New $Q$-conditional symmetries and exact solutions of some reaction–diffusion–convection equations arising in mathematical biology

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Abstract

A theorem giving a complete description of $Q$-conditional symmetries of a class of nonlinear reaction–diffusion–convection equations is proved. Furthermore the $Q$-conditional symmetries obtained and the method of additional generating conditions are applied for finding exact solutions of the generalized Fisher, Fitzhugh–Nagumo and Kolmogorov–Petrovskii–Piskunov equations. The symmetries and solutions constructed are compared with those obtained by other authors. In particular, it was established that the known travelling wave solutions of these equations are particular cases of more general (non-Lie) solutions. The relation between $Q$-conditional symmetries and generalized conditional symmetries is also shown.

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1. Introduction

Nonlinear reaction–diffusion–convection (RDC) equation

$$u_t = \left[A(u)u_x\right]_x + B(u)u_x + C(u),$$  \hspace{1cm} (1)
where $u = u(t, x)$ is the unknown function and $A(u), B(u), C(u)$ are arbitrary smooth functions and the subscripts $t$ and $x$ denote differentiation with respect to these variables, generalizes a great number of the known nonlinear second-order equations describing various processes in biology (see, e.g., the well-known books [1–3]). The well-known principle of linear superposition cannot be applied to generate new exact solutions to nonlinear partial differential equations (PDEs). Thus, the classical methods (the Fourier method, the method of the Laplace transformations, and so forth) are not applicable for solving nonlinear partial differential equations. While there is no existing general theory for integrating nonlinear RDC of form (1), construction of particular exact solutions for these equations is a non-trivial and important problem. Finding exact solutions that have a physical, chemical or biological interpretation is of fundamental importance. Of course, a change of variables can sometimes be found that transforms a nonlinear PDE into a linear equation, but finding exact solutions of most nonlinear partial differential equations generally requires new methods.

Now the most popular methods for construction of exact solutions to nonlinear PDEs are the method of inverse scattering problem and the Lie method (of course, there are many methods related to each of them). In this paper we do not consider the first one since it is not efficient for solving nonlinear RDC equations. The Lie method [4–7] is based on using the Lie symmetry of a given PDE for the construction of its exact solutions. Although the technique of this method is well known, new results are constantly obtained for nonlinear RDC equations with non-trivial Lie symmetries.

On the other hand, it is well known that some nonlinear RDC equations arising in applications have poor Lie symmetry. For example, the Fisher equation and Fitzhugh–Nagumo equation, which are widely used in mathematical biology, are invariant only under the time- and space translations. The Lie method is not efficient for such equations since it enables to construct only those ansätze and exact solutions, which can be obtained without using this cumbersome algorithm. Taking into account this fact, one needs to apply other approaches for solving such equations. One of them is the Bluman–Cole approach of non-classical symmetries [8] (i.e. $Q$-conditional symmetries in terminology of [7]). Nevertheless this approach was suggested about 35 years ago, its successful applications for solving nonlinear RDC equations were realized only in 1990s [9–12]. Several other approaches for solving nonlinear evolution PDEs were independently suggested in middle 1990s, particularly, the method of compatible differential constraints, the method of linear invariant subspaces, the method of generalized conditional symmetries, the method of heir-equations, the method of additional generating conditions [13–20].

Here we restrict ourselves on the exhibition of new results obtained for a particular case of Eq. (1), namely:

$$u_t = u_{xx} + \lambda u u_x + C(u), \quad \lambda \in \mathbb{R}.$$  

(2)

The motivation of this choice has two aspects. The first one is to consider a simple nonlinear RDC equation but involving three transport mechanisms (diffusion, reaction and convection); the second one is to deal with the equation, which arises in some applications (Eq. (2) contains as a particular cases the Fisher, Murray and Fitzhugh–Nagumo equations that are widely used for modelling in mathematical biology [1–3]; the Burgers and Newell–Whitehead equations arising in fluid dynamics [21,22]).

The paper is organized as follows. In Section 2, we present two theorems giving a complete description of Lie and $Q$-conditional symmetries of the nonlinear RDC equation (2). In Section 3, $Q$-conditional symmetries found are applied for finding exact solutions of the generalized Fisher, Fitzhugh–Nagumo and Kolmogorov–Petrovskii–Piskunov equations. Section 4 is devoted
to searching exact solutions of those equations using the method of additional generating conditions. The solutions obtained here have the essentially different structure than those constructed in Section 3. The main results of the paper are summarized and discussed in the last section.

2. Lie’s and $Q$-conditional symmetries of the RDC equation (2)

In papers [12,23], a complete description of Lie symmetries was obtained for the RDC equation (1). It follows from [23] (see Theorem 1) that Eq. (2) admits a non-trivial Lie algebra only in the cases listed below.

**Theorem 1.** All possible maximal algebras of invariance (MAI) (up to equivalent representations generated by transformations of form (3)) of the nonlinear equation (2) with non-zero $\lambda$ are exhausted by the algebras with the basic operators

\[(a)\quad P_t = \frac{\partial}{\partial t}, \quad P_x = \frac{\partial}{\partial x}, \quad D_1 = 2t P_t + x P_x - UP_U,
\]

if $C = \lambda_1 U^3$, $\lambda_1 \neq 0$;

\[(b)\quad P_t, P_x, \quad G_1 = \exp(\lambda_1 t) \left( P_x - \frac{\lambda_1}{\lambda} P_U \right),
\]

if $C = \lambda_1 U$, $\lambda_1 \neq 0$;

\[(c)\quad P_t, P_x, \quad G_1 = t P_x - \frac{1}{\lambda} P_U, \quad D_2 = 2t P_t + x P_x - UP_U,
\]

\[\Pi_1 = t^2 P_t + tx P_x - \left( \frac{x}{\lambda} + tU \right) P_U,
\]

if $C = 0$;

\[(d)\quad P_t, P_x,
\]

if $C(u)$ is an arbitrary function. Any other equation of form (2) admitting higher than two-dimensional MAI is reduced by a local substitution of the form

\[t \to t,
\]

\[x \to c_1 x + c_2 t + c_3 t^2,
\]

\[U \to c_4 + c_5 t + c_6 U,
\]

(3)

to one of those given above (the constants $c_1, \ldots, c_6$ are determined by the form of the equation in question).

**Remark 1.** There are three additional nonlinear equations

\[u_t = u_{xx} + \lambda uu_x + \lambda_1 (u + \alpha)^k, \quad k = 1 \text{ and } k = 3,
\]

(4)

and

\[u_t = u_{xx} + \lambda uu_x + \alpha,
\]

(5)

which admit non-trivial Lie algebras (here $\alpha$ is an arbitrary constant). However, Eqs. (4) and (5) can be reduced to those listed in Theorem 1 by the local substitutions of form (3):

\[t^* = t, \quad x^* = x - \lambda \alpha t, \quad U^* = U + \alpha
\]

(6)
and

\[ t^* = t, \quad x^* = x - \frac{1}{2} \lambda \alpha t^2, \quad U^* = U - \alpha t, \]

respectively [12].

