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Identities involving Frobenius–Euler polynomials arising from non-linear differential equations

Taekyun Kim

Department Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea

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ABSTRACT

In this paper we consider non-linear differential equations which are closely related to the generating functions of Frobenius–Euler polynomials. From our non-linear differential equations, we derive some new identities between the sums of products of Frobenius–Euler polynomials and Frobenius–Euler polynomials of higher order.

1. Introduction

Let $u \in \mathbb{C}$ with $u \neq 1$. Then the Frobenius–Euler polynomials are defined by generating function as follows:

$$F_u(t,x) = \frac{1-u}{e^t - u} e^{xt} = \sum_{n=0}^{\infty} H_n(x \mid u) \frac{t^n}{n!} \quad \text{(see [2,4])}. \tag{1}$$

In the special case, $x = 0$, $H_n(0 \mid u) = H_n(u)$ are called the n-th Frobenius–Euler numbers (see [4]). By (1), we get

$$H_n(x \mid u) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} H_l(u) \quad \text{for } n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}. \tag{2}$$

E-mail address: tkkim@kw.ac.kr.

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Thus, by (1) and (2), we get the recurrence relation for $H_n(u)$ as follows:

$$H_0(u)=1, \qquad \bigl(H(u)+1\bigr)^n-uH_n(u)=\begin{cases}1-u & \text{if } n=0,\\ 0 & \text{if } n>0,\end{cases} \tag{3}$$

with the usual convention about replacing $H(u)^n$ by $H_n(u)$ (see [4,10,12,15]).

The Bernoulli and Euler polynomials can be defined by

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \qquad \frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}.$$

In the special case, $x=0$, $B_n(0)=B_n$ are the n-th Bernoulli numbers and $E_n(0)=E_n$ are the n-th Euler numbers.

The formula for a product of two Bernoulli polynomials are given by

$$B_m(x)B_n(x)=\sum_{r=0}^{\infty}\left(\binom{m}{2r}n+\binom{n}{2r}m\right)\frac{B_{2r}B_{m+n-2r}(x)}{m+n-2r}+(-1)^{m+1}\frac{m!n!}{(m+n)!}B_{m+n}, \tag{4}$$

where $m+n\geqslant 2$ and $\binom{m}{n}=\frac{m!}{n!(m-n)!}=\frac{m(m-1)\cdots(m-n+1)}{n!}$ (see [2,3]).

From (1), we note that $H_n(x\mid -1)=E_n(x)$. In [10], Nielson also obtained similar formulas for $E_n(x)E_m(x)$ and $E_m(x)B_n(x)$.

In view point of (4), Carlitz has considered the following identities for the Frobenius–Euler polynomials as follows:

$$\begin{aligned}H_m(x\mid\alpha)H_n(x\mid\beta)&=H_{m+n}(x\mid\alpha\beta)\frac{(1-\alpha)(1-\beta)}{1-\alpha\beta}\\&\quad+\frac{\alpha(1-\beta)}{1-\alpha\beta}\sum_{r=0}^{m}\binom{m}{r}H_r(\alpha)H_{m+n-r}(x\mid\alpha\beta)\\&\quad+\frac{\beta(1-\beta)}{1-\alpha\beta}\sum_{s=0}^{n}\binom{n}{s}H_s(\beta)H_{m+n-s}(x\mid\alpha\beta),\end{aligned} \tag{5}$$

where $\alpha,\beta\in\mathbb{C}$ with $\alpha\neq 1$, $\beta\neq 1$ and $\alpha\beta\neq 1$ (see [4]).

In particular, if $\alpha\neq 1$ and $\alpha\beta=1$, then

$$\begin{aligned}H_m(x\mid\alpha)H_n\bigl(x\mid\alpha^{-1}\bigr)&=-(1-\alpha)\sum_{r=1}^{m}\binom{m}{r}H_r(\alpha)\frac{B_{m+n-r+1}(x)}{m+n-r+1}\\&\quad-\bigl(1-\alpha^{-1}\bigr)\sum_{s=1}^{n}\binom{n}{s}H_s\bigl(\alpha^{-1}\bigr)\frac{B_{m+n-s+1}(x)}{m+n-s+1}\\&\quad+(-1)^{n+1}\frac{m!n!}{(m+n+1)!}(1-\alpha)H_{m+n+1}(\alpha).\end{aligned}$$

For $r\in\mathbb{N}$, the n-th Frobenius–Euler polynomials of order r are defined by generating function as follows:

$$
\begin{aligned}
F_u^r(t,x) &= \underbrace{F_u(t,x)\times F_u(t,x)\times\cdots\times F_u(t,x)}_{r\text{-times}} \\
&= \underbrace{\left(\frac{1-u}{e^t-u}\right)\times\left(\frac{1-u}{e^t-u}\right)\times\cdots\times\left(\frac{1-u}{e^t-u}\right)}_{r\text{-times}} e^{xt} \\
&= \sum_{n=0}^{\infty} H_n^{(r)}(x\mid u)\frac{t^n}{n!} \quad \text{for } u\in\mathbb{C} \text{ with } u\neq 1.
\end{aligned} \tag{6}
$$

In the special case, $x=0$, $H_n^{(r)}(0\mid u)=H_n^{(r)}(u)$ are called the n-th Frobenius–Euler numbers of order r (see [1–14,16]).

