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Multidimensional cell lists for investigating 3-manifolds ☆

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Abstract

The paper presents a new method of investigating topological properties of three-dimensional manifolds by means of computers. Manifolds are represented as block complexes. The paper contains definitions and a theorem necessary to transfer some basic knowledge of the classical topology to finite topological spaces. The method is based on subdividing the given set into blocks of cells in such a way that a k-dimensional block be homeomorphic to a k-dimensional ball. The block structure is described by the data structure known as "cell list" which is generalized here for the multidimensional case. Results of computer experiments are presented.

1. Introduction

Topological knowledge plays an important role in computer graphics and image analysis. Images may be represented in computers only as finite sets. Therefore it is usual to perform topological investigations in a Hausdorff space and then to transfer the results to finite sets. One of the aims of the present investigation is to demonstrate that topological investigations may be performed *directly in finite sets* on which a T_0-topology is defined. Such a topological space can be represented in computers. We demonstrate here a new tool for investigating 3-manifolds by means of computers: the three-dimensional (3D) cell list. The same tool may be implemented for economically encoding and analyzing three-dimensional (3D) images, e.g. in computer tomography.

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2. State-of-the-Art

It is known from the topological literature that the problem of the complete classification of 3-manifolds is still unsolved while the classification of 2-manifolds is known since about 100 years [4]. In recent time some efforts have been made to use computers for investigating 3-manifolds. Matveev suggested the following method [15] based on the notion of a spine.

A spine [3] is some kind of a two-dimensional (2D) skeleton of the 3-manifold: if K is a polyhedron, if K collapses to L [21, p. 123], and if there is no elementary collapse of L, then L is a *spine* of K. A standard [3] or special [15] spine of a manifold has the same fundamental group as the manifold. In [15] the notion of complexity k of a 3-manifold was introduced. It is the number of vertices (0-cells) in the so-called *almost special spine* of the manifold under consideration. It has been shown that the singular graph of a special spine is a regular graph of degree 4. There are only finitely many different spines corresponding to a given regular graph of degree 4. Thus it is possible to enumerate all spines having a given number of vertices.

Matveev has also introduced the so-called T-transformation, which transforms a spine of a given manifold to another special spine of the same manifold, which may be simpler. He also uses topological invariants introduced in [24]. More than 1000 3-manifolds of complexity k up to nine have been analyzed by means of this method [15].

We suggest here another method of using computers for the investigation of 3-manifolds. The method was already reported in [13]. The recent presentation is an extended and revised version of [13]. According to our method a 3-manifold is represented as an abstract cell complex (AC complex) [9] with (approximately) minimum number of cells. It is encoded by the cell list as described in Section 5. A comparison of the cell list with other data structures may be found in Section 6. It is easy to see that isomorphic cell lists correspond to combinatorially homeomorphic manifolds. The question whether the minimum cell list of a 3-manifold is unique is yet open. There is the hope that in the case that it is not unique, the number of different cell lists with a minimum number of cells of a 3-manifold of a limited complexity is not too large, so that all such lists may be exhaustively tested by a computer whether they are combinatorially homeomorphic to a cell list of some already known manifold. In this presentation we describe our method of computing the cell list with an approximately minimum number of cells for a given 3-manifold.

3. Basic notions

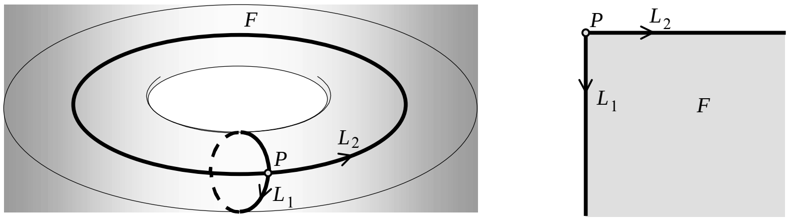
We refer here to the classical definition on an n-manifold [21, p. 13]. It is known [16] that any 3-manifold may be triangulated and that triangulations of homeomorphic 3-manifolds are combinatorially homeomorphic. Two complexes are called *combinatorially homeomorphic* if their simplicial schemata become isomorphic after finite sequences of elementary subdivisions. An *elementary subdivision* in general consists in dividing one m-dimensional cell c^m into two such cells by the introduction of a new interior $(m-1)$-dimensional cell spanning an $(m-2)$-sphere in the boundary of c^m [21, p. 24]. However, simplicial complexes contain too many elements and are therefore difficult to process. Simplices may be united to greater cells by an operation inverse to the subdivision: a subcomplex combinatorially homeomorphic to a k-simplex (or equivalently to a k-ball) may be declared to be a k-dimensional cell or a k-cell. In what follows we shall write "homeomorphic" for "combinatorially homeomorphic".

While simplices are mostly considered as subsets of a Euclidean space we prefer to work with AC complexes [9]. An AC complex is a set of *abstract cells*. A non-negative integer is assigned to each cell. It is called the *dimension* of the cell. The set is provided with an antisymmetric, irreflexive and transitive binary relation called *bounding relation* B. If c_1, c_2 are two cells and $(c_1, c_2) \in B$ then it is usual to write $c_1 < c_2$ and to say "c_1 bounds c_2". A cell can only bound another cell of higher dimension. Two cells of an AC complex are called *incident* to each other if either they are identical, or one of them bounds the other one.

AC complexes differ both from simplicial and Euclidean complexes [18, p. 313] in so far that an abstract cell is never a part of another such cell. This property makes it possible to easily introduce the notion of *open subsets* of an AC complex and thus to define a T_0-topology on it in accordance with classical axioms [9]. Although an AC complex is a quotient of some Hausdorff space we do not consider the cells as subsets of a Hausdorff space, which subsets are infinite ones and therefore not explicitly representable in computers. We rather consider cells as elements of an abstract finite set. This is another advantage of the AC complexes since the topological space of a finite AC complex may be *directly and completely represented in a computer*. Thus there is no necessity to consider theoretical problems in a Hausdorff space (which is not representable in computers) and then to transfer the results to a different set represented in the computer. This advantage of the AC complexes is widely used in the present investigation.

