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Relative controllability of fractional dynamical systems with distributed delays in control[☆]

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ABSTRACT

This paper deals with the global relative controllability of linear and nonlinear fractional dynamical systems with distributed delays in control for finite dimensional spaces. Sufficient conditions for controllability results are obtained using Schauder's fixed point theorem and the controllability Grammian matrix which is defined by Mittag Leffler matrix function. An example is given to illustrate the theory.

1. Introduction

Fractional differential equations have been of great interest recently among researchers. Fractional calculus, in allowing integrals and derivatives of any positive real order (the term "fractional" is kept only for historical reasons), can be considered as a branch of mathematical analysis which deals with integrodifferential operators and equations where the integrals are of convolution type and exhibit (weakly singular) kernels of power-law type. Until recent times, fractional calculus was considered as a rather esoteric mathematical theory without applications, but in the last few decades there has been an explosion of research activities on the application of fractional calculus in various fields of science such as physics, mechanics, chemistry, engineering, etc. For example, we may cite hydrology application [1], physics [2], polymer rheology [3] and system biology [4]. Some recent contributions to the theory of fractional differential equations can be found in [5–8].

Recently, the fractional dynamical systems have attracted lots of attention in control systems society even though fractional-order control problems were investigated as early as 1960s [9]. In the fractional order controller, the fractional order integration or derivative of the output error is used for the current control force calculation. Thus, the dynamical fractional systems oriented towards the field of control theory concerning heat transfer, lossless transmission lines [10,11], the use of discretizing devices supported by fractional calculus. There are few papers regarding the controllability analysis of the fractional systems in the literature (see, for example, [12–15]).

On the other hand, time delays are often present in various engineering systems such as biological, economical systems, chemical processes. For instance, they appear as transportation and communication lags and also arise as feedback delay in measurement and closed loop systems. Due to the transmission of the signal, the mechanical transmission needs a length of time. For clear idea for time delays, one can refer the monographs [16,17]. There are important practical applications for systems such as economic, biological, physiological and spaceflight industry systems, having the phenomena of distributed time delays in control [18]. Models for integer order systems with delays in the control parameters have been proposed in the literature. Klamka [19,20] studied the controllability of nonlinear dynamical systems with distributed delays in control with the aid of Schauder's fixed point theorem, whereas Balachandran and Somasundaram [21] obtained the relative

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controllability of nonlinear systems with distributed delays in control and implicit derivative using Darbo fixed point theorem and Balachandran [22] extended the results for the nonlinear perturbation system with distributed delays and without implicit derivatives. The results are derived by using the Schauder fixed point theorem. For more on controllability of nonlinear systems, see [23].

However, it should be emphasized that to the best of our knowledge, the relative controllability of fractional dynamical system with distributed delays in control variables has not been established yet. In order to fill this gap, in this paper, we study the relative controllability for both linear and nonlinear fractional dynamical systems with distributed delays in control. The rest of the paper is organized as follows: In Section 2, some well known fractional operators and special functions, along with a set of properties are defined which will be of use as we proceed in our discussion. In Section 3, the linear fractional system with distributed delay in control is presented and the controllability condition is established using the controllability Grammian matrix which is defined by means of Mittag Leffler matrix function. In Section 4, the corresponding nonlinear fractional system is considered and the controllability results are examined with the natural assumption that the linear fractional system is relatively controllable. The results are established by using the Schauder fixed point theorem and the fractional calculus. Finally, Section 5 ends up with an example to illustrate the theory.

2. Preliminaries

Let $\alpha, \beta > 0$, with $n-1 < \alpha < n, n-1 < \beta < n$, and $n \in \mathbb{N}$, D is the usual differential operator. Let $\mathbb{R}^m$ be the m-dimensional Euclidean space, $\mathbb{R}_+ = [0, \infty)$, and suppose $f \in L_1(\mathbb{R}_+)$. The following definitions and properties are well known, for $\alpha, \beta > 0$ and f as a suitable function (see, for instance, [24,25]):

(a) Riemann–Liouville fractional operators:

$$(I_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt$$

$$(D_{0+}^{\alpha} f)(x) = D^n (I_{a+}^{n-\alpha} f)(x)$$

(b) Caputo fractional derivative:

$$({}^C D_{0+}^{\alpha} f)(x) = (I_{0+}^{n-\alpha} D^n f)(x),$$

in particular $I_{0+}^{\alpha}\ {}^C D_{0+}^{\alpha} f(t) = f(t) - f(0)$ $(0 < \alpha < 1)$.

