Convergence to Equilibrium on Invariant $d$-Hypersurfaces for Strongly Increasing Discrete-Time Semigroups

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Convergence to equilibrium for every positive semi-orbit of strongly increasing discrete-time semigroups on strongly ordered spaces is proved. The domain of the semigroup is assumed to be slightly more general than a closed order interval in a strongly ordered Banach space which is continuously imbedded into a Banach lattice. The semigroup is assumed to be order-compact, and every positive semi-orbit is assumed to be order-bounded. The crucial hypothesis is the Ljapunov stability of all equilibria. It is also proved that the set of equilibria is a simply ordered arc. The key tools are invariant $d$-hypersurfaces.

INTRODUCTION

A common goal of a number of articles written in the last few years about continuous- and discrete-time semigroups of strongly increasing nonlinear operators (called also strongly monotone or strongly order-preserving) has been to show that (most of) the positive semi-orbits are either convergent (to an equilibrium) or at least quasi-convergent (i.e., asymptotic to a set of equilibria). For continuous-time, strongly increasing local semiflows this goal was achieved by M. W. Hirsch [10, 13] in a very general setting. The decisive result in his approach is the following $\omega$-limit set dichotomy theorem: if $x \prec y$, then either $\omega(x) \succ \omega(y)$ or else $\omega(x) = \omega(y) \subset E$, the set of equilibria. So far no comparable substitute for this theorem has been found for nonautonomous continuous-time or autonomous discrete-time dynamical processes. Instead, the hypothesis of orbital stability (for every positive semi-orbit) was added by N. D. Alikakos, P. Hess, and H. Matano [2] (see also N. D. Alikakos and P. Hess [1] and

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This orbital stability is realized in a Banach lattice setting which provides a "background" for the underlying strongly ordered Banach space. For autonomous discrete-time and time-periodic processes they showed convergence to an equilibrium for every relatively compact positive semi-orbit. They applied this result to obtain quasi-convergence for asymptotically autonomous discrete-time processes.

In this paper we will prove the same results (Theorem 1.5) under a considerably weaker stability hypothesis while assuming a slightly stronger order-compactness hypothesis instead of compactness of orbit closures. Namely, in most applications to differential equations the latter one is verified by showing the former one and the order-boundedness of positive semi-orbits. We will only assume that every equilibrium is Lyapunov stable. (Notice that, for an equilibrium, the notion of Lyapunov stability coincides with that of orbital stability.) Since the set of equilibria $E$ is usually very small compared to the underlying space for the entire process, our hypothesis is much easier to verify. Furthermore, since the domain of our semigroup is assumed to be order-convex in the underlying Banach space, we will show that $E$ coincides with the image of a strictly increasing continuous path. In particular, every positive semi-orbit is orbitally stable. Alikakos et al. [2] assume only that the domain of the semigroup is order-connected.

Our approach is motivated by (mostly) finite-dimensional results of M. W. Hirsch [13, 14] obtained for cooperative systems of ordinary differential equations, and is completely different from the methods employed by Alikakos et al. [2]. Our key tools are invariant $d$-hypersurfaces (determined by a Lipschitz continuous projection) containing no pair $x, y$ with $x < y$. Under mild hypotheses it is proved implicitly in M. W. Hirsch [13] that such an invariant $d$-hypersurface must contain an equilibrium provided the domain of the semigroup is a closed order interval. We show that invariant $d$-hypersurfaces are rather abundant. In fact, every $\omega$-limit set is contained in an invariant $d$-hypersurface. None of our results about invariant $d$-hypersurfaces requires any stability hypothesis. It would be interesting to study the asymptotic behavior of autonomous discrete-time semigroups restricted to a fixed invariant $d$-hypersurface as is done for ordinary differential equations (see, e.g., M. W. Hirsch, C. C. Pugh, and M. Shub [15] and M. W. Hirsch [14]). In this study one could perhaps drop any stability hypothesis in transversal directions.

In our applications we consider two time-periodic dynamical processes: the first one is described by a reaction-diffusion equation of parabolic type with $\tau$-periodic coefficients and reaction function, while the second one is described by a cooperative system of ordinary differential equations with $\tau$-periodic entries. Our results can also be applied to a number of other
time-periodic dynamical processes arising in chemical and biochemical reactions, epidemiology, and population biology. For some of these applications we refer to J. C. Frauenthal [7], H. I. Freedman [8], M. W. Hirsch [11], A. Lahmanovich and J. A. Yorke [16], H. Othmer [17], J. F. Selgrade [21], and H. L. Smith [22, 23].

This paper is organized as follows. In Section 1 we present our main results, Proposition 1.1 (existence of an invariant $d$-hypersurface containing a prescribed unordered, positively invariant set), Proposition 1.2 (analytic description of an invariant $d$-hypersurface), Theorem 1.3 (convergence in an order interval $[p, q]$ if $p$ and $q$ are connected by an increasing continuous path in $E$), Theorem 1.4 (convergence in $[p, q]$ if $\{p, q\} \cap E = \{p, q\}$), and finally Theorem 1.5 (generalization of a convergence theorem of N. D. Alikakos, P. Hess, and H. Matano [2]), which is the only result requiring the Lyapunov stability of equilibria. In Section 2 we prove Propositions 1.1 and 1.2. In Section 3 we prove Theorems 1.3, 1.4, and 1.5. The proof of Theorem 1.4 employs Lemma 3.1, the existence of an equilibrium on a given invariant $d$-hypersurface in a positively invariant, closed order interval (due to M. W. Hirsch [13, Proof of Theorem 10.5] in a special case). In Section 4 we present two applications of these results, Example 4.1 (a reaction-diffusion PDE) and Example 4.2 (a cooperative system of ODEs). Finally, in Section 5 we discuss possible generalizations of Theorem 1.5 by weakening the stability hypothesis, and pose several open problems thus arising.

1. MAIN RESULTS

We start with some notation and a few definitions. Throughout the entire paper we assume the following three hypotheses (X), (V), and (T):

(X): $X$ is an ordered, metrizable topological space, i.e., $X$ is a metrizable topological space with a closed (partial) order relation $\leq$ in $X \times X$ (shortly, $X$ is an ordered space). We write $x \preceq y$ if $(x, y)$ belongs to the interior of the order relation in $X \times X$, while $x < y$ means $x \leq y$, $x \neq y$.

(V): $V$ is a strongly ordered, metrizable topological vector space (shortly, strongly ordered vector space), which is equivalent to saying that the positive cone $V_+ = \{x \in V: x \geq 0\}$ of $V$ has nonempty interior denoted by $\text{Int}(V_+)$. (In some of our results we will assume that $X$ is a nonempty subset of $V$ with closure $\text{Cl}(X)$.)

(T): $T$ is a continuous, strongly increasing mapping of $X$ into itself, i.e., $x, y \in X$ and $x < y$ implies $Tx \leq Ty$.

An ordered space $X$ is called strongly ordered if every open subset $U$ of $X$ satisfies: (SO1) If $x \in U$ then $a \preceq x \preceq b$ for some $a, b \in U$. It is easy to see
that, for every open subset $U$ of $X$, (SO1) implies: (SO2) If $a, b \in U$ and $a < b$ then $a \leq x \leq b$ for some $x \in U$. (For example, every nonempty, open subset of $V$ is a strongly ordered space.)

The **positive semi-orbit** (shortly, *orbit*) of any $x \in X$ is defined by
\[ \gamma^+(x) = \{ T^n x : n \in \mathbb{Z}_+ \}, \]
where $\mathbb{Z}_+ = \{0, 1, 2, \ldots \}$, and the $\omega$-limit set of $x$ is defined by $\omega(x) = \{ y \in X : T^n x \to y \ (k \to \infty) \}$ for some sequence $n_k \to \infty$ in $\mathbb{Z}_+$. Notice that if $\gamma^+(x)$ is relatively compact in $X$, then $\omega(x) \neq \emptyset$. The set of all *equilibria* (i.e., fixed points) of $T$ is denoted by $E(T)$. A subset $Y$ of $X$ is called *positively invariant* (shortly, *invariant*) if $T(Y) \subseteq Y$, and *totally invariant* if $T(Y) = Y$. For instance, every $\gamma^+(x)$ is invariant, and every $\omega(x)$ is totally invariant.