One of our methods of representing some 3-manifolds in computers consists in constructing a four-dimensional (4D) AC complex in the computer, in defining a strongly connected subset of the AC complex and in calculating the boundary of the subset.

To make the number of cells as small as possible we partition the AC complex A representing a 3-manifold into subsets each of which is homeomorphic to an open k-ball with $k = 0, 1, 2, 3$ (a 0-ball is a single 0-cell). We call such a subset a *k-dimensional block cell* or a *k-block* of A. A block b_1 of A is said to bound another block b_2 of A if b_1 contains a cell of A which bounds another cell of A contained in b_2. In this way a *bounding relation* is defined on the set of blocks of A and the set becomes an AC complex B called the *block complex* of A. The blocks are cells of B. The topology of the block complex B is a quotient topology of that of the underlying AC complex A, however, there is no necessity to consider it as a quotient topology of a Euclidean space. The data structures for encoding block complexes are described in Sections 4–7, the algorithm of computing them in Section 7.



(a) (b)

Fig. 1. Representations of the surface of a torus (a) and of a simple complex (b).

4. Incidence structures

4.1. The main idea

In topological literature manifolds are often represented as cell complexes. Thus, e.g. the surface of a torus may be represented as a complex consisting of a 0-cell, two 1-cells and one 2-cell (Fig. 1a). This representation has the advantage of being very simple. However, if one would try to interpret this representation as a complex, difficulties would occur, e.g. the complexes corresponding to Fig. 1a and b are the same: since the same sets of four cells, the same bounding relation and the same dimensions of the cells. The difference between these two complexes is that each of the 1-cells L_1 and L_2 in Fig. 1a bounds the 2-cell *two times*, on both sides. This may be seen, if one considers the embedding of the complex in a Euclidean space: a neighborhood of a point on the 1-cell contains two half-disks each of which lies in one and the same 2-cell. However, there is no possibility to describe this relation in the language of classical complexes.

Since one of our aims is to consider a purely combinatorial approach with no relation to a Euclidean space we consider the possibility to overcome this difficulty by introducing the notion of the incidence structure.

Definition PB. A k-block is called *proper* if its closure is homeomorphic to a closed k-ball. A block complex is called proper if all its blocks are proper.

Thus when considering Fig. 1a as a representation of a block complex then it is not a proper one: though each k-block with $k > 0$ is homeomorphic to an open k-ball the closures of the blocks are not homeomorphic to closed k-balls. An example of a proper block complex of the surface of a torus ($n = 2$) is shown in Fig. 2. The only drawback of this representation is that it has too many blocks as compared to Fig. 1a.

It is possible to reduce the number of blocks of a proper block complex while uniting two adjacent k-blocks which are not mutually simple (see Section 6 for the definition of "simple") by an operation inverse to the elementary subdivision. It may happen, that each of the united blocks was incident to one and the same third block. Then the union of these two blocks would be incident to the third block twice, at two

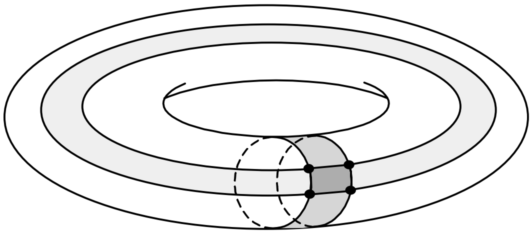


Fig. 2. A proper block complex of the surface of a torus.

different locations. In such a case we can loose some information about the topological structure of the set of blocks incident to the united block since the description of a block complex as an AC complex cannot indicate that a block is *multiply* incident to one and the same other block. To overcome this drawback we introduce for each block the so-called incidence structure.

Definition IS. The *incidence structure* of a block BC of a proper block complex K is a subcomplex of K containing all blocks incident to BC except BC itself.

This subcomplex may be described as an AC complex: by the set of its blocks and the bounding relation, each block being represented by its label.

To preserve the topological information about the set of blocks incident to a given block BC the incidence structure of BC must be stored *before the uniting* of blocks of the proper complex. Thus, e.g. the incidence structure of each 1-block of Fig. 2 consists of two different points (i.e. 0-blocks) and two different faces. During the uniting of two blocks the label of one of them in the incidence structure must be replaced by the label of another one. In this way it becomes possible that the label of one and the same block multiply occurs in the incidence structure of another block. In our example the complex of Fig. 2 will be converted to that of Fig. 1a. The incidence structure of each 1-block will still contain two points and two faces; however, their indices are no more different: a line starts and ends now at one and the same point and it has one and the same face on its left-hand and right-hand side.

The incidence structures of all blocks of a block complex must be stored in a data structure which is a generalization of the cell list [9]. The former cell list was designed to describe 2D Cartesian AC complexes [10,11] where a point may be incident to at most four lines. In the generalized 3D cell list the number of blocks incident to a point or to a curve in 3D is not limited. This property is important for enabling transformations of block complexes during topological investigations.

4.2. Incidence structures in multidimensional spaces

We will show in what follows that the incidence structure of any block of a multidimensional block complex representing a closed manifold is similar to the union of at most two topological spheres. This fact is the basis of the development of data structures enabling an economical representation of multidimensional block complexes in computers. To prove the necessary theorem we remind the reader some definitions.

An AC complex A is an *Alexandroff space* [1,8] and hence there exists in A the *smallest neighborhood* of each cell $c \in A$ while the notion "neighborhood" is to be understood in the classical topological sense [18,20] rather than in the sense usual in the context of neighborhood graphs [19]. It is the *open* set containing c and all cells of A bounded by c. We denote it by $\mathrm{SON}(c,A)$. For the incidence structure we need the set $\mathrm{SON}(c,A)$ without the cell c itself: we denote it by $\mathrm{SON}^*(c,A) = \mathrm{SON}(c,A) - \{c\}$.

