



A dynamical systems approach to the Kadison–Singer problem[☆]

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Abstract

In these notes we develop a link between the Kadison–Singer problem and questions about certain dynamical systems. We conjecture that whether or not a given state has a unique extension is related to certain dynamical properties of the state. We prove that if any state corresponding to a minimal idempotent point extends uniquely to the von Neumann algebra of the group, then every state extends uniquely to the von Neumann algebra of the group. We prove that if any state arising in the Kadison–Singer problem has a unique extension, then the injective envelope of a C^* -crossed product algebra associated with the state necessarily contains the full von Neumann algebra of the group. We prove that this latter property holds for states arising from rare ultrafilters and δ -stable ultrafilters, independent, of the group action and also for states corresponding to non-recurrent points in the corona of the group.

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1. Introduction

Let \mathcal{H} be a separable Hilbert space, and let $\mathcal{D} \subset B(\mathcal{H})$ be a discrete MASA. The Kadison–Singer problem [10] asks whether or not every pure state on \mathcal{D} has a unique extension to a state on $B(\mathcal{H})$. Without loss of generality, one can assume that the Hilbert space is $\ell^2(\mathbb{N})$, where \mathbb{N}

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denotes the natural numbers, with the canonical orthonormal basis, $\{e_n\}_{n \in \mathbb{N}}$ and that the MASA is the subalgebra of operators that are diagonal with respect to this basis.

However, since any two discrete MASA's on any two separable infinite-dimensional Hilbert spaces are conjugate, one may equally well assume that the Hilbert space is $\ell^2(G)$, where G is a countable, discrete group, with canonical orthonormal basis $\{e_g\}_{g \in G}$ and that the MASA is the set of operators that are diagonal with respect to this basis. Thus, $\mathcal{D} = \{M_f : f \in \ell^\infty(G)\}$, where M_f denotes the operator of multiplication by the function f . We let 1 denote the identity of G and let $U_g = \lambda(g)$ denote the unitary operators given by the left regular representation, so that $U_g e_h = e_{gh}$.

The reason that we prefer this slight change of perspective, is that we are interested in incorporating properties of the group action into results on the Kadison–Singer problem. Indeed, identifying the MASA, $\mathcal{D} \equiv \ell^\infty(G) \equiv C(\beta G)$, where βG denotes the Stone–Cech compactification of G , then pure states on \mathcal{D} correspond to the homomorphisms induced by evaluations at points in βG .

Moreover, we have that for each $g \in G$, the map $g_1 \rightarrow gg_1$ extends uniquely to a homeomorphism, $h_g : \beta G \rightarrow \beta G$ and this family of homeomorphisms satisfy, $h_{g_1} \circ h_{g_2} = h_{g_1 g_2}$, that is, they induce an action of G on the space βG and we set $h_g(\omega) = g \cdot \omega$.

In this paper we study the extent to which uniqueness or non-uniqueness of extensions of the pure state induced by a point $\omega \in \beta G$ is related to the dynamical properties of the point. In particular, we will be interested in the orbit $G \cdot \omega = \{g \cdot \omega : g \in G\}$ of the point.

We will use the fact that the map $g \rightarrow g \cdot \omega$ is one-to-one, which is a consequence of a theorem of Veech [16]. To see why this is so, note that this map being one-to-one is equivalent to requiring that the stabilizer subgroup, $G_\omega = \{g \in G : g \cdot \omega = \omega\}$ consist of the identity. When this is the case, we shall say that ω has *trivial stabilizer*.

If every point in βG has trivial stabilizer, then G acts freely (and continuously) on βG . Conversely, if there exists a compact, Hausdorff space X equipped with a continuous G -action that is free, then choosing any $x \in X$, the map $g \rightarrow g \cdot x$ extends to a continuous G -equivariant map $h : \beta G \rightarrow X$ and for any point $\omega \in \beta G$, the stabilizer of ω is contained in the stabilizer of $h(\omega)$ and hence is trivial. Thus, every point in βG has trivial stabilizer if and only if G can act freely and continuously on some compact Hausdorff space. Veech's theorem [16] shows that, in fact, every locally compact group acts freely on a compact, Hausdorff space. Thus, every point in βG has trivial stabilizer.

We begin by examining properties of any state extension of the pure state given by evaluation at a point ω . To this end, given $M_f \in \mathcal{D}$, we let $f \in C(\beta G)$ denote the corresponding continuous function on βG and let $s_\omega : \mathcal{D} \rightarrow \mathbb{C}$ denote the pure state given by evaluation at ω , that is, $s_\omega(f) = f(\omega)$.

Let $s : B(\ell^2(G)) \rightarrow \mathbb{C}$ be any state extension of s_ω and let $\pi : B(\ell^2(G)) \rightarrow B(\mathcal{H}_s)$ and $v_1 \in \mathcal{H}_s$ be the GNS representation of s , so that $s(X) = \langle \pi(X)v_1, v_1 \rangle$. We set $v_g = \pi(U_g)v_1$, $g \in G$, let $\mathcal{L}_s \subseteq \mathcal{H}_s$ denote the closed linear span of the v_g 's and let $\phi_s : B(\ell^2(G)) \rightarrow B(\mathcal{L}_s)$ denote the completely positive map given by $\phi_s(X) = P_{\mathcal{L}_s} \pi(X)|_{\mathcal{L}_s}$.

Note that for any $M_f \in \mathcal{D}$ we have that

$$\langle \pi(M_f)v_g, v_g \rangle = \langle \pi(U_g^{-1}M_fU_g)v_1, v_1 \rangle = f(g \cdot \omega).$$

Hence, these vectors are reducing for $\pi(\mathcal{D})$ and orthonormal. Hence, they are an orthonormal basis for \mathcal{L}_s and the map $W e_g = v_g$ is a Hilbert space isomorphism between \mathcal{L}_s and $\ell^2(G)$. Setting $\psi_s(X) = W^* \phi_s(X) W$ we obtain a completely positive map on $B(\ell^2(G))$.