Given $a, b \in X$, the set $[a, b] = \{ x \in X : a \leq x \leq b \}$ is called a **closed order interval**, and $[[a, b]] = \{ x \in X : a \ll x \ll b \}$ is called an **open order interval** in $X$. We write $[a, \infty] = \{ x \in X : x \gg a \}$, and similarly for $[\infty, b]$, etc. A subset $Y$ of $X$ is called: *order-convex* in $X$ if $[a, b] \subseteq Y$ whenever $a, b \in Y$ and $a < b$; *lower closed* if $[[-\infty, b]] \subseteq Y$ whenever $b \in Y$; and *upper closed* if $[a, \infty]] \subseteq Y$ whenever $a \in Y$.

We denote closed order intervals in $V$ by $[a, b]_V = \{ x \in V : a \leq x \leq b \}$, and similarly, all other concepts in $V$ will be marked by the subscript $V$ in case confusion might arise.

Now we are ready to introduce our crucial concept:

**Definition 1.0.** A pair $(A, B)$ of subsets $A, B$ of $X$ is called an **order decomposition** of $X$ if it has the following properties:

(i) $A \neq \emptyset$ and $B \neq \emptyset$; (ii) $A$ and $B$ are closed; (iii) $A$ is lower closed and $B$ is upper closed; (iv) $A \cup B = X$; and (v) $\text{Int}(A \cap B) = \emptyset$.

An order decomposition $(A, B)$ of $X$ is called *invariant* if $T(A) \subseteq A$ and $T(B) \subseteq B$. The set $H = A \cap B$ (possibly empty) is called the **boundary** of the order decomposition $(A, B)$ of $X$. A **d-hypersurface** is any nonempty subset $H$ of $X$ such that $H = A \cap B$ for some order decomposition $(A, B)$ of $X$.

Notice that the boundary $H$ of an order decomposition $(A, B)$ of $X$ satisfies $H = \partial A = \partial B$, where "$\partial$" is the boundary symbol in $X$, and $H$ is invariant whenever $(A, B)$ is invariant. It is also easy to see that a d-hypersurface $H$ never contains two strongly ordered points $x, y$ (with $x \ll y$). Consequently, if $H$ is invariant then it must be *unordered*, i.e., no pair of points $x, y \in H$ satisfies $x < y$.

If $X$ is a strongly ordered space it turns out to be very useful to work with the *order-topology* on $X$ whose neighborhood base is generated by all open order intervals $[[a, b]]$ with $a \ll b$. If $Y$ is any subset of $X$, we denote by $Y$ the set $Y$ endowed with the induced order topology. A subset $Y$ of $X$ is called *order-open* (order-closed, resp.) if it is open (closed, resp.) in $Y$. Notice that the identity mapping $i : X \to X$ is continuous, but in general not
homeomorphic. It is proved in Hirsch [12, 13] that if \( f: X_1 \rightarrow X_2 \) is a continuous, increasing mapping (i.e., \( x \leq x, y \) implies \( f(x) \leq f(y) \)) between two strongly ordered spaces, then \( f \) is continuous also in the order topologies, that is, the induced map \( f': X_1 \rightarrow X_2 \) is continuous. It is easy to see that the order topology on \( V \) is induced by any ordered norm \( |\cdot|_o \) on \( V \) defined by

\[
|x|_o = \inf \{ \lambda \in \mathbb{R}_+^1 : -\lambda e \leq x \leq \lambda e \}
\]

for some \( e \in \text{Int}(V_+) \), where \( \mathbb{R}_+^1 = [0, \infty) \).

Our first result guarantees the existence of invariant \( d \)-hypersurfaces, and our second result describes them as Lipschitz hypersurfaces:

**Proposition 1.1.** Let \( X \) and \( T \) satisfy (X) and (T), and let \( X \) be strongly ordered. Assume that \( G \) is a nonempty, unordered, invariant subset of \( X \) (for instance, \( G = \omega(x) \), for any relatively compact \( \gamma^+(x) \)). Then there exists an invariant order decomposition \((A, B)\) of \( X \) such that \( G \subseteq H = A \cap B \).

We recall that an everywhere defined linear mapping \( L: V_1 \rightarrow V_2 \) between two ordered vector spaces is called positive (strongly positive, resp.) if \( x, y \in V_1 \) and \( x \leq y \) implies \( Lx \leq Ly \) (\( Lx < Ly \), resp.) in \( V_2 \). We set \( I = \text{identity mapping on } V_1 \), and \( \mathbb{R}_1^1 = (-\infty, \infty) \).

**Proposition 1.2.** Let \( X \) be a nonempty, open subset of \( V_1 \), and let \((A, B)\) be an order decomposition of \( X \) with the boundary \( H = A \cap B \). Fix any vector \( v \in \text{Int}(V_+) \), and denote by \( R = \text{lin}\{v\} \) the linear subspace of \( V \) spanned by \( v \). Let \( Q \) be a positive continuous projection of \( V \) onto \( R \), which always exists, and set \( P = I - Q \) with \( W = P(V) \), the range of \( P \), so that \( V = W \oplus R \) is the direct algebraic and topological sum of \( W \) and \( R \). Then we have the following statements:

(i) The restriction \( P|_H \) of \( P \) to \( H \) is one-to-one, and both \( P|_H \) and its inverse \( \pi = (P|_H)^{-1} \): \( P(H) \rightarrow H \) are Lipschitz continuous in the ordered norm \( |\cdot|_V \) with a common Lipschitz constant 2.

(ii) \( P|_H \) is a homeomorphism of \( H \) onto \( P(H) \) in the topologies induced by that on \( V \).

(iii) Furthermore, set

\[
H \oplus R = \{ x \in V : x = x_0 + \tau v \text{ for some } x_0 \in H \text{ and } \tau \in \mathbb{R}_1^1 \},
\]

where \( x_0 \) and \( \tau \) are uniquely determined by \( Px = Px_0 \), and define a mapping \( h: H \oplus R \rightarrow V \) by

\[
h(x) = Px_0 + \tau v, \quad x = x_0 + \tau v \in H \oplus R,
\]
and similarly for $P(H) \oplus R$. Then also $h$ and its inverse $h^{-1} : P(H) \oplus R \to H \oplus R$ are Lipschitz continuous in the ordered norm $| \cdot |_v$ with a common Lipschitz constant $\gamma$, and $h$ is a homeomorphism of $H \oplus R$ onto $P(H) \oplus R$ in the topologies induced by that on $V$.

(iv) If, in addition, $X$ is order-open in $V$ (i.e., open in $\hat{V}$), then $P(H)$ is order-open in $W$, and $P(H) \oplus R$ is order-open in $V$.

Only part (i) of Proposition 1.2 will be needed for the proofs of Theorems 1.3, 1.4, and 1.5. We refer to M. W. Hirsch [14, Proposition 2.6] for a finite-dimensional analogue of part (i), i.e., $\dim(V) < \infty$.

We say that a subset $K$ of $X$ attracts another set $Y \subset X$ if $\gamma^+(x)$ is relatively compact in $X$ and $\omega(x) \subset K$ for all $x \in Y$. Let $X \subset V$. Let $| \cdot |_e$ be any ordered norm on $V$, where $e \in \text{Int}(V_+)$. An equilibrium $p \in E(T)$ is called Lyapunov order-stable (shortly, order-stable) if:

(OS) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in X$ and $|x - p|_e < \delta$ implies $|T^n x - p|_e < \varepsilon$ for all $n \in \mathbb{Z}_+$.

This definition is independent from the choice of $e$. If $p \in E(T)$ is not order-stable, then it is called order-unstable.

Our next two theorems generalize results due to M. W. Hirsch [13, Theorems 9.7(a), (b) and 10.5], who proved them for continuous-time local semiflows; the first one only for $V = \mathbb{R}^N$. Our proofs essentially follow Hirsch's methods, except for the use of invariant $d$-hypersurfaces.

**THEOREM 1.3.** Let $X$ and $T$ satisfy (X) and (T), and let $X$ be strongly ordered. Let $J : [0, 1] \to X$ be a strictly increasing continuous path (with its image) contained in $E(T)$ (i.e., $\tau_1 < \tau_2$ implies $J(\tau_1) < J(\tau_2)$) with endpoints $a = J(0)$ and $b = J(1)$. Assume that $\gamma^+$ is relatively compact in $X$ for every $x \in [a, b]$. Then we have

$$\omega(x) = J(\tau), \quad x \in [a, b], \quad \text{and} \quad T^n x \to J(\tau) \quad \text{as} \quad n \to \infty,$$

where $\tau = \tau(x) \in [0, 1]$ depends on $x$. Moreover, the mapping $\tau : [a, b] \to [0, 1]$ is increasing and order-continuous (hence continuous).

