or solvable. In particular, Castell [3] has expressed a universal and explicit formula in terms of Lie brackets and iterated stochastic Stratonovich integrals. This formula contains the above results in the nilpotent case and extends works studied by Ben Arous [2] to general diffusions.

Usual methods to get representations are essentially based on the well-known facts on a system of ordinary differential equations obtained by replacing the Brownian path in SDE (1) with the piecewise linear approximation. By using the Campbell–Hausdorff formula to the solution of this ordinary differential equation and taking the limit for the extension to the case of SDE, the above results can be obtained.

In this paper we consider the free algebra generated by $\mathcal{X} = \{X_0, X_1, \ldots, X_r\}$ consisting of vector fields being coefficients of equations and extend this algebra to the algebra of formal power series related to the solution of SDE (1). For convenience, we scale SDE (1) by introducing a small parameter $\epsilon \in (0, 1]$:

$$
\begin{align*}
    d\xi^\epsilon_t &= \sum_{i=1}^{r} \epsilon X_i(\xi^\epsilon_t) \circ dB^i_t + \epsilon^2 X_0(\xi^\epsilon_t) \, dt, \\
    \xi^\epsilon_0 &= x_0.
\end{align*}
$$

The main purpose of this work is to show that $\log \xi^\epsilon_t$ is an element of the free Lie algebra generated by $\mathcal{X} = \{X_0, X_1, \ldots, X_r\}$. By using this result, we directly obtain a representation in terms of Lie brackets and iterated Stratonovich integrals without appealing the Campbell–Hausdorff formula and the limit procedure. This representation can be used to derive various interesting formulas of the solution by using algebraic computations and combinatorial arguments.

Our approach consists of the following procedures. In Section 2, we introduce some notation and consider formal power series. In Section 3, we prove that the logarithm of the solution is a Lie element by using Friedrichs test (see, e.g., [5, p. 170]). In Section 4, we obtain an explicit formula for this element in terms of Lie brackets and iterated Stratonovich integrals by applying the Specht–Wever theorem (see, e.g., [5, p. 169]) to each term of the expansion of the logarithm of the solution. We also give some examples in order to show how our representations are useful for solving SDE (1).

In what follows we will use the summation convention, that is, we will omit the summation sign over repeated indices.

2. Preliminaries

We introduce the notations and give some assumptions:

(1) We define the following sets:

$$
E := \{0, 1, \ldots, r\},
E_a := \{ (j_1, \ldots, j_a) : j_1, \ldots, j_a \in E \} \quad \text{for } a \geq 1,
E(b) := \bigcup_{a=1}^{b} E_a \quad \text{for } 1 \leq b \leq \infty.
$$

Let $J = (j_1, \ldots, j_m)$ be a multi-index with $J \in E^{\otimes m}$, $m = 1, 2, \ldots$. We set:

$$
|J| = m \quad \text{(size of } J) \quad \text{and} \quad \|J\| = m + \# \{ \alpha \mid j_\alpha = 0 \}.
$$
(2) Let $\mathcal{X}(\mathbb{R}^d)$ be the totality of $C^\infty$ bounded Lipschitz vector fields on $\mathbb{R}^d$ with the commutator $[X, Y] = XY - YX$ for $X, Y \in \mathcal{X}(\mathbb{R}^d)$. Let $X_i \in \mathcal{X}(\mathbb{R}^d), i = 0, 1, \ldots, r$, be given. For $J = (j_1, \ldots, j_m)$, we define $X_J$ and $X^J$ to denote the $|J|$th order differential operator and the iterated Lie brackets, respectively:

$$X_J = X_{j_1} \cdots X_{j_m} \quad \text{and} \quad X^J = [X_{j_1} [X_{j_2} \cdots [X_{j_{m-1}} X_{j_m}]]].$$

(3) Let $(B^1_r, \ldots, B^r_r)$ be an $r$-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We write $B^J_t$ as the iterated Stratonovich integrals:

$$B^J_t = \int_0^t \circ dB^{j_m}_t \int_0^{t_m} \circ dB^{j_{m-1}}_t \cdots \int_0^{t_2} \circ dB^{j_1}_t.$$ 

Also we write $B^0_t = t$ for simplicity.

Let $\mathcal{X}$ be a set of vector fields $X_0, X_1, \ldots, X_r$ on $\mathbb{R}^d$. Then we define a vector space $\mathcal{M}$ with basis $\mathcal{X}$ by

$$\mathcal{M} = \mathbb{R}X_0 \oplus \mathbb{R}X_1 \oplus \cdots \oplus \mathbb{R}X_r,$$

and we form the tensor algebra based on $\mathcal{M}$,

$$\mathfrak{T} = \mathbb{R}1 \oplus \mathcal{M} \oplus (\mathcal{M} \otimes \mathcal{M}) \oplus \cdots.$$ 

This algebra $\mathfrak{T}$ is graded with $\mathcal{M}_m = \mathcal{M} \otimes \mathcal{M} \otimes \cdots \otimes \mathcal{M}$ ($m$-times) as the subspace of homogeneous elements of degree $m$ and $\mathcal{M}_m \mathcal{M}_n \subseteq \mathcal{M}_{m+n}$. A basis for this space is the set of monomials of the form $X_{j_1} X_{j_2} \cdots X_{j_m}, j_i = 0, 1, \ldots, r$. Let $\mathfrak{T}$ denote the algebra of formal power series in the $X_i$. Thus elements of $\mathfrak{T}$ are $\sum_{i=0}^\infty a_i = a_0 + a_1 + \cdots$ with $a_i \in \mathcal{M}_i$, where $\mathcal{M}_0 = \mathbb{R}1$.

