Available online at www.sciencedirect.com
SCIENCE DIRECT®
J. Math. Anal. Appl. 294 (2004) 206–214
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Strong convergence of averaging iterations of nonexpansive nonself-mappings

Shin-ya Matsushita [a,*] and Daishi Kuroiwa [b]

[a] *Graduate School of Mathematics and Computer Science, Faculty of Science and Engineering, Shimane University, Japan*
[b] *Department of Mathematics and Computer Science, Faculty of Science and Engineering, Shimane University, Japan*

Received 27 December 2001
Available online 2 April 2004
Submitted by A. Cellina

Abstract

Let C be a closed convex subset of Hilbert space H, T a nonexpansive nonself-mapping from C into H, and x_0, x, y_0, y elements of C. In this paper, we study the convergence of the two sequences generated by

$$x_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n}\bigl(\alpha_n x + (1-\alpha_n)(PT)^j x_n\bigr) \quad \text{for } n = 0, 1, 2, \ldots,$$
$$y_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n}P\bigl(\alpha_n y + (1-\alpha_n)(TP)^j y_n\bigr) \quad \text{for } n = 0, 1, 2, \ldots,$$

where $\{\alpha_n\}$ is a real sequence such that $0 \leqslant \alpha_n \leqslant 1$, and P is the metric projection from H onto C.

Keywords: Fixed point; Nonexpansive nonself-mapping; Strong convergence; Metric projection

* Corresponding author.
E-mail addresses: shin-ya@cis.shimane-u.ac.jp (S. Matsushita), kuroiwa@cis.shimane-u.ac.jp (D. Kuroiwa).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.02.010

1. Introduction

Let H be a Hilbert space, C a nonempty closed convex subset of H, S a nonexpansive mapping from C into itself such that the set $F(S)$ of all fixed points of S is nonempty. Shimizu and Takahashi [4] studied the convergence of iteration process for a family of nonexpansive mappings in a Hilbert space. Using an idea of Shimizu and Takahashi [4], Shioji and Takahashi [5] studied the strong convergence of the sequence $\{x_n\}$ in the framework of a Banach space. We restate the sequence $\{x_n\}$ as follows:

$$x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n} S^j x_n \quad \text{for } n = 0, 1, 2, \ldots, \tag{1}$$

where x_0, x are elements of C, and $\{\alpha_n\}$ is a sequence such that $0 \leqslant \alpha_n \leqslant 1$. They proved that $\{x_n\}$ converges strongly to an element of fixed point of S which is nearest to x in the framework of a Hilbert space. But this approximation method is not suitable for some nonexpansive nonself-mappings. On the other hand, the authors [2] studied the strong convergence of the two sequences generated by

$$x_1 = x \in C, \qquad x_{n+1} = \alpha_n x + (1-\alpha_n)PTx_n \quad \text{for } n = 1, 2, \ldots, \tag{2}$$

$$y_1 = y \in C, \qquad y_{n+1} = P\big(\alpha_n y + (1-\alpha_n)Ty_n\big) \quad \text{for } n = 1, 2, \ldots, \tag{3}$$

where P is the metric projection from H onto C, and T is a nonexpansive nonself-mapping from C into H. We proved that $\{x_n\}$ and $\{y_n\}$ converge strongly to fixed points of T when $F(T)$ is nonempty.

In this paper, we study the two type iteration processes which are mixed iteration processes of (1)–(3) as follows:

$$x_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n}\big(\alpha_n x + (1-\alpha_n)(PT)^j x_n\big) \quad \text{for } n = 0, 1, 2, \ldots, \tag{4}$$

$$y_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n} P\big(\alpha_n y + (1-\alpha_n)(TP)^j y_n\big) \quad \text{for } n = 0, 1, 2, \ldots, \tag{5}$$

where x_0, x, y_0, y are elements of C, P is the metric projection from H onto C, and T is a nonexpansive nonself-mapping from C into H. To prove our results, we use the nowhere normal outward condition. It was first presented by Halpern and Bergman [1]. This boundary condition is very simple but great importance in our proof. By using nowhere normal outward condition, we first consider the sequence $\{x_n\}$ generated by (4) and prove that $\{x_n\}$ converges strongly as $n \to \infty$ to an element of fixed point of T when $F(T)$ is nonempty, further we consider the sequence $\{y_n\}$ generated by (5) and prove that $\{y_n\}$ converges strongly as $n \to \infty$ to an element of fixed point of T when $F(T)$ is nonempty.