Also, we have that

$$\langle \psi_s(X)e_h, e_g \rangle = \langle \pi(X)v_h, v_g \rangle = \langle \pi(U_{g^{-1}}XU_h)v_1, v_1 \rangle = s(U_{g^{-1}}XU_h).$$

This shows that the correspondence $s \rightarrow \psi_s$ is one-to-one.

In particular, we have that $\psi_s(U_g) = U_g$ and that for any $M_f \in \mathcal{D}$, $\psi_s(M_f) = \pi_\omega(M_f)$, where $\pi_\omega : \mathcal{D} \rightarrow \mathcal{D}$ is the $*$ -homomorphism given by

$$\pi_\omega(M_f)(g) = f(g \cdot \omega).$$

From these two equations, we see that the restriction of ψ_s to the C^* -algebra generated by \mathcal{D} and the set $\{U_g : g \in G\}$, is a $*$ -homomorphism, which we will denote by $\pi_\omega \times \lambda$, satisfying,

$$\pi_\omega \times \lambda \left(\sum_n M_{f_n} U_{g_n} \right) = \psi_s \left(\sum_n M_{f_n} U_{g_n} \right) = \sum_n \pi_\omega(M_{f_n}) U_{g_n},$$

for every finite, or norm convergent sum. This algebra is, in fact, $*$ -isomorphic to the *reduced crossed-product C^* -algebra*, $\mathcal{D} \times_r G$ as defined in say [14], although we do not need that fact here, we shall adopt that notation for the algebra.

By Choi’s theory of multiplicative domains [7] of completely positive maps, we have that for any $A_1, A_2 \in \mathcal{D} \times_r G$ and $X \in B(\ell^2(G))$, we have that

$$\psi_s(A_1 X A_2) = \pi_\omega \times \lambda(A_1) \psi_s(X) \pi_\omega \times \lambda(A_2).$$

We now characterize the range of this correspondence.

Theorem 1. *Let $\omega \in \beta G$. If $\psi : B(\ell^2(G)) \rightarrow B(\ell^2(G))$ is any completely positive extension of $\pi_\omega \times \lambda$ and we set $s(X) = \langle \psi(X)e_1, e_1 \rangle$, then s is a state extension of s_ω and $\psi = \psi_s$. Consequently, the map, $s \rightarrow \psi_s$ is a one-to-one, onto affine map between the convex set of state extensions of s_ω and the convex set of completely positive extensions of $\pi_\omega \times \lambda$.*

Proof. It is clear that s is a state extension of s_ω . Now given any $X \in B(\ell^2(G))$ and $g, h \in G$, we have that $\langle \psi(X)e_h, e_g \rangle = \langle \psi(U_{g^{-1}}XU_h)e_1, e_1 \rangle = s(U_{g^{-1}}XU_h) = \langle \psi_s(X)e_h, e_g \rangle$ and, thus, $\psi(X) = \psi_s(X)$.

Hence, the map $\psi \rightarrow s$ defines an inverse to the map $s \rightarrow \psi_s$, so that these correspondences are one-to-one and onto. Finally, it is clear that both of these correspondences preserve convex combinations. \square

Corollary 2. *Let $\omega \in \beta G$. Then the following are equivalent:*

- $s_\omega : \mathcal{D} \rightarrow \mathbb{C}$ has a unique extension to a state on $B(\ell^2(G))$,
- $\pi_\omega \times \lambda : \mathcal{D} \times_r G \rightarrow B(\ell^2(G))$ has a unique extension to a completely positive map on $B(\ell^2(G))$.

There is, of course, always one distinguished state extension of s_ω . If we let $E_0 : B(\ell^2(\mathbb{Z})) \rightarrow \mathcal{D}$ be the canonical projection onto the diagonal, then the *regular extension* of s_ω is given by

$$X \rightarrow s_\omega(E_0(X)).$$

Corresponding to this regular extension is a unique completely positive map, $\psi_\omega : B(\ell^2(\mathbb{Z})) \rightarrow B(\ell^2(\mathbb{Z}))$ which we also call the *regular completely positive extension*. We wish to describe this map in some detail.

Every $X \in B(\ell^2(G))$ has a formal series, $X \sim \sum_{g \in G} M_{f_g} U_g$ where $M_{f_g} = E_0(XU_g^{-1})$. To compute the (g_i, g_j) th entry of $\psi_\omega(X)$, we note that

$$\psi_\omega(X)_{g_i, g_j} = \langle \psi_\omega(X)e_{g_j}, e_{g_i} \rangle = \langle \psi_\omega(U_{g_i}^{-1} X U_{g_j})e_1, e_1 \rangle = s_\omega(E_0(U_{g_i}^{-1} X U_{g_j})).$$

But, we have that

$$U_{g_i}^{-1} X U_{g_j} \sim \sum_{g \in G} U_{g_i}^{-1} M_{f_g} U_{g_i} (U_{g_i^{-1} g g_j}),$$

and hence,

$$E_0(U_{g_i}^{-1} X U_{g_j}) = U_{g_i}^{-1} M_{f_{g_i g_j^{-1}}} U_{g_i}.$$

Thus, $\psi_\omega(X)_{g_i, g_j} = f_{g_i g_j^{-1}}(g_i \cdot \omega)$.

These observations lead to the following result.