Now we consider the following formal power series in elements of the algebra $\mathfrak{T}$ generated by $\mathcal{X} = \{X_0, X_1, \ldots, X_r\}$:

$$\xi^\varepsilon_t := 1 + \varepsilon \int_0^t \circ dB^{j_1}_t X_{j_1} + \varepsilon^2 \int_0^t \circ dB^{j_1}_t X_0$$

$$+ \varepsilon^2 \int_0^t \circ dB^{j_2}_t \int_0^{t_2} \circ dB^{j_1}_t X_{j_1} X_{j_2} + \varepsilon^3 \int_0^t \circ dB^{j_3}_t \int_0^{t_2} \circ dB^{j_1}_t X_{j_1} X_0$$

$$+ \varepsilon^3 \int_0^t \circ dB^{j_2}_t \int_0^{t_2} \circ dB^{j_1}_t X_0 X_{j_2}$$

$$+ \varepsilon^3 \int_0^t \circ dB^{j_3}_t \int_0^{t_3} \circ dB^{j_2}_t \int_0^{t_2} \circ dB^{j_1}_t X_{j_1} X_{j_2} X_{j_3} + \cdots$$

$$= 1 + \sum_{p=1}^\infty \sum_{|J|=p} \varepsilon^{|J|} B^J_t X_J,$$

where $1$ denote the identity vector field, that is,

$$1 = x^i \frac{\partial}{\partial x^i}.$$
Here the equality (3) holds \( \mathbb{P} \)-a.s. as an identity between two formal power series. We reorder the right-hand side of (3) by increasing powers of \( \epsilon \), so that this series becomes

\[
1 + \sum_{p=1}^{\infty} \epsilon^p \sum_{\|J\|=p} B^J_t X_J.
\]

By the successive applications of the Ito formula, we have the following theorem (see [6, p. 416]):

**Theorem 1.** Let \( \xi_t(x_0) \) be the unique solution of SDE (2). Suppose that the infinite series (4) is absolutely convergent \( \mathbb{P} \)-a.s. for each \( t \in [0, T] \) and \( \epsilon \in (0, 1] \). Then it holds \( \mathbb{P} \)-a.s.

\[
\xi_{\epsilon t}(x_0) = x_0 + \sum_{p=1}^{\infty} \epsilon^p \sum_{\|J\|=p} B^J_t X_J(x_0).
\]

### 3. Algebraic lemmas and the main theorem

In this section we show that when (4) holds, the logarithm of \( \xi_{\epsilon t} \), regarded as an element of the algebra \( \mathfrak{g} \) generated by a set \( \mathfrak{X} \), is a Lie element. For a fixed positive integer \( b \), we identify the set \( \{ y = (y_J)_{J \in E(b)} : y_J \in \mathbb{R}, J \in E(b) \} \) with \( \mathbb{R}^{\sharp E(b)} \). The coordinate system on \( \mathbb{R}^{\sharp E(b)} \) is also denoted by \( y_J, J \in E(b) \). We define the vector field \( Q_i = Q_i^{(b)}, i \in E, \) on \( \mathbb{R}^{\sharp E(b)} \) by

\[
Q_i^{(b)} = \frac{\partial}{\partial y^i} + \sum_{a+1 \leq b \atop j_1, \ldots, j_a \in E} y_j^{j_1 \ldots j_a} \frac{\partial}{\partial y_j^{j_1 \ldots j_a}}.
\]

Then the following proposition has given by Yamato [9].

**Proposition 2.** The \( \sharp E(b) \)-dimensional process \( Y_t = (B^J_t, J \in E(b)), t \geq 0, \) is the unique solution of the following SDE:

\[
\begin{aligned}
&dY_t = Q_i^{(b)}(Y_t) \circ dB^i_t, \\
&Y_0 = 0 \in \mathbb{R}^{\sharp E(b)}.
\end{aligned}
\]

We introduce the notation \( \sum (j_1, \ldots, j_{\gamma}(j_{\gamma+1}, \ldots, j_l)) \) appearing in the following lemma. The sum \( \sum (j_1, \ldots, j_{\gamma}(j_{\gamma+1}, \ldots, j_l)) \) is taken in the following way: We indicate the \( l - \gamma \) numbers \( j_{\gamma+1}, \ldots, j_l \) by \( l - \gamma \) bars. Thus \( |j_1||j_2||j_3|| \) is used as a symbol for \( \gamma = 3 \) and \( l = 12 \), which represents

\[
\text{j4, j1, j5, j6, j2, j7, j8, j9, j3, j10, j11, j12.}
\]

The sum is taken over all the ways of this arrangement of equal to the ways of placing \( l - \gamma \) bars between \( \gamma \) numbers \( j_1, j_2, \ldots, j_{\gamma} \) including two ends. The number of arrangements is equal to the number of selecting \( \gamma \) places out of \( l \), that is, \( \left( \begin{array}{l} l \\ \gamma \end{array} \right) \).

**Lemma 3.** For any nonnegative integer \( \gamma \) and a positive integer \( l \) with \( \gamma \leq l \), we have

\[
\sum_{(j_1, \ldots, j_{\gamma})(j_{\gamma+1}, \ldots, j_l)} B_{i}^{(j_1, j_2, \ldots, j_{\gamma})} B_{i}^{(j_{\gamma+1}, \ldots, j_l)} = B_{i}^{(j_1, \ldots, j_{\gamma})} B_{i}^{(j_{\gamma+1}, \ldots, j_l)}, \quad \mathbb{P}-a.s.
\]
\textbf{Proof.} Set $J_{ij}^{p} := (j_{p}, \ldots, j_{q})$. For $p > q$ we put $y^{i}_{j_{p}} = 1$ and $B_{t}^{i}_{j_{p}} = 1$. When $\gamma = 0$ or $\gamma = 1$, it is obvious that (8) holds for all $l = 1, 2, \ldots$. From (6), we have that for $\gamma = 1, \ldots, l - 1$
\[ Q_{t}^{(l)}(y^{i}_{j_{1}}y^{i}_{j_{r+1}}) = y^{i}_{j_{1}}y^{i}_{j_{r+1}}\delta^{j_{l}}_{i} + y^{i}_{j_{1}}y^{i}_{j_{r+1}}\delta^{l}_{i}. \]
Hence applying the Ito’s formula to a function $f(y) = y^{i}_{j_{1}}y^{i}_{j_{r+1}}$ where $f : \mathbb{R}^{tE(l)} \to \mathbb{R}$, we obtain
\[ B_{t}^{i}_{j_{1}}B_{t}^{i}_{j_{r+1}} = \int_{0}^{t} dB_{t_{y_{j_{1}}}j_{1}}^{i}B_{t}^{i}_{j_{r+1}} + \int_{0}^{t} dB_{t_{y_{j_{r+1}}}j_{r+1}}^{i}B_{t}^{i}_{j_{1}}. \quad (9) \]
When $\gamma = 1$ and $l = 2$, the integration by parts gives
\[ \sum_{(j_{1},j_{2})} B_{t}^{i_{1}}B_{t}^{i_{2}} = B_{t}^{i_{1}}B_{t}^{i_{2}} + B_{t}^{i_{2}}B_{t}^{i_{1}} = B_{t}^{i_{1}}B_{t}^{i_{2}}. \quad (10) \]
Now we assume that (8) is true for $\gamma = 1$ and $l = n - 1$. Then from (9), we get
\[ B_{t}^{i_{1}}B_{t}^{i_{2}} \cdots B_{t}^{i_{n}} = \int_{0}^{t} dB_{t_{y_{j_{1}}}j_{1}}^{i}B_{t}^{i_{2}} + \int_{0}^{t} dB_{t_{y_{j_{2}}}j_{2}}^{i}B_{t}^{i_{3}} + \cdots + \int_{0}^{t} dB_{t_{y_{j_{n}}}j_{n}}^{i}B_{t}^{i_{1}}. \]
\[ = \sum_{(j_{1},j_{2}, \ldots,j_{n-1},j_{n})} B_{t}^{i_{j_{1}}j_{2} \cdots j_{n-1}j_{n}} + \sum_{(j_{1},j_{2}, \ldots,j_{n-1})} B_{t}^{i_{j_{1}}j_{2} \cdots j_{n-1}} = \sum_{(j_{1},j_{2}, \ldots,j_{n})} B_{t}^{i_{j_{1}}j_{2} \cdots j_{n}}. \quad (11) \]
Hence (8) holds for $\gamma = 1$ and $l = 1, 2, \ldots$. Suppose that (8) is true for $\gamma = m + 1, l = n$ and $\gamma = m, l = n + 1$ with $m + 1 \leq n$. Then by (9),
\[ B_{t}^{i_{m+1}}B_{t}^{i_{m+2}} = \int_{0}^{t} dB_{t_{y_{j_{m+1}}}j_{m+1}}^{i}B_{t}^{i_{m+2}} + \int_{0}^{t} dB_{t_{y_{j_{m+2}}}j_{m+2}}^{i}B_{t}^{i_{m+1}}. \]
\[ = \sum_{(j_{1},\ldots,j_{m})} B_{t}^{i_{j_{1}}j_{2} \cdots j_{m}j_{m+1}} + \sum_{(j_{1},\ldots,j_{m+1})} B_{t}^{i_{j_{1}}j_{2} \cdots j_{m+1}}. \quad (12) \]
Hence (8) holds for all nonnegative integers $\gamma$ and positive integers $l$ with $\gamma \leq l$. \hfill \Box