The following is a well known relation, for finite interval $[a, b] \in \mathbb{R}_+$

$$(D_{a+}^{\alpha} f)(x) = ({}^C D_{a+}^{\alpha} f)(x) + \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{\Gamma(1+j-\alpha)} (x-a)^{j-\alpha}, \quad n = \Re(\alpha) + 1.$$

The following properties of mentioned operators are specially interesting:

(i) $({}^C D_{0+}^{\alpha} 1) = 0$
(ii) $(D_{0+}^{\alpha} 1) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}$
(iii) $I_{0+}^{\alpha}(f(t) + g(t)) = I_{0+}^{\alpha} f(t) + I_{0+}^{\alpha} g(t)$
(iv) $I_{0+}^{\alpha} I_{0+}^{\beta} f(t) = I_{0+}^{\alpha+\beta} f(t) = I_{0+}^{\beta} I_{0+}^{\alpha} f(t)$
(v) $D_{0+}^{\alpha} I_{0+}^{\alpha} f(t) = f(t)$
(vi) $I_{0+}^{\alpha}\ {}^C D_{0+}^{\alpha} f(t) = f(t) - f(0), 0 < \alpha < 1$
(vii) In general, $D_{0+}^{\alpha} D_{0+}^{\beta} f(t) \neq D_{0+}^{\alpha+\beta} f(t)$ and $D_{0+}^{\alpha} D_{0+}^{\beta} f(t) \neq D_{0+}^{\beta}\ D_{0+}^{\alpha} f(t)$
(viii) The Laplace transform of the Caputo fractional derivative is

$$\mathcal{L}\left\{ {}^C D_{0+}^{\alpha} f(t) \right\}(s) = s^{\alpha} F(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\alpha-1-k}.$$

From the above we observe that, the Riemann–Liouville fractional derivatives have singularity at zero and the fractional differential equations in the Riemann–Liouville sense require initial conditions of special form lacking physical interpretation. To over come this difficulty Caputo [26] defined the Caputo derivative but in general, both the Riemann–Liouville and the Caputo fractional operators do not possess neither semigroup nor commutative properties, which are inherent to the derivatives on integer order. There are other types of fractional derivatives in the literature. The Caputo fractional derivative is more often used in the applied researches. For basic facts and other results on fractional differential equations one can refer the books [24,27,28,25]. For brevity of notation let us take I_{0+}^{α} as I^{α} and ${}^C D_{0+}^{\alpha}$ as ${}^C D^{\alpha}$ and the fractional derivative is taken as Caputo sense.

(c) Mittag Leffler Function

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)},\quad \text{for } \alpha,\beta>0.$$

The general Mittag Leffler function satisfies

$$\int_0^{\infty}e^{-t}t^{\beta-1}E_{\alpha,\beta}(t^{\alpha}z)dt=\frac{1}{1-z},\quad \text{for } |z|<1.$$

The Laplace transform of $E_{\alpha,\beta}(z)$ follows from the integral

$$\int_0^{\infty}e^{-st}t^{\beta-1}E_{\alpha,\beta}(\pm at^{\alpha})dt=\frac{s^{\alpha-\beta}}{(s^{\alpha}\mp a)}.$$

That is

$$\mathcal{L}\left\{t^{\beta-1}E_{\alpha,\beta}(\pm at^{\alpha})\right\}(s)=\frac{s^{\alpha-\beta}}{(s^{\alpha}\mp a)},$$

for $\Re(s)>|a|^{\frac{1}{\alpha}}$ and $\Re(\beta)>0$. In particular, for $\beta=1$,

$$E_{\alpha,1}(\lambda z^{\alpha})=E_{\alpha}(\lambda z^{\alpha})=\sum_{k=0}^{\infty}\frac{\lambda^k z^{k\alpha}}{\Gamma(\alpha k+1)},\quad \lambda,z\in\mathbb{C}$$

have the interesting property ${}^{C}D^{\alpha}E_{\alpha}(\lambda t^{\alpha})=\lambda E_{\alpha}(\lambda t^{\alpha})$ and

$$\mathcal{L}\left\{E_{\alpha}(\pm at^{\alpha})\right\}(s)=\frac{s^{\alpha-1}}{(s^{\alpha}\mp a)},\quad \text{for } \beta=1.$$

