Asymptotic Approximations for Multivariate Integrals with an Application to Multinormal Probabilities*

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The efficient computation of multinormal integrals is an important problem of multivariate statistics. In this paper it is shown, that using methods of asymptotic analysis, asymptotic expansions for multinormal integrals can be obtained. These results are an extension of a result obtained by Ruben (1964, J. Res. Nat. Bur. Standards B 68, No. 1 3-11). While the approximations of Ruben are valid only for domains bounded by hyperplanes, the results given here also apply to domains with nonlinear boundaries.


1. INTRODUCTION

In many practical applications of probability theory it is necessary to integrate $n$-dimensional probability densities over parts of the $n$-dimensional real space. The available numerical procedures, however, become extremely time-consuming if the dimension $n$ increases above moderate

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values (numerical integration) or if the value of the probability integral is close to zero or close to one (Monte Carlo methods). For the important case of the multinormal distribution, a survey of the available results can be found in Johnson and Kotz [8]. One such result is due to Ruben [9] who gave an asymptotic approximation for the multinormal integral over domains bounded by hyperplanes, the distance of the domain from the origin approaching infinity.

In general, an alternative way to deal with multidimensional integrals is provided by the asymptotic analysis studying the behaviour of integrals depending on parameters which approach certain limits. In a number of important examples this procedure allows the derivation of simple approximations. A review of those methods is presented in Bleistein and Handelsmann [1] and Fedoryuk [4]. The methods described there are restricted to domains whose boundary is smooth at the critical point while Ruben's solution covers only domains bounded by finitely many hyperplanes and to integrands being multinormal density functions.

In the present paper which summarizes and generalizes previous results of the authors (see Breitung [2, 3] and Hohenbichler [7]) a general theorem is derived about the asymptotic approximation of multidimensional integrals over domains which are given by intersections of smooth sets. This theorem generalizes Ruben's result to nonlinear boundaries and arbitrary density functions. It requires the determination of certain critical points on the boundary of the domain of integration and the computation of the first and second derivatives of the density function and the functions defining the boundary in those points.

2. Notation

In the following $\mathbb{R}^n$ denotes the $n$-dimensional real space and $\mathbb{R}^n_+$ the subset of $\mathbb{R}^n$ which consists of the vectors with non-negative components. Only measurable sets will be considered. The closure of a set $F \subset \mathbb{R}^n$ is denoted by $\overline{F}$. Column vectors in $\mathbb{R}^n$ are denoted by $x, y, \ldots$ and their transposes by $x', y', \ldots$. The origin in $\mathbb{R}^n$, i.e., the zero vector, is written as $0$ and the unit vector in the direction of the $k$th axis as $e_k$. The euclidian norm of a vector $x$ is denoted by $|x|$.

For a function $f: \mathbb{R}^n \to \mathbb{R}$ which is twice differentiable, the first and second derivatives with respect to $x_i$ and $x_j$ ($i, j = 1, ..., n$) at $x$ are denoted by $f''^{ij}(x)$ and $f''^{ij}(x)$. The gradient of $f(x)$ at $x$ is written as $\nabla f(x)$.

Let $h: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be two functions such that there exists a $x_0 > 0$ with $g(x) \neq 0$ for all $x > x_0$. Then, the relation

$$\lim_{x \to \infty} \frac{h(x)}{g(x)} = 1$$
is abbreviated as usual in asymptotic analysis by:

\[ g(x) \sim h(x) \quad (x \to \infty). \]

3. A Theorem about the Asymptotic Approximation of Multidimensional Integrals

In the books of Fedoryuk [4] and Bleistein and Handelsman [1] several results about the asymptotic behaviour of multidimensional integrals over domains bounded by a hypersurface are given. Here, we will derive a generalization of those theorems. For this purpose first some lemmas will be proved. In Lemma 1 it is shown that under certain conditions the asymptotic behaviour of a multidimensional integral depending on a parameter is completely determined by the values of the integral near the maximum of the function in the integration domain. Lemmas 2 and 3 give asymptotic approximations for integrals of a special form. Lemma 4 comprises the results of Lemmas 2 and 3. The general case covered by Theorem 1 is, by a suitable coordinate transformation, reduced to the case considered in Lemma 4.

**Lemma 1.** Let \( F \subseteq \mathbb{R}^n \) be a measurable set, \( y_0 \in F \), \( f: F \to \mathbb{R} \) and \( h: F \to \mathbb{R} \) measurable functions, \( f \) and \( h \) being continuous at \( y_0 \) with \( h(y_0) \neq 0 \). Let, further, the following conditions (A), (B), and (C) hold:

(A) For each neighbourhood \( V \) of \( y_0 \):

\[ f(y_0) > \sup\{ f(y); y \in F \setminus V \}. \]

(B) For each neighbourhood \( V \) of \( y_0 \):

\[ \int_{F \cap V} dy > 0. \]

(C) \[ \int_F |h(y)| \exp(f(y)) \, dy < \infty. \]

Then, for each \( \lambda \geq 1 \ (\lambda \in \mathbb{R}) \),

\[ \int_F |h(y)| \exp(\lambda f(y)) \, dy < \infty \]

and, for each neighbourhood \( V \) of \( y_0 \), the following asymptotic relation is valid:

\[ \int_F h(y) \exp(\lambda f(y)) \, dy - \int_{F \cap V} h(y) \exp(\lambda f(y)) \, dy \quad (\lambda \to \infty). \]
Remark 1. The condition (A) of Lemma 1 can be derived from condition (A*):

(A*) \( f \) is defined on the closure \( F^a \) of \( F \) and is continuous on \( F^a \), has an unique maximum at \( y_0 \) and for some \( \varepsilon > 0 \) the set \( \{ y \in F^a; f(y) \geq f(y_0) - \varepsilon \} \) is compact.