**THEOREM 1.4.** Let $T : [p, q]_V \to [p, q]_V$ be a continuous, compact, strongly increasing mapping where $[p, q]_V$ is an order interval in $V$. Assume that $p, q \in E(T)$ with $p \ll q$ are the only equilibria of $T$ in $[p, q]_V$. Then the entire set $(p, q) = \{x \in V : p < x < q\}$ is attracted by either $p$ or $q$. In particular, either $q$ or $p$, resp., is order-unstable.

A subset $Y$ of $X$ is called order-bounded if it is contained in a finite union of order intervals in $X$. A mapping $T : X \to X$ is called order-compact if $T([a, b])$ has compact closure for all $a, b \in X$ with $a \leq b$. If $Y \subset X$ and
Z \subset X$, we write $Y \preceq Z$ whenever $y \preceq z$ for all $y \in Y$ and $z \in Z$, and similarly for "<" and "\preceq."

Our last result is a consequence of Theorems 1.3 and 1.4. Here we assume that $V$ (see (V)) is continuously imbedded into a Banach lattice $V_0 = (V_0, \leq, \| \cdot \|)$ whose ordering extends that of $V$, i.e., $V_+ = (V_0)_+ \cap V$, and

$$(V_0) \text{ every nonempty, compact subset of } V \text{ has a supremum in } V_0 \text{ (e.g., } V_0 \text{ can be any reflexive Banach lattice, any AL-space, or any AM-space; see H. H. Schaefer [20, Theorem II.5.10 and Proposition II.7.6])}.$$

We further assume that $X_0$ is a nonempty, closed subset of $V_0$ satisfying:

$$(X_0.1) \text{ } X_0 \text{ is closed under the supremum and infimum operations; and}$$

$$(X_0.2) \text{ } X_0 \text{ is order-convex in } V_0.$$

Finally, let $T_0 : X_0 \to X_0$ be a mapping whose restriction to $X = X_0 \cap V$ is denoted by $T$, where $X_0$ and $X$ have the topologies induced from $V_0$ and $V$, respectively. We assume that $T_0$ satisfies:

$$(T.1) \text{ } T_0(X_0) \subset V \text{ and } T_0 : X_0 \to X \text{ is continuous;}$$

$$(T.2) \text{ } T : X \to X_0 \text{ is order-compact;}$$

$$(T.3) \text{ } T : X \to X \text{ is strongly increasing; and}$$

$$(T.4) \text{ every equilibrium of } T_0 \text{ is Ljapunov stable in } X_0 \text{ (i.e., for every } p \in E(T_0) \text{ and } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that } x \in X_0 \text{ and } \| x - p \| \leq \delta \text{ implies } \| T^n_0 x - p \| \leq \varepsilon \text{ for all } n \in \mathbb{Z}_+).$$

**THEOREM 1.5.** Let all hypotheses (V), (V_0), (X_0.1), (X_0.2), (T.1), (T.2), (T.3), and (T.4) be satisfied. Assume that $\gamma^+(x)$ is order-bounded for every $x \in X$. Then $E(T)$ is either a singleton or coincides with (the image of) a strictly increasing continuous path $J : [0, 1] \to X$, where $J_\gamma \subset [0, 1]$ is an interval with endpoints 0 and 1 which is closed if and only if $E(T)$ is compact in $X$. Moreover, for every $x \in X_0$, we have

$$\omega(x) = \{ p \}, p \in E(T_0) = E(T), \text{ and } T^n_0 x \to p \text{ as } n \to \infty.$$ 

If $E(T)$ is not a singleton, then also $p = J(\tau(x))$, where the mapping $\tau : X_0 \to [0, 1]$ is increasing and continuous.

It is clear that the order-compactness of $T$ and the order-boundedness of $\gamma^+(x)$ above imply that $\gamma^+(x)$ has compact closure in $X$. Hypotheses (X_0.1) and (X_0.2) are satisfied, for instance, if $X_0$ is a closed order interval in $V_0$, or the closure of the union of an increasing sequence of closed order intervals in $V_0$, which is the case in many applications (see Section 4).
The following result contains more information about the convergence to equilibria which are not the endpoints of (the image of) the path \( J \).

**Corollary 1.6.** Let all hypotheses of Theorem 1.5 be satisfied. Assume that \( E(T) \) is not a singleton. Then every \( x \in X \) such that \( \tau(x) \equiv \rho \in (0, 1) \) is contained in an invariant \( d \)-hypersurface \( H = H(x) \subset X \). If \( V \) is separable, this \( d \)-hypersurface is uniquely determined by \( \rho \in (0, 1) \) for all but countably many \( \rho \)'s from \( (0, 1) \).

2. **Invariant \( d \)-Hypersurfaces**

We need the following lemma.

**Lemma 2.1.** Let \( X \) be a strongly ordered space. If \( F \subset X \) is lower closed (upper closed, resp.), then so is its closure \( \text{Cl}(F) \). Its complement \( X - F \) is upper closed (lower closed, resp.). The union and intersection of any family of lower (upper) closed sets are also lower (upper) closed.

**Proof.** Consider \( x \in X - \text{Cl}(F) \). Then also \( y \in X - \text{Cl}(F) \) for some \( y \leq x \) because \( X \) is strongly ordered. Consequently, we have \([y, \infty]\) \cap \( F = \emptyset \) since \( F \) is lower closed. Hence \([x, \infty]\) \cap \text{Cl}(F) = \emptyset. So we have proved that \( X - \text{Cl}(F) \) is upper closed, which is equivalent to \( \text{Cl}(F) \) is lower closed. Q.E.D.

**Proof of Proposition 1.1.** Define

\[ G_- = \{ x \in X : x \leq y \text{ for some } y \in G \} \]

and

\[ G_+ = \{ x \in X : x \geq y \text{ for some } y \in G \}. \]

We claim that

\[ \text{Cl}(G_-) \cap \text{Int}(G_+) = \emptyset \quad (1) \]

and

\[ \text{Cl}(G_+) \cap \text{Int}(G_-) = \emptyset. \quad (2) \]

We prove only (1); the proof of (2) is similar. Suppose \( x \in \text{Cl}(G_-) \cap \text{Int}(G_+) \neq \emptyset \). Since \( G_+ \) is upper closed and \( X \) is strongly ordered, \( x \in \text{Int}(G_+) \) implies \( x \gg z \) for some \( z \in G_+ \). But then \([z, \infty]\) is an open order neighborhood of \( x \), and therefore \( x \in \text{Cl}(G_-) \) implies \( z \leq w \) for some \( w \in G_- \). It follows from the definition of \( G_- \) and \( G_+ \) that \( a \leq z \) and \( w \leq b \).
for some \( a, b \in G \). Hence, we obtain \( a \preceq b \), which contradicts our hypothesis that \( G \) is unordered. We conclude that both (1) and (2) must hold.

Denote by \( Y \) the set of all lower closed, closed, invariant subsets \( Y \) of \( X \) such that

\[
(i) \quad G_- \subset Y; \quad \text{and} \quad (ii) \quad Y \cap \text{Int}(G_+) = \emptyset.
\]

First note that Lemma 2.1 implies \( \text{Cl}(G_-) \in Y \) because \( G_- \) is lower closed and invariant, and \( T \) is increasing and continuous. Second we show that the ordered set \( Y \) endowed with the "\( \subset \)" ordering possesses a maximal element. Consider a nonempty, simply ordered subset \( Y' \) of \( Y \). Set

\[
Z = \text{Cl} \left( \bigcup \{ Y : Y \in Y' \} \right).
\]

It is easy to see that \( Z \in Y \). Hence, we may apply Zorn's lemma to conclude that \( Y \) possesses a maximal element, say, \( Y_0 \).

Set \( Y_0^* = \text{Cl}(X - Y_0) \). Hence, \( \partial Y_0 = \partial Y_0^* = Y_0 \cap Y_0^* \).

By (ii), \( Y_0^* \neq \emptyset \). Lemma 2.1 shows that \( Y_0^* \) is upper closed. We claim that the set \( Y_0^* = \text{Cl}(X - Y_0) \) is also invariant.

On the contrary, suppose there exists \( x \in Y_0^* \) such that \( Tx \notin Y_0^* \). Hence, \( Tx \in \text{Int}(Y_0) \) by (3). We show that also \( x \notin \text{Cl}(G_+) \). Suppose \( x \in \text{Cl}(G_+) \). Then every order neighborhood \( [[-\infty, b]] \) of \( x \) intersects the set \( G_+ \), and consequently, \( G_+ \) is upper closed implies \( b \in \text{Int}(G_+) \). In particular, \( b \gg b' \) for some \( b' \in G \). From the strong monotonicity of \( T \) we obtain \( Tb \gg Tb' \in G \), which entails also \( Tb \in \text{Int}(G_+) \). But \( Tx \in \text{Int}(Y_0) \) and the continuity of \( T \) enable us to choose \( b \in [[x, \infty]] \) such that \( Tb \in \text{Int}(Y_0) \).

