Strong convergence of averaging iterations of nonexpansive nonself-mappings

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Abstract

Let $C$ be a closed convex subset of Hilbert space $H$, $T$ a nonexpansive nonself-mapping from $C$ into $H$, and $x_0, x, y_0, y$ elements of $C$. In this paper, we study the convergence of the two sequences generated by

\begin{align*}
x_{n+1} &= \frac{1}{n+1} \sum_{j=0}^{n} (\alpha_n x + (1 - \alpha_n)(PT)^j x_n) \quad \text{for } n = 0, 1, 2, \ldots, \\
y_{n+1} &= \frac{1}{n+1} \sum_{j=0}^{n} P(\alpha_n y + (1 - \alpha_n)(TP)^j y_n) \quad \text{for } n = 0, 1, 2, \ldots,
\end{align*}

where $\{\alpha_n\}$ is a real sequence such that $0 \leq \alpha_n \leq 1$, and $P$ is the metric projection from $H$ onto $C$.

Keywords: Fixed point; Nonexpansive nonself-mapping; Strong convergence; Metric projection
1. Introduction

Let $H$ be a Hilbert space, $C$ a nonempty closed convex subset of $H$, $S$ a nonexpansive mapping from $C$ into itself such that the set $F(S)$ of all fixed points of $S$ is nonempty. Shimizu and Takahashi [4] studied the convergence of iteration process for a family of nonexpansive mappings in a Hilbert space. Using an idea of Shimizu and Takahashi [4], Shioji and Takahashi [5] studied the strong convergence of the sequence $\{x_n\}$ in the framework of a Banach space. We restate the sequence $\{x_n\}$ as follows:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^{n} S^j x_{n} \quad \text{for } n = 0, 1, 2, \ldots , (1)$$

where $x_0, x$ are elements of $C$, and $\{\alpha_n\}$ is a sequence such that $0 \leq \alpha_n \leq 1$. They proved that $\{x_n\}$ converges strongly to an element of fixed point of $S$ which is nearest to $x$ in the framework of a Hilbert space. But this approximation method is not suitable for some nonexpansive nonself-mappings. On the other hand, the authors [2] studied the strong convergence of the two sequences generated by

$$x_1 = x \in C, \quad x_{n+1} = \alpha_n x + (1 - \alpha_n) PT x_{n} \quad \text{for } n = 1, 2, \ldots , (2)$$

$$y_1 = y \in C, \quad y_{n+1} = P\left(\alpha_n y + (1 - \alpha_n) T y_n\right) \quad \text{for } n = 1, 2, \ldots , (3)$$

where $P$ is the metric projection from $H$ onto $C$, and $T$ is a nonexpansive nonself-mapping from $C$ into $H$. We proved that $\{x_n\}$ and $\{y_n\}$ converge strongly to fixed points of $T$ when $F(T)$ is nonempty.

In this paper, we study the two type iteration processes which are mixed iteration processes of (1)–(3) as follows:

$$x_{n+1} = \frac{1}{n+1} \sum_{j=0}^{n} (\alpha_n x + (1 - \alpha_n)(PT)^j x_{n}) \quad \text{for } n = 0, 1, 2, \ldots , (4)$$

$$y_{n+1} = \frac{1}{n+1} \sum_{j=0}^{n} P\left(\alpha_n y + (1 - \alpha_n)(TP)^j y_n\right) \quad \text{for } n = 0, 1, 2, \ldots , (5)$$

where $x_0, x, y_0, y$ are elements of $C$, $P$ is the metric projection from $H$ onto $C$, and $T$ is a nonexpansive nonself-mapping from $C$ into $H$. To prove our results, we use the nowhere normal outward condition. It was first presented by Halpern and Bergman [1]. This boundary condition is very simple but great importance in our proof. By using nowhere normal outward condition, we first consider the sequence $\{x_n\}$ generated by (4) and prove that $\{x_n\}$ converges strongly as $n \to \infty$ to an element of fixed point of $T$ when $F(T)$ is nonempty,

further we consider the sequence $\{y_n\}$ generated by (5) and prove that $\{y_n\}$ converges strongly as $n \to \infty$ to an element of fixed point of $T$ when $F(T)$ is nonempty.