We recall some related facts with our works about Lie algebra (see, e.g., [5]). Let $\widetilde{S}^{(i)}$ be a subset of elements of the form $a_{i} + a_{i+1} + \cdots$. We define a valuation in $\widetilde{S}$ by
\[ |a| = \begin{cases} 0 & \text{if } a = 0, \\ 2^{-i} & \text{if } a \neq 0 \text{ and } a \in \widetilde{S}^{(i)}, a \notin \widetilde{S}^{(i+1)}. \end{cases} \]
From this the series \( \sum_{i=1}^{\infty} x_i, x_i \in \mathfrak{F} \), converges if and only if \( |x_i| \to 0 \). If \( z \) has zero \( \mathbb{R} \) component, then
\[
\exp z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots
\]
(13) and
\[
\log(1 + z) = z - \frac{z^2}{2!} + \frac{z^3}{3!} - \cdots
\]
(14)
are well-defined elements of \( \mathfrak{F} \).

We denote by \( \mathfrak{F}_L \) Lie algebra of an associative algebra \( \mathfrak{F} \). Let \( \mathfrak{F}_L \) be the subalgebra of \( \mathfrak{F}_L \) generated by the elements of \( \mathfrak{X} \). An element \( a \in \mathfrak{F} \) is called a Lie element if \( a \in \mathfrak{F}_L \). Friedrichs’s theorem gives a criterion that an element of \( \mathfrak{F} \) is a Lie element.

**Theorem 4** (Friedrichs). Let \( \mathfrak{F} \) be the free algebra generated by \( X_0, \ldots, X_r \) over \( \mathbb{R} \). Let \( \delta \) be the diagonal mapping of \( \mathfrak{F} \), that is, the homomorphism of \( \mathfrak{F} \) into \( \mathfrak{F} \otimes \mathfrak{F} \) such that \( \delta(X_i) = X_i \otimes 1 + 1 \otimes X_i \). Then \( a \in \mathfrak{F} \) is a Lie element, that is, \( a \in \mathfrak{F}_L \) if and only if \( \delta(a) = a \otimes 1 + 1 \otimes a \).

We see that an element is a Lie element if and only if its homogeneous parts are Lie elements. Also if \( a \) is a Lie element which is homogeneous of degree \( m \), then \( a \) may be written as a linear combination of elements of the following form:
\[
[\cdots[[X_{i_1}, X_{i_2}], X_{i_3}]\cdots X_{i_m}] \quad \text{for } i_j = 0, 1, \ldots, r \quad \text{and } \quad m = 1, 2, \ldots.
\]
Let \( \mathfrak{F}_L \) denote the subset of \( \mathfrak{F} \) of elements of the form \( \sum_{i=1}^{\infty} a_i \) where \( a_i \) is a Lie element in \( \mathfrak{F}_i \).

It is obvious that \( \mathfrak{F}_L \) is a subalgebra of \( \mathfrak{F}_L \). We set \( \xi^\epsilon_t := \exp Z^\epsilon(t) \), where
\[
Z^\epsilon(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left( \sum_{l=1}^{\epsilon} \sum_{\|J\|=l} X_J B^J_l \right)^n.
\]
(15)

From the above argument, the two series (4) and (15) converges in \( \mathfrak{F} \) for each \( t \geq 0 \) and \( \omega \in \Omega_1 \subseteq \Omega \) with \( \mathbb{P}(\Omega_1) = 1 \).

We apply the criterion of Friedrichs to show that the element \( \log(\xi^\epsilon_t) \) is a Lie element.

**Theorem 5.** For fixed \( t \geq 0, \epsilon \in [0, 1] \) and \( \omega \in \Omega_1 \subseteq \Omega \) with \( \mathbb{P}(\Omega_1) = 1 \), the element \( \log(\xi^\epsilon_t(\omega)) \) is a Lie element, that is, \( Z^\epsilon(t, \omega) \in \mathfrak{F}_L \).