The *closure* $\mathrm{Cl}(c,A)$ is a notion dual to $\mathrm{SON}(c,A)$. It is the set containing c and all cells of A, bounding c. Again, we need the set without c itself: $\mathrm{Cl}^*(c,A){=}\mathrm{Cl}(c,A){-}\{c\}$.

The incidence structure of a proper block BC is the following union:

$$\mathrm{IS}(BC,A) = \mathrm{SON}^*(BC,A) \cup \mathrm{Cl}^*(BC,A). \tag{1}$$

The incidence structure of a non-proper block containing fewer cells must be computed while starting with that of the original proper block complex and uniting some blocks which are not mutually simple.

Definition BI. An isomorphism between two complexes, which retains the bounding relation, is called *B-isomorphism*. $\mathrm{BI} : A \to B$ is a B-isomorphism iff for any $a_1, a_2 \in A$, $a_1 < a_2$ implies $\mathrm{BI}(a_1) < \mathrm{BI}(a_2)$.

Theorem SN. *The set* $\mathrm{SON}^*(c^k, M^n)$ *of any k-cell c^k of an n-manifold M^n is* B-*isomorphic to an $(n-k-1)$-dimensional sphere if c^k does not belong to the boundary ∂M^n and if* $0 \leqslant k \leqslant n-1$. *The set* $\mathrm{Cl}^*(c^k, M^n)$ *is then* B-*isomorphic to an $(k-1)$-dimensional sphere*.

To prove the Theorem we prove at first the particular case of $k=0$, which is the content of the following:

Lemma. *The set* $\mathrm{SON}^*(c^0, M^n)$ *of a* 0-*cell* $c^0 \in M^n$ *is* B-*isomorphic to an $(n-1)$-dimensional sphere*.

Proof. According to the definition of an n-manifold M^n the SON of a point (i.e. of a 0-cell) $c^0 \in M^n$ is an open n-ball B^n. The frontier of B^n is then an $(n-1)$-sphere $\mathrm{S}^{(n-1)}$. Consider the set $V = \mathrm{SON}^*(c^0, M^n)$. Each cell $c^k \in V$ has some cells in $\mathrm{S}^{(n-1)}$ which bound c^k. Consider a partition of $\mathrm{S}^{(n-1)}$ into blocks $b^{(k-1)}(c^k)$ corresponding to the cells $c^k \in V$ such that all cells of $b^{(k-1)}(c^k)$ bound c^k:

$$b^{(k-1)}(c^k) = U(c^k) - \partial U(c^k), \tag{2}$$

where

$$U(c^k) = \mathrm{Cl}^*(c^k) \cap \mathrm{S}^{(n-1)}. \tag{3}$$

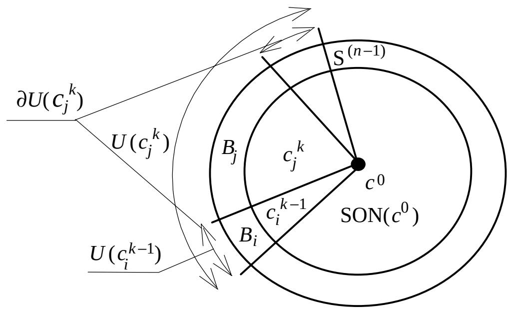


Fig. 3. A logical diagram of cells and blocks of the proof of the Lemma.

Let $c_i^{(k-1)}$ and c_j^k be two cells of V such that $c_i^{(k-1)} < c_j^k$. The corresponding blocks are

$$B_i = U(c_i^{(k-1)}) - \partial U(c_i^{(k-1)}) \quad \text{and} \quad B_j = U(c_j^k) - \partial U(c_j^k). \tag{4}$$

Fig. 3 shows a logical diagram of these cells and blocks.

The blocks compose a block complex $SB^{(n-1)}$. The subcomplex $U(c_j^k)$ has dimension $(k-1)$ since $\mathrm{Cl}^*(c_j^k)$ contains only cells of dimension less than k. The boundary $\partial U(c_j^k)$ contains cells of dimension $k-2$ while according to the definition of a boundary each of these cells bounds exactly one cell of $U(c_j^k)$.

The subcomplex B_i contains cells of $\mathrm{Cl}^*(c_j^k)$ since

$$B_i \subset \mathrm{Cl}^*(c_i^{(k-1)}) \subset \mathrm{Cl}^*(c_j^k), \tag{5}$$

because $c_i^{(k-1)} < c_j^k$. When intersecting each term of (5) with $\mathrm{S}^{(n-1)}$ we obtain

$$B_i = B_i \cap \mathrm{S}^{(n-1)} \subset \mathrm{Cl}^*(c_i^{(k-1)}) \cap \mathrm{S}^{(n-1)} \subset \mathrm{Cl}^*(c_j^k) \cap \mathrm{S}^{(n-1)} = U(c_j^k). \tag{6}$$

Thus B_i contains cells of $U(c_j^k)$ but no cells of $U(c_j^k) - \partial U(c_j^k) = B_j$, since $B_i \cap B_j = \emptyset$: the blocks compose a partition of $\mathrm{S}^{(n-1)}$. Therefore $B_i \subset \partial U(c_j^k)$ and B_i contains at least one cell $c_i^{(k-2)}$, which bounds exactly one $(k-1)$-cell of B_j. Thus B_i bounds B_j and hence the map $I : V \to \mathrm{SB}^{(n-1)}$ taking each k-cell $c^k \in V$ to a $(k-1)$-dimensional block of $\mathrm{SB}^{(n-1)}$ corresponding to c^k retains the bounding relation and is a B-isomorphism. □