(d) Solution Representation:
Consider the linear fractional differential equation of the form

$$\begin{aligned}&{}^{C}D^{\alpha}x(t)=Ax(t)+f(t),\quad t\in[0,T],\\&x(0)=x_0,\end{aligned}$$

where $0<\alpha<1$, $x\in\mathbb{R}^n$ and A is an $n\times n$ matrix. In order to find the solution, apply Laplace transform on both sides and use the property (viii), we get

$$\begin{aligned}&s^{\alpha}X(s)-s^{\alpha-1}x(0)=AX(s)+F(s)\\&X(s)=[s^{\alpha-1}(s^{\alpha}I-A)^{-1}]x_0+F(s)(s^{\alpha}I-A)^{-1}.\end{aligned}$$

Apply inverse Laplace transform on both sides and by we have

$$\mathcal{L}^{-1}\{X(s)\}=\mathcal{L}^{-1}\left\{s^{\alpha-1}(s^{\alpha}I-A)^{-1}\right\}x_0+\mathcal{L}^{-1}\{F(s)\}*\mathcal{L}^{-1}\left\{(s^{\alpha}I-A)^{-1}\right\}.$$

Finally, substituting Laplace transformation of Mittag Leffler function, we get the solution of the given system [29,30]

$$\begin{aligned}&x(t)=E_{\alpha}(At^{\alpha})x_0+f(t)*t^{\alpha}E_{\alpha,\alpha}(At^{\alpha})\\&x(t)=E_{\alpha}(At^{\alpha})x_0+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^{\alpha})f(s)ds.\end{aligned}$$

where $E_{\alpha}(At^{\alpha})$ is the matrix extension of the mentioned Mittag Leffler functions with the following representation:

$$E_{\alpha}(At^{\alpha})=\sum_{k=0}^{\infty}\frac{A^k t^{k\alpha}}{\Gamma(1+k\alpha)}$$

with the property ${}^{C}D^{\alpha}E_{\alpha}(At^{\alpha})=AE_{\alpha}(At^{\alpha})$.

3. Linear systems

Consider the linear fractional dynamical system with distributed delays in control represented by the fractional differential equation of the form

$$\begin{aligned}&{}^{C}D^{\alpha}x(t)=Ax(t)+\int_{-h}^{0}d_{\tau}B(t,\tau)u(t+\tau),\quad t\in[0,T]:=J,\ 0<\alpha<1 \qquad (1)\\&x(0)=x_0,\end{aligned}$$

where $x \in \mathbb{R}^n$ and the second integral term is in the Lebesgue Stieltjes sense with respect to τ. Let $h > 0$ be given. For function $u : [-h, T] \to \mathbb{R}^m$ and $t \in J$, we use the symbol u_t to denote the function on $[-h, 0]$, defined by $u_t(s) = u(t + s)$ for $s \in [-h, 0)$. A is a $n \times n$ matrix, $B(t, \tau)$ is an $n \times m$ dimensional matrix continuous in t for fixed τ and is of bounded variation in τ on $[-h, 0]$ for each $t \in J$ and continuous from left in τ on the interval $(-h, 0)$.

Let us assume the following definitions of complete state of the system (1) at time t and relative controllability [22,21].

Definition 3.1. The set $y(t) = \{x(t), u_t\}$ is the complete state of the system (1) at time t.

Definition 3.2. System (1) is said to be globally relatively controllable on J if for every complete state $y(0)$ and every vector $x_1 \in \mathbb{R}^n$ there exists a control $u(t)$ defined on J such that the corresponding trajectory of the system (1) satisfies $x(T) = x_1$.