Theorem 3. *Let $\omega \in \beta G$ and let ψ_ω be the regular completely positive extension corresponding to s_ω . If $X \sim \sum_{g \in G} M_{f_g} U_g$, then $\psi_\omega(X) \sim \sum_{g \in G} \pi_\omega(M_{f_g}) U_g$. Moreover, s_ω has a unique extension to a state on $B(\ell^2(G))$ if and only if ψ_ω is the unique completely positive map on $B(\ell^2(G))$ extending $\pi_\omega \times \lambda : \mathcal{D} \times_r G \rightarrow B(\ell^2(G))$.*

Note that $VN(G)$ is always contained in the range of ψ_ω , since $X \in VN(G)$ implies that there are scalars, $a_g \in \mathbb{C}$ such that $X \sim \sum_g a_g U_g$ and hence, $\psi_\omega(X) \sim \sum_g \pi_\omega(a_g I) U_g = \sum_g a_g U_g$.

It may seem paradoxical to attempt to make progress on the Kadison–Singer problem by replacing statements about uniqueness of the extension of a state to statements about uniqueness of the extension of a completely positive map, but something is gained by making the domain and range of the map the same space. We make this precise in the following results.

Definition 4. Let $\omega \in \beta G$, then we define the uniqueness set for ω to be the set,

$$\mathcal{U}(\omega) = \{X \in B(\ell^2(G)) : s(X) = s_\omega(E_0(X)) \forall s\},$$

where s denotes an arbitrary state extension of s_ω . We also define the uniqueness set for π_ω to be the set,

$$\mathcal{U}(\pi_\omega \times \lambda) = \{X \in B(\ell^2(G)) : \psi(X) = \psi_\omega(X) \forall \psi\},$$

where ψ denotes an arbitrary completely positive extension of $\pi_\omega \times \lambda$.

Proposition 5. *Let $\omega \in \beta G$, then*

$$\mathcal{U}(\pi_\omega \times \lambda) = \bigcap_{g, h \in G} U_g \mathcal{U}(\omega) U_h.$$

Proof. Since every completely positive extension of $\pi_\omega \times \lambda$ is of the form ψ_s for some state extension of s_ω , we have that $X \in \mathcal{U}(\pi_\omega)$, if and only if $\psi_s(X) = \psi_\omega(X)$ for every extension s . But this is equivalent to $s(U_{g^{-1}} X U_h) = \langle \psi_s(X) e_h, e_g \rangle = \langle \psi_\omega(X) e_h, e_g \rangle = s(E_0(U_{g^{-1}} X U_h))$ for every $g, h \in G$, which is equivalent to $U_{g^{-1}} X U_h \in \mathcal{U}(\omega)$ for every $g, h \in G$, and the result follows. \square

Of course, if s_ω has a unique extension, then the sets above are all equal to $B(\ell^2(G))$, but if some s_ω fails to have a unique extension, then it should be easier to show that $\mathcal{U}(\pi_\omega \times \lambda) \neq B(\ell^2(G))$, than to show that $\mathcal{U}(\omega) \neq B(\ell^2(G))$.

We now turn our attention to some results that relate uniqueness of extension to injective envelopes.

Definition 6. We let $\mathcal{A}_\omega = \pi_\omega \times \lambda(\mathcal{D} \times_r G)$ and we denote the von Neumann algebra generated by $\{U_g : g \in G\}$ by $VN(G)$.

Recall that a map ϕ is called a \mathcal{B} -bimodule map for an algebra \mathcal{B} if $\phi(b_1 x b_2) = b_1 \phi(x) b_2$, for every $b_1, b_2 \in \mathcal{B}$.

Proposition 7. Let $\omega \in \beta G$. If s_ω has a unique extension, then every completely positive map, $\phi : B(\ell^2(G)) \rightarrow B(\ell^2(G))$ that fixes \mathcal{A}_ω elementwise, also fixes the range of ψ_ω elementwise and is a $VN(G)$ -bimodule map.

Proof. If ϕ does not fix the range, then $\phi \circ \psi_\omega$ would be another completely positive map extending π_ω . Thus, ϕ must fix the range of ψ_ω elementwise. But $VN(G)$ is a subset of the range of ψ_ω and so must be fixed. By Choi’s [7] theory of multiplicative domains this implies that ϕ is a $VN(G)$ -bimodule map. \square

Remark 8. For many countable groups G , even when $G = \mathbb{Z}$, there exist completely positive maps, $\phi : B(\ell^2(G)) \rightarrow B(\ell^2(G))$ that fix $C^*(G)$ elementwise and hence are $C^*(G)$ -bimodule maps, but whose range does not contain $VN(G)$ and that are not $VN(G)$ -bimodule maps. Examples of such completely positive maps are constructed in [6]. However, any completely positive map $\phi : B(\ell^2(G)) \rightarrow B(\ell^2(G))$ that fixes $\mathcal{D} \times_r G$ elementwise is necessarily the identity map on all of $B(\ell^2(G))$, since $\mathcal{D} \times_r G$ contains the compact operators. Thus, since $C^*(G) \subseteq \mathcal{A}_\omega \subseteq \mathcal{D} \times_r G$, whether or not s_ω has a unique extension should be related to how small $\pi_\omega(\mathcal{D})$ can be made.

This last result can be interpreted in terms of injective envelopes. Recall, that given a unital C^* -subalgebra $\mathcal{A} \subseteq B(\mathcal{H})$ and a completely positive idempotent map $\phi : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ that fixes \mathcal{A} elementwise and is minimal among all such maps, then the range of ϕ , $\mathcal{R}(\phi)$ is completely isometrically isomorphic to the injective envelope of \mathcal{A} , $I(\mathcal{A})$. Such maps are called *minimal \mathcal{A} -projections*. Thus, in particular, the ranges of any two minimal \mathcal{A} -projections are completely isometrically isomorphic via a map that fixes \mathcal{A} elementwise. See [11] for further details. For this reason the collection of subspaces that are ranges of minimal \mathcal{A} -projections are called the *copies of the injective envelope of \mathcal{A}* . If we let $\mathcal{F}(\mathcal{A})$ denote the set of elements of $B(\mathcal{H})$ that are elementwise fixed by every completely positive map that fixes \mathcal{A} , then it is clear that $\mathcal{F}(\mathcal{A})$ is contained in every copy of $I(\mathcal{A})$. In [12] it is shown that $\mathcal{F}(\mathcal{A})$ is, in fact, the intersection of all copies of $I(\mathcal{A})$, but that is not necessary for the following result.