Proof of the theorem. Consider the SON of a 0-cell c^0 and a k-cell $c^k \in \mathrm{SON}^*(c^0, M^n)$, $1 \leqslant k \leqslant n-1$. According to the Lemma, c^k will be mapped (as an element of $\mathrm{SON}^*(c^0, M^n)$) by I onto a $(k-1)$-dimensional cell $a^{(k-1)}$ of an $(n-1)$-dimensional sphere $\mathrm{S}^{(n-1)}$. Suppose, the Theorem is true for a $(k-1)$-dimensional cell of a manifold. Since $\mathrm{S}^{(n-1)}$ is a manifold, $\mathrm{SON}^*(a^{(k-1)}, \mathrm{S}^{(n-1)})$ must be B-isomorphic to a sphere of the dimension:

$$(n-1) - (k-1) - 1 = n - 1 - k + 1 - 1 = n - k - 1. \tag{7}$$

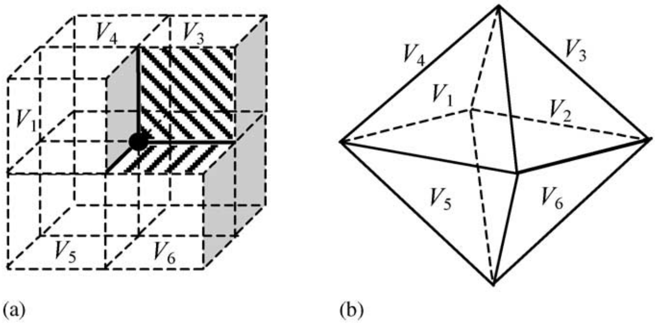


Fig. 4. The SON* of a point (a) and the octahedron's surface (b) B-isomorphic to the SON*.

However, I maps $\mathrm{SON}^*(c^k, M^n)$ onto $\mathrm{SON}^*(a^{(k-1)}, \mathrm{S}^{(n-1)})$ and the latter onto $\mathrm{S}^{(n-k-1)}$. Thus, if the Theorem is true for a $(k-1)$-dimensional cell it is also true for a k-dimensional one. According to the Lemma the Theorem is true for $k=1$ since in this case $a^{(k-1)}$ is a 0-cell. Therefore, the Theorem is true for any $1 \leqslant k \leqslant (n-1)$.

To prove the assertion concerning Cl* it is sufficient to consider a set dual to M^n, where each k-cell is replaced by an $(n-k)$-cell, the bounding relation is reversed and the SON of a cell c is replaced by its closure. □

Fig. 4 shows the SON* of a point in a 3D Cartesian AC complex and its B-isomorphic map onto the surface of an octahedron, which surface is a S^2.

The SON* of a point contains 8 cubes V_1–V_8 (V_2 is removed), 12 faces and 6 edges. The B-isomorphic surface of an octahedron contains 8 faces, 12 edges and 6 points. The cubes are mapped onto the faces, the faces of the SON* onto the edges and the edges onto the points. The above results are illustrated in Table 1 showing the incidence structures of interior cells (or blocks) of a 3D Cartesian AC complex A^3. In cases of spaces of dimension 2 and 3 the union of SON* with Cl* happens to be B-isomorphic to a 2D sphere for cells of any dimension (compare Table 1, column 6). It should be noted that this fact is of no importance for applications since the implementation of a data structure isomorphic to the union $\mathrm{S}^1 \cup \mathrm{S}^0$ is simpler than that of S^2.

We use the incidence structures to describe non-proper block complexes. Such a description is the list of incidence structures of all blocks of a complex. The list is called the *cell list* [9,10]. The cell list for 3D complexes is described in Section 5.

Table 1 shows the incidence structures of cells c^k of all dimensions $k = 0, 1, 2, 3$; and the 2-spheres B-isomorphic to them.

According to Theorem SN the incidence structures of a k-block in an n-dimensional manifold M^n consists of two complexes one of which is B-isomorphic to $\mathrm{S}^{(n-k-1)}$ and the other to $\mathrm{S}^{(k-1)}$. Thus the topological structure of M^n may be described as a list of descriptions isomorphic to spheres of lower dimensions. Therefore it may be recursively composed of structures isomorphic to S^0 and S^1 which are a pair of points and a cyclically closed sequence, respectively.

Table 1
Incidence structures in a 3D space

	$Cl^*(c^k, A^3)$		$SON^*(c^k, A^3)$		$Cl^* \cup SON^*$
dimension k	complex	min. sphere	complex	min. sphere	min. sphere
1	*2*	*3*	*4*	*5*	*6*
0	∅	∅	dim=3	dim=2	dim=2
1	dim=0	dim=0	dim=3	dim=1	dim=2
2	dim=1	dim=1	dim=3	dim=0	dim=2
3	dim=2	dim=2	∅	∅	dim=2

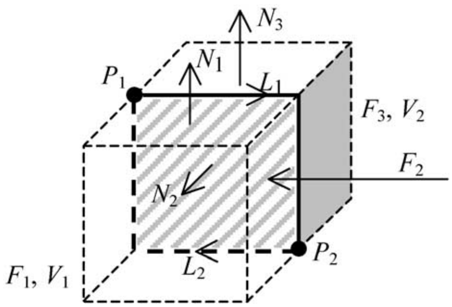


Fig. 5. Example of a simple 3-manifold with boundary.

5. The three-dimensional cell list

On the basis of Theorem SN it becomes possible to construct the 3D cell list as a set of tables while each row of a table describes the incidence structure of a block of the block complex of a 3-manifold. Each incidence structure is described as one or two AC complexes each of which is B-isomorphic to a k-sphere with $k \leqslant 2$.

Let us demonstrate an example. The tables below are constituents of the 3D *topological* cell list of a 3-manifold with boundary. The manifold is shown in Fig. 5. A topological list unlike a geometrical one contains no embedding data (no coordinates). Thus five squares of the surface of each cube in Fig. 5 are considered as *a single face*.