The solution of the system (1) is given by the following expression [29,30]

$$x(t) = E_\alpha(At^\alpha)x_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) \left[\int_{-h}^0 d_\tau B(s,\tau) u(s+\tau) \right] ds, \tag{2}$$

where $E_\alpha(At^\alpha)$ is the Mittag Leffler matrix function. Now using the well known result of unsymmetric Fubini theorem [31] and change of order of integration to the last term, we have

$$\begin{aligned} x(t) &= E_\alpha(At^\alpha)x_0 + \int_{-h}^0 dB_\tau \left[\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) u(s+\tau) B(s,\tau) ds \right] \\ &= E_\alpha(At^\alpha)x_0 + \int_{-h}^0 dB_\tau \left[\int_\tau^0 (t-(s-\tau))^{\alpha-1} E_{\alpha,\alpha}(A(t-(s-\tau))^\alpha) B(s-\tau,\tau) u_0(s) ds \right] \\ &\quad + \int_{-h}^0 dB_\tau \left[\int_0^{t+\tau} (t-(s-\tau))^{\alpha-1} E_{\alpha,\alpha}(A(t-(s-\tau))^\alpha) B(s-\tau,\tau) u(s) ds \right] \\ &= E_\alpha(At^\alpha)x_0 + \int_{-h}^0 dB_\tau \left[\int_\tau^0 (t-(s-\tau))^{\alpha-1} E_{\alpha,\alpha}(A(t-(s-\tau))^\alpha) B(s-\tau,\tau) u_0(s) ds \right] \\ &\quad + \int_0^t \left[\int_{-h}^0 (t-(s-\tau))^{\alpha-1} E_{\alpha,\alpha}(A(t-(s-\tau))^\alpha) d_\tau B_t(s-\tau,\tau) \right] u(s) ds, \end{aligned} \tag{3}$$

where

$$B_t(s,\tau) = \begin{cases} B(s,\tau), & s \le t \\ 0, & s > t \end{cases}$$

and dB_τ denotes the integration of Lebesgue Stieltjes sense with respect to the variable τ in the function $B(t, \tau)$.

For our convenience, let us introduce the notation

$$G(t,s) = \int_{-h}^0 (t-(s-\tau))^{\alpha-1} E_{\alpha,\alpha}(A(t-(s-\tau))^\alpha) d_\tau B_t(s-\tau,\tau). \tag{4}$$

Define the controllability Grammian matrix

$$W(0,T) = \int_0^T G(T,s) G^*(T,s) ds$$

where the $*$ indicates the matrix transpose.

Theorem 3.1. *The linear control system (1) is relatively controllable on* $[0, T]$ *if and only if the controllability Grammian matrix*

$$W = \int_0^T G(T,s) G^*(T,s) ds \tag{5}$$

is positive definite, for some $T > 0$.

Proof. Since W is positive definite, that is, it is non-singular and so its inverse is well-defined. Define the control function as,

$$\begin{aligned} u(t) &= G^*(T,t) W^{-1} \Bigg(x_1 - E_\alpha(AT^\alpha)x_0 \\ &\quad - \int_{-h}^0 dB_\tau \left[\int_\tau^0 (T-(s-\tau))^{\alpha-1} E_{\alpha,\alpha}(A(T-(s-\tau))^\alpha) B(s-\tau,\tau) u_0(s) ds \right] \Bigg), \end{aligned} \tag{6}$$

where the complete state $y(0)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrarily. Inserting (6) in (3) and using (4), we have

$$\begin{aligned} x(T) &= E_\alpha(AT^\alpha)x_0 + \int_{-h}^{0} dB_\tau \left[\int_{\tau}^{0} (T-(s-\tau))^{\alpha-1}E_{\alpha,\alpha}(A(T-(s-\tau))^\alpha)B(s-\tau,\tau)u_0(s)ds\right] \\ &\quad + \int_0^T \left[\int_{-h}^{0} (T-(s-\tau))^{\alpha-1}E_{\alpha,\alpha}(A(T-(s-\tau))^\alpha)d_\tau B_T(s-\tau,\tau)\right] \\ &\quad \times \left[\int_{-h}^{0} (T-(s-\tau))^{\alpha-1}E_{\alpha,\alpha}(A(T-(s-\tau))^\alpha)d_\tau B_T(s-\tau,\tau)\right]^* W^{-1}\Bigg(x_1 - E_\alpha(AT^\alpha)x_0 \\ &\quad - \int_{-h}^{0} dB_\tau \left[\int_{\tau}^{0} (T-(s-\tau))^{\alpha-1}E_{\alpha,\alpha}(A(T-(s-\tau))^\alpha)B(s-\tau,\tau)u_0(s)ds\right]\Bigg) d\tau \\ &= x_1. \end{aligned}$$

Thus the control $u(t)$ transfers the initial state $y(0)$ to the desired vector $x_1 \in \mathbb{R}^n$ at time T. Hence the system (1) is controllable.