One notes that the RDC equation arising in case (c) is the Burgers equation which can be reduced to the linear heat equation \( v_t = v_{xx} \) by the well-known Cole–Hopf substitution \( u = \frac{2}{\lambda} (\log v)_x \), while MAI arising in case (d) is the trivial Lie algebra which lead only to plane-wave solutions. So non-trivial Lie ansätze leading to exact solutions with new structure can be constructed only in cases (a) and (b).

It is known the Bluman–Cole method [8] gives a constructive algorithm to find non-classical symmetries which we call \( Q \)-conditional symmetries. In fact, non-classical symmetries are only a particular case of the conditional symmetry notion introduced in [7] (see Section 5.7). It should be stressed there is no any constructive algorithm to find all possible conditional symmetries for a given PDE while one exists for searching \( Q \)-conditional symmetries. In the case of the nonlinear equation (2), all \( Q \)-conditional symmetries are generated by the first-order operators of two types (up to equivalent representations generated by multiplying on the arbitrary smooth function \( M(t, x, u) \)):

\[ Q = \partial_t + \xi^1(t, x, u)\partial_x + \eta(t, x, u)\partial_u \]  

and

\[ Q = \partial_x + \eta(t, x, u)\partial_u, \]

where the functions \( \xi^1 \) and \( \eta \) must be found. Now let us formulate a theorem which gives complete information on \( Q \)-conditional symmetries of Eq. (2). Note that we do not consider the case \( \lambda = 0 \) since this case has been studied in [9–11] and we search for purely conditional symmetry operators, i.e. those that cannot be reduced to Lie symmetry operators.

**Theorem 2.** (i) Equation (2) is \( Q \)-conditional invariant under operator (8) if and only if it has one of the following forms (up to equivalent representations generated by multiplying on the arbitrary smooth function \( M(t, x, u) \)):

\[ Q = \partial_t + \frac{\lambda^2}{2} u + q \partial_x + \left( a + bu - \frac{\lambda q}{2} u^2 - \frac{\lambda^2}{4} u^3 \right) \partial_u, \]  

where triplet of the functions \( (a, b, q) \) is the general solution of the system

\[ a_t = a_{xx} - 2aq_x, \]

\[ b_t = b_{xx} - 2bq_x + \lambda ax, \]

\[ q_t = q_{xx} - 2qq_x - 2bx, \]
Case 2. The equation
\[ u_t = u_{xx} + \lambda uu_x + \lambda_0 + \lambda_2 u^2 \]  
(13)
is \(Q\)-conditional invariant under the operator
\[ Q = \partial_t + \left( -\lambda u + \frac{\lambda_2}{\lambda} \right) \partial_x + (\lambda_0 + \lambda_2 u^2) \partial_u, \]
(14)
where \(\lambda_0, \lambda_2 \neq 0\) are arbitrary constants.

Case 3. The equation
\[ u_t = u_{xx} + \lambda uu_x + \lambda_0 + \lambda_1 u + \lambda_3 u^3 \]  
(15)
is \(Q\)-conditional invariant under the operators
\[ Q_i = \partial_t + p_i u \partial_x + \frac{3p_i}{2p_i - \lambda} (\lambda_0 + \lambda_1 u + \lambda_3 u^3) \partial_u, \quad i = 1, 2, \]
(16)
where \(p_i\) are the roots of the quadratic equation \(2p^2 + \lambda p + 9\lambda_3 - \lambda^2 = 0\), while \(\lambda_0, \lambda_1, \lambda_3 \neq 0\) are arbitrary constants, and the operators
\[ Q = \partial_t + b \partial_x + (y b_{xx} - b_x u) \partial_u, \]
(17)
where the function \(b(t, x)\) is an arbitrary solution of the overdetermined system
\[
\begin{align*}
(\lambda \gamma - 3)b_{xx} + 2bb_x + b_t &= 0, \\
b_{xxx} - bb_{xx} + \lambda_1 b_x &= 0, \\
\lambda \gamma b_{xxx} + b_x^2 + 3\lambda_1 b_x + \frac{3\lambda_0}{\gamma} b &= c_0,
\end{align*}
\]
(18)
where \(\gamma = \frac{\lambda}{3\lambda_3}\), \(\lambda_0 \lambda_1 \lambda_3 \neq 0\) and \(c_0 \in \mathbb{R}\).

(ii) Equation (2) is \(Q\)-conditional invariant under operator (9) if and only if the function \(\eta\) is a solution of the three-dimensional nonlinear equation
\[ \eta_t + \eta_{xx} + 2\eta \eta_{xu} + \eta^2 \eta_{uu} + \lambda \eta^2 + \mu \eta_x + \gamma C_u - \eta_u C = 0. \]
(19)

Proof. The proof of the theorem is based on the determining equations that have been obtained in [12] (see p. 535) to find operators of the \(Q\)-conditional symmetries (8) and (9) for the RDC equation (1). The system of those equations is very complicated and it is a quite difficult task to construct its general solutions if \(A(u), B(u), C(u)\) are arbitrary functions. However, this system can be essentially simplified if one considers the RDC equation (2). In fact, we obtain the following system for the function \(C(u)\) and the coefficients \(\xi^1\) and \(\eta\) of operator (8):
\[
\begin{align*}
\xi^1_{uu} &= 0, \\
\eta_{uu} &= -2\xi^1_u (\lambda u + \xi^1) + 2\xi^1_{xu}, \\
\lambda \eta + \xi^1_t + 2\xi^1 \xi^1_x - 2\xi^1 \eta + \lambda u \xi^1_x + 2\eta \xi^1_u - \xi^1_{xx} + 3\xi^1 C(u) &= 0, \\
\eta (\eta - C(u))_u + (2\xi^1_x - \eta_u)(\eta - C(u)) + \eta_t - \lambda u \eta_x - \eta_{xx} &= 0.
\end{align*}
\]
(20)-(22)

Subsystem (20) is easily integrated and its general solutions has the form
\[
\begin{align*}
\xi^1 &= a(t, x)u + b(t, x), \\
\eta &= -\frac{1}{3} a(a + \lambda)u^3 + (a_x - ab)u^2 + d(t, x)u + e(t, x),
\end{align*}
\]
(23)
where \(a, b, d\) and \(e\) are arbitrary smooth functions. Obviously, one needs to consider two different cases, namely: (I) \(a \neq 0\) and (II) \(a = 0\).