2. Preliminaries and notations

Throughout this paper, we denote the set of all nonnegative integers by $\mathbb{N}$. Let $H$ be a real Hilbert space with norm $\| \cdot \|$ and inner product $\langle \cdot, \cdot \rangle$. Let $C$ be a closed convex subset
of $H$, and $T$ a nonself-mapping from $C$ into $H$. We denote the set of all fixed points of $T$ by $F(T)$. $T$ is said to nonexpansive if
\[\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C.\]

From condition of $C$, there is a mapping $P$ from $H$ onto $C$ which satisfies
\[\|x - Px\| = \min_{y \in C} \|x - y\| \quad \text{for all } x \in H.\]  

(6)

This mapping $P$ is said to be the metric projection from $H$ onto $C$. We know that the metric projection is nonexpansive and that a mapping $P$ from $H$ onto $C$ satisfies (6) if and only if
\[\langle x - Px, y - Px \rangle \leq 0 \quad \text{for each } y \in C \text{ and } x \in H.\]

$T$ is said to satisfy nowhere normal outward condition ((NNO) for short) if
\[Tx \in S_x \quad \text{for all } x \in C,\]
where $P$ is the metric projection from $H$ onto $C$ and $S_x = \{y \in H \mid y \neq x, Py = x\}$.

Concerning (NNO) condition, we know the following [2]:

**Proposition 1.** Let $H$ be a Hilbert space, $C$ a nonempty closed convex subset of $H$, $P$ the metric projection from $H$ onto $C$, and $T$ a nonself-mapping from $C$ into $H$. Suppose that $T$ satisfies (NNO) condition. Then $F(T) = F(PT)$.

**Proposition 2.** Let $H$ be a Hilbert space, $C$ a nonempty closed convex subset of $H$, and $T$ a nonexpansive nonself-mapping from $C$ into $H$. If $F(T)$ is nonempty then $T$ satisfies (NNO) condition.

The following lemma is a similar result which is proved by Shimizu and Takahashi [3]. In a similar way, we can show this lemma.

**Lemma 1.** Let $H$ be a Hilbert space, $C$ a nonempty closed convex subset of $H$, and $S$ a nonexpansive mapping from $C$ into itself such that $F(S)$ is nonempty. Let $\{x_n\}$ be a sequence in $C$ such that $\{x_{n+1} - \frac{1}{n+1} \sum_{j=0}^{n} S^j x_n\}$ converges strongly to $0$, and let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to $x \in C$. Then $x$ is a fixed point of $S$.

**Proof.** We show that $\{S^k x\}$ converges strongly to $x$. If not, there exist a positive number $\varepsilon$ and a subsequence $S^{k_i} x$ of $\{S^k x\}$ such that $\|S^{k_i} x - x\| \geq \varepsilon$ for each $k$. Since $\{x_{n_i}\}$ converges weakly to $x$, for each $y \in C$ with $y \neq x$, we have

\[\liminf_{i \to \infty} \|x_{n_i} - x\| < \liminf_{i \to \infty} \|x_{n_i} - y\|.\]

Let $r = \liminf_{i \to \infty} \|x_{n_i} - x\|$ and choose $\delta > 0$ such that $\delta < \sqrt{r^2 + \varepsilon^2} / 4 - r$. Then, there exists a subsequence $\{x_{m_i}\}$ of $\{x_{n_i}\}$ such that $\|x_{m_i} - x\| < r + \delta / 6$ for every $i$. On the other hand, we have

\[\|x_{m_i} - S^j x\| \leq \|x_{m_i} - \frac{1}{m_i} \sum_{j=0}^{m_i-1} S^j x_{m_i-1}\|.\]
\begin{align*}
&+ \left\| \frac{1}{m_i} \sum_{j=0}^{m_i-1} S^j x_{m_i-1} - S \left( \frac{1}{m_i} \sum_{j=0}^{m_i-1} S^j x_{m_i-1} \right) \right\| \\
&+ \left\| S \left( \frac{1}{m_i} \sum_{j=0}^{m_i-1} S^j x_{m_i-1} \right) - S^l x \right\| \\
\leq 2 \left\| x_{m_i} - \frac{1}{m_i} \sum_{j=0}^{m_i-1} S^j x_{m_i-1} \right\| + \| x_{m_i} - x \| \\
&+ \left\| \frac{1}{m_i} \sum_{j=0}^{m_i-1} S^j x_{m_i-1} - S \left( \frac{1}{m_i} \sum_{j=0}^{m_i-1} S^j x_{m_i-1} \right) \right\|.
\end{align*}