2. Preliminaries and notations

Throughout this paper, we denote the set of all nonnegative integers by $\mathbf{N}$. Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot\,,\cdot\rangle$. Let C be a closed convex subset of H, and T a nonself-mapping from C into H. We denote the set of all fixed points of T by $F(T)$. T is said to nonexpansive if

$$\|Tx - Ty\| \leqslant \|x - y\| \quad \text{for all } x, y \in C.$$

From condition of C, there is a mapping P from H onto C which satisfies

$$\|x - Px\| = \min_{y \in C} \|x - y\| \quad \text{for all } x \in H. \tag{6}$$

This mapping P is said to the metric projection from H onto C. We know that the metric projection is nonexpansive and that a mapping P from H onto C satisfies (6) if and only if $\langle x - Px, y - Px\rangle \leqslant 0$ for each $y \in C$ and $x \in H$. T is said to satisfy nowhere normal outward condition ((NNO) for short) if

$$Tx \in S_x^c \quad \text{for all } x \in C,$$

where P is the metric projection from H onto C and $S_x = \{y \in H \mid y \neq x, Py = x\}$. Concerning (NNO) condition, we know the following [2]:

Proposition 1. *Let H be a Hilbert space, C a nonempty closed convex subset of H, P the metric projection from H onto C, and T a nonself-mapping from C into H. Suppose that T satisfies* (NNO) *condition. Then $F(T) = F(PT)$.*

Proposition 2. *Let H be a Hilbert space, C a nonempty closed convex subset of H, and T a nonexpansive nonself-mapping from C into H. If $F(T)$ is nonempty then T satisfies* (NNO) *condition.*

The following lemma is a similar result which is proved by Shimizu and Takahashi [3]. In a similar way, we can show this lemma.

Lemma 1. *Let H be a Hilbert space, C a nonempty closed convex subset of H, and S a nonexpansive mapping from C into itself such that $F(S)$ is nonempty. Let $\{x_n\}$ be a sequence in C such that $\{x_{n+1} - \frac{1}{n+1}\sum_{j=0}^{n} S^j x_n\}$ converges strongly to 0, and let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to $x \in C$. Then x is a fixed point of S.*

Proof. We show that $\{S^l x\}$ converges strongly to x. If not, there exist a positive number ε and a subsequence $S^{l_k} x$ of $\{S^l x\}$ such that $\|S^{l_k} x - x\| \geqslant \varepsilon$ for each k. Since $\{x_{n_i}\}$ converges weakly to x, for each $y \in C$ with $y \neq x$, we have

$$\liminf_{i \to \infty} \|x_{n_i} - x\| < \liminf_{i \to \infty} \|x_{n_i} - y\|.$$

Let $r = \liminf_{i \to \infty} \|x_{n_i} - x\|$ and choose $\delta > 0$ such that $\delta < \sqrt{r^2 + \varepsilon^2/4} - r$. Then, there exists a subsequence $\{x_{m_i}\}$ of $\{x_{n_i}\}$ such that $\|x_{m_i} - x\| < r + \delta/6$ for every i. On the other hand, we have

$$\|x_{m_i} - S^l x\| \leqslant \left\| x_{m_i} - \frac{1}{m_i} \sum_{j=0}^{m_i - 1} S^j x_{m_i - 1} \right\|$$