Consider case (I). Substituting (23) into (21), we immediately establish that the function \(C(u)\) can be at maximum a cubic polynomial with respect to the variable \(u\), i.e.:

\[
C(u) = \lambda_0 + \lambda_1 u + \lambda_2 u^2 + \lambda_3 u^3, \tag{24}
\]

where the constants \(\lambda_i, i = 0, \ldots, 3\), are determined by the functions \(a, b, d\) and \(e\). The relevant formulas have the form

\[
\begin{align*}
2a^2 + a\lambda + 9\lambda_3 - \lambda^2 &= 0, \\
b(\lambda - 2a) &= 3\lambda_2, \\
d(\lambda - 2a) &= -3a\lambda_1, \\
e(\lambda - 2a) &= -3a\lambda_0, 
\end{align*} \tag{25}
\]

if \(\lambda_3 \neq 0\). If \(\lambda_3 = 0\) then the first equation (of course, with \(\lambda_3 = 0\)) of (25) is again obtained and then \(a = -\lambda\) or \(a = \frac{1}{2}\lambda\). The value \(a = -\lambda\) leads only to a particular case of system (25) with \(\lambda_3 = 0\) therefore there is no need to consider one separately. The value \(a = \frac{1}{2}\lambda\) leads to the requirement \(\lambda_2 = 0\) therefore the function \(C(u)\) (see (24)) is linear. According to Theorem 1, the RDC equation (2) with the linear \(C(u)\) can be reduced to the form

\[
\begin{align*}
&ut = u_{xx} + \lambda uux_x, \tag{26} \\
or \\
&ut = u_{xx} + \lambda uux_x + \lambda_1 u. \tag{27}
\end{align*}
\]

Equation (26) is, of course, the well-known Burgers equation and its \(Q\)-conditional symmetry is easily obtained by substitution \(C(u) = 0, a = \frac{1}{2}\lambda\) and (23) into the determining equations (21), (22). After the relevant calculations, we obtain their general solution leading to operator (11) with the coefficients satisfying (12). The analogous procedure was realized for Eq. (27), however, only Lie symmetry operators were found.

Now we consider the case \(\lambda_3 \neq 0\) and its subcase \(\lambda_3 = 0\) and \(a = -\lambda\). Solving the system of the algebraic equation (25) with respect to the \(a, b, d, e\) and substituting the expressions obtained into the determining equation (23), we arrive at the general solution (24) and

\[
\begin{align*}
\xi^1 &= -\lambda u + \frac{\lambda_2}{\lambda}, \\
\eta &= \lambda_2 u^2 + \lambda_1 u + \lambda_0, 
\end{align*} \tag{28}
\]

if \(\lambda_3 = 0\) and \(a = -\lambda\), and

\[
\begin{align*}
\xi^1 &= pu - \frac{3\lambda_2}{2p - \lambda}, \\
\eta &= \frac{3p}{2p - \lambda} (\lambda_3 u^3 + \lambda_2 u^2 + \lambda_1 u + \lambda_0), 
\end{align*} \tag{29}
\]

if \(\lambda_3 \neq 0\). Here the constant \(p\) is the solution of the quadratic equation \(2p^2 + p\lambda + 9\lambda_3 - \lambda^2 = 0\). So, operator (8) with coefficients (28) form the \(Q\)-conditional symmetry of the RDC equation

\[
\begin{align*}
&ut = u_{xx} + \lambda uux_x + \lambda_0 + \lambda_1 u + \lambda_2 u^2, \tag{30}
\end{align*}
\]
while this operator with coefficients (29) form two $Q$-conditional symmetries of the RDC equation

$$u_t = u_{xx} + \lambda uu_x + \lambda_0 + \lambda_1 u + \lambda_2 u^2 + \lambda_3 u^3. \quad (31)$$

It should be noted that there is only one $Q$-conditional symmetry, if $8\lambda_3 + \lambda_2^2 = 0$. Finally, we note that Eqs. (30) and (31) are reduced to the same those with $\lambda_1 = 0$ and $\lambda_2 = 0$, respectively, using the local substitutions of form (3) with $c_3 = c_5 = 0$. Thus, the examination of case (I) is now completed and cases 1, 2 and 3 (excepting operator (17)) of the theorem are obtained.

The examination of case (II) is rather similar and leads only to $Q$-conditional symmetry operator (17), where the function $b(t, x)$ is the general solution of the nonlinear system (18). It should be noted that this system is compatible (for example, $b = \text{const}$ is a solution, if $c_0 = \frac{3\lambda_0}{\gamma} b$), however, I was unable to construct its general solution in the case $\lambda_0 \lambda_1 \lambda_3 \neq 0$. The theoretical background of this difficulty follows from the recently published paper [24]. Indeed, one easily checks that both third-order ordinary differential equations arising in (18) cannot be linearized by point and contact transformations (see [24, Theorems 2.1 and 5.1]). In the case $\lambda_0 \lambda_1 \lambda_3 = 0$, the general solution was found but the operators obtained coincide with the Lie symmetry operators listed in Theorem 1.

Thus, the proof of part (i) of the theorem is now completed. The proof of part (ii) can be simply obtained from formula (2.39) [12] setting $F_0 = 1$, $F_1 = -\lambda u$, $F_2 = -C(u)$, $v = u$.  

**Remark 2.** Particular cases of Theorem 2 were derived in papers [25] (case 1), [12,26] (some subcases of cases 2, 3) however the complete description (with the proof) of $Q$-conditional symmetries of the RDC equation (2) is presented here the first time.

**Remark 3.** The nonlinear RDC equations (30) and (31) possess also non-trivial $Q$-conditional symmetries. However, Eqs. (30) and (31) are reduced to (13) and (15), respectively, therefore they are not listed in Theorem 2.

**Remark 4.** Equation (19) can be reduced to the initial equation (2) by a set of non-local substitutions [26] (see for details [27]). Hence the search of $Q$-conditional symmetries of form (9) is equivalent (up to non-local transformations) to solving (2).

### 3. Exact solutions of some RDC equations of form (2)

Let us apply the $Q$-conditional symmetry operators listed in Theorem 2 for finding exact solutions of some RDC equations arising in applications. Consider a particular case of Eq. (30) of the form

$$u_t = u_{xx} + \lambda uu_x + u(\lambda_1 - \lambda_2 u). \quad (32)$$

where $\lambda_1 \lambda_2 > 0$. Equation (32) was introduced in [1] as a generalization of the Fisher equation (i.e. (32) with $\lambda = 0$) by adding the simplest convection term therefore it can be named the Murray equation. As follows from Theorem 1 this equation is invariant with respect to the trivial Lie algebra spanned by the operators $P_t$ and $P_x$, i.e. the Lie method leads only to plane wave ansatz

$$u = \phi(\omega), \quad \omega = x - \alpha t, \quad \alpha \in \mathbb{R}, \quad (33)$$
which reduces (32) to the ordinary differential equation (ODE)
\[\phi_{\omega\omega} + (\alpha + \lambda \phi)\phi_{\omega} + \phi(\lambda_1 - \lambda_2 \phi) = 0. \tag{34}\]

The nonlinear ODE (34) with the arbitrary coefficients \(\lambda, \lambda_1, \lambda_2\) and \(\alpha\) cannot be integrated, however, the particular solutions
\[u = \frac{\lambda_1}{\lambda_2} + c_1 \exp(\gamma x + \gamma^2 t), \quad u = c_1 \exp(\gamma x + (\gamma^2 + \lambda_1)t), \quad \gamma = \frac{\lambda_2}{\lambda}, \quad c_1 \in \mathbb{R}, \tag{35}\]
were found. It turns out another solution can be constructed using the \(Q\)-conditional invariance of (32). In fact, according to Theorem 2 and Remark 3 this equation admits the \(Q\)-conditional symmetry operator
\[Q = \partial_t - \left(\lambda u + \frac{\lambda_2}{\lambda}\right)\partial_x + (\lambda_1 u - \lambda_2 u^2)\partial_u. \tag{36}\]

Using this operator one can construct the exact solution (for details see [12])
\[u = \frac{\lambda_1 + c_1 \exp(\gamma x + \gamma^2 t)}{\lambda_2 + c_0 \exp(-\lambda_1 t)}, \quad \gamma = \frac{\lambda_2}{\lambda}. \tag{37}\]

This solution at \(c_0 \neq 0\) is not of the plane wave form (32), i.e. it is so-called non-Lie solution because one cannot be found using the Lie method. Note (37) with \(c_0 > 0\) and \(\lambda_1 < 0\) is the blow-up solution because it increases infinitely for the finite time \(t_0 = |\lambda_1|^{-1} \ln|\lambda_2/c_0|\).