**Proof.** It is clear that we have the extension of Friedrichs’s theorem: \( a = \sum_{i=0}^{\infty} a_i \in \mathfrak{F}, a_i \in \mathfrak{F}_i \), is in \( \mathfrak{F}_L \) if and only if \( \delta(a) = a \otimes 1 + 1 \otimes a \) where \( \delta: a \to \delta(a) \) is the diagonal mapping from \( \mathfrak{F} \) into \( \mathfrak{F} \otimes \mathfrak{F} \). Hence if we have already prove that
\[
\delta(\exp Z^\epsilon(t)) = (\exp Z^\epsilon(t) \otimes 1)(1 \otimes \exp Z^\epsilon(t)),
\]
then
\[
\delta(\exp Z^\epsilon(t)) = \log(\exp Z^\epsilon(t))
\]
\[
= \log((\exp Z^\epsilon(t) \otimes 1)(1 \otimes \exp Z^\epsilon(t)))
\]
\[
= \log(\exp Z^\epsilon(t) \otimes 1) + \log(1 \otimes \exp Z^\epsilon(t))
\]
\[
= \log(\exp Z^\epsilon(t) \otimes 1) + 1 \otimes \log(\exp Z^\epsilon(t)).
\]
In order to apply the criterion of Friedrichs to prove that $Z^\varepsilon(t)$ is a Lie element, it suffices to show that $\delta(\varepsilon^\xi_t(x)) = \varepsilon^\xi_t(\omega) \otimes \varepsilon^\xi_t(\omega)$ for fixed $\varepsilon > 0$, $t \geq 0$, and $\omega \in \Omega_\varepsilon \subseteq \Omega$ with $\mathbb{P}_0(\Omega_\varepsilon) = 1$ (see, e.g., [5, p. 173]). Let us set $B_t^{(j_1,j_2,\ldots,j_m)} = B_t^{j_1j_2\ldots j_m}$. By the definition of $\delta$ and $X_{j_i} \in \mathcal{X}$ for $i = 1, \ldots, m$, we have

\[
\delta(\varepsilon^\xi_t) = 1 \otimes 1 + \varepsilon p \sum_{|\mathcal{J}|=p} B_t^{\mathcal{J}} \delta(X_{\mathcal{J}})
\]

\[
= 1 \otimes 1 + \varepsilon B_t^{(j_1)}(X_{j_1} \otimes 1 + 1 \otimes X_{j_1})
\]

\[
+ \varepsilon^2 [B_t^{(j_1,j_2)}(X_{j_1} \otimes 1 + 1 \otimes X_{j_1})(X_{j_2} \otimes 1 + 1 \otimes X_{j_2})
\]

\[
+ B_t^{(0)}(X_0 \otimes 1 + 1 \otimes X_0)]
\]

\[
+ \varepsilon^3 [B_t^{(j_1,j_2,j_3)}(X_{j_1} \otimes 1 + 1 \otimes X_{j_1})(X_{j_2} \otimes 1 + 1 \otimes X_{j_2})
\]

\[
\times (X_{j_3} \otimes 1 + 1 \otimes X_{j_3})
\]

\[
+ B_t^{(j_1,0)}(X_{j_1} \otimes 1 + 1 \otimes X_{j_1})(X_0 \otimes 1 + 1 \otimes X_0)
\]

\[
+ B_t^{(0,j_2)}(X_0 \otimes 1 + 1 \otimes X_0)(X_{j_2} \otimes 1 + 1 \otimes X_{j_2})
\]

\[
+ \cdots
\]

\[
+ \varepsilon^p \sum_{|\mathcal{J}|=p/2} \sum_{|\mathcal{J}|=p} B_t^{(j_1,j_2,\ldots,j_{p-q})} (X_{j_1} \otimes 1 + 1 \otimes X_{j_1}) \cdots (X_{j_{p-q}} \otimes 1 + 1 \otimes X_{j_{p-q}})
\]

\[
+ \cdots,
\]

where the sum $\sum_{|\mathcal{J}|=p} \sum_{|\mathcal{J}|=q} B_t^{(j_1,j_2,\ldots,j_{p-q})} (X_{j_1} \otimes 1 + 1 \otimes X_{j_1}) \cdots (X_{j_{p-q}} \otimes 1 + 1 \otimes X_{j_{p-q}})$ is taken over all the ways of choosing $q$'s 0 out of $j_1, \ldots, j_{p-q}$. When $\gamma = 0$, we set $X_{j_1} \cdots X_{j_{\gamma}} = 1$ and $B_t^{(j_1,\ldots,j_{\gamma})} = 1$. Then

\[
\varepsilon^\xi_t \otimes \varepsilon^\xi_t = 1 \otimes 1 + \varepsilon B_t^{(j_1)}(X_{j_1} \otimes 1 + 1 \otimes X_{j_1})
\]

\[
+ \varepsilon^2 [B_t^{(j_1,j_2)}(X_{j_1} X_{j_2} \otimes 1 + 1 \otimes X_{j_1} X_{j_2}) + B_t^{(j_1)} B_t^{(j_2)}(X_{j_1} \otimes 1 + 1 \otimes X_{j_1} X_{j_2})
\]

\[
+ B_t^{(0)}(X_0 \otimes 1 + 1 \otimes X_0)]
\]

\[
+ \varepsilon^3 [B_t^{(j_1,j_2,j_3)}(X_{j_1} X_{j_2} \otimes 1 + 1 \otimes X_{j_1} X_{j_2}) + B_t^{(j_1)} B_t^{(j_2)}(X_{j_1} \otimes 1 + 1 \otimes X_{j_1} X_{j_2})
\]

\[
+ B_t^{(j_1,0)}(X_{j_1} \otimes 1 + 1 \otimes X_{j_1})(X_0 \otimes 1 + 1 \otimes X_0)
\]

\[
+ B_t^{(0,j_2)}(X_0 \otimes 1 + 1 \otimes X_0)(X_{j_2} \otimes 1 + 1 \otimes X_{j_2})
\]

\[
+ \cdots
\]

\[
+ \varepsilon^p \sum_{|\mathcal{J}|=p/2} \sum_{|\mathcal{J}|=q} B_t^{(j_1,\ldots,j_{p-q})} (X_{j_1} \otimes 1 + 1 \otimes X_{j_1}) \cdots (X_{j_{p-q}} \otimes 1 + 1 \otimes X_{j_{p-q}})
\]

\[
+ \cdots,
\]

Now we will show that for $p = 1, 2, \ldots$ and $q = 0, 1, \ldots, \lfloor p/2 \rfloor$. 