In this paper we derive non-linear differential equations from (1) and we study the solutions of non-linear differential equations. Finally, we give some new and interesting identities and formulae for the Frobenius–Euler polynomials of higher order by using our non-linear differential equations.

2. Computation of sums of the products of Frobenius–Euler numbers and polynomials

In this section we assume that

$$F=F(t)=\frac{1}{e^t-u}, \quad \text{and} \quad F^N(t,x)=\underbrace{F\times\cdots\times F}_{N\text{-times}}\, e^{xt} \quad \text{for } N\in\mathbb{N}. \tag{7}$$

Thus, by (7), we get

$$F^{(1)}=\frac{dF(t)}{dt}=\frac{-e^t}{(e^t-u)^2}=-\frac{1}{e^t-u}+\frac{u}{(e^t-u)^2}=-F+uF^2. \tag{8}$$

By (8), we get

$$F^{(1)}(t,x)=F^{(1)}(t)e^{tx}=-F(t,x)+uF^2(t,x), \quad \text{and} \quad F^{(1)}+F=uF^2. \tag{9}$$

Let us consider the derivative of (8) with respect to t as follows:

$$2uFF'=F''+F'. \tag{10}$$

Thus, by (10) and (8), we get

$$2!u^2F^3-2uF^2=F''+F'. \tag{11}$$

From (11), we note that

$$2!u^2F^3=F^{(2)}+3F'+2F, \quad \text{where } F^{(2)}=\frac{d^2F}{dt^2}. \tag{12}$$

Thus, by the derivative of (12) with respect to t, we get

$$2!u^2 3F^2F'=F^{(3)}+3F^{(2)}+2F^{(1)}, \quad \text{and} \quad F^{(1)}=uF^2-F. \tag{13}$$

By (13), we see that

$$3!u^3F^4F = F^{(3)} + 6F^{(2)} + 11F^{(1)} + 6F. \tag{14}$$

Thus, from (14), we have

$$3!u^4F^4(t,x) = F^{(3)}(t,x) + 6F^{(2)}(t,x) + 11F^{(1)}(t,x) + 6F(t,x).$$

Continuing this process, we set

$$(N-1)!u^{N-1}F^N = \sum_{k=0}^{N-1} a_k(N)F^{(k)}, \tag{15}$$

where $F^{(k)} = \frac{d^kF}{dt^k}$ and $N \in \mathbb{N}$.

Now we try to find the coefficient $a_k(N)$ in (15). From the derivative of (15) with respect to t, we have

$$N!u^{N-1}F^{N-1}F^{(1)} = \sum_{k=0}^{N-1} a_k(N)F^{(k+1)} = \sum_{k=1}^{N} a_{k-1}(N)F^{(k)}. \tag{16}$$

By (8), we easily get

$$N!u^{N-1}F^{N-1}F^{(1)} = N!u^{N-1}F^{N-1}\left(uF^2 - F\right) = N!u^NF^{N+1} - N!u^{N-1}F^N. \tag{17}$$

From (16) and (17), we can derive the following equation (18):

$$\begin{aligned} N!u^NF^{N+1} &= N(N-1)!u^{N-1}F^N + \sum_{k=1}^{N} a_{k-1}(N)F^{(k)} \\ &= N\sum_{k=0}^{N-1} a_k(N)F^{(k)} + \sum_{k=1}^{N} a_{k-1}(N)F^{(k)}. \end{aligned} \tag{18}$$

In (15), replacing N by $N+1$, we have

$$N!u^NF^{N+1} = \sum_{k=0}^{N} a_k(N+1)F^{(k)}. \tag{19}$$

By (18) and (19), we get

$$\sum_{k=0}^{N} a_k(N+1)F^{(k)} = N!u^NF^{N+1} = N\sum_{k=0}^{N-1} a_k(N)F^{(k)} + \sum_{k=1}^{N} a_{k-1}(N)F^{(k)}. \tag{20}$$