$$
\begin{aligned}
&+\left\|\frac{1}{m_i}\sum_{j=0}^{m_i-1}S^j x_{m_i-1}-S^l\left(\frac{1}{m_i}\sum_{j=0}^{m_i-1}S^j x_{m_i-1}\right)\right\| \\
&+\left\|S^l\left(\frac{1}{m_i}\sum_{j=0}^{m_i-1}S^j x_{m_i-1}\right)-S^l x\right\| \\
\leqslant{}& 2\left\|x_{m_i}-\frac{1}{m_i}\sum_{j=0}^{m_i-1}S^j x_{m_i-1}\right\|+\|x_{m_i}-x\| \\
&+\left\|\frac{1}{m_i}\sum_{j=0}^{m_i-1}S^j x_{m_i-1}-S^l\left(\frac{1}{m_i}\sum_{j=0}^{m_i-1}S^j x_{m_i-1}\right)\right\|.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
&\left\|\frac{1}{m_i}\sum_{j=0}^{m_i-1}S^j x_{m_i-1}-S^l\left(\frac{1}{m_i}\sum_{j=0}^{m_i-1}S^j x_{m_i-1}\right)\right\| \\
&\quad\leqslant\left\|\frac{1}{m_i}\sum_{j=1}^{m_i}S^j x_{m_i-1}-S^l\left(\frac{1}{m_i}\sum_{j=1}^{m_i}S^j x_{m_i-1}\right)\right\|+\frac{1}{m_i}\|x_{m_i-1}-S^{m_i}x_{m_i-1}\| \\
&\quad\quad+\left\|S^l\left(\frac{1}{m_i}\sum_{j=1}^{m_i}S^j x_{m_i-1}\right)-S^l\left(\frac{1}{m_i}\sum_{j=0}^{m_i-1}S^j x_{m_i-1}\right)\right\| \\
&\quad\leqslant\left\|\frac{1}{m_i}\sum_{j=1}^{m_i}S^j x_{m_i-1}-S^l\left(\frac{1}{m_i}\sum_{j=1}^{m_i}S^j x_{m_i-1}\right)\right\|+\frac{2}{m_i}\|x_{m_i-1}-S^{m_i}x_{m_i-1}\|.
\end{aligned}
$$

Since $\{x_{m_i-1}\}$ and $\{S^{m_i}x_{m_i-1}\}$ are bounded, there exists a positive integer i_1 such that

$$
\frac{1}{m_i}\|x_{m_i-1}-S^{m_i}x_{m_i-1}\|<\frac{\delta}{6}
$$

for each $i\geqslant i_1$. Since $\{x_{m_i-1}\}$ is bounded, by Lemma 3 in [3] there exists a positive integer L_0, such that for every $l\geqslant L_0$, there exists a positive integer i_l satisfying

$$
\left\|\frac{1}{m_i}\sum_{j=1}^{m_i}S^j x_{m_i-1}-S^l\left(\frac{1}{m_i}\sum_{j=1}^{m_i}S^j x_{m_i-1}\right)\right\|<\frac{\delta}{6}
$$

for each $i\geqslant i_l$. Since $\lim_{n\to\infty}\|x_{n+1}-\frac{1}{n+1}\sum_{j=0}^{n}S^j x_n\|=0$, there exists a positive integer i_0 such that

$$
\left\|x_{m_i}-\frac{1}{m_i}\sum_{j=0}^{m_i-1}S^j x_{m_i-1}\right\|<\frac{\delta}{6}
$$

for all $i\geqslant i_0$. So, for any $l\geqslant L_0$ and $i\geqslant\max\{i_l,i_0,i_1\}$, we have

$$
\|x_{m_i}-S^l x\|<2\times\frac{\delta}{6}+r+\frac{\delta}{6}+\frac{\delta}{6}+2\times\frac{\delta}{6}=r+\delta.
$$