Thus \( Tb \in Y_0 \cap \text{Int}(G_+) \) contradicts (ii), and therefore \( x \notin \text{Cl}(G_+) \) must hold. Now we can choose \( b \in [[x, \infty]] \) such that \( b \notin \text{Cl}(G_+) \) and \( Tb \in \text{Int}(Y_0) \). Then \( Y_0 \) is lower closed and \( x \notin \text{Int}(Y_0) \) imply \( b \notin Y_0 \) and \( T[[-\infty, b]] \subset [[-\infty, Tb]] \subset Y_0 \). Also \( [[-\infty, b]] \cap G_+ = \emptyset \) since \( G_+ \) is upper closed.

Define the set \( Y_1 = Y_0 \cup [[-\infty, b]] \). Clearly \( Y_1 \in Y \), and \( Y_0 \) is properly contained in \( Y_1 \), which contradicts the maximality of \( Y_0 \). We conclude that \( Y_0^* \) must be invariant.

We observe that \((A, B) = (Y_0, Y_0^*)\) is an invariant order decomposition of \( X \) satisfying \( G_- \subset A \) and \( G_+ \subset B \). Finally, we have

\[
G = G_+ \cap G_- \subset A \cap B = H.
\]

Q.E.D.

In Proposition 1.1 we may take \( G = \omega(x) \) whenever \( \text{Cl}(\nu^+(x)) \) is compact, as it follows from the following lemma. We recall that a subset \( C \) of
$X$ is called a cycle (or $k$-cycle) of $T$ if $C = \{ p, Tp, \ldots, T^{k-1}p \}$ for some $k = \text{card}(C) \geq 2$ and $T^k p = p$. The point $p$ is called a $(k-\text{periodic point})$ of $T$.

**Lemma 2.2.** Let $X$ and $T$ satisfy $(X)$ and $(T)$. Let $x \in X$ be such that $\text{Cl}(\gamma^+(x))$ is compact. Assume that $\text{Cl}(\gamma^+(x))$ is not unordered. Then $\omega(x)$ is either a singleton or a cycle, and is unordered.

**Proof.** Let $y < z$ for some $y, z \in \text{Cl}(\gamma^+(x))$. Since $\text{Cl}(\gamma^+(x))$ is invariant and $Ty \subseteq Tz$, there exist $m, n \in \mathbb{Z}_+$, $m \neq n$, such that $T^m x \subseteq T^n x$. Setting $k = |m - n|$ we obtain a monotone sequence (increasing if $m < n$, and decreasing if $m > n$)

$$x_0, T^k x_0, T^{2k} x_0, \ldots,$$

with $x_0 = T^l x$ and $l = \min\{m, n\}$, whose limit $w \in \text{Cl}(\gamma^+(x))$ exists by compactness and monotonicity and satisfies $T^k w = w$. Hence $\omega(x) = \{ w, Tw, \ldots, T^{k-1}w \}$, $k \geq 1$. Since $T$ is strongly increasing, $\omega(x)$ must be unordered unless it is infinite. Q.E.D.

Let $A, B \subset V$. We set $A + B = \{ a + b : a \in A, b \in B \}$.

**Proof of Proposition 1.2.** The existence of the projection $Q$ is due to M. Krein and M. Rutman, see H. H. Schaefer [19, Sect. V.5.4, Corollary 2].

**Proof of (i).** (We follow M. W. Hirsch [14, Proposition 2.6].) To show that $\hat{P}|_H$ is one-to-one we take

$$x = x_0 + \tau v, y = y_0 + \sigma v \in H \oplus R$$

satisfying $Px = Py$, where $x_0, y_0 \in H$ and $\tau, \sigma \in \mathbb{R}^1$. Then $x_0 - y_0 \in \text{Ker}(P) = R$, the kernel of $P$, whence $x_0 - y_0 = \rho v$ for some $\rho \in \mathbb{R}^1$. Since $x_0, y_0 \in H$ and $H$ is unordered, we conclude that $\rho = 0$, i.e., $x_0 = y_0$.

Next we show that $\hat{P}|_H$ is Lipschitz continuous. Since $Q$ is a positive continuous projection of $V$ onto $R$, it is also order-continuous, and so is $P = I - Q$. Hence, $P = \hat{P}$ is a bounded linear operator on $V$ with respect to the ordered norm $|\cdot|_v$. The operator norm of $\hat{P}$ satisfies

$$|\hat{P}|_{v, \text{op}} = |I - \hat{Q}|_{v, \text{op}} \leq |I|_{v, \text{op}} + |\hat{Q}|_{v, \text{op}} = 1 + 1 = 2.$$

In particular, $\hat{P}|_H$ is Lipschitz continuous for $|\cdot|_v$ with a Lipschitz constant 2.

To prove the Lipschitz continuity of $\pi = (\hat{P}|_H)^{-1}$ we denote by $S = \{ x \in V : |x|_v = 1 \}$ the unit sphere in $V$. Notice that $S + v \subset V_+$, and $S + \rho v \subset \text{Int}(V_+)$ for each $\rho > 1$. Let $S_w = S \cap W$ be the unit sphere in $W$. Then, given $x \in S_w$ and $\rho \in \mathbb{R}^1$ arbitrary, $x + \rho v \notin \text{Int}(V_+)$ implies $\rho \leq 1$. 

Now take \( a, b \in P(H) \), \( a \neq b \). Set \( a - b = w \in W \) and \( \pi a - \pi b = z \in V \). Then \( P_z = w \) shows that \( z = w + \lambda v \) for some \( \lambda \in \mathbb{R}^1 \). We may assume that \( \lambda \geq 0 \). Notice that the points \( \pi a, \pi b \in H \) are not ordered by \( \leq \) or \( \geq \) which entails \( z \notin \text{Int}(V_+) \). Since also
\[
\frac{z}{|w|} = \frac{w}{|w|} + (\frac{\lambda}{|w|}) v \notin \text{Int}(V_+),
\]
it follows that \( \rho = \frac{\lambda}{|w|} \leq 1 \). From the triangle inequality and \( |v| = 1 \) we therefore get
\[
|z| / |w| \leq 1 + \rho \leq 2,
\]
which completes the proof of part (i).

**Proof of (ii).** Since \( Q : V \to V \) is continuous, so is \( P = I - Q \). It remains to prove that \( \pi : P(H) \to H \) is continuous. Consider the mapping \( \Pi : P(H) \to V \) defined by \( \Pi = \pi - P \). Take any \( y \in P(H) \). Then \( \Pi y = P \pi y - Py = y - y = 0 \) shows that \( \Pi : P(H) \to R \subset V \). By part (i), both \( P \) and \( \pi \) are order-continuous, and so is \( \Pi \). Hence, \( \dim(R) = 1 \) implies that \( \mathcal{R} = R \), and therefore \( \Pi \) is also continuous. We conclude that \( \pi = P + \Pi \) is continuous.

**Proof of (iii).** Since \( P \) is one-to-one, so is \( h \). The continuity of \( h \) and \( h^{-1} \) follows from that of \( P \) and \( \pi \) and the following identities, respectively:
\[
h(x) = Px + x - \pi Px, \quad x = x_0 + \tau v \in H \oplus R, \quad (1)
\]
and
\[
h^{-1}(y) = \pi Py + y - Py, \quad y = y_0 + \rho v \in P(H) \oplus R. \quad (2)
\]
Combining these two identities with part (ii) and the triangle inequality we obtain that both \( h \) and \( h^{-1} \) are also Lipschitz continuous in the norm \( |\cdot|_v \) with a common Lipschitz constant \( 7 \).

**Proof of (iv).** Let \( X \) be order-open in \( V \). This means that every \( x \in X \) has an order neighborhood \( \left[ [a, b] \right]_V \) in \( V \) satisfying \( x \in \left[ [a, b] \right]_V \subset X \).