Consider generalized Fitzhugh–Nagumo (FN) equation
\[u_t = u_{xx} + \lambda u u_x + \lambda_3 u(u - \delta)(1 - u), \quad 0 < \delta < 1, \quad \lambda_3 > 0, \tag{38}\]
and generalized Kolmogorov–Petrovskii–Piskunov (KPP) equation
\[u_t = u_{xx} + \lambda_3 u_x - \lambda_3 u(1 - u)^2, \quad \lambda_3 < 0, \tag{39}\]
which are particular cases of Eq. (31). We remained the reader that (38) with \(\lambda = 0\) is the famous FN equation [30] describing nerve impulse propagation. It can be also considered as a simplification of Hodgkin–Huxley model (see, e.g., [2]) describing the ionic current flows for axonal membranes. Equation (39) with \(\lambda = 0\) is the KPP equation, which firstly was introduced in [31] to describe the population dynamics under some restrictions on characteristic individuals.

Both Eqs. (38) and (39) can be reduced to the form
\[v_t = v_{yy} + \lambda v v_y + \lambda_0 + \lambda_1 v - \lambda_3 v^3, \tag{40}\]
where
\[\lambda_1 = \lambda_3 \left(\frac{1}{3} (\delta + 1)^2 - \delta\right), \quad \lambda_0 = \lambda_3 \frac{1}{9} (\delta + 1)^2 - \delta, \tag{41}\]
by the local substitution
\[v(t, y) = u - \frac{1}{3} (\delta + 1), \quad y = x + \frac{\lambda}{3} (\delta + 1)t. \tag{42}\]
Thus, Eq. (40) is locally equivalent to the generalized FN and KPP equations for \(0 < \delta < 1\) and \(\delta = 1\), respectively.
As follows from Theorem 1 Eq. (40) with \( \lambda_3 \neq 0 \) is invariant only with respect to the trivial Lie algebra \((P_t, P_y)\) and the plane wave ansatz

\[
v = \phi(\omega), \quad \omega = y - \alpha t, \quad \alpha \in \mathbb{R},
\]

reduces one to the nonlinear ODE

\[
\phi_{\omega\omega} + (\alpha + \lambda \phi)\phi_{\omega} + \lambda_0 + \lambda_1 \phi - \lambda_3 \phi^3 = 0,
\]

which cannot be integrated for the arbitrary coefficients \(\lambda, \lambda_0, \lambda_1, \lambda_3\) and \(\alpha\). The known particular solutions of (44) generate the plane wave solutions of (40), which will be presented later as particular cases of more general solutions.

It turns out, the \(Q\)-conditional symmetry of (40) leads to new families of non-Lie solutions. In fact, according to Theorem 2 this equation admits two \(Q\)-conditional symmetries

\[
Q_1 = \partial_t + \frac{3\kappa - \lambda}{4} v\partial_y + \frac{3\kappa - \lambda}{2(\kappa - \lambda)} (\lambda_0 + \lambda_1 v - \lambda_3 v^3) \partial_v,
\]

and

\[
Q_2 = \partial_t - \frac{3\kappa + \lambda}{4} v\partial_y + \frac{3\kappa + \lambda}{2(\kappa + \lambda)} (\lambda_0 + \lambda_1 v - \lambda_3 v^3) \partial_v,
\]

where \(\kappa = \sqrt{\lambda^2 + 8\lambda_3}\). Using these operators, one can construct the non-Lie ansätze

\[
\phi(\omega) = \int \frac{v \, dv}{\lambda_0 + \lambda_1 v - \lambda_3 v^3} - \frac{2}{\kappa - \lambda} y, \quad \omega = \int \frac{dv}{\lambda_0 + \lambda_1 v - \lambda_3 v^3} - \frac{3\kappa - \lambda}{2(\kappa - \lambda)} t,
\]

and

\[
\phi(\omega) = \int \frac{v \, dv}{\lambda_0 + \lambda_1 v - \lambda_3 v^3} + \frac{2}{\kappa + \lambda} y, \quad \omega = \int \frac{dv}{\lambda_0 + \lambda_1 v - \lambda_3 v^3} - \frac{3\kappa + \lambda}{2(\kappa + \lambda)} t,
\]

by solving the first-order PDEs \(Q_1(v) = 0\) and \(Q_2(v) = 0\), respectively. Since these ansätze cannot be written in an explicit form (with respect to the variable \(v\)), we prefer to solve directly the overdetermined systems

\[
v_t = v_{yy} + \lambda v v_y + \lambda_0 + \lambda_1 v - \lambda_3 v^3,
\]

\[
Q_1(v) \equiv v_t + \frac{3\kappa - \lambda}{4} v v_y - \frac{3\kappa - \lambda}{2(\kappa - \lambda)} (\lambda_0 + \lambda_1 v - \lambda_3 v^3) = 0
\]

and

\[
v_t = v_{yy} + \lambda v v_y + \lambda_0 + \lambda_1 v - \lambda_3 v^3,
\]

\[
Q_2(v) \equiv v_t - \frac{3\kappa + \lambda}{4} v v_y - \frac{3\kappa + \lambda}{2(\kappa + \lambda)} (\lambda_0 + \lambda_1 v - \lambda_3 v^3) = 0,
\]

what is equivalent to the substitution (47) and (48) into (40).

Consider firstly the overdetermined system (49). Eliminating \(u_t\) in the first equation using the second one, we arrive at the second-order ODE

\[
v_{yy} + \frac{3(\kappa + \lambda)}{4} v v_y - \frac{\kappa + \lambda}{2(\kappa - \lambda)} (\lambda_0 + \lambda_1 v - \lambda_3 v^3) = 0,
\]

which can be reduced to the form

\[
V_{y^*y^*} + 3V V_{y^*} + V^3 - \frac{\lambda_1}{\lambda_3} V - \frac{\lambda_0}{\lambda_3} = 0
\]
by the local substitution
\[ v(t, y) = V(t, y^*), \quad y^* = \frac{\kappa + \lambda}{4} y. \] (53)

It is known that ODE (52) is reduced to the linear third-order ODE
\[ W_{y^*y^*y^*} - \frac{\lambda_1}{\lambda_3} W_{y^*} - \frac{\lambda_1}{\lambda_3} W = 0 \] (54)

by the non-local substitution [28] (see item (6.38))
\[ V = \frac{W_{y^*}}{W}. \] (55)

To construct the general solution of (54), one needs to solve the cubic equation
\[ \lambda_0 + \lambda_1 \alpha - \lambda_3 \alpha^3 = 0, \] (56)
which has three different roots \( \alpha_1, \alpha_2, \alpha_3 \) in the case of the generalized FN equation and two different roots \( \alpha_1 \) and \( \alpha_2 \) in the case of the generalized KPP equation. The relevant general solutions take the form
\[ W(t, y^*) = \phi_1(t) \exp(\alpha_1 y^*) + \phi_2(t) \exp(\alpha_2 y^*) + \phi_3(t) \exp(\alpha_3 y^*) \] (57)

and
\[ W(t, y^*) = \phi_1(t) \exp(\alpha_1 y^*) + \phi_2(t) \exp(\alpha_2 y^*) + \phi_3(t) y^* \exp(\alpha_2 y^*), \] (58)

where \( \phi_i(t), i = 1, 2, 3 \), are arbitrary (at the moment) functions.