Proof of Lemma 1. For \( \lambda \geq 1 \) and \( f(y) \geq f(y_0) - \delta \), we have

\[
\exp((\lambda - 1) f(y)) \geq \exp((\lambda - 1)(f(y_0) - \delta))
\]

or

\[
\exp(\lambda f(y)) \geq \exp((\lambda - 1)(f(y_0) - \delta)) \exp(f(y)).
\] (1)

In the same way we have for \( \lambda \geq 1 \) and \( f(y) \leq f(y_0) - \varepsilon \):

\[
\exp(\lambda f(y)) \leq \exp((\lambda - 1)(f(y_0) - \varepsilon)) \exp(f(y)).
\] (2)

In the sequel the following notations are used:

\[
G(\lambda, V) := \int_{F \cap V} h(y) \exp(\lambda f(y)) \, dy
\]

\[
G_a(\lambda, V) := \int_{F \cap V} |h(y)| \exp(\lambda f(y)) \, dy
\]

\[
\bar{G}(\lambda, V) := \int_{F \setminus V} h(y) \exp(\lambda f(y)) \, dy
\]

\[
\bar{G}_a(\lambda, V) := \int_{F \setminus V} |h(y)| \exp(\lambda f(y)) \, dy.
\]

Since \( f(y) \leq f(y_0) \) for all \( y \in F \), (2) implies (with \( \varepsilon = 0 \)) \( G_a(\lambda, W) \leq \exp((\lambda - 1) f(y_0)) G_a(1, W) < \infty \) for each \( \lambda \geq 1 \) and each \( W \subseteq \mathbb{R}^n \). Without loss of generality we assume now that \( h_0 := h(y_0) > 0 \).

At first it is shown

\[
(*) \quad \text{If } V \text{ is a neighbourhood of } y_0 \text{ with } h(y) > 0 \text{ for all } y \in V, \text{ then for each neighbourhood } W \text{ of } y_0 \text{ with } V \subset W:
\]

\[
G(\lambda, V) \sim G(\lambda, W) \quad (\lambda \to \infty).
\]

For a proof of (*) we notice that \( \varepsilon := f(y_0) - \sup \{ f(y); y \in F \setminus V \} > 0 \). Due to the continuity of \( f \) at \( y_0 \) there exists a neighbourhood \( V_1 \subset V \) of \( y_0 \) such, that for all \( y \in V_1 \):

\[
f(y) \geq f(y_0) - \varepsilon / 2.
\]
Since \( h(y) > 0 \) for \( y \in V_1 \), this gives with (1):

\[
|G(\lambda, V_1)| = G_d(\lambda, V_1) \geq \exp((\lambda - 1)(f(y_0) - \epsilon/2)) G_d(1, V_1). \tag{3}
\]

(Note that \( G_d(1, V_1) > 0 \) due to condition (B).) Further, due to (2),

\[
\tilde{G}_d(\lambda, V) \leq \exp((\lambda - 1)(f(y_0) - \epsilon)) \tilde{G}_d(1, V) \tag{4}
\]

(with \( \tilde{G}_d(1, V) < \infty \) due to condition (C)).

Combining the last two inequalities (3) and (4) yields:

\[
\lim_{\lambda \to \infty} \frac{\tilde{G}_d(\lambda, V)}{G(\lambda, V_1)} = 0. \tag{5}
\]

Since \( h(y) > 0 \) for all \( y \in V \),

\[
0 \leq G(\lambda, V_1) \leq G(\lambda, V). \tag{6}
\]

Further,

\[
0 \leq G_d(\lambda, W \setminus V) \leq G_d(\lambda, \mathbb{R}^n \setminus V) = \tilde{G}_d(\lambda, V). \tag{7}
\]

The relations (5), (6), and (7) imply together:

\[
\lim_{\lambda \to \infty} \frac{G_d(\lambda, W \setminus V)}{G(\lambda, V)} = 0.
\]

This proves (*).

Let now \( V \) be an arbitrary neighbourhood of \( y_0 \). Then there is a neighbourhood \( V_1 \subset V \) of \( y_0 \) with \( h(y) > 0 \) for all \( y \in V_1 \) due to the assumption. From (*) it follows then:

\[
G(\lambda, V_1) \sim G(\lambda, V) \quad (\lambda \to \infty)
\]

and

\[
G(\lambda, V_1) \sim G(\lambda, \mathbb{R}^n) \quad (\lambda \to \infty).
\]

Combining these asymptotic relations finally proves Lemma 1.

**Lemma 2.** Let \( D \subset \mathbb{R}^n \) be a bounded closed set containing the origin in its interior. If

(a) \( f: D \to \mathbb{R} \) and \( h: D \to \mathbb{R} \) are continuous functions and \( h(0) \neq 0 \).

(b) \( f(x) < f(0) \) for all \( x \in D \cap \mathbb{R}^n_+ \) with \( x \neq 0 \).
(c) There is an open neighbourhood $V \subset D$ of the origin such, that $f(x)$ is continuously differentiable in $V$.

(d) $f'(0) < 0$ for $i = 1, \ldots, n$.