Fix any \( x \in H \subset X \). We want to show that \( y = Px \) is in the interior of \( P(H) \subset \mathcal{W} \). Using a shift if necessary, we may assume that \( x = 0 \). Then also \( y = 0 \). Choose \( \rho > 0 \) so small that \( \left[ [-\rho v, \rho v] \right]_V \subset X \). Now it is clear that we may restrict ourselves to the case when \( X = \left[ [-\rho v, \rho v] \right]_V \). Since \( P \) is a projection of \( V \) onto \( W \), we obtain
\[
P(\left[ [0, \rho v] \right]_V) = \left[ [-\left(\frac{\rho}{2}\right) \rho v, \left(\frac{\rho}{2}\right) \rho v] \right]_V \cap W, \quad (3)
\]
which is a neighborhood of \( 0 \in \mathcal{W} \). Hence, it suffices to prove that
\[
P(\left[ [0, \rho v] \right]_V) \subset P(H). \quad (4)
\]
Take any \( z \in [[0, \rho v]]_\nu \). Then \( z - \rho v \in [[-\rho v, 0]]_\nu \). In particular, the line segment \( L \) with endpoints \( z \) and \( z - \rho v \) intersects both sets \( \mathcal{A}_1 = [[-\rho v, 0]]_\nu \) and \( \mathcal{B}_1 = [[0, \rho v]]_\nu \). Also \( L \subseteq [[-\rho v, \rho v]]_\nu \). Now recall that \( 0 \in H = \mathcal{A} \cap \mathcal{B} \), where \( \mathcal{A} \) is lower closed and \( \mathcal{B} \) is upper closed, and both are closed in \( X = [[-\rho v, \rho v]]_\nu \). Consequently, \( \mathcal{A}_1 \subseteq \mathcal{A} \) and \( \mathcal{B}_1 \subseteq \mathcal{B} \), and in particular, \( L_A = L \cap \mathcal{A} \neq \emptyset \) and \( L_B = L \cap \mathcal{B} \neq \emptyset \). Moreover, \( L_A \) and \( L_B \) are closed in \( L \) with \( L_A \cup L_B = L \). Since \( L \) is connected, we must have \( L_A \cap L_B = \emptyset \), which shows that \( L \cap H = \emptyset \). Thus, we get \( P_L = P(L) \subset P(H) \), and so (4) is valid. We have proved that \( P(H) \) is open in \( \hat{W} \).

Finally, \( \hat{V} \) is a topological product of \( \hat{W} \) and \( R \). Hence, \( P(H) \oplus R \) is an open subset of \( \hat{V} \) because \( P(H) \) is open in \( \hat{W} \). Q.E.D.

3. Proofs of Theorems

Proof of Theorem 1.3. Fix \( x \in [a, b] \) and denote by \( \alpha \) and \( \beta \), resp., the largest and the smallest numbers in \( [0, 1] \subset \mathbb{R}^1 \) satisfying

\[
J(x) \leq \omega(x) \leq J(\beta). \tag{1}
\]

Since \( \text{Cl}(\gamma^+(x)) \) is compact, the topologies of \( X \) and \( \hat{X} \) coincide on \( \text{Cl}(\gamma^+(x)) \). Thus, our goal is to prove that \( \alpha = \beta \). Suppose not, i.e., \( \alpha < \beta \). By Lemma 2.2 the set \( \omega(x) \) is unordered. Hence \( J(\alpha) \notin \omega(x) \) or \( J(\beta) \notin \omega(x) \). Since \( \omega(x) \) is compact and totally invariant, and \( T \) is strongly increasing, it follows from (1) that \( J(\alpha) \leq \omega(x) \) or \( \omega(x) \leq J(\beta) \). But this contradicts the definition of \( \alpha \) and \( \beta \) since \( X \) is strongly ordered and \( J \) is strictly increasing and continuous with \( J([0, 1]) \subset E(T) \). We conclude that \( \alpha = \beta = \tau(x) \in [0, 1] \).

The mapping \( \tau: [a, b] \to [0, 1] \) is clearly increasing because \( x, y \in X \) and \( x \leq y \) implies \( T^n x \leq T^n y \) for all \( n \in \mathbb{Z}_+ \). To prove that \( \tau \) is order-continuous consider any \( x \in [a, b] \). Since \( \tau \) is increasing we have to show only the following two claims:

\begin{enumerate}

\item [(L)] If \( 0 \leq \alpha < \tau(x) \) then there exists \( y \in [[-\infty, x]] \) such that \( \alpha < \tau(w) \leq \tau(x) \) for all \( w \in [y, x] \cap [a, b] \).

\item [(R)] If \( \tau(x) < \beta \leq 1 \) then there exists \( z \in [[x, \infty]] \) such that \( \tau(x) \leq \tau(w) < \beta \) for all \( w \in [x, z] \cap [a, b] \).

\end{enumerate}

We prove only (L) (the left-order-continuity of \( \tau \)); the proof of (R) is analogous. Choose \( x, x' \in \mathbb{R}^1 \) with \( 0 \leq x < x' < \tau(x) \). Let \( H = \mathcal{A} \cap \mathcal{B} \) be the boundary of an invariant order decomposition \( (\mathcal{A}, \mathcal{B}) \) of \( X \) satisfying
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J(\alpha') \in H. Such \((A, B)\) exists by Proposition 1.2. Then obviously \(x \in \text{Int}(B)\), since otherwise \(x \in A\) would imply \(\tau(x) \leq \alpha'\), a contradiction. Hence, there exists \(y \in \text{Int}(B)\) such that \(y \leq x\) because \(X\) is strongly ordered. Since \(B\) is upper closed, we obtain \([y, \infty] \subset B\). Finally, \(\tau\) is increasing and \(B\) is invariant imply \(\tau(w) \geq \alpha'\) for all \(w \in [y, \infty] \cap [a, b]\). In particular, (L) is valid. So we have proved that \(\tau\) is order-continuous. Q.E.D.

The proof of Theorem 3.1 makes use of the following result whose special case is due to M. W. Hirsch [13, Proof of Theorem 10.5].

**Lemma 3.1.** Let \(T: [p, q] \to [p, q]\) be a continuous, compact, strongly increasing mapping where \([p, q]\), \(p < q\), is an order interval in \(V\). Hence \(T([p, q] \setminus \{p, q\}) \subset ([p, q])\). Assume that \((A, B)\) is an invariant order decomposition of \(X = ([p, q])\) with the boundary \(H = A \cap B\). Then \(H \cap E(T) \neq \emptyset\).

**Proof.** Set \(v = q - p \in \text{Int}(V_+).\) Using a shift if necessary, we may assume that \(p = 0\), and so \(q = v\). Let \(P\) and \(\pi\) be as in Proposition 1.2. Since both \(P\) and \(\pi\) are Lipschitz continuous in the order norm \(\|\cdot\|\), they can be uniquely extended to Lipschitz continuous mappings \(P_0: H_0 \to \hat{V}\), where \(H_0 = \text{Cl}_{\hat{v}}(H)\), and \(\pi_0: H_1 \to \hat{V}\), where \(H_1 = \text{Cl}_{\hat{v}}(P(H))\), respectively. Notice that \(P_0 = P|_{H_0}\) and \(\pi_0\) are each other's inverses. Also \(H_0 \subset [0, v] \setminus \{0, v\}\) and \(H_1 \subset [0, v] \cap \hat{W}\). By an argument similar to (3) in the proof of Proposition 1.2, the set \(P(H)\) is star-shaped from 0, and so is \(H_1 \subset \hat{W}\). Furthermore, \(P(H)\) is a bounded order neighborhood of 0 in \(\hat{W}\), and so is \(H_1\). It follows that \(H_1\) is homeomorphic to the unit ball in \(\hat{W}\) centered at 0. Now consider the composition \(T_0 = P_0 \circ T \circ \pi_0\) as a mapping from \(H_1\) into itself. Then \(T_0\) is order-continuous, because \(T\) is continuous and increasing and hence order-continuous. The compactness of \(T\) implies that of \(T_0\). Thus, we may apply the fixed point theorem of J. Schauder and A. Tichonov, see K. Deimling [6, Theorem 10.1], to conclude that \(T_0(x_1) = x_1\) for some \(x_1 \in H_1\). Hence, \(x_0 = \pi_0(x_1) \in H_0\) satisfies \(x_0 \in E(T)\). Since 0, \(v \notin H_0\), we have 0 < \(x_0 < v\). Then 0 < \(T_0 < x_0 < Tv < v\) by the strong monotonicity of \(T\), and so \(x_0 \in X\). But \(H_0 \cap X = H\) since \(H\) is order-closed in \(X\), and \(X\) is order-open in \(V\). This entails \(x_0 \in H\) as desired. To see that \(H\) is order-closed in \(X\) observe that \(H = X - (\text{Int}(A) \cup \text{Int}(B))\), where the sets \(\text{Int}(A)\) and \(\text{Int}(B)\) are order-open in \(X\), since they are open and order-convex in \(X\). So the proof of the lemma is complete. Q.E.D.