Corollary 9. Let $\omega \in \beta G$. If s_ω has a unique extension, then $VN(G) \subseteq \mathcal{R}(\psi_\omega) \subseteq \mathcal{F}(\mathcal{A}_\omega)$.

Thus, if the Kadison–Singer problem has an affirmative answer, then necessarily $VN(G) \subseteq I(\mathcal{A}_\omega)$, and this inclusion is as a subalgebra for every ω . However, we know that, generally, $VN(G)$ is *not* a subalgebra of $I(C^*(G))$.

Problem 10. Let $D_g \in \pi_\omega(\mathcal{D})$ be chosen such that $Y \sim \sum_{g \in G} D_g U_g$ is a bounded operator. Is Y necessarily in the range of ψ_ω ? Can conditions on the orbit of ω be given that guarantee that this is the case?

2. Dynamics and algebra

In this section we begin to look at how dynamical properties of points and their behavior with respect to a natural semigroup structure on βG can be related to uniqueness of extension. For more on this structure see [9], but we recall a few basic definitions.

Given $\omega \in \beta G$ we let $\rho_\omega : \beta G \rightarrow \beta G$ be the unique continuous function satisfying, $\rho_\omega(g) = g \cdot \omega$, for all $g \in G$ so that $\rho_\omega(\beta G)$ is the closure of the orbit of ω . Since $\rho_\omega \circ h_g(g_1) = \rho_\omega(gg_1) = gg_1 \cdot \omega = h_g \circ \rho_\omega(g_1)$, we have that $\rho_\omega \circ h_g = h_g \circ \rho_\omega$, i.e., the map ρ_ω is equivariant for the action of G on βG . This map also defines a semigroup structure on βG by setting, $\alpha \cdot \omega \equiv \rho_\omega(\alpha)$. We caution that in spite of the notation, this operation is *not* abelian even when the underlying group is abelian (except for finite groups).

However, it is associative and continuous in the left variable and so it gives βG the structure of a *compact right topological semigroup*. We refer the reader to [9] for these and other basic facts about this algebraic structure on βG . One fact that we shall use is that the corona, $G^* = \beta G \setminus G$ is a two-sided ideal in βG .

We now wish to relate dynamical properties of a point ω , to the structure of \mathcal{A}_ω and to the semigroup properties of ω .

Proposition 11. Let $\omega, \alpha \in \beta G$. Then $\rho_\omega \circ \rho_\alpha = \rho_{\alpha \cdot \omega}$, $\pi_\alpha \circ \pi_\omega = \pi_{\alpha \cdot \omega}$ and $\psi_\alpha \circ \psi_\omega = \psi_{\alpha \cdot \omega}$.

Proof. We have that $\rho_\omega \circ \rho_\alpha(g) = \rho_\omega(h_g(\alpha)) = h_g \circ \rho_\omega(\alpha) = h_g(\alpha \cdot \omega) = \rho_{\alpha \cdot \omega}(g)$, and hence, $\rho_\omega \circ \rho_\alpha = \rho_{\alpha \cdot \omega}$.

The second equality comes from the fact that after identifying $\mathcal{D} = C(\beta G)$, then $\pi_\omega(f) = f \circ \rho_\omega$, and hence, $\pi_\alpha \circ \pi_\omega(f) = f \circ \rho_\omega \circ \rho_\alpha = f \circ \rho_{\alpha \cdot \omega} = \pi_{\alpha \cdot \omega}(f)$. The proof of the third identity is similar. \square

Definition 12. A point $\omega \in \beta G$ is called *idempotent* if $\omega \cdot \omega = \omega$.

Non-zero idempotent points are known to exist [9], since the corona is a compact right continuous semigroup. As we shall see below they play a special role in the Kadison–Singer problem.

Proposition 13. Let $\omega \in \beta G$. Then the following are equivalent:

- ω is idempotent,
- $\rho_\omega \circ \rho_\omega = \rho_\omega$,
- $\pi_\omega : \mathcal{D} \rightarrow \mathcal{D}$ is idempotent,
- $\pi_\omega \times \lambda : \mathcal{D} \times_r G \rightarrow \mathcal{D} \times_r G$ is idempotent,
- ψ_ω is an idempotent completely positive map.

Theorem 14. *Let $\omega \in \beta G$ be an idempotent. If s_ω has a unique state extension, then $\mathcal{R}(\psi_\omega)$ is completely isometrically isomorphic to the injective envelope of \mathcal{A}_ω . Moreover, $\mathcal{R}(\psi_\omega)$ is the unique copy of the injective envelope inside $B(\ell^2(G))$ and ψ_ω is the unique projection onto it. Thus, the identity map on \mathcal{A}_ω extends uniquely to an embedding of $I(\mathcal{A}_\omega)$ into $B(\ell^2(G))$.*

Proof. We have seen earlier that for any $\omega \in \beta G$ the range of ψ_ω is contained in any copy of the injective envelope. Thus, when ω is idempotent, since ψ_ω is already a completely positive projection, its range must be a minimal completely positive projection and its range must be the unique copy of the injective envelope. \square