Also we can write the summands of the right-hand side in (17) as
\[
\sum_{\gamma=0}^{p-q} B_t^{(j_1, j_2, \ldots, j_{p-q})} B_t^{(j_{\gamma+1}, \ldots, j_{p-q})} X_{j_1} \cdots X_{j_{\gamma}} \otimes X_{j_{\gamma+1}} \cdots X_{j_{p-q}}.
\]
(16)

We write the summands of the left-hand side in (16) as follows:
\[
B_t^{(j_1, j_2, \ldots, j_{p-q})} (X_{j_1} \otimes 1 + 1 \otimes X_{j_1}) \cdots (X_{j_{p-q}} \otimes 1 + 1 \otimes X_{j_{p-q}})
\]
\[
= \sum_{\gamma=0}^{p-q} \left( \sum_{i_1, \ldots, i_{p-q}} B_t^{(j_1, j_2, \ldots, j_{p-q})} X_{j_1} \cdots X_{j_{\gamma}} \otimes X_{j_{\gamma+1}} \cdots X_{j_{p-q}} \right),
\]
(17)

where the sum \(\sum_{i_1, \ldots, i_{p-q}}\) is over the values of \(i_1, \ldots, i_{p-q}\) such that
\[i_1 < i_2 < \cdots < i_{\gamma} \quad \text{and} \quad i_{\gamma+1} < i_{\gamma+2} < \cdots < i_{p-q}.
\]

Also we can write the summands of the right-hand side in (17) as
\[
B_t^{(j_1, j_2, \ldots, j_{p-q})} \sum_{i_1, \ldots, i_{p-q}} X_{j_1} \cdots X_{j_{\gamma}} \otimes X_{j_{\gamma+1}} \cdots X_{j_{p-q}}
\]
\[
= \left( \sum_{(j_1, \ldots, j_{j_{\gamma}}, (j_{\gamma+1}, \ldots, i_{p-q})} B_t^{(j_1, j_2, \ldots, j_{p-q})} \right) X_{j_1} \cdots X_{j_{\gamma}} \otimes X_{j_{\gamma+1}} \cdots X_{j_{p-q}}.
\]
(18)

For example, when \(\gamma = 2\) and \(p - q = 4\), by the change of variables,
\[
B_t^{(j_1, j_2, j_3, j_4)} \sum_{i_1, \ldots, i_4} (X_{j_1} X_{j_2} \otimes X_{j_3} X_{j_4})
\]
\[
= B_t^{(j_1, j_2, j_3, j_4)} (X_{j_1} X_{j_2} \otimes X_{j_3} X_{j_4} + X_{j_1} X_{j_3} \otimes X_{j_2} X_{j_4} + X_{j_1} X_{j_4} \otimes X_{j_2} X_{j_3})
\]
\[
= B_t^{(j_1, j_2, j_3, j_4)} + B_t^{(j_1, j_3, j_2, j_4)} + B_t^{(j_1, j_3, j_4, j_2)} + B_t^{(j_1, j_2, j_2, j_4)} + B_t^{(j_3, j_1, j_3, j_2)}
\]
\[
+ B_t^{(j_3, j_4, j_1, j_2)} X_{j_1} X_{j_2} \otimes X_{j_3} X_{j_4}
\]
\[
= \left( \sum_{(i_1, i_2, i_3, i_4)} B_t^{(i_1, i_2, i_3, i_4)} \right) X_{j_1} X_{j_2} \otimes X_{j_3} X_{j_4}.
\]

To show that (16) holds, we need from (18) to prove that for \(\gamma = 0, 1, \ldots, p - q\),
\[
\sum_{i_{a_1}, \ldots, i_{a_q} \in [1, \ldots, p-q]} \left( \sum_{j_{a_1} = 0, \ldots, j_{a_q} = 0}^{p-q} B_t^{(j_1, j_2, \ldots, j_{p-q})} \right) X_{j_1} \cdots X_{j_{\gamma}} \otimes X_{j_{\gamma+1}} \cdots X_{j_{p-q}}
\]
\[
= \sum_{i_{a_1}, \ldots, i_{a_q} \in [1, \ldots, p-q]} \sum_{j_{a_1} = 0, \ldots, j_{a_q} = 0}^{p-q} B_t^{(j_1, \ldots, j_{j_{\gamma}}, (j_{\gamma+1}, \ldots, j_{p-q})} B_t^{(j_{\gamma+1}, \ldots, j_{p-q})} \right) X_{j_1} \cdots X_{j_{\gamma}} \otimes X_{j_{\gamma+1}} \cdots X_{j_{p-q}}.
\]
Hence it suffices to show that for $\gamma = 0, 1, \ldots, p - q$,
\[
\sum_{(j_1, \ldots, j_p)} B_t^{(j_1, \ldots, j_p-q)} B_t^{(j_1, \ldots, j_{p-q})} = B_t^{(j_1, \ldots, j_{p-q})} B_t^{(j_1, \ldots, j_{p-q})}.
\]  
(19)

It follows from Lemma 3 that (19) holds. Hence $Z^\epsilon(t, \omega)$ is a Lie element for each $t \geq 0$ and a.s. $\omega \in \Omega$. □

4. Applications

In this section, using the result in Section 3, we directly derive a representation of a solution without appealing an associated ordinary differential equation.

We reorder (3) by the homogeneous elements of the degree $|J|$, so that
\[
\sum_{p=1}^\infty \epsilon^p \sum_{|J|=p} B_t^J X_J = \sum_{p=1}^\infty \epsilon^p \sum_{|J|=p} B_t^J X_J.
\]
(20)

Here we understand the equality (20) as an identity in $\mathfrak{F}$. Hence by expanding out the powers in (15), we write $Z^\epsilon(t)$ as follows:
\[
Z^\epsilon(t) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} \sum_{p_1=1}^\infty \cdots \sum_{p_n=1}^\infty \sum_{|J_1|=p_1}^\infty \cdots \sum_{|J_n|=p_n}^\infty \epsilon^n \sum_{|J_1|=p_1}^\infty \cdots \sum_{|J_n|=p_n}^\infty \epsilon^n X_{J_1} \cdots X_{J_n}.
\]
(21)

For fixed positive integers $k$ we consider all ways of choosing positive integers $p_1, \ldots, p_n$ satisfying $\sum_{i=1}^n p_i = k$. Let us set $q_0 = 0$ and $q_j = p_1 + \cdots + p_j$ for $j \geq 1$. We denote $J_{i+1} = (j_{q_0+1}, j_{q_0+2}, \ldots, j_{q_{i+1}})$ for $i = 0, 1, \ldots, n - 1$, and $X^\epsilon_{J_i} = \epsilon^{|J_i|} X_{J_i}$, where the notation $|J|$ is defined by analogy with $|J|$,
\[
|J_i| = \begin{cases} 
1 & \text{if } j_i \neq 0, \\
2 & \text{if } j_i = 0.
\end{cases}
\]