Consider firstly solution (57). Taking into account formulas (52), (53) and (55), one easily obtains the general solution of the nonlinear second-order ODE (51)
\[ v(t, y) = \frac{\alpha_1 \phi_1(t) \exp(\gamma_1 y) + \alpha_2 \phi_2(t) \exp(\gamma_2 y) + \alpha_3 \phi_3(t) \exp(\gamma_3 y)}{\phi_1(t) \exp(\gamma_1 y) + \phi_2(t) \exp(\gamma_2 y) + \phi_3(t) \exp(\gamma_3 y)}, \] (59)

where \( \gamma_i = \alpha_i \frac{\kappa + \lambda}{4}, i = 1, 2, 3 \). Finally, to obtain the general solution of the overdetermined system (49), it is sufficiently to substitute (59) into the second equation of this system. After the relevant calculations a cumbersome expression is obtained, however, one splits into separate parts for the functions \( \exp((\gamma_i + \gamma_j) y), j < i = 1, 2, 3 \), and we arrive at the ODE system
\[
\frac{\dot{\phi}_1 \phi_2 - \dot{\phi}_2 \phi_1}{\phi_1} = \frac{\lambda_3(3\kappa - \lambda)}{2(\kappa - \lambda)} \left( \alpha_1^2 - \alpha_2^2 \right) \phi_1 \phi_2, \\
\frac{\dot{\phi}_2 \phi_3 - \dot{\phi}_3 \phi_2}{\phi_2} = \frac{\lambda_3(3\kappa - \lambda)}{2(\kappa - \lambda)} \left( \alpha_2^2 - \alpha_3^2 \right) \phi_2 \phi_3, \\
\frac{\dot{\phi}_3 \phi_1 - \dot{\phi}_1 \phi_3}{\phi_3} = \frac{\lambda_3(3\kappa - \lambda)}{2(\kappa - \lambda)} \left( \alpha_3^2 - \alpha_1^2 \right) \phi_1 \phi_3 \] (60)

(hereafter the dot next to \( \phi_i(t), i = 1, 2, 3 \), denotes differentiation with respect to the variable \( t \)).

This system is integrable and its general has the form
\[
\phi_1 = c_1 \phi(t) \exp(\beta_1 \alpha_1^2 t), \quad \phi_2 = c_2 \phi(t) \exp(\beta_1 \alpha_2^2 t), \quad \phi_3 = c_3 \phi(t) \exp(\beta_1 \alpha_3^2 t), \] (61)
where \( c_i, i = 1, 2, 3, \) are arbitrary constants, \( \beta_1 = \frac{\lambda_3(3\kappa - \lambda)}{2(\kappa - \lambda)} \) and \( \phi(t) \) is an arbitrary function. Finally, substituting (61) into (59), we obtain three parameter family of exact solutions of nonlinear RDC equation (40)

\[
v(t, y) = \frac{\alpha_i c_i \exp(\gamma_i y + \beta_1 \alpha_i^2 t)}{c_i \exp(\gamma_i y + \beta_1 \alpha_i^2 t)}
\]

(hereafter summation is assumed from 1 to 3 over the repeated index \( i \)). Assuming \( c_1 \neq 0 \) this solution family can be reduced to the form

\[
v(t, y) = \frac{\alpha_1 + \alpha_2 c_2 \exp((\gamma_2 - \gamma_1)y + \beta_1(\alpha_2^2 - \alpha_1^2)t) + \alpha_3 c_3 \exp((\gamma_3 - \gamma_1)y + \beta_1(\alpha_3^2 - \alpha_1^2)t)}{1 + c_2 \exp((\gamma_2 - \gamma_1)y + \beta_1(\alpha_2^2 - \alpha_1^2)t) + c_3 \exp((\gamma_3 - \gamma_1)y + \beta_1(\alpha_3^2 - \alpha_1^2)t)}
\]

(63)

In the quite similar way one can solve the overdetermined system (50), if the algebraic equation (56) has three different roots \( \alpha_1, \alpha_2, \alpha_3 \) (in fact, the first equation can be also reduced to form (52)). Finally, the following family of exact solutions of Eq. (40)

\[
v(t, y) = \frac{\alpha_i c_i \exp(\sigma_i y + \beta_2 \alpha_i^2 t)}{c_i \exp(\sigma_i y + \beta_2 \alpha_i^2 t)}
\]

is obtained. In the case \( c_1 \neq 0 \), this formula can be rewritten as

\[
v(t, y) = \frac{\alpha_1 + \alpha_2 c_2 \exp((\sigma_2 - \sigma_1)y + \beta_2(\alpha_2^2 - \alpha_1^2)t) + \alpha_3 c_3 \exp((\sigma_3 - \sigma_1)y + \beta_2(\alpha_3^2 - \alpha_1^2)t)}{1 + c_2 \exp((\sigma_2 - \sigma_1)y + \beta_2(\alpha_2^2 - \alpha_1^2)t) + c_3 \exp((\sigma_3 - \sigma_1)y + \beta_2(\alpha_3^2 - \alpha_1^2)t)}
\]

(65)

Here \( \sigma_i = \alpha_i - \frac{\kappa + \lambda}{4}, i = 1, 2, 3, \kappa = \sqrt{\lambda^2 + 8\lambda_3} \) and \( \beta_2 = \frac{\lambda_3(3\kappa + \lambda)}{2(\kappa + \lambda)} \).