In particular,
\begin{align*}
&\left\| \frac{1}{m_i} \sum_{j=0}^{m_i-1} S^j x_{m_i-1} - S \left( \frac{1}{m_i} \sum_{j=0}^{m_i-1} S^j x_{m_i-1} \right) \right\| \\
&\leq \left\| \frac{1}{m_i} \sum_{j=0}^{m_i-1} S^j x_{m_i-1} - S \left( \frac{1}{m_i} \sum_{j=0}^{m_i-1} S^j x_{m_i-1} \right) \right\| + \frac{1}{m_i} \| x_{m_i-1} - S^{m_i} x_{m_i-1} \| \\
&+ \left\| S \left( \frac{1}{m_i} \sum_{j=0}^{m_i-1} S^j x_{m_i-1} \right) - S \left( \frac{1}{m_i} \sum_{j=0}^{m_i-1} S^j x_{m_i-1} \right) \right\| \\
&\leq \left\| \frac{1}{m_i} \sum_{j=0}^{m_i-1} S^j x_{m_i-1} - S \left( \frac{1}{m_i} \sum_{j=0}^{m_i-1} S^j x_{m_i-1} \right) \right\| + \frac{2}{m_i} \| x_{m_i-1} - S^{m_i} x_{m_i-1} \|.
\end{align*}

Since \( \{ x_{m_i-1} \} \) and \( \{ S^{m_i} x_{m_i-1} \} \) are bounded, there exists a positive integer \( i_1 \) such that
\[
\frac{1}{m_i} \| x_{m_i-1} - S^{m_i} x_{m_i-1} \| < \frac{\delta}{6}
\]
for each \( i \geq i_1 \). Since \( \{ x_{m_i-1} \} \) is bounded, by Lemma 3 in [3] there exists a positive integer \( L_0 \), such that for every \( l \geq L_0 \), there exists a positive integer \( i_i \) satisfying
\[
\left\| \frac{1}{m_i} \sum_{j=0}^{m_i} S^j x_{m_i-1} - S^l \left( \frac{1}{m_i} \sum_{j=0}^{m_i} S^j x_{m_i-1} \right) \right\| < \frac{\delta}{6}
\]
for each \( i \geq i_i \). Since \( \lim_{n \to \infty} \| x_{n+1} - \frac{1}{n+1} \sum_{j=0}^{n} S^j x_n \| = 0 \), there exists a positive integer \( i_{i_0} \) such that
\[
\left\| x_{m_i} - \frac{1}{m_i} \sum_{j=0}^{m_i-1} S^j x_{m_i-1} \right\| < \frac{\delta}{6}
\]
for all \( i \geq i_{i_0} \). So, for any \( l \geq L_0 \) and \( i \geq \max \{ i_i, i_0, i_1 \} \), we have
\[
\| x_{m_i} - S^l x \| < 2 \times \frac{\delta}{6} + r + \frac{\delta}{6} + 2 \times \frac{\delta}{6} = r + \delta.
\]
Choose \( l_k \geq L_0 \). Then for \( i \geq \max\{i_k, i_0, i_1\} \), we have

\[
\left\| x_{m_i} - \frac{S^{k}x + x}{2} \right\|^2 = 2\left(\left\| x_{m_j} - \frac{S^{j}x}{2} \right\|^2 + \left\| x_{m_j} - x \right\|^2 - \left\| \frac{S^{j}x}{2} - x \right\|^2\right) < \frac{(r + \delta)^2}{2} + \frac{(r + \delta/2)^2}{2} - \frac{\varepsilon^2}{4} < (r + \delta)^2 - \frac{\varepsilon^2}{4} < r^2.
\]

This implies

\[
\lim_{i \to \infty} \left\| x_{m_i} - \frac{S^{k}x + x}{2} \right\| < r.
\]

However, we have \( \frac{S^{k}x + x}{2} \neq x \) from \( \left\| S^{k}x - x \right\| \geq \varepsilon \), then we obtain \( \lim_{i \to \infty} \left\| x_{m_i} - x \right\| < \lim_{i \to \infty} \left\| \frac{S^{k}x + x}{2} \right\| \), this is a contradiction. Therefore, \( \{S^{j}x\} \) converges strongly to \( x \). This implies, for each \( \varepsilon > 0 \), there exists a positive number \( l_0 \) such that

\[
\left\| S^{l}x - x \right\| \leq \frac{\varepsilon}{2}
\]

for each \( l \geq l_0 \). So, we have that if \( l \geq l_0 + 1 \), then

\[
\left\| S^{l+1}x - x \right\| \leq \left\| S^{l}x - x \right\| + \left\| S^{l+1}x - S^{l}x \right\| \leq \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, we get that \( Sx = x \). \( \square \)

3. Strong convergence theorems

In this section, we prove two strong convergence theorems. To prove our results, we use the method employed in [6,7].