Then the following asymptotic relation is valid (with $\lambda \geq 1$ being a real parameter):

$$\int_{D \cap \mathbb{R}^n_+} h(x) \exp(\lambda f(x)) \, dx \sim h(0) \exp(\lambda f(0)) \left( \prod_{i=1}^{n} |f'(0)|^{-1} \right) \lambda^{-n} \quad (\lambda \to \infty).$$

(8)

Proof of Lemma 2. Let

$$I(\lambda) := \int_{D \cap \mathbb{R}^n_+} h(x) \exp(\lambda f(x)) \, dx$$

$$\tilde{I}(\lambda) := h(0) \exp(\lambda f(0)) \left( \prod_{i=1}^{n} |f'(0)|^{-1} \right) \lambda^{-n}.$$

Due to condition (c), $f(x)$ is continuously differentiable in a convex open neighbourhood $V$ of the origin and, by a Taylor expansion, we obtain for all $x$ in this neighbourhood:

$$f(x) = f(0) + (\nabla f(\theta(x) x))' x \quad \text{with } 0 \leq \theta(x) \leq 1.$$  

(9)

Due to the conditions (a)–(c) and Eq. (9) there exists an $\varepsilon_0 > 0$ with $0 < \varepsilon_0 < 1$ such that for all $\varepsilon$ with $0 < \varepsilon < \varepsilon_0$ some $\delta(\varepsilon)$ exists with:

(α) $V_\varepsilon \subset V$, where $V_\varepsilon := \{ x \in \mathbb{R}^n : |x_i| < \delta(\varepsilon) \text{ for } 1 \leq i \leq n \}$

(β) $f(x) \leq f(0) + (1 - \varepsilon)(\nabla f(0))' x$ for all $x \in V_\varepsilon \cap \mathbb{R}^n_+$

(γ) $h(x) \leq h(0) + \varepsilon$ for all $x \in V_\varepsilon$.

Let now $F := D \cap \mathbb{R}^n_+$ and

$$I_\varepsilon(\lambda) := \int_{F \cap V_\varepsilon} h(x) \exp(\lambda f(x)) \, dx.$$  

Due to (α)–(γ) we obtain

$$I_\varepsilon(\lambda) \leq \int_{F \cap V_\varepsilon} (h(0) + \varepsilon) \exp(\lambda f(0) + (1 - \varepsilon)(\nabla f(0))' x) \, dx$$

$$= (h(0) + \varepsilon) \exp(\lambda f(0)) \prod_{i=1}^{n} \int_{0}^{\delta(\varepsilon)} \exp(\lambda (1 - \varepsilon) f'(0) x_i) \, dx_i.$$
By applying Lemma 1,
\[ \int_0^{\delta(x)} \exp(\lambda(1 - \varepsilon)f'(0)x_i) \, dx \sim \int_0^{\infty} \exp(\lambda(1 - \varepsilon)f'(0)x_i) \, dx, \]
\[ = \lambda^{-1} |(1 - \varepsilon)f'(0)|^{-1}, \]
we find that for \( \lambda \to \infty \):
\[ \limsup_{\lambda \to \infty} \frac{I_\varepsilon(\lambda)}{\bar{I}(x)} \leq \left( 1 + \frac{\varepsilon}{h(0)} \right) (1 - \varepsilon)^{-n} \quad \text{for each } \varepsilon \text{ with } 0 < \varepsilon < \varepsilon_0. \]

On the other hand, with \( y_0 = 0 \) the conditions of Lemma 1 are fulfilled and thus
\[ I_\varepsilon(\lambda) \sim I(\lambda) \quad \text{for each } \varepsilon \text{ with } 0 < \varepsilon < \varepsilon_0. \]

The last two relations together imply
\[ \limsup_{\lambda \to \infty} \frac{I(\lambda)}{\bar{I}(\lambda)} \leq 1. \]

By bounding \( f(x) \) from below in a suitable set it can be shown in the same way that
\[ \liminf_{\lambda \to \infty} \frac{I(\lambda)}{\bar{I}(\lambda)} \geq 1. \]

Therefore:
\[ I(\lambda) \sim \bar{I}(\lambda) \quad (\lambda \to \infty). \]

This proves Lemma 2.

**Lemma 3.** Let \( D \subset \mathbb{R}^n \) be a bounded closed set containing the origin in its interior. If

(a) \( f: D \to \mathbb{R} \) and \( h: D \to \mathbb{R} \) are continuous functions with \( h(0) \neq 0 \);
(b) \( f(x) < f(0) \) for all \( x \in D, x \neq 0 \);
(c) There is an open neighbourhood \( U \subset D \) of the origin such that \( f(x) \) is twice continuously differentiable in \( U \);
(d) the Hessian \( H(0) \) of \( f(x) \) at the origin is negative definite;

then the following asymptotic relation is valid:
\[ \int_D h(x) \exp(\lambda f(x)) \, dx \]
\[ \sim (2\pi)^{n/2} h(0) \exp(\lambda f(0)) |\det(H(0))|^{-1/2} \lambda^{-n/2} \quad (\lambda \to \infty). \]
**Proof of Lemma 3.** A proof of this lemma can be found in Bleistein and Handelsman [1, Chap. 8.3], but in that text the conditions under which the lemma is valid are not exactly specified. For an exact proof see Fedoryuk [4, Chap. 2, paragraph 4, Theorem 4.1 and Remark 4.1, p. 74–75].

**Remark 2.** The result of Lemma 3 has been derived by several authors independently; see the references in the cited books. The conditions can be weakened. For example, it is not necessary that \( f(x) \) and \( h(x) \) are continuous in the whole domain \( D \), see Fulks and Sather [5] or Lemma 1.