Then the expansion (21) can be written as
\[
Z^\epsilon(t) = \sum_{k=1}^\infty \sum_{n=1}^k \sum_{p_1, \ldots, p_n = 1}^{p_k = k} \frac{(-1)^{n-1}}{n} \int_0^t \circ dB_{t_j_{q_1}}^{j_{q_1}} \cdots \int_0^t \circ dB_{t_j_{q_n}}^{j_{q_n}} \cdots \int_0^t \circ dB_{t_j_{q_1+1}}^{j_{q_1+1}} \cdots \int_0^t \circ dB_{t_j_{q_n+1+1}}^{j_{q_n+1+1}}
\]
\[
\times X^\epsilon_{J_1} \cdots X^\epsilon_{J_1} X^\epsilon_{J_1+1} \cdots X^\epsilon_{J_n+1} \cdots X^\epsilon_{J_n}
\]
\[
= \sum_{k=1}^\infty \sum_{n=1}^k \sum_{p_1, \ldots, p_n = 1}^{p_k = k} \frac{(-1)^{n-1}}{n} \int_0^t \circ dB_{t_j_{1}}^{j_{1}} \circ dB_{t_j_{2}}^{j_{2}} \cdots \circ dB_{t_j_{2}}^{j_{2}}
\]
\[
\times X^\epsilon_{J_1} X^\epsilon_{J_2} \cdots X^\epsilon_{J_n}.
\]
(22)
where the integral in the second equation is taken over the following region:

\[ 0 < t_1 < \cdots < t_{q_1} < t, \ldots, 0 < t_{q_{n-1}+1} < \cdots < t_{q_n} < t. \]

Let \( \mathfrak{g}' \) denote the ideal \( \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots \) in \( \mathfrak{g} \). Then we define a linear mapping \( \tau \) of \( \mathfrak{g}' \) into \( \mathfrak{g}^L \) such that

\[
\tau X_i = X_i, \quad \tau (X_{j_1} \cdots X_{j_m}) = \left[ \cdots [X_{j_1}, X_{j_2}] \cdots X_{j_m} \right], \quad \text{for } m > 1.
\]

We use the following Specht–Wever theorem to obtain a representation in terms of Lie brackets and iterated Stratonovich integrals.

**Theorem 6** (Specht–Wever). If a field \( \Phi \) is of characteristic 0, then a homogeneous element \( a \) of degree \( m > 0 \) is a Lie element if and only if \( \tau a = ma \).

Since \( Z^e(t) \) is a Lie element, if we apply the operator \( \tau \) to the homogeneous part of degree \( k \), we have the following expression of \( Z^e(t) \) as a Lie element by Specht–Wever theorem:

\[
Z^e(t) = \sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{(-1)^{n-1}}{kn} \sum_{p_1, \ldots, p_n = k} \int o d B_{t_1}^{j_1} o d B_{t_2}^{j_2} \cdots o d B_{t_{qn}}^{j_{qn}} \times \left[ \cdots [X_{j_1}, X_{j_2}] \cdots X_{j_{qn}} \right].
\]

Hence the solution \( \xi_t(x_0) \) of SDE (1) is represented, in algebra of formal power series, as

\[
\xi_t(x_0) = \exp(Z(t))(x_0), \tag{23}
\]

where

\[
Z(t) = \sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{(-1)^{n-1}}{kn} \sum_{p_1, \ldots, p_n = k} \int o d B_{t_1}^{j_1} o d B_{t_2}^{j_2} \cdots o d B_{t_{qn}}^{j_{qn}} \times \left[ \cdots [X_{j_1}, X_{j_2}] \cdots X_{j_{qn}} \right].
\]

If Lie algebra generated by \( X_0, \ldots, X_r \) is nilpotent of order \( p \), that is,

\[
\left[ \cdots [X_{j_1}, X_{j_2}] \cdots X_{j_m} \right] = 0 \quad \text{for } j_1, \ldots, j_m \in \{0, 1, \ldots, r\} \text{ and } m > p,
\]

it follows from (23) that we obtain an explicit expression of the solution of SDE (1),

\[
\xi_t(x_0) = \exp \left( \sum_{k=1}^{p} Z_k(t) \right)(x_0), \tag{24}
\]

where

\[
Z_k(t) = \sum_{n=1}^{k} \frac{(-1)^{n-1}}{kn} \sum_{p_1, \ldots, p_n = k} \int o d B_{t_1}^{j_1} o d B_{t_2}^{j_2} \cdots o d B_{t_{qn}}^{j_{qn}} \times \left[ \cdots [X_{j_1}, X_{j_2}] \cdots X_{j_{qn}} \right].
\]

Now we provide some examples in order to show how the above representations (23) and (24) are useful for solving SDE (1).
Example 7 (Commutative case). Suppose that $X_0, X_1, \ldots, X_r$ are commutative, that is, $[X_i, X_j] = 0$ for all $i, j \in \{0, 1, \ldots, r\}$. Then the solution $\xi_t(x_0)$ is represented as

$$\xi_t(x_0) = \exp \left( tX_0 + \sum_{i=1}^{r} B^i_t X_i \right)(x_0).$$

(25)

In this commuting case, the solution $\xi_t(x_0)$ of SDE (1) is equal to $\phi_1(x_0)$ a.s., which is the solution of the ordinary differential equation:

$$\frac{d\phi_s}{ds} = \left( tX_0 + \sum_{i=1}^{r} B^i_s X_i \right)(\phi_s) \quad \text{and} \quad \phi_0(x_0) = x_0.
$$

Example 8 (Yamato [9]). Let us consider SDE on $\mathbb{R}^3$

$$\begin{cases}
    d\xi_t = X_1(\xi_t) \circ dB^1_t + X_2(\xi_t) \circ dB^2_t, \\
    \xi_0 = x_0,
\end{cases}
$$

where $x_0 = (x^1_0, x^2_0, x^3_0)$ and the vector fields $X_1$ and $X_2$ on $\mathbb{R}^3$ are given by

$$X_1 = \frac{\partial}{\partial x^1} + 2x^2 \frac{\partial}{\partial x^3} \quad \text{and} \quad X_2 = \frac{\partial}{\partial x^2} - 2x^1 \frac{\partial}{\partial x^3}.
$$

(27)

Then it is easy to see that

$$[X_1, X_2] = -4 \frac{\partial}{\partial x^3} \quad \text{and} \quad [[X_1, X_2], X_1] = [X_1, [X_1, X_2]] = 0.
$$

Hence the Lie algebra generated by $X = \{X_0, X_1, X_2\}$ is nilpotent of order $p = 2$.