On the other hand, if it is not positive definite, there exists a nonzero y such that

$$\begin{aligned} & y^*Wy = 0 \\ & y^* \int_0^T G(T,s)G^*(T,s)yds = 0 \\ & y^*G(T,s) = 0, \quad \text{on } [0,T]. \end{aligned}$$

Let $x_0 = [E_\alpha(AT^\alpha)]^{-1}y$. By the assumption, there exists a control u such that it steers the complete initial state $y(0) = \{x_0, u_0(s)\}$ to the origin in the interval $[0, T]$. It follows that

$$\begin{aligned} x(T) &= E_\alpha(AT^\alpha)x_0 + \int_{-h}^{0} dB_\tau \left[\int_{\tau}^{0} (T-(s-\tau))^{\alpha-1}E_{\alpha,\alpha}(A(T-(s-\tau))^\alpha)B(s-\tau,\tau)u_0(s)ds\right] \\ &\quad + \int_0^t \left[\int_{-h}^{0} (T-(s-\tau))^{\alpha-1}E_{\alpha,\alpha}(A(T-(s-\tau))^\alpha)d_\tau B_T(s-\tau,\tau)\right] u(s)ds \\ &= y + \int_{-h}^{0} dB_\tau \left[\int_{\tau}^{0} (T-(s-\tau))^{\alpha-1}E_{\alpha,\alpha}(A(T-(s-\tau))^\alpha)B(s-\tau,\tau)u_0(s)ds\right] \\ &\quad + \int_0^t \left[\int_{-h}^{0} (T-(s-\tau))^{\alpha-1}E_{\alpha,\alpha}(A(T-(s-\tau))^\alpha)d_\tau B_T(s-\tau,\tau)\right] u(s)ds \\ &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} 0 &= y^*y + \int_0^T y^*G(T,s)u(s)ds \\ &\quad + y^* \int_{-h}^{0} dB_\tau \left[\int_{\tau}^{0} (T-(s-\tau))^{\alpha-1}E_{\alpha,\alpha}(A(T-(s-\tau))^\alpha)B(s-\tau,\tau)u_0(s)ds\right]. \end{aligned}$$

But the second and third term are zero leading to the conclusion $y^*y = 0$. This is a contradiction to $y \neq 0$. Thus W is positive definite. Hence the desired result. □

4. Nonlinear systems

Consider the nonlinear fractional dynamical system with distributed delays in control represented by the fractional differential equation of the form

$$\begin{aligned} & {}^C D^\alpha x(t) = Ax(t) + \int_{-h}^{0} d_\tau B(t,\tau)u(t+\tau) + f(t, x(t), u(t)), \quad t \in [0,T] := J,\ 0 < \alpha < 1, \\ & x(0) = x_0, \end{aligned} \tag{7}$$

where A and B are as above and $f : J \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function. Assume the following space:

Denote Q as the Banach space of continuous $\mathbb{R}^n \times \mathbb{R}^m$ valued functions defined on the interval J with the uniform norm $\|(z, v)\| = \|z\| + \|v\|$ where

$$\|z\| = \sup\{|z(t)| : t \in J\}.$$

That is, $Q = C_n(J) \times C_m(J)$, where $C_n(J)$ is the Banach space of continuous $\mathbb{R}^n$ valued functions defined on the interval J with the sup norm. For each $(z, v) \in Q$, consider the linear fractional dynamical system

$$
\begin{aligned}
{}^C D^\alpha x(t) &= Ax(t) + \int_{-h}^{0} d_\tau B(t, \tau)u(t+\tau) + f(t, z(t), v(t)), \quad t \in J,\ 0 < \alpha < 1 \\
x(0) &= x_0.
\end{aligned} \tag{8}
$$

Then the solution of the system (8) can be expressed in the following form [29,30]

$$
\begin{aligned}
x(t) = {} & E_\alpha(At^\alpha)x_0 + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)f(s, z(s), v(s))ds \\
& + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)\left[\int_{-h}^{0} d_\tau B(s,\tau)u(s+\tau)\right]ds.
\end{aligned}
$$