**Lemma 4.** Let \( D \subset \mathbb{R}^n \) be a bounded closed set containing the origin in its interior and let \( k \in \{0, \ldots, n\} \). If

(a) \( f: D \to \mathbb{R} \) and \( h: D \to \mathbb{R} \) are continuous functions with \( h(0) \neq 0 \);
(b) \( f(x) < f(0) \) for all \( x \in D \cap (\mathbb{R}^k_+ \times \mathbb{R}^{n-k}) \), \( x \neq 0 \);
(c) there is a neighbourhood \( V \subset D \) of the origin such that \( f(x) \) is twice continuously differentiable in \( V \);
(d) \( f'(0) < 0 \) for \( i = 1, \ldots, k \);
(e) the matrix \( \tilde{H}(0) := (f''(0))_{l,m = k+1, \ldots, n} \) is negative definite;

then the following asymptotic relation is valid:

\[
\int_{D \cap (\mathbb{R}^k_+ \times \mathbb{R}^{n-k})} h(x) \exp(\lambda f(x)) \, dx \\
\sim (2\pi)^{(n-k)/2} h(0) \exp(\lambda f(0)) \left( \sum_{i=1}^{k} |f''(0)|^{-1} \right) \\
\times |\det(\tilde{H}(0))|^{-1/2} \lambda^{-(n+k)/2} \quad (\lambda \to \infty). \tag{11}
\]

**Proof of Lemma 4.** Define

\[
I(\lambda) := \int_{D \cap (\mathbb{R}^k_+ \times \mathbb{R}^{n-k})} h(x) \exp(\lambda f(x)) \, dx
\]

\[
\tilde{I}(\lambda) := (2\pi)^{(n-k)/2} h(0) \exp(\lambda f(0)) \prod_{i=1}^{k} |f''(0)|^{-1} |\det(\tilde{H}(0))|^{-1/2} \lambda^{-(n+k)/2}.
\]

Due to assumption (c), in a sufficiently small convex neighbourhood of the origin, \( f(x) \) allows for a second-order Taylor expansion:

\[
f(x) = f(0) + \sum_{i=1}^{n} f'(0) x_i + \frac{1}{2} \sum_{l,j=1}^{n} f''(\theta(x) x_i x_j \quad (0 \leq \theta(x) \leq 1). \tag{12}
\]
Due to assumption (b) we have
\[ f'(0) = 0 \text{ for } j = k + 1, \ldots, n. \]

Rewriting Eq. (12):
\[
f(x) = f(0) + \sum_{i=1}^{k} x_i \left[ f'(0) + \frac{1}{2} \sum_{j=1}^{k} f''(\theta(x) x) x_j + \sum_{j=k+1}^{n} f''(\theta(x) x) x_j x_m \right] + \sum_{l,m=k+1}^{n} f''(\theta(x) x) x_l x_m.
\]

Consider now, for \( x \in V \) and \( y \in E_{n-k} := \{(y_{k+1}, \ldots, y_n) \in \mathbb{R}^{n-k} : |y| = 1\} \), the functions
\[
M_1(y) := y^\top \hat{A}(0) y
\]
\[
M_2(x, y) := y^\top \hat{A}(x) y,
\]
where \( \hat{A}(x) := (f''(x(x))_{l,m=k+1,\ldots,n}. \)

Choosing \( \delta \) sufficiently small so that for \( W_\delta := \{x \in \mathbb{R}^n : |x| < \delta\} \) the closure \( \overline{W_\delta} \) is contained in \( V \), \( M_2 \) is uniformly continuous in \( W_\delta \times E_{n-k}. \)

On the other hand since \( \hat{A}(0) \) is negative definite, there is
\[
c := \inf\{-M_2(0, y) : y \in E_{n-k}\} = \inf\{-M_1(y) : y \in E_{n-k}\} > 0.
\]

It follows that for each \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that
\[
|M_2(x, y) - M_2(x_1, y_1)| < \varepsilon c \quad \text{for } |(x - x_1, y - y_1)| < \delta
\]
and, in particular,
\[
-M_2(x, y) \geq -M_1(y) - \varepsilon c \geq -(1 - \varepsilon) M_1(y) \quad \text{for } |x| < \delta.
\]

This implies again
\[
\hat{x}^\top \hat{A}(\theta(x) x) \hat{x} \leq (1 - \varepsilon) \hat{x}^\top \hat{A}(0) \hat{x} \quad \text{for } |x| < \delta, \quad (13)
\]
where \( \hat{x} = (x_{k+1}, \ldots, x_n). \)

Now, since
\[
\frac{1}{2} \sum_{j=1}^{k} f''(\theta(x) x) x_j + \sum_{j=k+1}^{n} f''(\theta(x) x) x_j \to 0 \quad \text{for } x \to 0,
\]
there exists an \( \varepsilon_0 > 0 \) such that, for all \( \varepsilon > 0 \) with \( 0 < \varepsilon < \varepsilon_0 \) there
exists a \( \delta(\varepsilon) > 0 \) such that for \( V_\varepsilon := \{ x \in \mathbb{R}^n : |x_i| < \delta(\varepsilon) \text{ for } i = 1, \ldots, n \} \) and all \( x \in V_\varepsilon \) the conditions (\( \alpha \))–(\( \delta \)) are satisfied:

\[
(\alpha) \quad V_\varepsilon \subset V;
\]

\[
(\beta) \quad \frac{1}{2} \sum_{i=1}^k f'^u(\theta(x) x) x_i + \sum_{j=k+1}^n f'^u(\theta(x) x) x_j \leq -\varepsilon f'(0)
\]

for \( i = 1, \ldots, k \) and due to Eq. (13),

\[
\bar{x}^T \bar{A}(\theta(x) x) \bar{x} \leq (1 - \varepsilon) \bar{x}^T \bar{A}(0) \bar{x}
\]

and therefore

\[
f(x) \leq f(0) + (1 - \varepsilon) \sum_{i=1}^k f'(0) x_i \\
\quad + \frac{1}{2} (1 - \varepsilon) \sum_{i=1}^n f''(0) x_i x_i;
\]

\[
(\gamma) \quad h(x) \leq h(0) + \varepsilon;
\]

\[
(\delta) \quad (1 - \varepsilon) \bar{A}(0) \text{ is negative definite}.
\]