Let us consider now solution (58). Taking into account formulas (52), (53) and (55), the general solution of the nonlinear second-order ODE (51)

\[
v(t, y) = \frac{\alpha_1 \phi_1(t) \exp(\gamma_1 y) + \alpha_2 \phi_2(t) \exp(\gamma_2 y) + \phi_3(t) \exp(\gamma_2 y)(1 + y \gamma_2)}{\phi_1(t) \exp(\gamma_1 y) + \phi_2(t) \exp(\gamma_2 y) + \frac{\kappa + \lambda}{4} \phi_3(t) y \exp(\gamma_2 y)}
\]

(66)

is obtained (we use here again the notation \( \gamma_i = \alpha_i - \frac{\kappa + \lambda}{4}, i = 1, 2 \)). To obtain the general solution of the overdetermined system (49), it is sufficiently to substitute (66) into the second equation of this system. It is important to note at this step that the cubic equation (56) has two different roots \( \alpha_1 \) and \( \alpha_2 \) only under the condition \( \alpha_1 + 2\alpha_2 = 0 \). After the relevant simplifications an expression is obtained, which splits into separate parts for the functions \( \exp((\gamma_1 + \gamma_2)y), y \exp((\gamma_1 + \gamma_2)y) \) and \( \exp(2(\gamma_2 y)) \). Finally, we arrive at the ODE system

\[
3\alpha_2(\dot{\phi}_2 \phi_2 - \dot{\phi}_1 \phi_1) + \dot{\phi}_1 \phi_3 - \dot{\phi}_3 \phi_1 = \frac{\lambda_3(3\kappa - \lambda)}{2(\kappa - \lambda)}(9\alpha^3_2 \phi_1 \phi_2 - 3\alpha^3_2 \phi_1 \phi_3),
\]

\[
\dot{\phi}_1 \phi_3 - \dot{\phi}_3 \phi_1 = 3\alpha^2_2 \frac{\lambda_3(3\kappa - \lambda)}{2(\kappa - \lambda)} \phi_1 \phi_3,
\]

\[
\dot{\phi}_2 \phi_3 - \dot{\phi}_3 \phi_2 = 2\alpha_2 \frac{\lambda_3(3\kappa - \lambda)}{2(\kappa - \lambda)} \phi_3^2.
\]

(67)

In the case \( \phi_3 = 0 \), solution (66) is reduced to a particular case of (59) so that we assume \( \phi_3 \neq 0 \). The ODE system (67) is integrable under this restriction and its general has the form

\[
\phi_1 = c_1 \exp(3\beta_1 \alpha_1^2 t) \phi(t), \quad \phi_2 = (c_2 + 2\beta_1 \alpha_2 t) \phi(t), \quad \phi_3 = \phi(t).
\]

(68)
Finally, substituting (68) into (66), we obtain two parameter family of exact solutions
\[
v(t, y) = \frac{1 + \alpha_2 c_2 + \gamma y + 2\beta_1 \alpha_2^2 t - 2\alpha_2 c_1 \exp(3(\beta_1 \alpha_2^2 t - \gamma y))}{c_2 + \frac{\kappa + \lambda}{4} y + 2\beta_1 \alpha_2 t + c_1 \exp(3(\beta_1 \alpha_2^2 t - \gamma y))}
\]  
(69)
of the nonlinear RDC equation (40) when this equation takes the form
\[
v_t = v_{yy} + \lambda v v_y + \lambda_3 (\alpha - v)(v - \alpha_2)^2, \quad \alpha_1 + 2\alpha_2 = 0.
\]  
(70)
In the quite similar way the overdetermined system (50) has been also solved and the following family of exact solutions of the nonlinear RDC equation (70)
\[
v(t, y) = \frac{1 + \alpha_2 c_2 + \gamma y + 2\beta_2 \alpha_2^2 t - 2\alpha_2 c_1 \exp(3(\beta_2 \alpha_2^2 t - \gamma y))}{c_2 + \frac{\kappa - \lambda}{4} y + 2\beta_2 \alpha_2 t + c_1 \exp(3(\beta_2 \alpha_2^2 t - \gamma y))}
\]  
(71)
has been found (the values \(\kappa, \beta_2\) and \(\sigma_2\) are the same as in (65)).

One observes that the exact solution families constructed above contains several plane wave solutions of form (43). In fact, vanishing one of the constants \(c_i, i = 1, 2, 3\), in (62) and (64), we obtain travelling wave solutions
\[
v(t, y) = \frac{\alpha_i + \alpha_j c \exp((\gamma_j - \gamma_i)(y + \beta_{ij} t))}{1 + c \exp((\gamma_j - \gamma_i)(y + \beta_{ij} t))}
\]  
(72)
and
\[
v(t, y) = \frac{\alpha_i + \alpha_j c \exp((\sigma_j - \sigma_i)(y + \beta_{ij} t))}{1 + c \exp((\sigma_j - \sigma_i)(y + \beta_{ij} t))},
\]  
(73)
respectively. Here any summation over \(i\) or \(j\) is not assumed, \(\beta_{ij}^1 = \frac{4\beta_1}{\kappa + \lambda} (\alpha_i + \alpha_j)\), \(\beta_{ij}^2 = \frac{4\beta_2}{\kappa - \lambda} (\alpha_i + \alpha_j)\), \(i \neq j, i, j = 1, 2, 3\), and \(c\) is a positive constant. Obviously, each travelling wave solution of form (72)–(73) tends to \(\alpha_i\) or to \(\alpha_j\) if \(\omega = y + \beta_{ij}^k t \to \pm \infty\), \(k = 1, 2\) (we remind the reader that the roots \(\alpha_1, \alpha_2, \alpha_3\) call the steady-state points of the nonlinear equation (40)).

One note that each travelling wave has the speed \(\beta_{ij}^k\), therefore, taking into account the identity \(\beta_{ki}^j = \beta_{ki}^i\), six different travelling wave solutions are generated by formulae (72), (73).

In the case of Eq. (70), two plane wave solutions with different structure follows from (69) and (71), namely:
\[
v(t, y) = \alpha_2 + \frac{1}{c + \frac{\kappa + \lambda}{4} y + 2\beta_1 \alpha_2 t}
\]  
(74)
and
\[
v(t, y) = \alpha_2 + \frac{1}{c + \frac{\kappa - \lambda}{4} y + 2\beta_2 \alpha_2 t}.
\]  
(75)

**Remark 5.** Any solution of form (62)–(65), (69) and (71) with \(c_1 c_2 \neq 0\) cannot be found using the Lie ansatz (43) so that they are non-Lie solutions of the nonlinear RDC equation (40).

Now we present the families of exact solutions of the generalized FN and KPP equations, which are easily constructed using the solutions found above and formulae (41) and (42). The generalized FN equation (38) possesses the solutions
\[
u(t, x) = u_0 + \frac{\alpha_i c_i \exp(\gamma_i x + (u_0 \lambda \gamma_i + \beta_1 \alpha_i^2) t)}{c_i \exp(\gamma_i x + (u_0 \lambda \gamma_i + \beta_1 \alpha_i^2) t)}
\]  
(76)
and
\[ u(t, x) = u_0 + \frac{\alpha_i c_i \exp(\sigma_i x + (u_0 \lambda \sigma_i + \beta_2 \alpha_i^2)t)}{c_i \exp(\sigma_i x + (u_0 \lambda \sigma_i + \beta_2 \alpha_i^2)t)}, \quad (77) \]
where \(c_i, i = 1, 2, 3\), are arbitrary constants, \(\gamma_i, \sigma_i, \beta_1\) and \(\beta_2\) are defined above, \(u_0 = \frac{1}{3}(\delta + 1)\), \(\alpha_1 = -\frac{2}{3}(\delta + 1)\), \(\alpha_2 = \frac{1}{3}(2\delta - 1)\), \(\alpha_3 = \frac{1}{3}(2 - \delta)\) and the summation is assumed over the repeated index \(i\). In the case \(c_i > 0, i = 1, 2, 3\), this type of exact solutions is known in applications as two-shock waves (see, e.g., [16]).