Conjecture 15. *We conjecture that if ω is idempotent, then s_ω possesses non-unique extensions. That is, the Kadison–Singer conjecture is false and idempotent points provide counterexamples. In fact, we believe that idempotent points fail to satisfy the condition of Corollary 9, $VN(G) \subseteq \mathcal{F}(\mathcal{A}_\omega)$.*

The following result lends some credence to the above conjecture, at least for *minimal idempotents*. An idempotent ω is minimal if it is minimal in any of several different orders ([9, Definition 1.37] and [4]) and this is shown to be equivalent to the left ideal generated by ω , i.e., $\{\alpha \cdot \omega : \alpha \in \beta G\}$ being a minimal left ideal [9, Theorem 2.9]. Moreover, by [9, Theorem 19.23c], a minimal idempotent is *uniformly recurrent* (see also [5] where this is proved for the case of \mathbb{N}). Recall that ω uniformly recurrent means [9, Definition 19.1] that for every neighborhood U of ω we have that the set $S = \{g \in G : g \cdot \omega \in U\}$ is *syndetic*, i.e., there exists a finite set $g_1, \dots, g_m \in G$ such that $g_1 \cdot S \cup g_2 \cdot S \cup \dots \cup g_m \cdot S = G$.

Theorem 16. *Let G be a countable, abelian, discrete group and let $X \in VN(G)$. If there exists a minimal idempotent ω such that $X \in \mathcal{U}(\omega)$, then $X \in \mathcal{U}(\alpha)$ for every $\alpha \in \beta G$.*

Thus, if there exists *any* state s_α which fails to have unique extension for some $X \in VN(G)$, then every minimal idempotent fails to have unique extension for that X .

Proof. Since $\mathcal{U}(\omega)$ is an operator system, $X \in \mathcal{U}(\omega)$ if and only if $\text{Re}(X) = (X + X^*)/2$ and $\text{Im}(X) = (X - X^*)/(2i)$ are both in $\mathcal{U}(\omega)$. Thus, it will be sufficient to assume that $X = X^*$. Moreover, since $I \in \mathcal{U}(\omega)$, it is sufficient to assume also that $E(X) = 0$. Now by Anderson’s paving results [1] and [2] (see also [13, Theorem 2.7]), $X \in \mathcal{U}(\omega)$ if and only if for each $\epsilon > 0$, there exists A in the ultrafilter corresponding to ω such that $-\epsilon P_A \leq P_A X P_A \leq +\epsilon P_A$ where $P_A \in \mathcal{D}$ is the diagonal projection $P_A = \text{diag}(a_g)$ with

$$a_g = \begin{cases} 1, & g \in A, \\ 0, & g \notin A. \end{cases}$$

Let $U \subseteq \beta G$ be the clopen neighborhood of ω satisfying $A = U \cap G$, so that under the identification of \mathcal{D} with $C(\beta G)$ the projection P_A is identified with the characteristic function of U , χ_U . We have that $S = \{g \in G : g \cdot \omega \in U\}$ is syndetic, so let g_1, \dots, g_m be as in the definition of syndetic.

Note that $\psi_\omega(P_A) = \pi_\omega(P_A) = \pi_\omega(\chi_U) = \text{diag}(b_g)$ where

$$b_g = \begin{cases} 1, & g \cdot \omega \in U, \\ 0, & g \cdot \omega \notin U, \end{cases}$$

that is $\psi_\omega(P_A) = P_S$. Also, since $X \in VN(G)$ we have that $\psi_\omega(P_A X P_A) = \pi_\omega(P_A)\psi_\omega(X)\pi_\omega(P_A) = P_S X P_S$. Thus, applying ψ_ω to the above inequality, we have that $-\epsilon P_S \leq P_S X P_S \leq +\epsilon P_S$.

Next notice that conjugating the first inequality by $\lambda(g)$, we have that $\lambda(g)P_A\lambda(g^{-1}) = P_{gA}$ and

$$\lambda(g)P_A X P_A \lambda(g^{-1}) = \lambda(g)P_A \lambda(g^{-1})\lambda(g)X\lambda(g^{-1})\lambda(g)P_A \lambda(g^{-1}) = P_{gA} X P_{gA}.$$

Thus, $-\epsilon P_{gA} \leq P_{gA} X P_{gA} \leq +\epsilon P_{gA}$. Applying the map ψ_ω to this inequality and observing that $\psi_\omega(P_{gA}) = P_{gS}$, we have that

$$-\epsilon P_{g_i S} \leq P_{g_i S} X P_{g_i S} \leq +\epsilon P_{g_i S},$$

for $i = 1, \dots, m$.

Since, $G = g_1 S \cup \dots \cup g_m S$, and $\epsilon > 0$ was arbitrary, this shows that X is pavalable in Anderson’s sense and so, applying Anderson’s [3] paving results, we have that every pure state on \mathcal{D} has a unique extension to X . \square

Conjecture 17. *The same result holds for non-abelian groups.*

Conjecture 18. *Assuming that a minimal idempotent has a unique extension, should imply that every point has a unique extension. That is, if we fix any minimal idempotent ω , then the Kadison–Singer conjecture is true if and only if s_ω has a unique state extension.*

If ω is any point in the corona, then ψ_ω annihilates any operator all of whose “diagonals” are c_0 . Thus, when ω is idempotent, $\mathcal{R}(\psi_\omega)$ can contain no operators with any c_0 “diagonals” and so in particular, no compact operator. In this sense, $\pi_\omega(\mathcal{D})$ is a “small” subset of $\ell^\infty(G)$, which is one of the reasons that we conjecture that these points are potential counterexamples. Thus, in this case $\mathcal{R}(\psi_\omega) \cap \mathcal{K} = (0)$, where \mathcal{K} denotes the compacts. Since any copy of the injective envelope of \mathcal{A}_ω is contained in $\mathcal{R}(\psi_\omega)$ it also follows that for ω idempotent, $I(\mathcal{A}_\omega) \cap \mathcal{K} = (0)$ and hence cannot be all of $B(\ell^2(G))$.