Table 2
List of the branch points (0-blocks)

Label	N_{lin}	Lines
P_1	2	$-L_1, +L_2$;
P_2	2	$+L_1, -L_2$;

The version of the cell list presented here is slightly simplified as compared to that of [13]: the data contained in the lists of 1- and 2-blocks are not repeated in the lists of 0- and 3-blocks, as this was the case in [13].

Each row of Table 2 describes the SON* of the corresponding point. The description is reduced to the enumeration of oriented lines (1-blocks) incident to the point (the lines are not necessarily straight ones). This is sufficient to specify the complete set SON* since it is the union of SONs of all lines bounded by the point, the SONs being specified in the list of the lines shown below. The value N_{lin} denotes the number of lines incident to P_j, $j = 1, 2$.

Each row in the following list of the lines (1-blocks) contains the descriptions of the sets Cl* and SON* of an oriented line L_i, $i = 1, 2$. The set Cl$^*(L_i)$ consists of the starting and the end point of the line L_i. The set SON$^*(L_i)$ is described as a cyclic sequence of oriented faces and volumes incident to L_i. The value N_{SON} denotes the number of blocks in the SON* of L_i. These blocks compose the chained list where they are represented in the order of the right-handed rotation around L_i. The sign "−" before the label of an oriented face F_k shows that the normal to F_k points against the rotation. The pointer Z_m points to the first element of the chained list. A zero symbol at the end of the chained list denotes that the list is not cyclically closed which may be the case for manifolds with boundary. Otherwise the first element is repeated at the end of the chained list to indicate that the list is cyclically closed.

The list of faces (2-blocks) has a similar structure: the incidence structure of an oriented face F_i contains the descriptions of the sets SON* and Cl* of F_i, $i = 1, 2, 3$. The set SON$^*(F_i)$ contains the indices of at most two 3-blocks incident to F_i. The normal to F_i points to the 3-block indicated as "+Vol". The set Cl$^*(F_i)$ is described as a cyclic sequence of N_{Cl} 0- and 1-blocks. The first symbol is repeated at the end of the chained list to show that the sequence is cyclically closed.

The list of the 3-blocks has a structure dual to that of the 0-blocks: each row contains the description of the set Cl* of the corresponding 3-block.

The value N_f denotes the number of faces F_j incident to V_i, $i = 1, 2$. This number is followed by the enumeration of the oriented faces: a positive index $+F_j$ in the row V_i denotes that the normal to F_j points to V_i. These data are sufficient to specify the complete set Cl$^*(V_i)$ which is the union of the sets Cl$^*(F_j)$ described in the list of the faces.

A presentation of the fundamental group of a given complex may be computed from its cell list by the method suggested by Poincaré [17] and proved by Tietze [23]. According to this method it is necessary to find the spanning tree of the 1-dimensional (1D) skeleton of the complex and ignore all 1-cells in the tree. Each of the remaining 1-cells is a generator, the concatenation of the generators in the perimeter of each 2-cell, being equated to identity, is a relation of the presentation of the fundamental group.

Table 3
List of the lines (1-blocks)

Label	Start	End	N_{SON}	Pointer	Chained list
L_1	P_1	P_2	5	$Z_5 \rightarrow$	$-F_1 \rightarrow V_1 \rightarrow -F_2 \rightarrow V_2 \rightarrow +F_3 \rightarrow 0$
L_2	P_2	P_1	5	$Z_6 \rightarrow$	$-F_1 \rightarrow V_1 \rightarrow -F_2 \rightarrow V_2 \rightarrow +F_3 \rightarrow 0$

Table 4
List of faces (2-blocks)

Label	+Vol	− Vol	N_{Cl}	Pointer	Chained list
F_1	–	V_1	4	$Z_7 \rightarrow$	$P_1 \rightarrow -L_2 \rightarrow P_2 \rightarrow -L_1 \rightarrow P_1$
F_2	V_1	V_2	4	$Z_8 \rightarrow$	$P_1 \rightarrow -L_2 \rightarrow P_2 \rightarrow -L_1 \rightarrow P_1$
F_3	–	V_2	4	$Z_9 \rightarrow$	$P_1 \rightarrow +L_1 \rightarrow P_2 \rightarrow +L_2 \rightarrow P_1$

Table 5
List of volumes (3-blocks)

Label	N_f	Faces
V_1	2	$-F_1, +F_2;$
V_2	2	$-F_2 - F_3,$

Table 6
Incidence structures in a 4D space

Dimension of the block	SON*	Cl*	IS	Representation
0	S^3	$\emptyset$	S^3	L^3
1	S^2	S^0	$S^2 + S^0$	L^2 + point pair
2	S^1	S^1	$S^1 + S^1$	cycle + cycle
3	S^0	S^2	$S^0 + S^2$	point pair + L^2
4	$\emptyset$	S^3	S^3	L^3

In a similar way cell lists for manifolds (with and without boundary) of greater dimension may be constructed. The list of an n-manifold consists of incidence structures each of which consists of at most two complexes B-isomorphic to a sphere of some lower dimension. Such a complex may be described by a cell list of dimension less than n. Thus, e.g. in the cell list of a 5-manifold, a full description of the incidence structure of a 0- and of a 5-block is a 4D cell list. However, it may be replaced, as in the 3D case, by an enumeration of 1- and 4-blocks, correspondingly. The incidence structures of 1- and 4-blocks contain complexes B-isomorphic to S^3. They may be described as 3D cell lists. The incidence structures of 2- and 3-blocks may be described as 2D cell lists.

The following tables illustrate this for $n = 4$ and 5. The notation L^k stays for an k-dimensional cell list, "cycle" stays for a closed chained list, like that used, e.g. in Tables 2–7. A cycle is an S^1.

The gained understanding shows that cell lists for block complexes of manifolds of any dimension may be constructed by means of a recursion: the cell list of an n-manifold consists of lists of lower dimensions.