**Theorem 1.** Let \( H \) be a Hilbert space, \( C \) a nonempty closed convex subset of \( H \), \( P_1 \) the metric projection from \( H \) onto \( C \), \( T \) a nonexpansive nonself-mapping from \( C \) into \( H \) such that \( F(T) \) is nonempty, and \( \{\alpha_n\} \) a sequence of real numbers such that \( 0 \leq \alpha_n \leq 1 \), \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty \). Suppose that \( \{x_n\} \) is given by \( x_0, x \in C \) and

\[
x_{n+1} = \frac{1}{n+1} \sum_{j=0}^{n} \left( \alpha_n x + (1 - \alpha_n)(P_1 T)^j x_n \right) \quad \text{for } n = 0, 1, 2, \ldots
\]

Then \( \{x_n\} \) converges strongly to \( P_2 x \in F(T) \), where \( P_2 \) is the metric projection from \( C \) onto \( F(T) \).

**Proof.** Since \( F(T) \) is nonempty, then \( T \) satisfies (NNO) condition. Let \( z \in F(T) \), \( M = \max\{\|x - z\|, \|x_0 - z\|\} \). Then we have

\[
\|x_1 - z\| = \|\alpha_0 x + (1 - \alpha_0)x_0 - z\| \leq \alpha_0 \|x - z\| + (1 - \alpha_0)\|x_0 - z\| \leq M.
\]

If \( \|x_n - z\| \leq M \) for some \( n \in \mathbb{N} \), then we can show that \( \|x_{n+1} - z\| \leq M \) similarly. Therefore, by induction, we obtain \( \|x_n - z\| \leq M \) for all \( n \in \mathbb{N} \) and hence \( \{x_n\} \) is bounded. Also, from
\[
\begin{aligned}
\|x_{n+1} - \frac{1}{n+1} \sum_{j=0}^{n} (P_1 T)^j x_n\| & = \left\| \frac{1}{n+1} \sum_{j=0}^{n} (\alpha_n x + (1-\alpha_n)(P_1 T)^j x_n) - \frac{1}{n+1} \sum_{j=0}^{n} (P_1 T)^j x_n \right\| \\
& \leq \alpha_n \left\| x - \frac{1}{n+1} \sum_{j=0}^{n} (P_1 T)^j x_n \right\|,
\end{aligned}
\]

we have \(\{x_{n+1} - \frac{1}{n+1} \sum_{j=0}^{n} (P_1 T)^j x_n\}\) converges strongly to 0. Next, we shall prove that

\[
\limsup_{n \to \infty} (P_2 x - x_n, P_2 x - x) \leq 0. \tag{7}
\]

Let \(\{x_{n_i}\}\) be a subsequence of \(\{x_n\}\) such that

\[
\lim_{i \to \infty} (P_2 x - x_{n_i}, P_2 x - x) = \limsup_{n \to \infty} (P_2 x - x_n, P_2 x - x),
\]

and there exists \(\{x_{n_{ij}}\}\), which is a subsequence of \(\{x_{n_i}\}\) converging weakly as \(j \to \infty\) to \(w \in C\). From Lemma 1 and Proposition 1, we obtain \(w \in F(T)\). Then we have

\[
\limsup_{n \to \infty} (P_2 x - x_n, P_2 x - x) = \lim_{j \to \infty} (P_2 x - x_{n_{ij}}, P_2 x - x) = (P_2 x - w, P_2 x - x) \leq 0.
\]