**Proof of Theorem 1.4.** Suppose that neither \(p\) nor \(q\) attracts the entire set \((p, q)\). Denote by \(L^0\) the open line segment with endpoints \(p\) and \(q\). We claim that there exists a point \(c \in L^0\) satisfying \(p, q \notin \omega(c)\).

Assume the contrary, i.e., either \(p \in \omega(x)\) or \(q \in \omega(x)\), for every \(x \in L^0\). Consequently, we have either \(\omega(x) = p\) or \(\omega(x) = q\), for every \(x \in L^0\), since
ω(x) is unordered by Lemma 2.2. Moreover, T is increasing implies that there exists c ∈ L₀ such that

\[ \omega(x) = p \text{ for } x \in L₀ \text{ with } x < c, \text{ and } \omega(x) = q \text{ for } x \in L₀ \text{ with } x > c. \]

Since either ω(c) = p or ω(c) = q, we will assume that ω(c) = p; the case ω(c) = q is similar. Thus, there exists n ∈ T⁺ such that Tⁿc ≤ c. Since V is strongly ordered and Tⁿ is continuous, there exists c' ∈ L₀, c' > c, such that Tⁿc' ≤ c. But then T is increasing implies ω(c') = ω(Tⁿc') = p, which contradicts ω(c') = q. We conclude that there exists a point c ∈ L₀ satisfying p ≠ ω(c).

Since the set ω(c) is nonempty and unordered, it follows from Proposition 1.1 that there exists an invariant order decomposition (A, B) of X = \([p, q]\) satisfying ω(c) ⊆ H = A ∩ B. Hence, Lemma 3.1 implies H ∩ E(T) ≠ ∅, which contradicts our hypothesis that \((p, q) \cap E(T) = ∅\). We conclude that the entire set \(\{p, q\}\) is attracted by either p or q. Q.E.D.

An ordered space X is called order-connected if for each pair x, y ∈ X, x < y, there exists an increasing continuous path J: \([0, 1]\) → X such that J(0) = x and J(1) = y.

Proof of Theorem 1.5. We first show that E(T) is order-connected. Fix any a, b ∈ E(T) with a < b. Hence a ≤ b since T is strongly increasing. Notice that \([a, b] = [a, b]_V\) since X is order-convex in V by (X₀, 2). Denote by Y the set of all simply ordered subsets Y ⊆ E(T) ∩ [a, b]. Let Y be ordered by inclusion: "⊂." It is easy to see that we may apply Zorn's lemma to conclude that Y possesses a maximal element, say, Y*.

We show that Y* has the following properties:

(i) \(a, b \in Y^*\);

(ii) \(p, q \in Y^* \text{ and } p < q \text{ implies } p \equiv c \equiv q \text{ for some } c \in Y^*\);

(iii) Y* is compact;

(iv) Y* is connected; and

(v) Y* is the image of a strictly increasing continuous path J with endpoints a and b which attracts the order interval \([a, b]\).

(i) is trivial. To prove (ii) we first have to realize that (T.1) and (T.2) imply the order-compactness of T²: X → X. Now suppose that (ii) is false, i.e., the set \((p, q) - \{x ∈ V: p < x < q\}\) contains no equilibrium of T. Obviously p ≡ q since T is strongly increasing. Applying Theorem 1.4 with T² in place of T we observe that the entire set \((p, q)\) is attracted by either p or q. In particular, either q or p, resp., is Lyapunov unstable in X₀, thus contradicting (T.4). So (ii) must hold.

(iii): Since T²: X → X is order-compact, it follows that Y* is relatively compact. Hence, it suffices to show that Y* is closed. Suppose
not; then there exist \( y \in [a, b] - Y^* \) and a sequence \( \{y_n\} \) in \( Y^* \) such that \( y_n \to y \) as \( n \to \infty \). Take any such sequence \( \{y_n\} \). If the sequence \( \{y_n\} \) is not already monotone (i.e., either decreasing or increasing) for all \( n \geq n_0 \), where \( n_0 \in \mathbb{N} \), it can be split into two subsequences, one decreasing and the other one increasing. It follows that \( y \geq \sup(Y^* \cap [a, y]) \) and \( y \leq \inf(Y^* \cap [y, b]) \). The continuity of \( T \) entails \( y \in E(T) \). Hence, the set \( Y' = Y^* \cup \{y\} \) is simply ordered, and so \( Y' \in Y \). This contradicts the maximality of \( Y^* \).

(iv) follows from (ii), (iii), and the fact that \( Y^* \) is simply ordered.

(v): By (i), (iii), and (iv), the set \( Y^* \) is a simply ordered continuum which is not a point. It follows that \( Y^* \) is the image of a strictly increasing continuous path \( J : [0, 1] \to X \) with endpoints \( a \) and \( b \); see R. Wilder [25, Chap. I, Theorem 11.12]. By Theorem 1.3, the image of \( J \) attracts every \( x \in [a, b] \). Notice that \( T^2 : X \to X \) is order-compact and \( y^+(x) \) is order-bounded in \( X \) imply that \( y^+(x) \) is relatively compact in \( X \). So (i)-(v) are valid.

We deduce from (v) that \( E(T) \) is order-connected. Furthermore, let \( S \) denote the set of all maximal simply ordered subsets \( S \) of \( E(T) \). Notice that \( S \neq \emptyset \) by Zorn's lemma. Take any \( S \in S \). Then (v) implies \( S = \text{Image}(J_S) \), where \( J_S : [0, 1] \) (or \( (0, 1), [0, 1), (0, 1) \)) \( \to X \) is a strictly increasing continuous path (possibly not closed). Note that \( S \) is either compact or order-unbounded. Moreover, \( S \) attracts every order interval \([a, b]\) where \( a, b \in S \) and \( a \leq b \). We want to show \( E(T) = S \).

Suppose not; fix any \( u \in E(T) - S \). Choose \( c \in S \) arbitrary, and set \( x = \inf\{u, c\} \) and \( y = \sup\{u, c\} \) in \( V_0 \). Then \( (X_0, 1) \) implies \( x, y \in X_0 \). Notice that \( \{u, c\} = T(\{u, c\}) \) gives \( Tx \leq x \leq y \leq Ty \). Since \( T \) is increasing, the sequence \( \{T^n x\} \) is decreasing, while \( \{T^n y\} \) is increasing. Hence, compactness implies that \( T^n x \downarrow p \in E(T) \) and \( T^n y \uparrow q \in E(T) \) as \( n \to \infty \). Consequently, (v) and \( u, c \in [p, q] \) show that both \( u \) and \( c \) are attracted by the image of a strictly increasing continuous path \( J : [0, 1] \to E(T) \) with endpoints \( p \) and \( q \). Hence \( u, c \in E(T) \) forces \( u, c \in \text{Image}(J) \). Since \( \text{Image}(J) \) is simply ordered, we obtain either \( u < c \) or else \( u > c \). But \( c \in S \) was arbitrary, and therefore also \( S' = S \cup \{u\} \) is a simply ordered subset of \( E(T) \), a contradiction to the maximality of \( S \). Thus, we have proved that \( S = E(T) \).

To prove convergence to equilibrium, fix any \( x \in X_0 \). Since \( \omega(x) = \omega(T_0 x) \) and \( T_0 x \in X \) by (T.1), from now on we may assume that \( x \in X \). As above, \( y^+(x) \) is relatively compact in \( X \). Set \( a = \inf(\omega(x)) \) and \( b = \sup(\omega(x)) \) in \( V_0 \), by \((V_0)\). Then \( (X_0, 1) \) implies \( a, b \in X_0 \). Notice that \( \omega(x) = \omega(T x) \) gives \( Ta \leq a \leq b \leq Tb \). Since \( T \) is increasing, the sequence \( \{T^n a\} \) is decreasing, while \( \{T^n b\} \) is increasing. Hence, compactness implies that \( T^n a \downarrow p \in E(T) \) and \( T^n b \uparrow q \in E(T) \) as \( n \to \infty \). If \( p = q \), we are
done. So we may assume that \( p \leq q \). Then \( p \leq \omega(x) \leq q \). If \( p \in \omega(x) \) or \( q \in \omega(x) \), then \( \omega(x) = p \) or \( \omega(x) = q \), resp., since \( \omega(x) \) is unordered. Finally, if \( p < \omega(x) < q \) then also \( p \leq \omega(x) \leq q \), since \( \omega(x) \) is totally invariant and compact and \( T \) is strongly increasing. Consequently, \( p \leq T^n x \leq q \) for some \( n \in \mathbb{Z}_+ \). It follows from the proof of (v) above and Theorem 1.3 that \( \omega(x) = J(\tau(T^n x)) \). The continuity of \( \tau: X_0 \rightarrow [0, 1] \) follows from (T.1) and the proof of Theorem 1.3. Q.E.D.