Table 7
Incidence structures in a 5D space

Dimension of the block	SON*	Cl*	IS	Representation
0	S^4	$\emptyset$	S^4	L^4
1	S^3	S^0	$S^3 + S^0$	L^3 + point pair
2	S^2	S^1	$S^2 + S^1$	L^2 + cycle
3	S^1	S^2	$S^1 + S^2$	cycle + L^2
4	S^0	S^3	$S^0 + S^3$	point pair + L^3
4	$\emptyset$	S^4	S^4	L^4

6. A comparison of data structures using lists of space elements

A data structure designed to *efficiently* represent topological information must satisfy the following two demands:

(1) The structure must contain *complete* topological information sufficient to get knowledge about topological relations among the subsets of the space *without a search*. To the topological relations belong primarily the incidence and the adjacency relations (two distinct subsets are adjacent if there is a space element incident to both of them).
(2) The structure must be able to correctly represent non-proper complexes (Section 4.1) which are often used in topological investigations because they contain much less elements than the corresponding proper complexes.

Data structures known from the literature do not fulfil these demands. The classical incidence matrix (see, e.g. [18,20]) enables one to encode any *proper* cell complex. It contains complete topological information. However, it is not suitable to encode non-proper complexes, as explained above. Besides that, it is not economical: it contains in the case of an n-dimensional complex

$$\sum_{k=1}^{n} N_{k-1}N_k,$$

elements where N_k is the number k-dimensional cells. This number is in practically relevant cases too large. Because of these reasons data structures using "linear" rather then "quadratic" lists of space elements are preferable.

Most data structures of this kind suggested for 2D images can be hardly generalized for the 3D case. So the structures using the notion of "half-edges", e.g. the frontier topological graph structure (FTG) [6] would need in the 3D case the introduction of "half-faces". Then each edge would occur in so many copies as twice the number of faces bounded by it. The structure would be no more a graph as this is the case for the 2D version of an FTG: a complete FTG structure would be needed for each 3D region. This is not economical.

Structures using "darts" [14,2] contain too many elements: some hundred thousands for relatively simple images [2]. They have no explicit entries for such space elements

Table 8
The cell list of the 3-manifold $S^1 \times S^1 \times S^1$ computed by identification

```
The List contains 1 points, 3 lines, 3 faces and 1 volumes
- - - - - - Partial list of points - - - - - -
Point 1 is incident with 6 lines.
  The lines:    -1;  3;  -3;  2;  -2;  1;
- - - - - - Partial list of lines - - - - - -
line 1; StartP= 1 EndP= 1; Inc. with 4 pairs (F,U) of faces and volumes
  The SON: (F  -2;  V  1)(F  1;  V  1)(F  2;  V  1) (F  1;  V  1)
line 2; StartP= 1 EndP= 1; Inc. with 4 pairs (F,U) of faces and volumes
  The SON: (F  -3;  V  1)(F  1;  V  1)(F  3;  V  1)(F  -1;  V  1)
line 3; StartP= 1 EndP= 1; Inc. with 4 pairs (F,U) of faces and volumes
  The SON: (F  -2;  V  1)(F  3;  V  1)(F  2;  V  1))(F  -3;  V  1)
- - - - - - Partial list of faces - - - - - -
Face 1; Boundary:  (p 1; L  1) (p 1; L  2) (p 1; L  -1) (p 1; L  -2)
Face 2; Boundary:  (p 1; L  3) (p 1; L  1) (p 1; L  -3) (p 1; L  -1)
Face 3; Boundary:  (p 1; L  3) (p 1; L  2) (p 1; L  -3) (p 1; L  -2)
- - - - - - Partial list of volumes - - - - - -
Volume 1 is incident with 6 faces.
  The faces:    1;  2;  3;  -2;  -3;  -1;
```

as volumes, regions, edges and vertices: they contain only the "darts" whose interpretation is not obvious: in [14, p. 36] the semantics of darts is specified by means of a rather complicated relation of connected components of $(n-1)$-G-maps to cells; in the FTGs [6, p. 154] darts are interpreted as "half-edges", however, in the description of border maps [2, Fig. 5] an edge is associated to *four* rather than to two half-edges! In no one of the cited publications the advantages of darts as compared to cells are specified. Data structures using darts do not correspond to our desire to represent a manifold by as few as possible space elements.

In the area of computer graphics and geometric modeling 3D list data structures are known since many years. One of the most popular is the "boundary representation" (see, e.g. [5]). This structure enables one to easily trace the boundary of a 2D face of a body. However, to find which bodies in a 3D scene are adjacent to each other demands an *exhaustive search* through the descriptions of *all vertices of all bodies in the scene*. Even simpler questions, as e.g. which edges are incident to a given vertex, demand an exhaustive search to be answered. This is true for all 3D data structures known to the author, besides the 3D cell list.

As far as we know, the possibility to represent non-proper complexes was not discussed in the literature before the author's publication [13]. Non-proper complexes are important for our investigations since they contain extremely few elements. Thus even in the simplest case of a torus the non-proper complex of Fig. 1a contains 4 elements while the *simplest* proper complex of Fig. 2 contains $4 \times 2^{\dim} = 16$ elements. One of the simplest 3-manifolds represented below (Table 8 and Fig. 7) is described by 8 elements. When representing it as a proper complex one would need at least 2^3 times more, i.e. 64 elements.

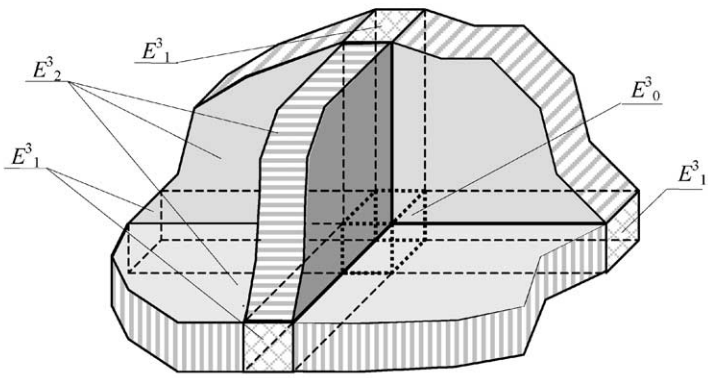


Fig. 6. An example of elementary handles of indices 0–2 glued together.