Using the well known result of unsymmetric Fubini theorem [31] and change of order of integration to the last term, we have

$$
\begin{aligned}
x(t) = {} & E_\alpha(At^\alpha)x_0 + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)f(s, z(s), v(s))ds \\
& + \int_{-h}^{0} dB_\tau\left[\int_\tau^0 (t-(s-\tau))^{\alpha-1}E_{\alpha,\alpha}(A(t-(s-\tau))^\alpha)B(s-\tau,\tau)u_0(s)ds\right] \\
& + \int_0^t \left[\int_{-h}^{0} (t-(s-\tau))^{\alpha-1}E_{\alpha,\alpha}(A(t-(s-\tau))^\alpha)d_\tau B_t(s-\tau,\tau)\right]u(s)ds
\end{aligned} \tag{9}
$$

where

$$
B_t(s,\tau) = \begin{cases} B(s,\tau), & s \le t \\ 0, & s > t \end{cases}
$$

and dB_τ denotes the integration of Lebesgue Stieltjes sense with respect to the variable τ in the function $B(t, \tau)$. For brevity, let us introduce the notation:

$$
\begin{aligned}
\eta(y(0), x_1; z, v) = {} & x_1 - E_\alpha(AT^\alpha)x_0 - \int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(A(T-s)^\alpha)f(s, z(s), v(s))ds \\
& - \int_{-h}^{0} dB_\tau\left[\int_\tau^0 (T-(s-\tau))^{\alpha-1}E_{\alpha,\alpha}(A(T-(s-\tau))^\alpha)B(s-\tau,\tau)u_0(s)ds\right].
\end{aligned} \tag{10}
$$

Define the control function

$$
u(t) = G^*(T,t)W^{-1}\eta(y(0), x_1; z, v), \tag{11}
$$

where the complete state $y(0)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrarily and $*$ denotes the matrix transpose.

Theorem 4.1. *Let the continuous function f satisfies the condition*

$$
\lim_{|(x,u)| \to \infty} \frac{|f(t,x,u)|}{|(x,u)|} = 0
$$

uniformly in $t \in J$, and suppose that the linear fractional system (1) is globally relatively controllable. Then the nonlinear system (7) is globally relatively controllable on J.

Proof. Define the operator $\Psi : Q \to Q$ by

$$
\Psi(z, v) = (x, u),
$$

where

$$
\begin{aligned}
u(t) = {} & G^*(T,t)W^{-1}\eta(y(0), x_1; z, v) \\
= {} & G^*(T,t)W^{-1}\Bigg(x_1 - E_\alpha(AT^\alpha)x_0 - \int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(A(T-s)^\alpha)f(s, z(s), v(s))ds \\
& - \int_{-h}^{0} dB_\tau\left[\int_\tau^0 (T-(s-\tau))^{\alpha-1}E_{\alpha,\alpha}(A(T-(s-\tau))^\alpha)B(s-\tau,\tau)u_0(s)ds\right]\Bigg)
\end{aligned}
$$

and

$$\begin{aligned} x(t) &= E_\alpha(At^\alpha)x_0 + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)f(s,z(s),v(s))ds \\ &\quad + \int_{-h}^0 dB_\tau\left[\int_\tau^0 (t-(s-\tau))^{\alpha-1}E_{\alpha,\alpha}(A(t-(s-\tau))^\alpha)B(s-\tau,\tau)u_0(s)ds\right] \\ &\quad + \int_0^t\left[\int_{-h}^0 (t-(s-\tau))^{\alpha-1}E_{\alpha,\alpha}(A(t-(s-\tau))^\alpha)d_\tau B_t(s-\tau,\tau)\right]u(s)ds. \end{aligned}$$

Now let us introduce the following constants,

$$\begin{aligned} &a_1 = \sup\|E_\alpha(AT^\alpha)x_0\|; \qquad a_2 = \sup\|E_{\alpha,\alpha}(A(T-s)^\alpha)\|; \qquad a_3 = \sup\|E_\alpha(A(T-(s-\tau))^\alpha)\|; \\ &a_4 = \left\|\int_\tau^0 (t-(s-\tau))^{\alpha-1}E_{\alpha,\alpha}(A(t-(s-\tau))^\alpha)B(s-\tau,\tau)u_0(s)ds\right\|; \qquad a_5 = \sup\|G^*(T,t)\|; \\ &a = \max\left\{a_3T\|G(T,s)\|, 1\right\}; \qquad d_1 = 4a_5|W^{-1}|\alpha^{-1}\left[|x_1| + a_1 + a_4\right]; \qquad d_2 = 4[a_1+a_4]; \\ &c_1 = 4a_2a_5T^\alpha|W^{-1}|\alpha^{-1}; \qquad c_2 = 4a_2T^\alpha\alpha^{-1}; \qquad c = \max\left\{c_1, c_2\right\}; \qquad d = \max\left\{d_1, d_2\right\}; \\ &\sup|f| = \sup\left\{|f(s,z(s),v(s))|;\ s\in J\right\}. \end{aligned}$$