Define now

\[
I^*_\varepsilon(\lambda) := \int_{V_\varepsilon \cap \mathbb{R}_+^k \times \mathbb{R}^{n-k}} h(x) \exp(\lambda f(x)) \, dx.
\]

Since the assumptions of Lemma 1 are fulfilled, we have

\[
I(\lambda) \sim I^*_\varepsilon(\lambda) \quad (\lambda \to \infty).
\] (14)

Due to the properties (\( \alpha \))–(\( \delta \)) of \( V_\varepsilon \) the integral \( I^*_\varepsilon(\lambda) \) can be bounded from the above by

\[
I^*_\varepsilon(\lambda) \leq (h(0) + \varepsilon) \exp(\lambda f(0)) \prod_{i=1}^k \int_0^{\delta(\varepsilon)} \exp(\lambda (1 - \varepsilon) f'(0) x_i) \, dx_i \\
\quad \times \left( \int_{-\delta(\varepsilon)}^{\delta(\varepsilon)} \cdots \int_{-\delta(\varepsilon)}^{\delta(\varepsilon)} \exp \left( \frac{\lambda}{2} (1 - \varepsilon) \sum_{i,m=k+1}^n f''(0) x_i x_m \right) \, dx_{k+1} \cdots dx_n \right)
\]

\[
=: S(\lambda),
\] (15)

where due to Lemmas 2 and 3,

\[
S(\lambda) \sim (h(0) + \varepsilon) \exp(\lambda f(0)) \prod_{i=1}^k |f'(0)(1 - \varepsilon)|^{-1} \lambda^{-k} \\
\quad \times (2\pi)^{(n-k)/2} \left( (1 - \varepsilon)^{-(n-k)/2} |\det(\bar{A}(0))|^{-1/2} \right) \lambda^{-(n-k)/2} \quad (\lambda \to \infty).
\]
Dividing by \( \bar{I}(\lambda) \):

\[
\frac{S_\varepsilon(\lambda)}{\bar{I}(\lambda)} \sim \left( 1 + \frac{\varepsilon}{h(0)} \right) (1 - \varepsilon)^{-(n+k)/2}.
\]

Since this is valid for all \( \varepsilon \) with \( 0 < \varepsilon < \varepsilon_0 \) and due to relations (14) and (15), there is in the limit

\[
\limsup_{\lambda \to \infty} \frac{I(\lambda)}{\bar{I}(\lambda)} \leq 1.
\]

In the same way \( I(\lambda) \) can be bounded from below. Therefore,

\[
I(\lambda) \sim \bar{I}(\lambda) \quad (\lambda \to \infty).
\]

This proves Lemma 4.

We are now ready to prove the general theorem. The following assumptions (V1–V8) are made:

(V1) There is a closed set \( V \subseteq \mathbb{R}^n \) such that \( F \subseteq V \) and \( y^* \) is an interior point of \( V \). Let \( k \in \{1, \ldots, n\} \) and the functions \( g_i : V \to \mathbb{R} \), \( i = 1, \ldots, k \), and \( f : V \to \mathbb{R} \) be twice continuously differentiable in a neighbourhood of \( y^* \). The function \( h : V \to \mathbb{R} \) is assumed to be continuous at \( y^* \) with \( h(y^*) \neq 0 \).

(V2) The gradients \( a_i := \nabla g_i(y^*) \) \( (i = 1, \ldots, k) \) are linearly independent and there is \( a_{ij} = 0 \) for \( i = 1, \ldots, k \) and \( j = k + 1, \ldots, n \) \( (a_U \text{ being the } j \text{th component of } a_i) \). In particular, \( A := (a_{ij})_{i,j=1,\ldots,k} \) is non-singular.

(V3) There is a neighbourhood \( W \subseteq V \) of \( y^* \) with

\[
W \cap F = W \cap \bigcap_{i=1}^k \{ x ; g_i(x) \geq 0 \}.
\]

(V4) There is \( g_i(y^*) = 0 \) for \( i = 1, \ldots, k \).

(V5) There is \( f(y^*) > f(y) \) for each \( y \in F^a \), \( f \) is continuous on \( F^a \) and for some \( e > 0 \) the set \( \{ y \in F^a : f(y) > f(y^*) - e \} \) is compact (compare Remark 1).

(V6) There is a unique representation of the gradient \( \nabla f(y^*) \) in the form: \( \nabla f(y^*) = \sum_{i=1}^k \gamma_i a_i \) with \( \gamma_i < 0 \) for \( i = 1, \ldots, k \). In particular, due to (V2) there is \( f^{k+1}(y^*) = \ldots = f^n(y^*) = 0 \).

(V7) The matrix \( D = (d_{ij})_{i,j=k+1,\ldots,n} \) defined by \( d_{ij} := f^{ij}(y^*) - \sum_{i=1}^k \gamma_i g^{ij}(y^*) \) is non-singular. In the case \( k = n \) let \( \det(D) = 1 \).

(V8) \( \int_F |h(y)| \exp(f(y)) \, dy < \infty \).
Remark 3. (a) It can always be achieved by a suitable rotation of the coordinates that $a_{ij} = 0$ for $i = 1, ..., k$ and $j = k + 1, ..., n$ (compare (V2)).

(b) Under the given conditions (V1)–(V5) it is always possible to represent $\nabla f(y^*)$ as a unique linear combination of the $\nabla g_j(y^*)$ (see Hestenes [6]). In the proof of Theorem 1 it is, as a side-result, also shown that $\gamma_i \leq 0$ for $1 \leq i \leq k$. Therefore, the only essential condition required in (V6) is that the $\gamma_i$'s are non-zero.