There is another, may be more important, argument speaking for non-proper complexes: the classical procedure of generating 2D and 3D manifolds by identifying the sides of polygons or polyhedrons *always produces non-proper complexes*. To produce proper complexes the procedure should be replaced by an essentially more complicated one. Our computer experiments described in Section 8.1 implement the classical procedure and hence they could be performed only by means of cell lists.

7. The cell list of a 3-manifold represented as an AC complex

Consider a strongly connected subset of a four-dimensional AC complex. The boundary of the subset is known to be 3-manifold. It is a 3D AC complex. The block complex of a given 3D AC complex may be computed as follows: Consider two closed n-balls whose boundary intersection is an $(n-1)$-ball. Then the union of the n-balls is again an n-ball since uniting is a procedure inverse to the elementary subdivision of an n-cell. We call such two n-balls *mutually simple* or *simple relative to each other*. The union of the closures of two mutually simple n-cells or n-blocks is a closed subcomplex homeomorphic to a closed n-ball B^n.

Let us select an arbitrary n-cell of the given n-dimensional complex ($n = 2, 3$) as the seed of B^n. Then all n-cells, which are simple relative to the growing ball B^n, can be *sequentially* united with it, one cell at each step. The closures of the united n-cells must be labeled as belonging to the closure of the n-block. When there are no more simple cells, the rest consisting of n-cells which are not simple relative to B^n can be subdivided into handles of indices 0–2 [7, p. 27, 166], as shown in Fig. 6 below.

An n-dimensional handle of index λ is defined [7, p. 28] as a direct product $H^n_\lambda = \mathrm{B}^\lambda \times \mathrm{B}^{n-\lambda}$, where B^m is an m-ball. We introduce here the combinatorial notion of an n-dimensional *elementary handle* E^n_λ of index λ as an n-dimensional complex homeomorphic to $\mathrm{B}^\lambda \times c^{n-\lambda}$, where $c^{n-\lambda}$ is an $(n-\lambda)$-dimensional cell. An E^n_λ of any λ is homeomorphic to an n-ball. The difference between the handles (in what follows we write “handle” for “elementary handle”) of different indices is that E^n_λ may be glued to a set S of other handles along a certain embedding $\varphi : (\partial \mathrm{B}^\lambda) \times c^{n-\lambda} \to \partial S$.

Any 3D complex may be partitioned into elementary handles. The first step of partitioning a given 3-manifold consists in sequentially labeling the closures of all simple 3-cells. They compose the maximum closed 3-ball contained in the manifold. The ball is an elementary handle of index 3. After that it is possible to recognize the index of the handle which any given not labeled 3-cell belongs to. Let us call it the *index of a 3-cell*.

For labeling the cells of handles of lower indices it is necessary to find a still not labeled 3-cell of index 2. It is the seed of the actual handle of index 2. Now it is necessary to label all 3-cells of index 2 being simple relative to the actual handle. Also the non-labeled cells of lower dimensions in the closure of the 3-cells must be labeled. When there are no more simple 3-cells, the seed of the next handle must be found. When no more 3-cells with index 2 can be found, the labeling of handles of index 1 may be started, etc. The result of this procedure is a complete partition of the given 3-manifold into elementary handles. Each cell of any dimension gets the label of the handle to which it belongs.

The next step consists in transforming the handles into blocks of a block complex. It may be easily seen that each handle of index m is 3-dual to an m-block. This means that when replacing each k-cell of a handle of index m by a $(3-k)$-cell and inverting the bounding relation, one obtains an m-block. It can be demonstrated that this procedure is inverse to an elementary subdivision of cells incident to the cells of a block. Thus the transformed manifold is homeomorphic to the original one.

The blocks obtained as the result of the transformation can be now recorded in the three-dimensional cell list described in Section 5. The incidence structure of any k-block b^k can be directly read from the closure of the n-cell c^n of the handle E_0 corresponding to a 0-block incident to b^k: due to the labeling procedure described above, the cells of $\mathrm{Cl}^*(c^n)$ contain labels of other handles having a common frontier with E_0. The bounding relations of the blocks are inverse to the bounding relations of the cells of $\mathrm{Cl}^*(c^n)$. Thus all data necessary for a cell list can be obtained.

The described procedure was implemented in a computer program. Some experimental results are reported below, in Section 8.2.

8. Computer experiments

8.1. Generating cell lists of 3-manifolds by identification of polyhedron faces

We use two methods of producing block complexes of 3-manifolds and their cell lists in the computer. The first method implements the classical idea of gluing (or identifying) the faces of a polyhedron. The description of a polyhedron must be input into the computer manually, just in the form of a three-dimensional cell list containing a single 3-block and as many 2-blocks as the number N_f of faces. Also a *list of desired identifications* of the faces and their boundaries, specifying the homeomorphism of the gluing, must be input. The list contains for each face F of the first $N_{\mathrm{f}}/2$ faces the index of the oriented face F_1 which must be identified with F, and the indices of the point and of the oriented edge (line) in the boundary of F_1 which must be identified with the first point and the first edge in the record of the boundary of F. The computer program developed by the author for this purpose replaces the labels of some blocks by that of the identified blocks and calculates the new incidence structures which are unions of the initial ones. The result is a cell list of a 3-manifold.