We now consider the opposite case, points for which $\pi_\omega(\mathcal{D})$ is a “large” subset of $\ell^\infty(\mathcal{D})$. We believe that these points are good candidates for having unique extensions.

The following result characterizes the points for which $\mathcal{K} \cap \mathcal{R}(\psi_\omega) \neq (0)$, at least for many groups and, in particular, for these points, we shall see that the injective envelope of \mathcal{A}_ω is all of $B(\ell^2(G))$.

Recall that given a dynamical system, a point ω is *non-recurrent* if there is an open neighborhood U of the point such that $g \cdot \omega \notin U$ for all $g \neq 1$. Also, given a semigroup, an element ω is *right cancellative*, if $\alpha \cdot \omega = \beta \cdot \omega$ implies that $\alpha = \beta$. Portions of the following result can also be deduced from [9, Theorem 8.11].

Theorem 19. *Let $\omega \in \beta G$ and assume that G is torsion free. Then the following are equivalent:*

- (i) ω is non-recurrent,
- (ii) $\mathcal{K} \cap \mathcal{A}_\omega \neq (0)$,
- (iii) $\mathcal{K} \subseteq \mathcal{A}_\omega$,
- (iv) $\pi_\omega : \mathcal{D} \rightarrow \mathcal{D}$ is onto,

- (v) $\mathcal{A}_\omega = \mathcal{D} \times_r G$,
- (vi) ρ_ω is one-to-one,
- (vii) ω is right cancellative.

Proof. (i) implies (iii). If ω is non-recurrent there is a clopen neighborhood U of ω containing no other $g \cdot \omega$, $g \neq 1$. Hence, $\pi_\omega(\chi_U)$ is the rank one projection onto the span of e_1 .

Since $U_g \in \mathcal{A}_\omega$, we have the matrix units, $E_{g_i, g_j} = U_{g_i} \pi_\omega(\chi_U) U_{g_j}^{-1} = \pi_\omega(U_{g_i} \chi_U U_{g_j}^{-1}) \in \mathcal{A}_\omega$ for all $g_i, g_j \in G$ and hence $\mathcal{K} \subseteq \mathcal{A}_\omega$

(ii) implies (i). Suppose \mathcal{A}_ω contains a nonzero compact $K = \sum \pi_\omega(M_{f_g}) U_g$ (formal sum). Then all $\pi_\omega(M_{f_g})$ are compact. So there is a non-zero $\pi_\omega(M_f) \in \mathcal{K} \cap \mathcal{A}_\omega$. Applying a sequence p_n of polynomials to this, tending to the characteristic functions of some eigenspace you find a finite rank “diagonal” projection $Q = \chi_S$ in \mathcal{A}_ω with S a finite set. Choose such a Q with $S = \{g_1, \dots, g_n\}$ of minimal non-zero cardinality. Conjugating Q by U_g , we obtain another such projection in \mathcal{A}_ω corresponding to the set $g \cdot S$ and so we may assume that $1 \in S$. Also, the product of two such projections is also in \mathcal{A}_ω and is the finite rank projection corresponding to $S \cap gS$. But since S is of minimal non-zero cardinality, either $S \cap g \cdot S$ is empty or $S \cap g \cdot S = S$. Hence, for each $g \in S$, $g^{-1} \cdot S = S$, and so S is a finite subgroup, but then any non-zero element of S would be a torsion element. Thus, $S = \{1\}$, and we have that $E_{1,1} \in \mathcal{A}_\omega$.

Hence, there exists $f \in C(\beta G)$, such that $E_{1,1} = \pi_\omega(M_f)$. Now $f(\omega) = 1$ and $f(g \cdot \omega) = 0$ for all $g \neq 1$. Thus, $U = \{\alpha : |f(\alpha)| > 1/2\}$ is an open set containing ω but no other $g \cdot \omega$ and so ω is non-recurrent.

Clearly, (iii) implies (ii) and so (i), (ii), and (iii) are equivalent.

Also, it is clear that (v) implies (iv) implies (ii). Moreover, since π_ω is given by composition with ρ_ω , we have that (iv) and (vi) are equivalent. Since, $\alpha \cdot \omega = \rho_\omega(\alpha)$, it is also clear that ω is right cancellative is equivalent to ρ_ω being one-to-one. Thus, (vi) and (vii) are equivalent.

(i) implies (iv). Since ω is non-recurrent there is a neighborhood U_1 of ω that contains no other point on its orbit. Hence, $g \cdot U_1$ is a neighborhood of $g \cdot \omega$ containing no other point on the orbit. Thus, non-recurrent is equivalent to the set $\{g \cdot \omega\}$ being discrete. Now using the fact that βG is a compact, Hausdorff space and hence normal, and that the set of points is countable, one can choose neighborhoods, V_g of $g \cdot \omega$ such that $V_g \cap V_h$ is empty for $g \neq h$, i.e., the points are what is called *strongly discrete*. To recall the construction, first enumerate the points, $\omega_i = g_i \cdot \omega$, then using normality, choose for each i disjoint open sets U_i, V_i such that $\omega_i \in U_i$ and the closure of $\{\omega_j : j \neq i\}$ is contained in V_i . Then set $W_1 = U_1, W_i = U_i \cap V_1 \cap \dots \cap V_{i-1}, i \geq 2$.