By (7), for any \(\varepsilon > 0\), there exists \(m \in \mathbb{N}\) such that

\[
(P_2 x - x_n, P_2 x - x) < \varepsilon \tag{8}
\]

for all \(n \geq m\). On the other hand, from

\[
x_{n+1} - P_2 x + \alpha_n (P_2 x - x) = \frac{1}{n+1} \sum_{j=0}^{n} (\alpha_n x + (1-\alpha_n)(P_1 T)^j x_n) - \left(\alpha_n x + (1-\alpha_n)P_2 x\right),
\]

this implies

\[
\|x_{n+1} - P_2 x\|^2 \leq \left\| \frac{1}{n+1} \sum_{j=0}^{n} (\alpha_n x + (1-\alpha_n)(P_1 T)^j x_n) - \left(\alpha_n x + (1-\alpha_n)P_2 x\right) \right\|^2 + 2\alpha_n \langle x_{n+1} - P_2 x, x - P_2 x \rangle \\
\leq \left\{ \left(1 - \alpha_n\right) \frac{1}{n+1} \sum_{j=0}^{n} \| (P_1 T)^j x_n - P_2 x \|^2 \right\}^2 + 2\alpha_n \langle x_{n+1} - P_2 x, x - P_2 x \rangle \\
\leq (1 - \alpha_n)^2 \| x_n - P_2 x \|^2 + 2\alpha_n \langle x_{n+1} - P_2 x, x - P_2 x \rangle
\]

for all \(n \in \mathbb{N}\). By (8),
\[ \|x_{n+1} - P_2x\| \leq 2\alpha_n \epsilon + (1 - \alpha_n)\|x_n - P_2x\|^2 \]
\[ = 2\epsilon (1 - (1 - \alpha_n)) + (1 - \alpha_n)\|x_n - P_2x\|^2 \]
\[ \leq 2\epsilon (1 - (1 - \alpha_n)) + (1 - \alpha_n)(2\epsilon (1 - (1 - \alpha_{n-1}) + (1 - \alpha_{n-1})\|x_{n-1} - P_2x\|^2) \]
\[ = 2\epsilon (1 - (1 - \alpha_n)(1 - \alpha_{n-1})) + (1 - \alpha_n)(1 - \alpha_{n-1})\|x_{n-1} - P_2x\|^2 \]
for all \( n \geq m \). By induction, we obtain
\[ \|x_{n+1} - P_2x\|^2 \leq 2\epsilon \left( 1 - \prod_{k=m}^{n} (1 - \alpha_k) \right) + \prod_{k=m}^{n} (1 - \alpha_k)\|x_m - P_2x\|. \]

Therefore, from \( \sum_{n=0}^{\infty} \alpha_n = \infty \), we have
\[ \limsup_{n \to \infty} \|x_{n+1} - P_2x\| \leq 2\epsilon. \]

Since \( \epsilon \) is arbitrary, we can conclude that \( \{x_n\} \) converges strongly to \( P_2x \). \( \square \)

**Theorem 2.** Let \( H \) be a Hilbert space, \( C \) a nonempty closed convex subset of \( H \), \( P_1 \) the metric projection from \( H \) onto \( C \), and \( T \) a nonexpansive nonself-mapping from \( C \) into \( H \)

such that \( F(T) \) is nonempty, and \( \{\alpha_n\} \) a sequence of real numbers such that \( 0 \leq \alpha_n \leq 1 \), \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty \). Suppose that \( \{y_n\} \) is given by \( y_0, y \in C \) and

\[ y_{n+1} = \frac{1}{n+1} \sum_{j=0}^{n} P_1(\alpha_n y + (1 - \alpha_n)(TP_1)^j y_n) \quad \text{for } n = 0, 1, 2, \ldots \]

Then \( \{y_n\} \) converges strongly to \( P_2y \in F(T) \), where \( P_2 \) is the metric projection from \( C \) onto \( F(T) \).

**Proof.** Since \( F(T) \) is nonempty, then \( T \) satisfies (NNO) condition. Let \( z \in F(T) \), \( M = \max\{\|y - z\|, \|y_0 - z\|\} \). Then we have

\[ \|y_1 - z\| = \|P_1(\alpha_0 y + (1 - \alpha_0)y_0 - z)\| \leq \|\alpha_0 y + (1 - \alpha_0)y_0 - z\| \]
\[ \leq \alpha_0 \|y - z\| + (1 - \alpha_0)\|y_0 - z\| \leq M. \]