Choose $l_k \geqslant L_0$. Then for $i \geqslant \max\{i_{l_k}, i_0, i_1\}$, we have

$$\left\| x_{m_i} - \frac{S^{l_k}x + x}{2} \right\|^2 = 2\left\| \frac{x_{m_i} - S^{l_k}x}{2} \right\|^2 + 2\left\| \frac{x_{m_i} - x}{2} \right\|^2 - \left\| \frac{S^{l_k}x - x}{2} \right\|^2$$
$$< \frac{(r+\delta)^2}{2} + \frac{(r+\delta/2)^2}{2} - \frac{\varepsilon^2}{4} < (r+\delta)^2 - \frac{\varepsilon^2}{4} < r^2.$$

This implies

$$\liminf_{i\to\infty} \left\| x_{m_i} - \frac{S^{l_k}x + x}{2} \right\| < r.$$

However, we have $\frac{S^{l_k}x+x}{2} \neq x$ from $\|S^{l_k}x - x\| \geqslant \varepsilon$, then we obtain $\liminf_{i\to\infty} \|x_{m_i} - x\| < \liminf_{i\to\infty} \|x_{m_i} - \frac{S^{l_k}x+x}{2}\|$, this is a contradiction. Therefore, $\{S^l x\}$ converges strongly to x. This implies, for each $\varepsilon > 0$, there exists a positive number l_0 such that

$$\|S^l x - x\| \leqslant \frac{\varepsilon}{2}$$

for each $l \geqslant l_0$. So, we have that if $l \geqslant l_0 + 1$, then

$$\|Sx - x\| \leqslant \|S^{l-1}x - x\| + \|S^l x - x\| \leqslant \varepsilon.$$

Since ε is arbitrary, we get that $Sx = x$. □

3. Strong convergence theorems

In this section, we prove two strong convergence theorems. To prove our results, we use the method employed in [6,7].

Theorem 1. *Let H be a Hilbert space, C a nonempty closed convex subset of H, P_1 the metric projection from H onto C, T a nonexpansive nonself-mapping from C into H such that $F(T)$ is nonempty, and $\{\alpha_n\}$ a sequence of real numbers such that $0 \leqslant \alpha_n \leqslant 1$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Suppose that $\{x_n\}$ is given by $x_0, x \in C$ and*

$$x_{n+1} = \frac{1}{n+1} \sum_{j=0}^{n} \bigl(\alpha_n x + (1-\alpha_n)(P_1T)^j x_n\bigr) \quad \text{for } n = 0, 1, 2, \ldots.$$

Then $\{x_n\}$ converges strongly to $P_2 x \in F(T)$, where P_2 is the metric projection from C onto $F(T)$.

Proof. Since $F(T)$ is nonempty, then T satisfies (NNO) condition. Let $z \in F(T)$, $M = \max\{\|x - z\|, \|x_0 - z\|\}$. Then we have

$$\|x_1 - z\| = \bigl\|\alpha_0 x + (1-\alpha_0)x_0 - z\bigr\| \leqslant \alpha_0\|x - z\| + (1-\alpha_0)\|x_0 - z\| \leqslant M.$$

If $\|x_n - z\| \leqslant M$ for some $n \in \mathbf{N}$, then we can show that $\|x_{n+1} - z\| \leqslant M$ similarly. Therefore, by induction, we obtain $\|x_n - z\| \leqslant M$ for all $n \in \mathbf{N}$ and hence $\{x_n\}$ is bounded. Also, from

$$\begin{aligned}
&\left\| x_{n+1} - \frac{1}{n+1} \sum_{j=0}^{n} (P_1 T)^j x_n \right\| \\
&\quad = \left\| \frac{1}{n+1} \sum_{j=0}^{n} \bigl(\alpha_n x + (1-\alpha_n)(P_1 T)^j x_n\bigr) - \frac{1}{n+1} \sum_{j=0}^{n} (P_1 T)^j x_n \right\| \\
&\quad \leqslant \alpha_n \left\| x - \frac{1}{n+1} \sum_{j=0}^{n} (P_1 T)^j x_n \right\|,
\end{aligned}$$

we have $\{x_{n+1} - \frac{1}{n+1} \sum_{j=0}^{n} (P_1 T)^j x_n\}$ converges strongly to 0. Next, we shall prove that

$$\limsup_{n\to\infty} \langle P_2 x - x_n, P_2 x - x\rangle \leqslant 0. \tag{7}$$

Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{i\to\infty} \langle P_2 x - x_{n_i}, P_2 x - x\rangle = \limsup_{n\to\infty} \langle P_2 x - x_n, P_2 x - x\rangle,$$

and there exists $\{x_{n_{i_j}}\}$ which is a subsequence of $\{x_{n_i}\}$ converging weakly as $j \to \infty$ to $w \in C$. From Lemma 1 and Proposition 1, we obtain $w \in F(T)$. Then we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle P_2 x - x_n, P_2 x - x\rangle &= \lim_{j\to\infty} \langle P_2 x - x_{n_{i_j}} P_2 x - x\rangle \\
&= \langle P_2 x - w P_2 x - x\rangle \leqslant 0.
\end{aligned}$$

By (7), for any $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that

$$\langle P_2 x - x_n, P_2 x - x\rangle < \varepsilon \tag{8}$$

for all $n \geqslant m$. On the other hand, from

$$\begin{aligned}
x_{n+1} - P_2 x + \alpha_n (P_2 x - x) = {}& \frac{1}{n+1} \sum_{j=0}^{n} \bigl(\alpha_n x + (1-\alpha_n)(P_1 T)^j x_n\bigr) \\
& - \bigl(\alpha_n x + (1-\alpha_n) P_2 x\bigr),
\end{aligned}$$

this implies

$$\begin{aligned}
\|x_{n+1} - P_2 x\|^2 &\leqslant \left\| \frac{1}{n+1} \sum_{j=0}^{n} \bigl(\alpha_n x + (1-\alpha_n)(P_1 T)^j x_n\bigr) - \bigl(\alpha_n x + (1-\alpha_n) P_2 x\bigr) \right\|^2 \\
&\quad + 2\alpha_n \langle x_{n+1} - P_2 x, x - P_2 x\rangle \\
&\leqslant \left\{ (1-\alpha_n) \frac{1}{n+1} \sum_{j=0}^{n} \bigl\| (P_1 T)^j x_n - P_2 x \bigr\| \right\}^2 \\
&\quad + 2\alpha_n \langle x_{n+1} - P_2 x, x - P_2 x\rangle \\
&\leqslant (1-\alpha_n)^2 \|x_n - P_2 x\|^2 + 2\alpha_n \langle x_{n+1} - P_2 x, x - P_2 x\rangle
\end{aligned}$$

for all $n \in \mathbf{N}$. By (8),

$$\begin{aligned}\|x_{n+1} - P_2x\| &\leqslant 2\alpha_n\varepsilon + (1-\alpha_n)\|x_n - P_2x\|^2 \\ &= 2\varepsilon\bigl(1-(1-\alpha_n)\bigr) + (1-\alpha_n)\|x_n - P_2x\|^2 \\ &\leqslant 2\varepsilon\bigl(1-(1-\alpha_n)\bigr) \\ &\quad + (1-\alpha_n)\bigl(2\varepsilon\bigl(1-(1-\alpha_{n-1})\bigr) + (1-\alpha_{n-1})\|x_{n-1} - P_2x\|^2\bigr) \\ &= 2\varepsilon\bigl(1-(1-\alpha_n)(1-\alpha_{n-1})\bigr) + (1-\alpha_n)(1-\alpha_{n-1})\|x_{n-1} - P_2x\|\end{aligned}$$

for all $n \geqslant m$. By induction, we obtain

$$\|x_{n+1} - P_2x\|^2 \leqslant 2\varepsilon\left(1 - \prod_{k=m}^{n}(1-\alpha_k)\right) + \prod_{k=m}^{n}(1-\alpha_k)\|x_m - P_2x\|.$$