**Theorem 1.** Under the conditions (V1)–(V8) the following asymptotic relation is valid:

\[
\int_{F} h(y) \exp(\lambda f(y)) \, dy \\
\sim (2\pi)^{(n-k)/2} h(y^*) \exp(\lambda f(y^*)) |\det(A)|^{-1} \left( \prod_{i=1}^{k} |\gamma_i|^{-1} \right) |\det(D)|^{-1/2} \\
\times \lambda^{-(n+k)/2} \quad (\lambda \to \infty).
\]  

(14)

Remark 4. This theorem is a generalization of the multidimensional Laplace method in the case of a boundary maximum (see [1, Chap. 8.3; 4 Chap. 2, paragraph 43]). Whereas in these references it is assumed that the boundary is smooth near the maximum point, the boundary here is allowed to be not differentiable at the maximum point.

**Proof of Theorem 1.** Define the function $u: V \to \mathbb{R}^n$ for $y = (y_1, ..., y_n)'$ by

\[ u(y) := (g_1(y), ..., g_k(y), y_{k+1} - y^*_{k+1}, ..., y_n - y^*)_'. \]

The functional matrix $D_u(y^*)$ of $u$ at $y^*$ writing $a_i := \nabla g_i(y^*)$ is given by

\[ D_u(y^*) = (a_1, ..., a_k, e_{k+1}, ..., e_n)' \]

and so due to (V2) and the standard rules for evaluating determinants,

\[ \det(D_u(y^*)) = \det(A) \neq 0. \]

Since the functional determinant of $u$ is non-zero at $y^*$, there exist neighbourhoods $V_1 \subset W$ of $y^*$ and $U_1$ of $x^* = u(y^*) = 0$ such that the restriction $u|_{V_1}: V_1 \to U_1$ of $u$ to $V_1$ is a bijective and twice continuously differentiable function with $\det(D_u(y)) \neq 0$ for all $y \in V_1$. Therefore, the inverse mapping $t: U_1 \to V_1$ is also bijective and twice continuously differentiable and its functional matrix $D_t(x^*)$ at $x^*$ is given by:

\[ D_t(x^*) = ((a_1, ..., a_k, e_{k+1}, ..., e_n)')^{-1}. \]

(16)

Without loss of generality, $U_1$ can be assumed to be bounded and closed.
Since $V_1 \cap F = V_1 \cap \bigcap_{i=1}^{k} \{g_i \geq 0\}$ due to condition (V3) it follows that

$$U_1 \cap (\mathbb{R}_+^k \times \mathbb{R}^{n-k}) = u(V_1 \cap F).$$

Using the theorem for integral transformations we obtain:

$$\int_{V_1 \cap F} h(y) \exp(\lambda f(y)) \, dy$$

$$= \int_{U_1 \cap (\mathbb{R}_+^k \times \mathbb{R}^{n-k})} h(t(x)) \det(D_t(x)) \exp(\lambda f(t(x))) \, dx. \quad (17)$$

Now it is demonstrated that Lemma 4 can be applied to the last integral. The condition (a) of the lemma is fulfilled due to assumption (V1) and since $|\det(D_t(x))|$ is a positive continuous function in the integration domain. Due to assumption (V5) at $y^*$ there is the unique maximum of $f(y)$ with respect to the set $V_1 \cap F$. Therefore, due to the definition of $t$ at $0$ there is the unique maximum of the function $f(t(x))$ with respect to the set $U_1 \cap (\mathbb{R}_+^k \times \mathbb{R}^{n-k})$; hence condition (b) is satisfied. Condition (c) is fulfilled due to the choice of $U_1$. In order to show that conditions (d) + (e) are fulfilled also, we have to compute the needed derivatives of $f(t(x))$ at $x = 0$.

Since $t$ is the inverse mapping of $u$ we have for the functional matrix $D_t(x)$ at $x \in U_1$:

$$D_t(x) = D_u^{-1}(t(x)).$$

This yields for $i, j = 1, ..., n$:

$$\sum_{v=1}^{n} u_{ij}^v(t(x)) t_i^v(x) = \delta_{ij}. \quad (18a)$$

$$\sum_{v=1}^{n} u_{ji}^v(t(x)) t_j^v(x) = \delta_{ij}. \quad (18b)$$

Due to condition (V2) and the definition of $u$ we have

$$u_{ji}^\mu(y^*) = 0 \quad \text{for} \quad j = 1, ..., k; \quad \mu = k + 1, ..., n \quad (19a)$$

$$u_{ji}^\mu(y^*) = \delta_{j\mu} \quad \text{for} \quad j = k + 1, ..., n; \quad \mu = 1, ..., n \quad (19b)$$

and

$$t_i^v(0) = \delta_{iv} \quad \text{for} \quad v = k + 1, ..., n; \quad i = 1, ..., n. \quad (20a)$$

Using Eqs. (18b), (19a), and (19b), we obtain for $v = 1, ..., k$ and $i = k + 1, ..., n$,

$$0 = \sum_{j=1}^{n} u_j(y^*) t_j^v(0) = \sum_{j=k+1}^{n} u_j(y^*) t_j^v(0) = t_i^v(0).$$
Together with (20a) this gives
\[ t'(0) = \delta_i, \quad \text{for } \nu = 1, \ldots, n; \ i = k + 1, \ldots, n. \]  
\tag{20b}

For \( i = 1, \ldots, n \) the first derivatives \( \tilde{f}'(x) \) of the function \( \tilde{f}(x) := f(t(x)) \) are given by:
\[ \tilde{f}'(x) = \sum_{\nu = 1}^{n} f'(t(x)) t'_\nu(x). \]  
\tag{21}

Due to Eq. (20a) this is, for \( x = 0 \) and \( i = 1, \ldots, k \),
\[ \tilde{f}'(0) = \sum_{\nu = 1}^{k} f'(y^*) t'_\nu(0) \]
and, due to assumption (V6),
\[ \tilde{f}'(0) = \sum_{\nu = 1}^{k} \sum_{s = 1}^{k} \gamma_s a_{\nu s} t'_\nu(0) = \sum_{s = 1}^{k} \gamma_s \left( \sum_{\nu = 1}^{k} a_{\nu s} t'_\nu(0) \right). \]