- For $k = 1, n = 1 \ (p_1 = 1)$, then $B^1_t X_1 + B^2_t X_2$.
- For $k = 2, n = 1 \ (p_1 = 2)$, then

$$\frac{1}{2} \left( \int_0^t B^1_s \circ dB^2_s [X_1, X_2] + \int_0^t B^2_s \circ dB^1_s [X_2, X_1] \right)
$$

$$= \frac{1}{2} \left( \int_0^t B^1_s \circ dB^2_s - \int_0^t B^2_s \circ dB^1_s \right)[X_1, X_2].
$$

- For $k = 2, n = 2 \ (p_1 = 1, p_2 = 1)$, then $B^1_t B^2_t [X_1, X_2] + B^2_t B^1_t [X_2, X_1] = 0$.

By using the formula (24) with $p = 2$, the solution $\xi_t(x_0)$ of SDE (26) is then explicitly expressed as

$$\xi_t(x_0) = \exp \left\{ B^1_t \frac{\partial}{\partial x^1} + B^2_t \frac{\partial}{\partial x^2} + 2 \left[ x^2_0 B^1_t - x^1_0 B^2_t - \left( \int_0^t B^1_s \circ dB^2_s - \int_0^t B^2_s \circ dB^1_s \right) \right] \frac{\partial}{\partial x^3} \right\}(x_0).
$$

(28)

Therefore it follows from (13) that the solution of SDE (26) is given by $\xi_t(x_0) = (\xi^1_t(x_0), \xi^2_t(x_0), \xi^3_t(x_0))$, where
\( \xi^1_t(x_0) = x_0^1 + B_t^1, \)
\( \xi^2_t(x_0) = x_0^2 + B_t^2, \)
\( \xi^3_t(x_0) = x_0^3 + 2 \left[ x_0^2 B_t^1 - x_0^1 B_t^2 - \left( \int_0^t B_s^1 \circ dB_s^2 - \int_0^t B_s^2 \circ dB_s^1 \right) \right]. \)

We can directly solve SDE (26). From (27), we obtain the following equation:
\[
\begin{aligned}
\begin{cases}
\frac{d\xi^1_t(x_0)}{dt} &= dB_t^1, \\
\frac{d\xi^2_t(x_0)}{dt} &= dB_t^2, \\
\frac{d\xi^3_t(x_0)}{dt} &= 2\xi^2_t(x_0) \circ dB_t^1 - 2\xi^1_t(x_0) \circ dB_t^2.
\end{cases}
\end{aligned}
\]

Hence we get
\( \xi^1_t(x_0) = x_0^1 + B_t^1, \)
\( \xi^2_t(x_0) = x_0^2 + B_t^2, \)
\( \xi^3_t(x_0) = x_0^3 + 2 \left[ x_0^2 B_t^1 - x_0^1 B_t^2 - \left( \int_0^t B_s^1 \circ dB_s^2 - \int_0^t B_s^2 \circ dB_s^1 \right) \right]. \)

In the next two examples, we give comparisons with the results obtained by Castell [3]. As mentioned in Introduction, we directly derive the formula given in [3] by using our representations without appealing a limit procedure and an ordinary differential equation. In [3], Castell has obtained the asymptotic expansion in small time of the solution of SDE (1). When the vector fields \( \mathcal{X} = \{X_0, X_1, \ldots, X_r\} \) generates a nilpotent Lie algebra, the formula of the solution is not asymptotic but exact. First we derive the exact formula obtained by Castell [3] in case of a nilpotent Lie algebra.

Let us denote by \( e(\sigma) \) the cardinality of the set \( \{ j \in \{1, \ldots, m - 1\} \mid \sigma(j) > \sigma(j+1) \} \), where \( \sigma \in \mathbb{S}_k \), the group of all permutations of \( k \) letters. If \( J = (j_1, \ldots, j_m) \), let us set \( J \circ \sigma = (j_{\sigma(1)}, \ldots, j_{\sigma(m)}) \).

**Example 9** (Nilpotent case, Castell [3]). Let \( X_0, \ldots, X_r \) be vector fields on \( \mathbb{R}^d \) such that the Lie algebra generated by \( \mathcal{X} = \{X_0, X_1, \ldots, X_r\} \) is the nilpotent of order \( p \). Then Castell [3] has obtained the following explicit formula of the solution of SDE (1):
\[
\xi_t(x_0) = \exp \left( \sum_{k=1}^p \sum_{|J|=k} C^J_t X^J \right)(x_0),
\]
where
\[
C^J_t = \sum_{\sigma \in \mathbb{S}_|J|} \frac{(-1)^e(\sigma)}{|J|^2 \cdot |\sigma|} B_t^{J \circ \sigma^{-1}}.
\]

Let us define
\[
\mathbb{S}_{q_n}(q_1, \ldots, q_n) = \{ \sigma \in \mathbb{S}_{q_n} \mid \sigma(q_j + 1) < \cdots < \sigma(q_{j+1}) \text{ for } 0 \leq j \leq n - 1 \}.
\]

Now we give the representation (30) by using our formula (24). First note that
\[ \int \circ dB^{j_1}_{t_1} \circ dB^{j_2}_{t_2} \cdots \circ dB^{j_{qn}}_{t_{qn}} \]
\[ = \sum_{\sigma} \int \circ dB^{j_1}_{t_{\sigma(1)}} \circ dB^{j_2}_{t_{\sigma(2)}} \cdots \circ dB^{j_k}_{t_{\sigma(k)}}, \quad \text{(31)} \]

the summation \( \sum_{\sigma} \) being over all \( \sigma \in S_{qn} (q_1, \ldots, q_n) \). It follows from (31) that

\[ Z_k(t) = \sum_{n=1}^{k} \frac{(-1)^{n-1}}{kn} \sum_{p_1, \ldots, p_n} \sum_{p_1 + \cdots + p_n = k} \int_{0 < t_1 < \cdots < t_k < t} \circ dB^{j_1}_{t_1} \circ dB^{j_2}_{t_2} \cdots \circ dB^{j_k}_{t_k} \]
\[ \cdots \circ dB^{j_{\sigma(k)}}_{t_{\sigma(k)}} \left[ \cdots [X_{j_1}, X_{j_2}] \cdots X_{j_k} \right], \quad \text{(32)} \]

By using algebraic computations in the proof in [3, Proposition 3.2], the right-hand side of (32) becomes

\[ Z_k(t) = \sum_{\sigma} \frac{(-1)^{e(\sigma)}}{k^{2(k-1)/e(\sigma)}} \int_{0 < t_1 < \cdots < t_k < t} \circ dB^{j_{\sigma(1)}}_{t_1} \circ dB^{j_{\sigma(2)}}_{t_2} \cdots \circ dB^{j_{\sigma(k)}}_{t_k} \left[ \cdots [X_{j_1}, X_{j_2}] \cdots X_{j_k} \right] \]

which states (30).