Therefore, from $\sum_{n=0}^{\infty}\alpha_n = \infty$, we have

$$\limsup_{n\to\infty}\|x_{n+1} - P_2x\| \leqslant 2\varepsilon.$$

Since ε is arbitrary, we can conclude that $\{x_n\}$ converges strongly to P_2x. □

Theorem 2. *Let H be a Hilbert space, C a nonempty closed convex subset of H, P_1 the metric projection from H onto C, and T a nonexpansive nonself-mapping from C into H such that $F(T)$ is nonempty, and $\{\alpha_n\}$ a sequence of real numbers such that $0 \leqslant \alpha_n \leqslant 1$, $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Suppose that $\{y_n\}$ is given by $y_0, y \in C$ and*

$$y_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n} P_1\bigl(\alpha_n y + (1-\alpha_n)(TP_1)^j y_n\bigr) \quad \textit{for } n = 0, 1, 2, \ldots.$$

Then $\{y_n\}$ converges strongly to $P_2y \in F(T)$, where P_2 is the metric projection from C onto $F(T)$.

Proof. Since $F(T)$ is nonempty, then T satisfies (NNO) condition. Let $z \in F(T)$, $M = \max\{\|y - z\|, \|y_0 - z\|\}$. Then we have

$$\begin{aligned}\|y_1 - z\| &= \bigl\|P_1\bigl(\alpha_0 y + (1-\alpha_0)y_0\bigr) - z\bigr\| \leqslant \bigl\|\alpha_0 y + (1-\alpha_0)y_0 - z\bigr\| \\ &\leqslant \alpha_0\|y - z\| + (1-\alpha_0)\|y_0 - z\| \leqslant M.\end{aligned}$$

If $\|y_n - z\| \leqslant M$ for some $n \in \mathbf{N}$. Then we can show that $\|y_{n+1} - z\| \leqslant M$ similarly. Therefore, by induction, we obtain $\|y_n - z\| \leqslant M$ for all $n \in \mathbf{N}$ and hence $\{y_n\}$ is bounded. Also, from

$$\begin{aligned}&\left\|y_{n+1} - \frac{1}{n+1}\sum_{j=0}^{n}(P_1T)^j y_n\right\| \\ &\quad = \left\|\frac{1}{n+1}\sum_{j=0}^{n} P_1\bigl(\alpha_n y + (1-\alpha_n)(TP_1)^j y_n\bigr) - \frac{1}{n+1}\sum_{j=0}^{n}(P_1T)^j y_n\right\| \\ &\quad \leqslant \frac{1}{n+1}\sum_{j=0}^{n}\bigl\|P_1\bigl(\alpha_n y + (1-\alpha_n)(TP_1)^j y_n\bigr) - (P_1T)^j y_n\bigr\|\end{aligned}$$

$$\leqslant \frac{1}{n+1}\sum_{j=0}^{n}\left\|\alpha_n y + (1-\alpha_n)(TP_1)^j y_n - (TP_1)^j y_n\right\|$$
$$\leqslant \alpha_n \frac{1}{n+1}\sum_{j=0}^{n}\left\|y - (TP_1)^j y_n\right\|,$$

we obtain $\{y_{n+1} - \frac{1}{n+1}\sum_{j=0}^{n}(P_1T)^j y_n\}$ converges strongly to 0. Next, we shall prove that

$$\limsup_{n\to\infty}\langle P_2y - y_n, P_2y - y\rangle \leqslant 0. \tag{9}$$

Let $\{y_{n_i}\}$ be a subsequence of $\{y_n\}$ such that

$$\lim_{i\to\infty}\langle P_2y - y_{n_i}, P_2y - y\rangle = \limsup_{n\to\infty}\langle P_2y - y_n, P_2y - y\rangle,$$