In the particular cases, we obtain the travelling wave solutions
\[ u(t, x) = u_0 + \frac{\alpha_i + \alpha_j c \exp((\gamma_j - \gamma_i)(x + (u_0 \lambda + \beta_1^j t)))}{1 + c \exp((\gamma_j - \gamma_i)(x + (u_0 \lambda + \beta_1^j t)))}, \quad i \neq j, i, j = 1, 2, 3, \quad (78) \]
and
\[ u(t, x) = u_0 + \frac{\alpha_i + \alpha_j c \exp((\gamma_j - \gamma_i)(x + (u_0 \lambda + \beta_2^j t)))}{1 + c \exp((\gamma_j - \gamma_i)(x + (u_0 \lambda + \beta_2^j t)))}, \quad i \neq j, i, j = 1, 2, 3 \quad (79) \]
(here any summation over \(i\) or \(j\) is not assumed!). It should be noted that solutions (78) and (79) generate the known solutions of the FN equation, i.e. (38) with \(\lambda = 0\), which have been constructed in paper [32].

Two families of exact solutions of the generalized KPP equation (39)
\[ u(t, x) = \frac{8 + c_2 + 2(\lambda + \kappa)x + (\lambda + \kappa)^2 t}{c_1 \exp(-\kappa + \frac{\lambda}{4}(x + \frac{\lambda - \kappa}{4}t)) + c_2 + 2(\lambda + \kappa)x + (\lambda + \kappa)^2 t}, \quad (80) \]
and
\[ u(t, x) = \frac{8 + c_2 + 2(\lambda - \kappa)x + (\lambda - \kappa)^2 t}{c_1 \exp(\frac{\lambda - \kappa}{4}(x + \frac{\lambda - \kappa}{4}t)) + c_2 + 2(\lambda - \kappa)x + (\lambda - \kappa)^2 t}, \quad (81) \]
are obtained from (69) and (71) at \(\alpha_1 = -\frac{2}{3}, \alpha_2 = \frac{1}{3}\) using formulae (41) and (42) with \(\delta = 1\). Moreover, the travelling wave solutions (78) and (79) with \(u_0 = \frac{2}{3}, \alpha_i = -\frac{2}{3}, \alpha_j = \frac{1}{3}\) are also the solutions of Eq. (39). It should be noted that the generalized KPP equation does not possess the two-shock wave solutions in contrary to the generalized FN equation.

4. Application of the method of additional generating conditions

Here we establish that it is possible to construct new non-Lie ansätze and to find new exact solutions for some nonlinear RDC equations of form (2) using additional generating conditions [19,20]. Note, we can examine only those RDC equations which are reducible to the quadratic form. Consider the equation
\[ u_t = u_{xx} + \lambda uu_x + \lambda_1 u - \lambda_2 u^2 - \lambda_3 u^3, \quad (82) \]
containing the Murray and generalized Fitzhugh–Nagumo equations as particular cases. The local substitution
\[ u = \frac{1}{v}, \quad v = v(t, x), \quad (83) \]
reduces (82) to the quadratic form
\[ vv_t = v v_{xx} - 2 v_x^2 + \lambda v_x - \lambda_1 v^2 + \lambda_2 v + \lambda_3. \] (84)
Consider an additional generating condition in the form of the linear third-order ODE [19,20]
\[ \alpha_1(t) \frac{d v}{d x} + \alpha_2(t) \frac{d^2 v}{d x^2} + \alpha_3(t) \frac{d^3 v}{d x^3} = 0, \] (85)
where \( \alpha_i(t), i = 1, 2, 3, \) is an arbitrary smooth functions and the variable \( t \) is considered as a parameter. Depending on \( \alpha_i(t) \), condition (85) generates a chain of ansätze, in the particular case, setting \( \alpha_i(t) = \alpha_i \in \mathbb{R}, i = 1, 2, \alpha_3(t) = 1, \) one obtains
\[ v(t, x) = \varphi_0(t) + \varphi_1(t) \exp(\gamma_1 x) + \varphi_2(t) \exp(\gamma_2 x), \quad \gamma_{1,2} = -\frac{1}{2} \left( \alpha_2 \pm \sqrt{\alpha_2^2 - 4\alpha_1} \right). \] (86)
Since this ansatz contains three yet-to-be determined functions, this enables us to reduce the given nonlinear PDE to a nonlinear system of first-order ODEs for the unknown functions \( \varphi_i, \) \( i = 1, 2, 3, \) and constants \( \gamma_1 \) and \( \gamma_2. \) In the particular case, substituting (86) with \( \varphi_2(t) = 0 \) and \( \gamma_1 = \gamma \neq 0 \) into (84), one arrives at the ODEs
\[ \varphi_0 \varphi_0 + \lambda_1 \varphi_0^2 - \lambda_2 \varphi_0 - \lambda_3 = 0, \]
\[ \varphi_1 + (\gamma^2 + \lambda_1) \varphi_1 = 0, \]
\[ \frac{d}{d t} (\varphi_0 \varphi_1) + (2\lambda_1 - \gamma^2) \varphi_0 \varphi_1 = (\lambda \gamma + \lambda_2) \varphi_0. \] (87)
In the case \( \varphi_0 = \text{const}, \) \( \gamma_1 = \gamma \neq 0 \) and \( \gamma_2 = 2\gamma, \) ansatz (86) leads to the ODE system
\[ \varphi_1 - (\gamma^2 - \lambda_1) \varphi_1 = 0, \]
\[ \varphi_2 + (4\gamma^2 + \lambda_1) \varphi_2 = 0, \]
\[ \varphi_0 \varphi_2 + 2\gamma^2 \varphi_1^2 + (2\lambda_1 - 4\gamma^2) \varphi_0 \varphi_2 - (2\lambda \gamma + \lambda_2) \varphi_2 = 0 \] (88)
and the algebraic conditions \( \gamma^2 = \lambda_1 / 6, \) \( \lambda_1 \varphi_0^2 = \lambda_2 \varphi_0 + \lambda_3, \) \( \lambda_1 \varphi_0 = \lambda \gamma + \lambda_2. \) Nevertheless systems (87) and (88) are overdetermined, they have non-trivial solutions under some restrictions on the coefficients.
Consider the case \( \lambda_3 = 0 \) and \( \lambda_1 \lambda_2 > 0, \) which corresponds to the Murray equation. The general solution of the first and second equations of (87) has the form
\[ \varphi_0(t) = \frac{\lambda_2}{\lambda_1} + c_0 \exp(-\lambda_1 t), \quad \varphi_1(t) = c_1 \exp(-\left(\lambda_1 + \gamma^2\right)t). \] (89)
where \( c_0 \) and \( c_1 \) are arbitrary constants. Substituting (89) into the third equation of (87), one arrives at the restriction \( c_0 = 0 \) and \( \gamma = \frac{\lambda \lambda_1}{2\lambda_3}. \) Taking into account formulae (86) with \( \varphi_2(t) = 0 \) and (83), we obtain finally the new exact solution
\[ u(t, x) = \left[ \frac{\lambda_2}{\lambda_1} + c_1 \exp(-\left(\gamma x + (\gamma^2 + \lambda_1) t\right)) \right]^{-1} \] (90)
of the Murray equation (32). In the case \( \lambda < 0, \lambda_1 = \lambda_2 = 1 \) and \( c_1 > 0, \) this travelling wave solution coincides with the numerical solution pictured in Murray’s book [2, Fig. 11.5(b)].
Similarly, a non-constant solution of the overdetermined system (88) was found, therefore the new exact solution
\[ u(t, x) = \left[ c_1 \exp(-\gamma (\mp x + 5\gamma t)) - \frac{2\lambda_2}{\lambda^2} c_1^2 \exp(-2\gamma (\mp x + 5\gamma t)) \right]^{-1}, \quad \gamma = \sqrt{\frac{\lambda_1}{6}}, \]