Thus, using the fact that βG is extremally disconnected, we may choose disjoint clopen sets $U_g, g \in G$, with $g \cdot \omega \in U_g$. Thus, if we let $A_g = U_g \cap G$, then $\pi_\omega(\chi_{A_g})$ is the rank one projection onto the span of e_g . Given any $f_1 \in \ell^\infty(G)$ define $f_2 \in \ell^\infty(G)$, by $f_2(h) = f_1(g)$ if and only if $h \in A_g$, and when $h \notin \bigcup_{g \in G} A_g$ set $f_2(h) = \lambda$ where $\lambda \in \mathbb{C}$ is any number. Then $f_2 \in \ell^\infty(G)$ and if \hat{f}_2 denotes the continuous extension of f_2 to βG , then $\hat{f}_2(g \cdot \omega) = f_1(g)$. Thus, $\pi_\omega(f_2) = f_1$ and so (iv) holds.

(iv) implies (v). Given any finite sum, $B = \sum_{j=1}^n M_{f_j} U_{g_j}$, we may pick continuous functions h_j , such that $\pi_\omega(\sum_{j=1}^n M_{h_j} U_{g_j}) = B$. Thus, the range of the *-homomorphism, $\pi_\omega : \mathcal{D} \times_r G \rightarrow \mathcal{D} \times_r G$ is dense in $\mathcal{D} \times_r G$ and hence is onto. \square

Corollary 20. Let G be a countable torsion free group. If $\omega_1, \omega_2 \in \beta G$ are non-recurrent, then $\omega_1 \cdot \omega_2$ is non-recurrent.

Corollary 21. *Let G be a countable torsion free group and let $\omega \in \beta G$. Then $I(\mathcal{A}_\omega) = B(\ell^2(G))$ if and only if ω is non-recurrent.*

Proof. If ω is non-recurrent, then \mathcal{A}_ω contains all the compacts and hence any completely positive map that fixes \mathcal{A}_ω fixes all the compacts and hence is the identity map. To see this note that if it fixes every diagonal matrix unit, then it is a Schur product map, which fixes every matrix unit and so is the identity map.

Conversely, if ω is not non-recurrent, then \mathcal{A}_ω contains no non-zero compacts and hence the quotient map to the Calkin algebra is an isometry on \mathcal{A}_ω . But the injective envelope is an essential extension of \mathcal{A}_ω and, hence, any embedding of $I(\mathcal{A}_\omega)$ into $B(\ell^2(G))$, composed with the quotient map must also be an isometry on $I(\mathcal{A}_\omega)$ and hence, $I(\mathcal{A}_\omega) \neq B(\ell^2(G))$. \square

The above result shows that non-recurrent points all satisfy the condition of Corollary 9 that is necessary for s_ω to have a unique extension. This lends credence to the following conjectures.

Conjecture 22. *We conjecture that if $\omega \in \beta G$ is non-recurrent, then the state s_ω extends uniquely to $B(\ell^2(G))$.*

Conjecture 23. *Let G be a countable torsion free group. If $\omega \in \beta G$ is non-recurrent, then we conjecture that $\mathcal{R}(\psi_\omega) = B(\ell^2(G))$.*

Assuming the last conjecture, one can prove that if ω is non-recurrent and s_α has a unique extension, then $s_{\alpha \cdot \omega}$ has a unique extension. So these conjectures might shed some light on the algebraic properties of points with unique extensions.

3. Dynamical properties of ultrafilters

We now examine the dynamical properties of various classes of ultrafilters that have been studied in relation to the Kadison–Singer problem.

An ultrafilter β on a countable set N is called *selective* [9] if given any partition of $N = \bigcup P_i$ into subsets, either P_i is in the ultrafilter for some i or there exists a set B in the ultrafilter, such that for every i , $B \cap P_i$ has cardinality at most 1. An ultrafilter is called *rare* [8] if for each partition $N = \bigcup P_i$ into finite subsets, there exists a set B in the ultrafilter such that for every i , $B \cap P_i$ has cardinality at most 1. An ultrafilter is called δ -stable [8] if for each partition $N = \bigcup P_i$, into sets of arbitrary sizes, then either one of the P_i 's is in the ultrafilter or there exists a set B in the ultrafilter such that for every i , $B \cap P_i$ is finite. Note that an ultrafilter is selective if and only if it is rare and δ -stable.

Finally, a point in a topological space is called a *P-point* [17] if every G_δ that contains the point contains an open neighborhood of the point.

Note that if X is a compact, Hausdorff space, $x \in X$ is a P-point and $f \in C(X)$, then $\{y \in X: f(y) = f(x)\}$ is a G_δ set and hence contains an open neighborhood of x ! Thus, this definition of P-point is antithetical to the concept of p-point that appears in function theory. Choquet [8, Proposition 1] proves that for any discrete set N , ω is a δ -stable if and only if ω is a P-point in the corona $\beta N \setminus N$. For this reason the term δ -stable has fallen into disuse and such ultrafilters are, generally, called P-points without reference to the topological space.

Finally, we recall a stronger notion than non-recurrent. A point ω in a dynamical system is *wandering*, if there exists a neighborhood, U of ω such that $g \cdot U \cap h \cdot U$ is empty for any $g \neq h$ in the group.

In an earlier version of this paper, we proved that rare ultrafilters are wandering points in \mathbb{N} and \mathbb{Z} and conjectured that they were wandering in every group. Since then, Ken Davidson has verified this conjecture. We present our proof of the case of \mathbb{N} and Davidson’s proof for general groups below.