Then

$$\begin{aligned} |u(t)| &\le \|G^*(T,t)\|\,|W^{-1}|\left[|x_1| + a_1 + a_4\right] + \|G^*(T,t)\|\,|W^{-1}|a_2T^\alpha\alpha^{-1}\sup|f| \\ &\le \left[\frac{d_1}{4a} + \frac{c_1}{4a}\sup|f|\right] \\ &\le \frac{1}{4a}[d + c\sup|f|], \end{aligned}$$

and

$$\begin{aligned} |x(t)| &\le (a_1+a_4) + a_3\int_0^t \|G(t,s)\|\,\|u(s)\|ds + a_2\int_0^t (t-s)^{\alpha-1}\sup|f|ds \\ &\le \frac{d}{4} + \frac{1}{4}[d + c\sup|f|] + \frac{c}{4}\sup|f| \\ &\le \frac{d}{2} + \frac{c}{2}\sup|f| \end{aligned}$$

By hypothesis the function f satisfies the following condition [32]. For each pair of positive constants c and d, there exists a positive constant r such that, if $|p| \le r$, then

$$c|f(t,p)| + d \le r, \quad \text{for all } t \in J. \tag{12}$$

Also for given c and d, if r is a constant such that the inequality (12) is satisfied, then any r_1 such that $r < r_1$ will also satisfy (12). Now, take c and d as given above, and let r be chosen so that (12) is satisfied. Therefore, if $\|z\| \le \frac{r}{2}$ and $\|v\| \le \frac{r}{2}$, then $|z(s)| + |v(s)| \le r$, for all $s \in J$. It follows that $d + c\sup|f| \le r$. Therefore, $|u(s)| \le \frac{r}{4a}$, for all $s \in J$, and hence $\|u\| \le \frac{r}{4a}$, which gives $\|x\| \le \frac{r}{2}$. Thus, we have proved that, if $Q(r) = \{(z,v) \in Q : \|z\| \le \frac{r}{2}$ and $\|v\| \le \frac{r}{2}\}$, then Ψ maps $Q(r)$ into itself. Since f is continuous, it implies that the operator is continuous, and hence is completely continuous by the application of Arzela–Ascoli's theorem. Since $Q(r)$ is closed, bounded and convex, the Schauder fixed point theorem guarantees that Ψ has a fixed point $(z,v) \in Q(r)$ such that $\Psi(z,v) = (z,v) \equiv (x,u)$. Hence $x(t)$ is the solution of the system (7), and it is easy to verify that $x(T) = x_1$. Further the control function $u(t)$ steers the system (7) from initial complete state $y(0)$ to x_1 on J. Hence the system (7) is globally relatively controllable on J. □

5. Example

In this section we apply the results obtained in the previous section for the following fractional dynamical systems with distributed delays in control. Consider the nonlinear fractional dynamical system

$$\left.\begin{aligned}{}^{C}D^{\alpha}x_1(t) &= x_2(t) + \int_{-1}^{0} e^{\tau}[\cos tu_1(t+\tau) + \sin tu_2(t+\tau)]d\tau + \frac{x_1(t)}{1+x_2^2(t)}\\ {}^{C}D^{\alpha}x_2(t) &= -x_1(t) + \int_{-1}^{0} e^{\tau}[-\sin tu_1(t+\tau) + \cos tu_2(t+\tau)]d\tau + \frac{x_2(t)}{1+x_1^2(t)}\end{aligned}\right\} \tag{13}$$

for $t \in J$ and $0 < \alpha < 1$. In matrix form

$$A = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}, \qquad B(t,\tau) = \begin{pmatrix} e^{\tau}\cos t & e^{\tau}\sin t\\ -e^{\tau}\sin t & e^{\tau}\cos t\end{pmatrix} \quad \text{and} \quad f(t,x(t)) = \begin{pmatrix} \dfrac{x_1}{1+x_2^2(t)}\\ \dfrac{x_2}{1+x_1^2(t)}\end{pmatrix}.$$