The program works as follows: First of all the program performs the identification of all points and edges. For this purpose the so-called analysis of equivalence must be done: given is a list of pairs of indices of elements which must be declared equivalent, required is a look-up-table containing for each original index a new index being one and the same for all equivalent elements. The solution of this problem is well-known [22]. However, the equivalence problem for *oriented edges* is a more complicated one: for example, from "A is equivalent to $-B$" must follow "$-A$ is equivalent to B". The relation "A is equivalent to $-A$" must be forbidden. We have found a rather simple solution of this problem and implemented it.

The equivalence of points and edges is calculated during the simultaneous tracing of the boundaries of two faces to be identified. The tracing must be "synchronized" according to the list of desired identifications mentioned above.

As the next step the program allocates new lists of points and lines. The program scans the old list of the lines, converts the old indices into the new ones and inserts the indices of the new oriented lines into the SON* structures of the new points. It also takes from each old line L the indices of the two faces incident to it and inserts them into the SON* structure of the new line in such a way that the faces interleaved by the repeated index of the single volume form a closed cycle. Then the old lists of the points and lines are replaced by the new ones. After this stage the indices in the incidence structures in the lists of the faces and volumes must be replaced by their new values.

Table 8 shows the cell list of the 3-manifold $S^1 \times S^1 \times S^1$ obtained as the result of identifying the faces of a cube. It is represented in the form as it was printed by the computer.

8.2. Generating cell lists of 3-manifolds specified as boundaries of 4D sets

We have developed a computer program which automatically calculates according to the algorithm described in Section 7 the handle decomposition and the cell list of a n-dimensional orientable manifold without boundary, $n = 2, 3$. The manifold must be defined as the boundary of a strongly connected subsets of an $(n + 1)$-dimensional Cartesian AC complex. The program also minimizes the number of blocks while uniting pairs of 0-blocks (points) incident to a line (1-block) until a block complex with a single 0-block, a single 3-block, m 1-blocks and m 2-blocks is obtained (the Euler number $N^0 - N^1 + N^2 - N^3 = 1 - m + m - 1$ must be 0).

Several examples of manifolds were successfully tested. As an example we show the results of investigating the well-known 3-manifold $S^1 \times S^1 \times S^1$ which also was obtained by identifying opposite faces of a cube as described above. The minimized cell list of this manifold contains a single point, three closed curves, three faces each spanned by two curves and a single volume. Since the cell list is redundant (its redundancy is necessary for the purpose of a fast search) its contents may be represented by that of the incidence structure of the single volume.

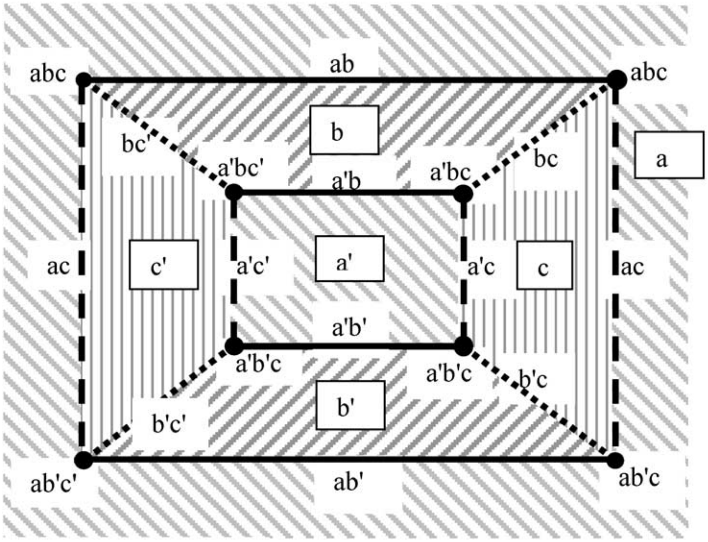


Fig. 7. Incidence structure of the 3-block of the cube with glued opposite faces.

As demonstrated above, the incidence structure of a 3-block of a 3-manifold is B-isomorphic to a 2-sphere and hence may be projected onto the plane. Fig. 7 shows a planar projection of the incidence structure of the 3-block. In Fig. 7 identified blocks are represented by graphical elements of the same hatching. The 2-blocks are denoted by a, b and c. Primed symbols correspond to opposite orientations. The 0- and 1-dimensional blocks are denoted by combinations of the symbols of the bounded 2-blocks.

A presentation of the fundamental group of this manifold may be found as follows: there is a single vertex which is the spanning tree. Therefore, all three 1-blocks ab, ac and bc are generators. Let us denote them by $x = ab$, $y = ac$ and $z = bc$. Then the perimeter of the face a (the exterior area) contains the sequence $xyx'y' = 1$, the perimeter of the face b the sequence $xzx'z' = 1$ and the perimeter of the face c the sequence $yzy'z' = 1$. Thus the fundamental group is the free abelian group of rank 3.

9. Conclusion

The described method gives the possibility to compute automatically a representation of a 2D or 3D manifold as a cell complex with the (almost) minimum number of cells. This evokes the hope to test by means of a fast computer the combinatorial homeomorphism of 3-manifolds as the *isomorphism* of cell complexes. Algorithms for testing the isomorphism are known. The difficulty of using them consists often in a very large number of combinations to be tested. Since our technique gives the possibility to generate representations with a small number of space elements, a search for the isomorphism may be successful. There remains of course a difficult problem: how can one prove that two manifolds are not homeomorphic when no isomorphism has been found?

The method also makes it possible to compute automatically a presentation of the fundamental group of the given manifold. The method may be useful for further investigations of 3-manifolds and may be a contribution for the solution of the still unsolved problem of classifying 3-manifolds.

The 3D cell list developed here may be also used for economically encoding and for analyzing 3D images, e.g. computer tomograms or time sequences of 2D images in digital television. For this purpose a geometric cell list is necessary. The author works now on a method of efficiently encoding surfaces by digital plane patches. This will lead to a precise and economic encoding of 3D scenes. The similar 2D problem of encoding 2D gray value images by digital straight segments is already solved [12].

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