Proposition 24. *Let $\omega \in \beta\mathbb{N}$ be rare, then ω is wandering in the corona \mathbb{N}^* .*

Proof. Let ω be a rare ultrafilter on \mathbb{N} . Consider the following partition of \mathbb{N} into finite sets:

$$\mathbb{N} = \{1\} \cup \{2\} \cup \{3, 4\} \cup \{5, 6\} \cup \{7, 8, 9\} \cup \{10, 11, 12\} \cup \{13, 14, 15, 16\} \dots$$

Let

$$\sigma = \{1\} \cup \{3, 4\} \cup \{7, 8, 9\} \cup \{13, 14, 15, 16\} \dots$$

(half the sets in the partition) and let σ_j denote the j th subset in this list.

By general properties of ultrafilters, either σ or its complement is in ω . We assume, without loss of generality, that $\sigma \in \omega$.

Since ω is rare, there exists $B \in \omega$ such that $B \cap \sigma_j$ has at most one element for all j .

Define $\gamma = \sigma \cap B \in \omega$, then $|\gamma \cap (\gamma + n)| \leq 2n \quad \forall n \in \mathbb{N}$ which can be seen by looking at how the σ_j ’s in σ are spread out.

Thus, if U denotes the clopen set in $\beta\mathbb{N}$ corresponding to γ , then $(n + U) \cap (m + U) \subset \mathbb{N}$ and is finite. Thus, the relatively open set $U \cap \mathbb{N}^*$ is wandering in \mathbb{N}^* . \square

We now present Davidson’s proof for general groups.

Theorem 25. *Let G be a countable, discrete group and let $\omega \in \beta G$ be a rare ultrafilter; then ω is wandering in the corona G^* , and, hence $\mathcal{A}_\omega = \mathcal{D} \times_r G$.*

Proof. Let e denote the identity of G and choose finite subsets, $\{e\} = G_0 \subseteq G_1 \subseteq \dots$ with $G_n = G_n^{-1}$ and $G = \bigcup_{n=0}^\infty G_n$. Set $P_n = G_n \cdots G_1 \cdot G_0$ and let $A_n = P_n \setminus P_{n-1}, n \geq 1$, and $A_0 = \{e\}$. We have that each A_n is finite, $A_n \cap A_m$ is empty for $n \neq m$, and $G = \bigcup_{n=0}^\infty A_n$.

If $g \in G_k$ and $n > k$, we claim that

$$gA_n \subseteq A_{n-1} \cup A_n \cup A_{n+1} = P_{n+1}/P_{n-2}.$$

To see this note that $gP_n \subseteq P_{n+1}$. If $p \in P_n$ and $gp \in P_{n-2}$, then $p = g^{-1}(gp) \in P_{n-1}$ and, hence $p \notin A_n$. Thus, if $p \in A_n$, then $gp \notin P_{n-2}$, and the claim follows.

Now since ω is rare, we may pick a set U in the ultrafilter ω , such that $U \cap A_n$ has at most one element for all n . Let $E = \bigcup A_n, n$ even and $O = \bigcup A_n, n$ odd. Then $E \cap O$ is empty and $E \cup O = G$. Hence either E is in the ultrafilter, or O is in the ultrafilter.

If E is in the ultrafilter ω , let $V = U \cap E$ which is in the ultrafilter ω . If $g \in G_k$, $g \neq e$ we claim that the cardinality of $gV \cap V$ is at most $k/2$. Assuming the claim, we see that the clopen neighborhood \mathcal{V} of ω corresponding to V , has the property that,

$$\mathcal{W} = \mathcal{V} \cap \mathbb{G}^*$$

is an open neighborhood of ω with $g\mathcal{W} \cap \mathcal{W}$ empty for all $g \neq e$.

To see the claim, write $V = \{a_{2m} : a_{2m} \in A_{2m}, m \geq 0\}$, so that $x \in gV \cap V$ if and only if $x = ga_{2m} = a_{2j}$. But if $2m > k$, then

$$ga_{2m} \in A_{2m-1} \cup A_{2m} \cup A_{2m+1},$$

and so $j = m$, forcing $g = e$. Thus, $ga_{2m} = a_{2j}$ has no solutions for $2m > k$ when $g \neq e$.

The proof for the case that O is in the ultrafilter ω , is identical. \square

Corollary 26. *Let G be a countable discrete group. Assuming the continuum hypothesis, G^* contains a dense set of wandering points.*

Proof. By [8], if we assume the continuum hypothesis, then the rare ultrafilters are dense in G^* . \square

We do not know if it is necessary to assume the continuum hypothesis to conclude that the wandering points are dense in G^* .

Problem 27. Let ω be an ultrafilter on \mathbb{N} , then for every countable, discrete group G and every labeling, $G = \{g_n : n \in \mathbb{N}\}$, we have that ω determines a point in G^* . The above result shows that when ω is rare then the point obtained in this fashion is wandering for the action of G on G^* . Conversely, if ω is an ultrafilter on \mathbb{N} with this property, then must ω be rare?

By the result of Reid [15], we know that the state given by any rare ultrafilter has a unique extension to $B(\ell^2(G))$. This suggests the following conjecture:

Conjecture 28. *If $\omega \in G^*$ is wandering for the G action on G^* , then the state corresponding to evaluation at ω extends uniquely to a state on $B(\ell^2(G))$.*

Proposition 29. *Let G be a countable, discrete group. Then δ -stable ultrafilters are non-recurrent in βG .*

Proof. Let ω be a δ -stable ultrafilter, we have that $\{g \cdot \omega : g \in G\}$ is a distinct set of points.

For each $g \in G$ that is not equal to the identity, the complement of $\{g \cdot \omega\}$ is an open neighborhood of ω . The intersection of these sets is a G_δ containing ω and hence, applying the equivalence of δ -stable to P-point, contains an open neighborhood of ω . No point on the orbit of ω returns to this open neighborhood. \square

Combining the above result with Theorem 19, we see that δ -stable ultrafilters satisfy the condition of Corollary 9 that is necessary for s_ω to have a unique extension.

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