If \( \|y_n - z\| \leq M \) for some \( n \in \mathbb{N} \). Then we can show that \( \|y_{n+1} - z\| \leq M \) similarly. Therefore, by induction, we obtain \( \|y_n - z\| \leq M \) for all \( n \in \mathbb{N} \) and hence \( \{y_n\} \) is bounded. Also, from

\[ y_{n+1} = \frac{1}{n+1} \sum_{j=0}^{n} (P_1 T)^j y_n \]
\[ = \frac{1}{n+1} \sum_{j=0}^{n} P_1(\alpha_n y + (1 - \alpha_n)(TP_1)^j y_n) - \frac{1}{n+1} \sum_{j=0}^{n} (P_1 T)^j y_n \]
\[ \leq \frac{1}{n+1} \sum_{j=0}^{n} \|P_1(\alpha_n y + (1 - \alpha_n)(TP_1)^j y_n) - (P_1 T)^j y_n\| \]
\[
\leq \frac{1}{n+1} \sum_{j=0}^{n} \| \alpha_n y + (1 - \alpha_n)(T P_1)^j y_n - (T P_1)^j y_n \|
\]
\[
\leq \alpha_n \frac{1}{n+1} \sum_{j=0}^{n} \| y - (T P_1)^j y_n \|,
\]
we obtain \( \{y_{n+1} - \frac{1}{n+1} \sum_{j=0}^{n} (P_1 T)^j y_n \} \) converges strongly to 0. Next, we shall prove that
\[
\limsup_{n \to \infty} (P_2 y - y_n, P_2 y - y) \leq 0. \tag{9}
\]
Let \( \{y_{n_i}\} \) be a subsequence of \( \{y_n\} \) such that
\[
\lim_{i \to \infty} \langle P_2 y - y_{n_i}, P_2 y - y \rangle = \limsup_{n \to \infty} \langle P_2 y - y_n, P_2 y - y \rangle,
\]
and there exists \( \{y_{n_{ij}}\} \) which is a subsequence of \( \{y_{n_i}\} \) converging weakly as \( j \to \infty \) to \( w \in C \). From Lemma 1 and Proposition 1, we obtain \( w \in F(T) \). Then we have
\[
\limsup_{n \to \infty} (P_2 y - y_n, P_2 y - y) = \lim_{j \to \infty} \langle P_2 y - y_{n_{ij}}, P_2 y - y \rangle = \langle P_2 y - w, P_2 y - y \rangle. \]
By (9), for any \( \varepsilon > 0 \), there exists \( m \in \mathbb{N} \) such that
\[
\langle P_2 y - y_n, P_2 y - y \rangle < \varepsilon \tag{10}
\]
for all \( n \geq m \). On the other hand, from
\[
y_{n+1} - P_2 y + \alpha_n(P_2 y - y) = \frac{1}{n+1} \sum_{j=0}^{n} P_1 (\alpha_n y + (1 - \alpha_n)(T P_1)^j y_n)
\]
\[
- P_1 (\alpha_n y + (1 - \alpha_n)P_2 y),
\]
this implies
\[
\| y_{n+1} - P_2 y \|^2 \leq \left\| \frac{1}{n+1} \sum_{j=0}^{n} P_1 (\alpha_n y + (1 - \alpha_n)(T P_1)^j y_n)
\]
\[
- P_1 (\alpha_n y + (1 - \alpha_n)P_2 y) \right\|^2 + 2\alpha_n (y_{n+1} - P_2 y, y - P_2 y)
\]
\[
\leq \left\{ \frac{1}{n+1} \sum_{j=0}^{n} \| P_1 (\alpha_n y + (1 - \alpha_n)(T P_1)^j y_n)
\]
\[
- P_1 (\alpha_n y + (1 - \alpha_n)P_2 y) \right\|^2 + 2\alpha_n (y_{n+1} - P_2 y, y - P_2 y)
\]
\[
\leq \left\{ (1 - \alpha_n)\frac{1}{n+1} \sum_{j=0}^{n} \| (T P_1)^j y_n - P_2 y \| \right\}^2
\]
\[
+ 2\alpha_n (y_{n+1} - P_2 y, y - P_2 y)
\]\
for all $n \in \mathbb{N}$. By (10), $\sum_{n=0}^{\infty} \alpha_n = \infty$, and in the proof of Theorem 1, we can conclude that \( \{y_n\} \) converges strongly to \( P_2y \). \( \square \)

References