**Proof of Corollary 1.6.** Fix \( x \in X \) with \( \tau(x) \equiv \rho \in (0, 1) \), By Proposition 1.1 it suffices to show that \( \text{Cl}(\gamma^+(x)) \) is unordered. Suppose this is not the case, i.e., \( y < z \) for some \( y, z \in \text{Cl}(\gamma^+(x)) \). By the proof of Lemma 2.2, \( \gamma^+(x) \) contains a strongly monotone (increasing or decreasing) subsequence \( x_0, T^k x_0, T^{2k} x_0, \ldots \).

By Theorem 1.5, this subsequence converges to \( p = J(\rho) \). Moreover, either \( x_0 \geq p \) or else \( x_0 < p \). For definiteness assume that \( x_0 \geq p \). Since \( J \) is continuous, there exists \( \sigma \in (p, 1) \) such that \( q = J(\sigma) \leq x_0 \). But then the monotonicity of \( \tau \) entails \( \sigma = \tau(q) \leq \tau(x_0) = p \), a contradiction to \( \sigma > p \).

We conclude from Proposition 1.1 that \( x \) is contained in an invariant \( d \)-hypersurface \( H = H(x) \subset X \).

Let \( \rho \in (0, 1) \). Then the set \( \tau^{-1}(\rho) = \{ x \in X_0 : \tau(x) = \rho \} \) is invariant, and so is the set \( \Sigma(\rho) = X \cap \tau^{-1}(\rho) \). If \( \Sigma(\rho) \) is unordered, then we may apply Proposition 1.1 to conclude that \( \Sigma(\rho) \subset H \) for some invariant \( d \)-hypersurface \( H \subset X \). If \( x < y \) for some \( x, y \in \Sigma(\rho) \), then also \( Tx \leq Ty \) and \( Tx, Ty \in \Sigma(\rho) \), so we may assume that \( x \leq y \). Hence, \( (X, \rho, 2) \) and the monotonicity of \( \tau \) give also \( [x, y] \subset \Sigma(\rho) \). In particular, \( \Sigma(\rho) \) has non-empty interior in \( V \). But the sets \( \Sigma(\rho) \) are pairwise disjoint as \( \rho \) ranges over the interval \( (0, 1) \), and therefore the separability of \( V \) implies that only countably many of them can have nonempty interior. This implies the second conclusion of Corollary 1.6. Q.E.D.

4. **Examples**

We present the following two examples:

**Example 4.1.** (We follow Alikakos et al. [2].) Consider the initial-boundary value problem

\[
\begin{align*}
\partial_t u + A(t) u &= f(x, t, u) \quad \text{in} \quad \Omega \times (0, \infty) \\
Bu &= 0 \quad \text{on} \quad \partial \Omega \times (0, \infty) \\
u(0) &= u_0 \quad \text{in} \quad \Omega.
\end{align*}
\] (IVP)
Here

\[ A(t) = - \sum_{i,j=1}^{N} a_{ij}(x, t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} a_i(x, t) \frac{\partial}{\partial x_i} + a_0(x, t) \]

is a uniformly elliptic linear operator over an open, bounded domain \( \Omega \subset \mathbb{R}^N \) of class \( C^{2+\mu} \), for some \( 0 < \mu < 1 \), whose coefficient functions \( a_{ij}, a_i, \) and \( a_0 \) belong to \( C^{\mu,2}(\overline{\Omega} \times \mathbb{R}^1) \) and are \( \tau \)-periodic in time \( t \), where \( \tau > 0 \) is fixed. \( B \) is a time-independent, linear boundary operator of Dirichlet, Neumann, or regular oblique derivative type whose coefficient functions are of class \( C^{1+\mu}(\partial \Omega) \). The reaction function \( f: \overline{\Omega} \times \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R} \) is assumed to be \( \tau \)-periodic in time and continuous with \( f(\cdot, \cdot, \xi) \in C^{\mu,2}(\overline{\Omega} \times \mathbb{R}^1) \) uniformly for \( \xi \) in compact intervals in \( \mathbb{R}^1 \), and differentiable with respect to \( \xi \), where \( \partial f/\partial \xi \) satisfies the same continuity hypotheses as \( f \).

A function \( u \in C^{1,1}(\overline{\Omega} \times (0, \tau]) \cap C^{1,0}(\overline{\Omega} \times [0, \tau]) \) is called a \textit{subsolution} of the (IVP) on the interval \([0, \tau]\) if it satisfies

\[
\begin{align*}
\partial_t u + A(t) u &\leq f(x, t, u) \quad \text{in } \Omega \times (0, \tau] \hfill \\
Bu &< 0 \quad \text{on } \partial \Omega \times (0, \tau];
\end{align*}
\]

a \textit{supersolution} \( \bar{u} \) is defined with reversed inequality signs.

We assume that the (IVP) has a subsolution \( u \) and a supersolution \( \bar{u} \) on \([0, \tau]\) satisfying (pointwise)

\[
y(0) < u(0) \quad \text{in } \Omega \tag{1}
\]

and

\[
y(0) \leq \bar{u}(\tau) \quad \text{and} \quad \bar{u}(0) \geq \bar{u}(\tau) \quad \text{in } \Omega. \tag{2}
\]

Set \( u_0 = u(0) \) and \( \bar{u}_0 = \bar{u}(0) \). We choose the following space setting for Theorem 1.5:

- either \( V_0 = L^p(\Omega) \) (for \( N < p < \infty \)) or \( V_0 = C^0(\overline{\Omega}) \) or \( V_0 = C^0_0(\overline{\Omega}) \) (for zero Dirichlet boundary conditions); \( X_0 = [u_0, \bar{u}_0]_{V_0} \); and either \( V = C^1(\Omega) \) or \( V = C^1_0(\overline{\Omega}) \) (for zero Dirichlet boundary conditions).

The strong parabolic maximum and boundary point principles together with (1) and (2) entail also

\[
y(0) \leq u(\tau) \leq \bar{u}(\tau) \leq \bar{u}(0) \quad \text{in } \Omega. \tag{3}
\]

Let \( u(t) \in V_0, t \geq 0, \) be the mild solution of the (IVP), where \( u_0 \in V_0 \) (see H. Amann [3] and A. Pazy [18]). Set \( T(t) u_0 = u(t), t \geq 0, \) and define \( T_0 \) by \( T_0 = T(\tau), \) the period mapping. Then \( T_0 \) maps \( X_0 \) into itself, by the
parabolic maximum principle, and satisfies the hypotheses (T.1), (T.2), and (T.3) of Theorem 1.5, by the smoothing property of parabolic evolution processes, by the compactness of the Sobolev imbedding $V \to V_0$, and by the strong parabolic maximum and boundary point principles, respectively. Notice that $u_0 \in E(T_0)$ if and only if $u(t)$ is $\tau$-periodic. Hence, the hypothesis (T.4) is equivalent to:

(T.4') Every $\tau$-periodic solution $u(t)$ of the (IVP) in $X = X_0 \cap V$ is Ljapunov stable in $X_0$.

Let $\Pi$ denote the set of all $\tau$-periodic solutions $u(t)$, $t \geq 0$, of the (IVP) in $V$ with $u(0) \in X$. Consider any $u'' \in \Pi$. Define the $\tau$-periodic function $g: \overline{Q} \times R^1_+ \to R^1$ by

$$g(x, t) = \frac{\partial f}{\partial \xi}(x, t, u''(x, t)).$$

It was proved by A. Beltramo and P. Hess [5] that the periodic eigenvalue problem

$$\begin{align*}
\partial_\tau w + A(t)w - g(t)w &= \lambda w & \text{in } & \Omega \times (0, \infty) \\
Bw &= 0 & \text{on } & \partial\Omega \times (0, \infty) \\
w(t) &= w(t + \tau) & \text{in } & \Omega \times [0, \infty) \\
\end{align*}$$

(PEP)

in $V_0$ for $\lambda$ has a real principal eigenvalue $\lambda_1 = \lambda_1(u''')$ which is characterized as the unique eigenvalue having an eigenfunction $w_1 \in V_0$ with $w_1 > 0$ (hence $w_1 \in \text{Int}(V_+)$). It follows from Alikakos et al. [2, Proof of Lemma 4.1] that the following hypothesis implies the asymptotic stability of each $u'' \in \Pi$:

$$(\lambda_1) \quad \lambda_1(u''') > 1 \quad \text{for all } u''' \in \Pi.$$ 

Recall that $u'' \in \Pi$ (more precisely, $u'''_0 = u''(0) \in X_0$) is called asymptotically stable in $X_0$ if, with respect to the period mapping $T_\tau$, $u'''_0$ is Ljapunov stable and $u'''_0$ attracts an open neighborhood of itself.