Here $x(t) = \binom{x_1(t)}{x_2(t)}$ with $x_1(t) = x(t)$; $D^{\frac{q}{2}}x_1(t) = x_2(t)$. The Mittag Leffler matrix function is given by [29]

$$E_{\alpha}(At^{\alpha}) = \begin{pmatrix} \displaystyle\sum_{j=0}^{\infty}\frac{(-1)^j t^{2j\alpha}}{\Gamma(1+2j\alpha)} & \displaystyle\sum_{j=0}^{\infty}\frac{(-1)^j t^{(2j+1)\alpha}}{\Gamma(1+(2j+1)\alpha)}\\ \displaystyle -\sum_{j=0}^{\infty}\frac{(-1)^j t^{(2j+1)\alpha}}{\Gamma(1+(2j+1)\alpha)} & \displaystyle\sum_{j=0}^{\infty}\frac{(-1)^j t^{2j\alpha}}{\Gamma(1+2j\alpha)}\end{pmatrix}.$$

Further

$$E_{\alpha,\alpha}(A(T-(s-\tau))^{\alpha}) = \begin{pmatrix} \displaystyle\sum_{j=0}^{\infty}\frac{(-1)^j (T-(s-\tau))^{2j\alpha}}{\Gamma[(2j+1)\alpha]} & \displaystyle\sum_{j=0}^{\infty}\frac{(-1)^j (T-(s-\tau))^{(2j+1)\alpha}}{\Gamma[(j+1)2\alpha]}\\ \displaystyle -\sum_{j=0}^{\infty}\frac{(-1)^j (T-(s-\tau))^{(2j+1)\alpha}}{\Gamma[(j+1)2\alpha]} & \displaystyle\sum_{j=0}^{\infty}\frac{(-1)^j (T-(s-\tau))^{2j\alpha}}{\Gamma[(2j+1)\alpha]}\end{pmatrix},$$

and

$$(T-(s-\tau))^{\alpha-1}E_{\alpha,\alpha}(A(T-(s-\tau))^{\alpha}) = \begin{pmatrix} \cos_{\alpha}(t) & \sin_{\alpha}(t)\\ -\sin_{\alpha}(t) & \cos_{\alpha}(t)\end{pmatrix},$$

where $\cos_{\alpha}(t)$ and $\sin_{\alpha}(t)$ are given by [24]

$$\begin{aligned}\cos_{\alpha}(t) &= \sum_{j=0}^{\infty}\frac{(-1)^j (T-(s-\tau))^{(2j+1)\alpha-1}}{\Gamma[(2j+1)\alpha]}\\ \sin_{\alpha}(t) &= \sum_{j=0}^{\infty}\frac{(-1)^j (T-(s-\tau))^{(j+1)2\alpha-1}}{\Gamma[(j+1)2\alpha]}\\ G(T,s) &= \int_{-1}^{0}(T-(s-\tau))^{\alpha-1}E_{\alpha,\alpha}(A(T-(s-\tau))^{\alpha})d_{\tau}B_T(s-\tau,\tau)\\ &= \begin{pmatrix} p(s) & q(s)\\ -q(s) & p(s)\end{pmatrix},\\ p(s) &= \int_{-1}^{0} e^{\tau}[\cos_{\alpha}(T-(s-\tau))\cos(s-\tau) - \sin_{\alpha}(T-(s-\tau))\sin(s-\tau)]d\tau\\ q(s) &= \int_{-1}^{0} e^{\tau}[\sin_{\alpha}(T-(s-\tau))\cos(s-\tau) - \cos_{\alpha}(T-(s-\tau))\sin(s-\tau)]d\tau.\end{aligned}$$

By simple matrix calculation one can see that the controllability matrix

$$\begin{aligned} W(0,T) &= \int_0^T G(T,s)G^*(T,s)ds\\ &= \int_0^T [p^2(s)+q^2(s)]\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix} ds\end{aligned}$$

is positive definite for any $T > 0$. Further the nonlinear function $f(t, x(t))$ is bounded and continuous and all conditions stated in Theorem 4.1 are satisfied. Hence the fractional system (13) is globally relatively controllable on $[0, T]$.

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