In the general case, we derive the asymptotic expansion of the solution of SDE (1) that was proven by Castell [3].

Example 10 (General case, Castell [3]). In the general case, Castell [3] has obtained the following asymptotic expansion of the stochastic flow. For all integer \( p \geq 2 \), we define the process \( R_p(t) \) on \( \mathbb{R}^d \) by

\[ \xi^{\epsilon}_t (x_0) = x_0 + \sum_{k=1}^{p-1} \sum_{\|J\| = k} C^J_t X^J (x_0) + \epsilon^{p/2} R_p(t)(x_0). \quad \text{(33)} \]

Then there exist \( \alpha > 0 \) and \( \beta > 0 \) such that for every \( c > \beta \),

\[ \lim_{t \to 0} \mathbb{P} \left[ \sup_{0 \leq s \leq t} \left\| R_p(s) \right\| \geq ct^{p/2} \right] \leq \exp \left\{ -\frac{c^\alpha}{\beta} \right\}. \quad \text{(34)} \]

Let \( \xi^{\epsilon}_t (x_0) \) be the solution of SDE (2). For fixed \( T > 0 \), we define \( H_p(\epsilon, t), t \leq T \) and \( \epsilon \in (0, 1] \), by

\[ \xi^{\epsilon}_t (x_0) = x_0 + \sum_{k=1}^{p-1} \epsilon^k \sum_{\|J\| = k} X^J (x_0) B^J_t + \epsilon^p H_p(\epsilon, t) \quad \text{a.s.} \quad \text{(35)} \]
Then Azencott [1] has shown that there exist \(\alpha > 0\) and \(\beta > 0\) such that for all \(c > \beta\),

\[
\lim_{\epsilon \to 0} P \left[ \sup_{0 \leq t \leq T} \| H_p(\epsilon, t)(x_0) \| \geq c \right] \leq \exp \left\{ -\frac{c^\alpha}{\beta T} \right\}.
\]

If we define \(U_{p-1}^\epsilon(t) = \sum_{k=1}^{p-1} \epsilon^k \sum_{\|J\| = k} C_t^J X_J^J\), then

\[
\exp(U_{p-1}^\epsilon(t))
\]

\[
= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k_1, \ldots, k_n} \epsilon^{k_1 + \cdots + k_n} \sum_{\|J_1\| = k_1, \ldots, \|J_n\| = k_n} C_t^{J_1} X_{J_1} \cdots C_t^{J_n} X_{J_n}
\]

\[
= 1 + \sum_{k=1}^{\infty} \epsilon^k \sum_{n=\left[\frac{k+p-2}{p-1}\right]} \frac{1}{n!} \sum_{\|J_1\| + \cdots + \|J_n\| = m} C_t^{J_1} X_{J_1} \cdots C_t^{J_n} X_{J_n}
\]

\[
= 1 + \sum_{k=1}^{p-1} \epsilon^k \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\|J_1\| + \cdots + \|J_n\| = k} C_t^{J_1} X_{J_1} \cdots C_t^{J_n} X_{J_n}
\]

\[
+ \epsilon^p Q_p^{(1)}(t) + \epsilon^p Q_p^{(2)}(\epsilon, t),
\]

where \(Q_p^{(1)}(t)\) and \(Q_p^{(2)}(\epsilon, t)\) are given by

\[
Q_p^{(1)}(t) = \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{\|J_1\| + \cdots + \|J_n\| = p} C_t^{J_1} X_{J_1} \cdots C_t^{J_n} X_{J_n},
\]

\[
Q_p^{(2)}(\epsilon, t) = \sum_{k=p+1}^{\infty} \epsilon^{k-p} \sum_{n=\left[\frac{k+p-2}{p-1}\right]} \frac{1}{n!} \sum_{\|J_1\| + \cdots + \|J_n\| = k} C_t^{J_1} X_{J_1} \cdots C_t^{J_n} X_{J_n}.
\]

From (13) and (23), it follows that

\[
\sum_{\|J\| = k} X_J B_J^J = \sum_{n=1}^{k} \frac{1}{n!} \sum_{\|J_1\| + \cdots + \|J_n\| = k} C_t^{J_1} X_{J_1} \cdots C_t^{J_n} X_{J_n}. \tag{36}
\]

By (35) and (36), we get

\[
\xi^\epsilon_t(x_0) = \exp(U_{p-1}^\epsilon(t))(x_0) + \epsilon^p R_p(\epsilon, t)(x_0), \tag{37}
\]

where \(R_p(\epsilon, t)(x_0)\) is given by

\[
R_p(\epsilon, t)(x_0) = H_p(\epsilon, t)(x_0) - Q_p^{(1)}(t)(x_0) + Q_p^{(2)}(\epsilon, t)(x_0).
\]

We recall the following definition, introduced by Azencott [1] (or see [3]):

**Definition 11.** Let \(\tau\) be a stopping time and let \(X = (X_t), \ t \geq 0\), be a continuous stochastic process on \([0, \tau)\) with values in \(\mathbb{R}^p\). Then \(X\) is said to be in \(\mathcal{W}(\alpha, \beta, \tau)\) if and only if for all \(t \geq 0\) and \(\epsilon \geq \beta\),

\[
P \left[ \sup_{0 \leq s \leq t} \| X_s \| \geq c; \ t < \tau \right] \leq \exp \left\{ -\frac{c^\alpha}{\beta t} \right\}.
\]
According to the properties (P1) and (P2) in [3], we have that for fixed $T > 0$, 
$$Q_p^{(1)}(t)(x_0) \in \mathcal{W}(\alpha, \beta, T).$$
It is also obvious that \(\lim_{\epsilon \to 0} \sup_{0 \leq t \leq T} \|Q_p^{(2)}(\epsilon, t)(x_0)\| = 0\) a.s. From (37), it follows that there exist $\alpha > 0$ and $\beta > 0$ such that for all $c > \beta$,
$$\lim_{\epsilon \to 0} P\left[ \sup_{0 \leq t \leq T} \|R_p(\epsilon, t)(x_0)\| \geq c \right] \leq \exp \left\{-\frac{c^\alpha}{\beta T}\right\},$$
which deduce (34) by the remarks given below Theorem 4.1 in [3].

References