Applying Theorem 1.5 to our present situation we obtain:

**Theorem 4.1'.** Under the hypothesis (T.4'), for every $u_0 \in X_0$ there exists a unique $u'' \in \Pi$ such that

$$\|T(t)u_0 - u''(t)\|_V \to 0 \quad \text{as } \quad t \to \infty.$$ 

Moreover, the initial value $u''(0) \in X$ of $u''$ depends continuously on $u_0 \in X_0$. 

Under the stronger hypothesis $(\lambda_1)$, the set $\Pi$ is a singleton, i.e., for every $u_0 \in X_0$, $T(t)u_0$ converges to the unique $\tau$-periodic solution of the (IVP) as $t \to \infty$.

The hypothesis $(\lambda_1)$ is rather strong. Namely, in many interesting situations when $\Pi$ is not a singleton (T.4') still holds, while $(\lambda_1)$ obviously does not. It can be shown (see P. Takáč [24]) that this is the case if, for instance, $u = 0$ in $\Omega \times [0, \tau]$ and $f$ is sublinear in $\xi \in \mathbb{R}^1_+$:

$$\alpha f(x, t, \xi) \leq f(x, t, \xi)$$

for all $x \in [0, 1]$, $\xi \in \mathbb{R}^1_+$, and $(x, t) \in \Omega \times \mathbb{R}^1_+$.

An analogous situation occurs in our second example:

**Example 4.2.** We consider the gonorrhea model of A. Lajmanovich & J. A. Yorke [16] generalized by M. W. Hirsch [11, Chap. III, Example 1.8]:

$$\frac{dx_i}{dt} = R_i(t, x) - C_i(t, x_i) = F_i(t, x), \quad i = 1, 2, \ldots, N. \quad \text{(Gon)}$$

Here $x = (x_1, x_2, \ldots, x_N) \equiv x(t) \in \mathbb{R}^N$ is the unknown function of time $t \in \mathbb{R}^1_+$, and each $F_i : \mathbb{R}^1_+ \times \mathbb{R}^N \to \mathbb{R}^1$ is continuous and $\tau$-periodic in time $t$, where $\tau > 0$. We assume that all partial derivatives $\frac{\partial F_i}{\partial x_j}$, $1 \leq i, j \leq N$, exist and are continuous in $\mathbb{R}^1_+ \times \mathbb{R}^N$. The time periodicity (seasonal influences) in this model was added by G. Aronsson and I. Mellander [4] and investigated also by H. L. Smith [22].

Biologically, we consider $N$ disjoint population classes. Let $P_i$ be the number of individuals in class $i$, assumed constant in time $t$, and $x_i$ the number of infecteds in class $i$. Then $y_i = P_i - x_i$ is the number of susceptibles in class $i$. Let $R_i$ and $C_i$, resp., be the infection and cure rates of class $i$. Intuitively, the order interval $[0, P] = [0, P_1] \times \cdots \times [0, P_N]$ in $\mathbb{R}^N$ should be invariant for the dynamical process generated by (Gon). We also assume that (a) no class can stay completely infected during any time interval, and (b) each class $j$ can directly or indirectly infect every other class $i$ ($i \neq j$), where the probability of infection increases with $x_j$ (cooperativeness).

To guarantee the invariance of the order interval $[0, P]$ in $\mathbb{R}^N$ and (a) we assume that, for all $t \in \mathbb{R}^1_+$ and $x \in [0, P]$,

$$F_i(t, x) \geq 0 \quad \text{if} \quad x_i = 0 \quad \text{for some} \quad i, \quad 1 \leq i \leq N, \quad \text{(1)}$$

and

$$F_i(t, x) < 0 \quad \text{if} \quad x_i = P_i \quad \text{for some} \quad i, \quad 1 \leq i \leq N. \quad \text{(2)}$$
To guarantee (b) we assume that
\[ \frac{\partial F_i}{\partial x_j} \geq 0 \quad \text{if} \quad i \neq j, \ 1 \leq i, j \leq N, \] (3)
for every \( x \in [0, P] \), and the \( N \times N \) Jacobian matrix
\[ D_x F = \left[ \frac{\partial F_i}{\partial x_j} \right]_{1 \leq i, j \leq N} \]
is irreducible \( (4) \)
for all \( t \in \mathbb{R}_+^1 \) and \( x \in [0, P] \) = \{ \( x \in \mathbb{R}^N : 0 \leq x \leq P \) \}.

Let \( x(t) \in \mathbb{R}^N, \ t \geq 0, \) be the solution of (Gon) for a given \( x(0) \in \mathbb{R}^N \). Set \( T(t) x(0) = x(t), \ t \geq 0, \) and define \( T \) by \( T = T(\tau) \), the period mapping. Then \( T \) maps \( [0, P] \) into itself with \( T(P) \leq P \), by (1) and (2), and is strongly increasing, by (3), (4), and Kamke’s theorem. It follows that all hypotheses of Theorem 1.5, except for (T.4), are satisfied with \( V_0 = V = \mathbb{R}^N, \ X_0 = X = [0, P], \) and \( T_0 = T \). Finally, we assume that each \( F_i \) \((i = 1, 2, \ldots, N)\) is sublinear in \( x \in [0, P] \):
\[ aF_i(t, x) \leq F_i(t, ax) \quad \text{for all} \quad a \in [0, 1], \ x \in [0, P], \) and \( t \in \mathbb{R}_+^1. \) (5)

It is proved in P. Takáč [24] that \( E(T) \) must be either a line segment \( l \cdot p = \{ \alpha p : \alpha \in l \} \), where \( l = [\alpha_0, 1] \) for some \( \alpha_0 \in [0, 1] \) and \( p \in E(T) \), or the union of \( l \cdot p \) with \( \{ 0 \} \). Moreover, \( l \cdot p \) attracts every \( x \in [0, P] \) with \( x \neq 0 \), and every equilibrium in \( l \cdot p \) is Ljapunov stable. Hence, if either \( 0 \notin E(T) \) or \( 0 \in E(T) \) is Ljapunov stable, then (T.4) holds, and we may apply Theorem 1.5 directly to this problem. Let now \( 0 \in E(T) \) be Ljapunov unstable. Then, instead of taking \( X = [0, P] \), we have to take \( X = [\varepsilon p, P] \) with any \( \varepsilon \in (0, \alpha_0) \).

Again, let \( \Pi \) denote the set of all \( \tau \)-periodic solutions of (Gon) in \( [0, P] \). Applying Theorem 1.5 to this problem we obtain:

**Theorem 4.2**'. Under the hypotheses (1)--(5), for every \( x(0) \in [0, P] \) there exists a unique \( x'' \in \Pi \) such that
\[ \| T(t) x(0) - x''(t) \|_{\mathbb{R}^N} \to 0 \quad \text{as} \quad t \to \infty. \]

A similar result was proved by H. L. Smith [22] under a stronger concavity hypothesis than (5).

### 5. Discussion

It is clear from the proof of Lemma 2.2 that if \( \text{Cl}(\gamma^+(x)) \) is not unordered, then the convergence of the positive semi-orbit of \( x \) towards the
cycle $\omega(x)$ is monotone under the mapping $T^k$. Hence, no stability of $x$ or $\omega(x)$ is needed. On the other hand, if $\text{Cl}(\gamma^+(x))$ is unordered, then it must be contained in an invariant $d$-hypersurface $H$. In Lemma 3.1 the geometry of the domain $X$ for $T$ played an important role. Again, no stability was assumed. Thus, to determine the asymptotic behavior of $T^n x$ as $n \to \infty$ we have to study the flow of the mapping $T$ restricted to $H$. The main problem is to find "reasonable" hypotheses on the geometry of $X$ and the stability of $T|_H$ under which one could study the asymptotic behavior of $T^n x$. Also, it is not clear under which hypotheses a continuous mapping $S: H \to H$ can be extended from $H$ to a strongly increasing mapping $T$ on $X$. Consequently, the impact of monotonicity on $T|_H$ is rather unclear.

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