Since \( a_{\nu s} = u'_j(y^*) \), for \( s = 1, \ldots, k \) and by Eq. (19a),
\[ \tilde{f}'(0) = \sum_{s = 1}^{k} \gamma_s \left( \sum_{\nu = 1}^{k} u'_j(y^*) t'_\nu(0) \right) = \sum_{s = 1}^{k} \gamma_s \left( \sum_{\nu = 1}^{n} u'_j(y^*) t'_\nu(0) \right). \]

Due to Eq. (18a) and assumption (V6),
\[ \tilde{f}'(0) = \sum_{\nu = 1}^{k} \gamma_i \delta_{i\nu} = \gamma_i < 0. \]

Note in particular that \( \tilde{f}'(0) = \gamma_i \) which without making use of \( \gamma_i < 0 \) implies that \( \gamma_i < 0 \), since \( \tilde{f} \) has a local maximum at \( x = 0 \) in \( U \cap (\mathbb{R}^k_+ \times \mathbb{R}^{n-k}) \); compare Remark 3(b).

The last series of equations is summarized in
\[ \tilde{f}'(0) = \gamma_i < 0 \quad \text{for } \ i = 1, \ldots, k, \]  
\tag{22}

whence condition (d) of Lemma 4 is fulfilled for \( \tilde{f}(x) \).

Using Eq. (21) and assumption (V6) we obtain for \( i, j = k + 1, \ldots, n \):
\[ \tilde{f}''(0) = \sum_{\nu = 1}^{n} \sum_{\mu = 1}^{n} f''(y^*) t'_\nu(0) t'_\mu(0) \]
\[ + \sum_{\nu = 1}^{n} \sum_{s = 1}^{k} \gamma_s g'_j(y^*) t'_\nu(0). \]  
\tag{23}
By differentiating Eq. (18a) with respect to $x_j$ and noting that $u^*_j = g^*_j$ for $s = 1, \ldots, k$, we obtain for $s = 1, \ldots, k$:

$$
\sum_{\nu = 1}^{n} \sum_{\mu = 1}^{n} g^*_\nu(t(x)) t'_\nu(x) t'_\mu(x) = - \sum_{\nu = 1}^{n} g^*_\nu(t(x)) t'_\nu(x).
$$

This yields, inserted in Eq. (23),

$$
\mathcal{J}^u(0) = \sum_{\nu = 1}^{n} \sum_{\mu = 1}^{n} t'_\nu(0) t'_\mu(0) \left( f^{*(u)}(y^*) - \sum_{s = 1}^{k} \gamma_s g^*_s(y^*) \right).
$$

(24)

Since $i, j = k + 1, \ldots, n$ by using Eq. (20b), we obtain finally:

$$
rij(0) = f^{*(u)}(y^*) - \sum_{s = 1}^{k} \gamma_s g^*_s(y^*).
$$

(25)

Therefore, the matrix $D = (rij(0))_{i = k + 1, \ldots, n}$ equals the matrix $D$. Furthermore, $\mathcal{J}(x)$ attains at $x = 0$ a maximum with respect to $U_1 \cap (\mathbb{R}^k_+ \times \mathbb{R}^{n-k})$ whence $\mathcal{B}$ is negative semidefinite: since due to assumption (V7) there is $\det(D) \neq 0$, $\mathcal{B}$ is even negative definite and so condition (e) of Lemma 4 is fulfilled.

We can now apply Lemma 4 to the integral on the right side of Eq. (17). Using the quantities given in Eqs. (16), (22), and (25), we obtain finally

$$
\int_{F \cap V_1} h(y) \exp(\lambda f(y)) \, dy
$$

$$
= \int_{U_1 \cap (\mathbb{R}^k_+ \times \mathbb{R}^{n-k})} h(t(x)) \left| \det(D_0(x)) \right| \exp(\lambda f(t(x))) \, dx
$$

$$
\sim (2\pi)^{(n-k)/2} h(y^*) \exp(\lambda f(y^*)) \left| \det(A) \right|^{-1} \left( \prod_{i = 1}^{k} |y_i|^{-1} \right)
$$

$$
\times \left| \det(D) \right|^{-1/2} \lambda^{-(n+k)/2} \quad (\lambda \to \infty).
$$

Due to conditions (V1) to (V8) and since $y^*$ is an interior point of $V_1$ we can use Lemma 1 and obtain

$$
\int_{F \cap V_1} h(y) \exp(\lambda f(y)) \, dy \sim \int_{F} h(y) \exp(\lambda f(y)) \, dy \quad (\lambda \to \infty).
$$

Combining the last two relations proves the theorem.

Remark 5. In the case that there are several points $y_1, \ldots, y_l (l \geq 1)$ in $F$ where the function $f(y)$ achieves its maximum with respect to $F$, a similar result can be obtained. The set $F$ is partitioned in $l$ sets $F_1, \ldots, F_l$ such that
ASYMPTOTIC APPROXIMATIONS

$F_v \cap F_\mu = \emptyset$ for $v \neq \mu$ and $\bigcup_{v=1}^t F_v = F$ and $y_v$ is an interior point of $F_v$. Then, for each set $F_v$ Theorem 1 will be applied and the sum of the approximations will give an approximation for the integral over $F$.

4. APPLICATION TO MULTINORMAL INTEGRALS

In this section, Theorem 1 will be applied to the special case of the $n$-dimensional multinormal density function:

$$\varphi_n(y) := -(2\pi)^{-n/2} \exp(f(y))$$

with $f(y) := -\frac{1}{2} |y|^2$.