and there exists $\{y_{n_{i_j}}\}$ which is a subsequence of $\{y_{n_i}\}$ converging weakly as $j \to \infty$ to $w \in C$. From Lemma 1 and Proposition 1, we obtain $w \in F(T)$. Then we have

$$\begin{aligned}\limsup_{n\to\infty}\langle P_2y - y_n, P_2y - y\rangle &= \lim_{j\to\infty}\langle P_2y - y_{n_{i_j}}, P_2y - y\rangle\\ &= \langle P_2y - w, P_2y - y\rangle \leqslant 0.\end{aligned}$$

By (9), for any $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that

$$\langle P_2y - y_n, P_2y - y\rangle < \varepsilon \tag{10}$$

for all $n \geqslant m$. On the other hand, from

$$\begin{aligned}y_{n+1} - P_2y + \alpha_n(P_2y - y) &= \frac{1}{n+1}\sum_{j=0}^{n}P_1\big(\alpha_n y + (1-\alpha_n)(TP_1)^j y_n\big)\\ &\quad - P_1\big(\alpha_n y + (1-\alpha_n)P_2y\big),\end{aligned}$$

this implies

$$\begin{aligned}\|y_{n+1} - P_2y\|^2 &\leqslant \left\|\frac{1}{n+1}\sum_{j=0}^{n}P_1\big(\alpha_n y + (1-\alpha_n)(TP_1)^j y_n\big)\right.\\ &\quad \left. - P_1\big(\alpha_n y + (1-\alpha_n)P_2y\big)\right\|^2 + 2\alpha_n\langle y_{n+1} - P_2y, y - P_2y\rangle\\ &\leqslant \left\{\frac{1}{n+1}\sum_{j=0}^{n}\left\|P_1\big(\alpha_n y + (1-\alpha_n)(TP_1)^j y_n\big)\right.\right.\\ &\quad \left.\left. - P_1\big(\alpha_n y + (1-\alpha_n)P_2y\big)\right\|\right\}^2 + 2\alpha_n\langle y_{n+1} - P_2y, y - P_2y\rangle\\ &\leqslant \left\{(1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}\left\|(TP_1)^j y_n - P_2y\right\|\right\}^2\\ &\quad + 2\alpha_n\langle y_{n+1} - P_2y, y - P_2y\rangle\\ &\leqslant (1-\alpha_n)^2\|y_n - P_2y\|^2 + 2\alpha_n\langle y_{n+1} - P_2y, y - P_2y\rangle\end{aligned}$$

for all $n \in \mathbf{N}$. By (10), $\sum_{n=0}^{\infty} \alpha_n = \infty$, and in the proof of Theorem 1, we can conclude that $\{y_n\}$ converges strongly to $P_2 y$. □

References

[1] B.R. Halpern, G.M. Bergman, A fixed-point theorem for inward and outward maps, Trans. Amer. Math. Soc. 130 (1968) 353–358.
[2] S. Matsushita, D. Kuroiwa, Approximation of fixed points of nonexpansive nonself-mappings, Sci. Math. Jpn. 57 (2003) 171–176.
[3] T. Shimizu, W. Takahashi, Strong convergence theorem for asymptotically nonexpansive mappings, Nonlinear Anal. 26 (1996) 265–272.
[4] T. Shimizu, W. Takahashi, Strong convergence to common fixed points of families of nonexpansive mappings, J. Math. Anal. Appl. 211 (1997) 71–83.
[5] N. Shioji, W. Takahashi, A strong convergence theorem for asymptotically nonexpansive mappings in Banach spaces, Arch. Math. 72 (1999) 354–359.
[6] W. Takahashi, Convex Analysis and Approximation of Fixed Points, Yokohama Publishers, Yokohama, 2000 (in Japanese).
[7] R. Wittmann, Approximation of fixed points of nonexpansive mappings, Arch. Math. 58 (1992) 486–491.