(91)
of the Murray equation (32) with the restriction on the coefficients \(6\lambda_2^2 = \lambda_1^2\) was constructed. This solution with the upper sign ‘−’ is valid if \(\lambda \lambda_2 < 0\) while one with the lower sign ‘+’ is valid if \(\lambda \lambda_2 > 0\). It should be noted that solution (91) with \(\lambda_1 = 1\) and arbitrary \(\lambda \lambda_2 < 0\) can be considered as a wave with the velocity \(5\gamma = 5/\sqrt{6}\) and this velocity coincides with that of the well-known travelling wave of the Fisher equation.

Analogously the case \(\lambda_2 = -\lambda_3(1 + \delta)\) and \(\lambda_1 = -\lambda_3 \delta\), \(0 < \delta < 1\), which corresponds to the generalized Fitzhugh–Nagumo equation (38), can be examined. The general solution of the first equation in (87) (with \(\lambda_1\) and \(\lambda_2\) listed above) takes the form

\[ |\varphi_0 - 1|^{\frac{1}{\delta}} = |\varphi_0 - 1|c_0 \exp(\lambda_3(1 - \delta)t). \]

(92)

Unfortunately, \(\varphi_0(t)\) cannot be expressed from (92) in terms of elementary functions, excepting a few values of the parameter \(\delta\). However, assuming \(\varphi_0(t) = \text{const}\), one can again find non-trivial solutions of (87) and finally construct exact solutions of (38). The solutions obtained are of form (78) and (79) and omitted here.

Finally, the general solution of the first equation in (87) with \(\lambda_1 = -\lambda_3\) and \(\lambda_2 = -2\lambda_3\) (the case of the generalized KPP equation) has the form

\[ |\varphi_0 - 1| \exp\left(-\frac{1}{\varphi_0 - 1}\right) = c_0 \exp(\lambda_3 t), \]

(93)

so that \(\varphi_0(t)\) again cannot be expressed in an explicit form and the solutions with \(\varphi_0(t) = \text{const}\) can be only constructed.

5. Conclusions

In this paper, the theorem giving a complete description of \(Q\)-conditional symmetries of the nonlinear RDC equation (2) is presented. It should be stressed that all \(Q\)-conditional symmetry operators listed in Theorem 2 contains at maximum the cubic nonlinearities with respect to the dependent variable \(u\). Analogous operators were earlier found for single reaction–diffusion (RD) equations [7,9–11], single RDC equations [12] and diffusion–convection systems [29]. However, we note that there is the essential difference between RDC equation (2) and the relevant RD equation (i.e. (2) with \(\lambda = 0\)). Firstly, RDC equation (2) with the logistic term \(C = u(\lambda_1 - \lambda_2 u)\) admits the \(Q\)-conditional symmetry of form (8), while the RD equation with this term, i.e. the Fisher equation

\[ u_t = u_{xx} + u(\lambda_1 - \lambda_2 u), \quad \lambda_1 \lambda_2 \neq 0, \]

does not possess one. Secondly, it is still open question whether one can construct the \(Q\)-conditional symmetry operator (17) in the explicit form while it has been done in the case of the relevant RD equation [11]. Note the first two equations of (18) at \(\lambda = 0\) and \(b_t = 0\) induce the equation \(6b_x - 2b^2 + 9\lambda_1 = 0\) derived in [11] (see p. 264) to construct a \(Q\)-conditional symmetry operator of form (17) for Eq. (15) with \(\lambda = 0\).
It is quite interesting to show the clear relation between \(Q\)-conditional symmetries and generalized conditional symmetries introduced in [15]. In fact, the left-hand side of (51) is nothing else but the generalized conditional symmetry

\[
\sigma_1 = v_{yy} + \frac{3(\kappa + \lambda)}{4} v_{yy} - \frac{\kappa + \lambda}{2(\kappa - \lambda)} \left( \lambda_0 + \lambda_1 v - \lambda_3 v^3 \right)
\]

of RDC equation (40). Using system (50), one easily obtains the second generalized conditional symmetry

\[
\sigma_2 = v_{yy} + \frac{3(\lambda - \kappa)}{4} v_{yy} - \frac{\kappa - \lambda}{2(\kappa + \lambda)} \left( \lambda_0 + \lambda_1 v - \lambda_3 v^3 \right), \quad \kappa = \sqrt{\lambda^2 + 8\lambda_3}.
\]

In other words, generalized conditional symmetries are immediately obtained from \(Q\)-conditional symmetries (at least in the case of RDC equations with the cubic nonlinearity). Note the only symmetry \(\sigma_2\) and the relevant family of solutions has been found in [15] (see formulas (1.6), (1.7) and (4.5), (4.6) therein).

The \(Q\)-conditional symmetries found in Section 2 are further applied for finding exact solutions of the Murray equation, the generalized FN and KPP equations that arise in mathematical biology and are extensively investigated. Moreover the method of additional generating conditions is also used to construct exact solutions for these equations. As result several two- and three-parameter families of exact solutions were found. To my best knowledge, many of them are absolutely new solutions and presented the first time here. Thus, \(Q\)-conditional symmetries and additional generating conditions lead to new types of exact solutions and they cannot be obtained by the classical Lie method. Note that those non-Lie solutions as particular cases contain the travelling wave solutions that are obtainable by the classical Lie method. On the other hand, the solutions obtained by \(Q\)-conditional symmetry approach (i.e. the Bluman–Cole approach of non-classical symmetries) and the method of additional generating conditions have the essentially different structure. It means that any nonlinear RDC equation arising in applications should be investigated by several approaches if one aims to construct a wide class of exact solutions.

The exact solutions obtained can be further applied for solving the nonlinear boundary-value problem with zero Neumann conditions (zero flux on boundaries is a typical condition in biologically motivated problems) and compared with the relevant numerical simulations. One can expect the numerical simulations will show that the numerical solutions of a class of boundary-value problems approximately coincide with the relevant exact solutions in the case of arbitrary sufficiently small perturbations of the initial profiles generated by the exact solutions. Note a such behavior of numerical solutions was recently established in the case of the diffusive Lotka–Volterra system [33].

References