For an $n$-dimensional Borel set $F \subset \mathbb{R}^n$ define:

$$P(F) := \int_F \varphi_n(y) \, dy.$$

Let $\lambda \geq 1$ be a real parameter. Then

$$P(\lambda F) = \int_{\lambda F} \varphi_n(y) \, dy = \lambda^n \int_F \varphi_n(\lambda y) \, dy. \quad (26)$$

In the following corollaries for several cases, the asymptotic form of $P(\lambda F)$ is given which can easily be derived by applying Theorem 1 to the right-hand integral in Eq. (26).

**Corollary 1.** Let $g_i(y) := a_i(y-y^*)$, $i = 1, \ldots, n$ be $n$ linear functions where $a_i$ and $y^*$ are constant vectors with $\det((a_1, \ldots, a_n)) \neq 0$ and $y^* = -\sum_{i=1}^n \gamma_i a_i$, with $\gamma_i < 0$ for $i = 1, \ldots, n$. Then, for the set $F$ defined by $F = \bigcap_{i=1}^n \{ y; g_i(y) \geq 0 \}$, the following asymptotic relation holds:

$$P(\lambda F) \sim |\det((a_1, \ldots, a_n))|^{-1} \left( \prod_{i=1}^n |\gamma_i|^{-1} \right) \lambda^{-n} \varphi_n(\lambda y^*).$$

(Of applying Theorem 1 note that $\varphi_n(\lambda y) = (2\pi)^{-n/2} \exp(\lambda^2 f(y))$).

**Remark 6.** This corollary gives just the first term of Ruben's result for the multinormal distribution function.

**Corollary 2.** Let a continuous function $g_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ with $g_1(0) < 0$ and $F := \{ y; g_1(y) \geq 0 \}$ be given. If there is a unique point $y^*$ on the set $S = \{ y; g_1(y) = 0 \}$, where the function $\varphi_n(y)$ achieves its maximum with
respect to the set \( F \) and the other conditions (V1)–(V8) for Theorem 1 are fulfilled for \( P(\lambda F) \) the following asymptotic relation is valid:

\[
P(\lambda F) \sim |y^*|^{-1} |\det(D)|^{-1/2} \lambda^{-1} \varphi_1(\lambda |y^*|) \quad (\lambda \to \infty)
\]

with \( D = (\delta_{ij} - (|y^*|/|\nabla g(y^*)|) g_i(y^*) g^i_j(y^*))_{i, j = 2, \ldots, n} \).

**Proof.** Application of Theorem 1. Due to conditions (V2) and (V6) the point \( y^* \) lies on the \( x_i \)-axis.

**Remark 7.** The result of Corollary 2 is stated in a form invariant under orthogonal transformations and holds therefore also for \( y^* \) not lying on the \( x_i \)-axis. This result can be derived also directly by known results of asymptotic analysis (see [3]).

**Corollary 3.** Let be given \( m \) twice continuously differentiable functions \( g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R} \) with \( g_i(0) < 0 \) for at least one \( i = 1, \ldots, m \). Let there be an unique point \( y^* \) where the function \( \varphi_n(y) \) achieves its maximum with respect to the set \( \{ y; g_i(y) \geq 0 \text{ for } i = 1, \ldots, m \} \). Without loss of generality let \( g_1, \ldots, g_k (1 \leq k \leq m) \) denote the set of active constraints, i.e., \( g_i(y^*) = 0 \) for \( i = 1, \ldots, k \). Then, under the assumptions and notations (V1)–(V8) and with \( h_i(y) = (\nabla g_i(y^*))' (y - y^*) \) the following asymptotic relation is valid for

\[
P(\lambda F) \sim |\det(D)|^{-1/2} \varphi_k(\lambda) \quad (\lambda \to \infty).
\]

(Note that the last probability is just the \( k \)-dimensional multivariate distribution function.)

**Proof.** Application of Theorem 1; compare the asymptotic solutions for the probabilities \( P(\lambda F) \) and \( P(\lambda \bar{F}) \).

**Remark 8.** This is the main result about approximations for multinormal integrals. It shows how the second derivatives influence the probability content of the domain \( \lambda F \).

**Remark 9.** (a) The condition "\( g_i(0) < 0 \) for at least one of \( i = 1, \ldots, m \)" guarantees that \( \varphi_n(y) \) attains its maximum on the boundary of \( F \), and that \( y^* \neq 0 \).

(b) If \( \varphi_n(y) \) attains its maximum in the interior of \( F \) then \( y^* = 0 \). Now, passing to the complement \( \bar{F} \) it is observed that \( \varphi_n(y) \) attains its maximum with respect to \( \bar{F} \) at the boundary of \( \bar{F} \) so that Corollary 3 possibly in connection with Remark 5 can be applied to \( F \), implying in turn also a result for \( P(\lambda \bar{F}) = 1 - P(\lambda F) \). Note that direct application of Lemma 3 to \( P(F) \) would only give the trivial result \( P(\lambda F) \sim 1 \).
5. SUMMARY AND CONCLUSIONS

In this paper a method for approximating multidimensional integrals has been derived which is a generalization of known methods of asymptotic analysis. All those methods are based on the evaluation of the local behaviour of the integrand at certain critical points. While, however, the known results are restricted to the case that critical points either lie in the interior of the integration domain or, if they lie on the boundary, the boundary is there twice continuously differentiable, the main result of the present paper allows the critical points also to lie on an edge or corner of the integration domain.

The aim of the authors was the computation of multinormal probabilities rather than the extension of the theory of asymptotic analysis. In view of Remarks 3(b), 5, and 9, Corollaries 1 and 3 cover most of the practical cases with extreme probabilities.

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