By comparing coefficients on the both sides of (20), we obtain the following equations:

$$Na_0(N) = a_0(N+1), \qquad a_N(N+1) = a_{N-1}(N). \tag{21}$$

For $1 \leqslant k \leqslant n-1$, we have

$$a_k(N+1) = Na_k(N) + a_{k-1}(N), \tag{22}$$

where $a_k(N)=0$ for $k \geqslant N$ or $k<0$. From (21), we note that

$$a_0(N+1)=Na_0(N)=N(N-1)a_0(N-1)=\cdots=N(N-1)\cdots 2a_0(2). \tag{23}$$

By (8) and (15), we get

$$F+F'=uF^2=\sum_{k=0}^{1}a_k(2)F^{(k)}=a_0(2)F+a_1(2)F^{(1)}. \tag{24}$$

By comparing coefficients on the both sides of (24), we get

$$a_0(2)=1, \quad \text{and} \quad a_1(2)=1. \tag{25}$$

From (23) and (25), we have $a_0(N)=(N-1)!$. By the second term of (21), we see that

$$a_N(N+1)=a_{N-1}(N)=a_{N-2}(N-1)=\cdots=a_1(2)=1. \tag{26}$$

Finally, we derive the value of $a_k(N)$ in (15) from (22).

Let us consider the following two variable function with variables s,t:

$$g(t,s)=\sum_{N\geqslant 1}\sum_{0\leqslant k\leqslant N-1}a_k(N)\frac{t^N}{N!}s^k, \quad \text{where } |t|<1. \tag{27}$$

By (22) and (27), we get

$$\begin{aligned}
&\sum_{N\geqslant 1}\sum_{0\leqslant k\leqslant N-1}a_{k+1}(N+1)\frac{t^N}{N!}s^k\\
&\quad=\sum_{N\geqslant 1}\sum_{0\leqslant k\leqslant N-1}Na_{k+1}(N+1)\frac{t^N}{N!}s^k+\sum_{N\geqslant 1}\sum_{0\leqslant k\leqslant N-1}a_k(N)\frac{t^N}{N!}s^k\\
&\quad=\sum_{N\geqslant 1}\sum_{0\leqslant k\leqslant N-1}Na_{k+1}(N)\frac{t^N}{N!}s^k+g(t,s).
\end{aligned} \tag{28}$$

It is not difficult to show that

$$\begin{aligned}
&\sum_{N\geqslant 1}\sum_{0\leqslant k\leqslant N-1}Na_{k+1}(N)\frac{t^N}{N!}s^k\\
&\quad=\frac{1}{s}\sum_{N\geqslant 1}\sum_{0\leqslant k\leqslant N-1}Na_{k+1}(N)\frac{t^N}{N!}s^{k+1}=\frac{1}{s}\sum_{N\geqslant 1}\sum_{1\leqslant k\leqslant N}a_k(N)\frac{t^N}{(N-1)!}s^k\\
&\quad=\frac{1}{s}\sum_{N\geqslant 1}\left(\sum_{0\leqslant k\leqslant N}a_k(N)\frac{t^Ns^k}{(N-1)!}-\frac{a_0(N)t^N}{(N-1)!}\right)=\frac{1}{s}\sum_{N\geqslant 1}\left(\sum_{0\leqslant k\leqslant N}a_k(N)\frac{t^N}{(N-1)!}s^k-t^N\right)\\
&\quad=\frac{t}{s}\left(\sum_{N\geqslant 1}\sum_{0\leqslant k\leqslant N}a_k(N)\frac{t^{N-1}s^k}{(N-1)!}-\frac{1}{1-t}\right)=\frac{t}{s}\left(g'(t,s)-\frac{1}{1-t}\right).
\end{aligned} \tag{29}$$

From (28) and (29), we can derive the following equation:

$$\sum_{N\geqslant 1}\sum_{0\leqslant k\leqslant N-1} a_{k+1}(N+1)\frac{t^N s^k}{N!} = \frac{t}{s}\left(g'(t,s) - \frac{1}{1-t}\right) + g(t,s). \tag{30}$$

The left hand side of (13)

$$\begin{aligned}
&= \sum_{N\geqslant 2}\sum_{1\leqslant k\leqslant N-2} a_{k+1}(N)\frac{t^{N-1}}{(N-1)!}s^k \\
&= \sum_{N\geqslant 2}\sum_{1\leqslant k\leqslant N-1} a_{k}(N)\frac{t^{N-1}s^{k-1}}{(N-1)!} = \frac{1}{s}\left(\sum_{N\geqslant 2}\sum_{1\leqslant k\leqslant N-1} a_{k}(N)\frac{t^{N-1}}{(N-1)!}s^k\right) \\
&= \frac{1}{s}\left(\sum_{N\geqslant 2}\left(\sum_{0\leqslant k\leqslant N-1} a_{k}(N)\frac{t^{N-1}}{(N-1)!}s^k - a_0(N)\frac{t^{N-1}}{(N-1)!}\right)\right) \\
&= \frac{1}{s}\left(\sum_{N\geqslant 2}\sum_{0\leqslant k\leqslant N-1} a_{k}(N)\frac{t^{N-1}}{(N-1)!}s^k - \frac{t}{1-t}\right) \\
&= \frac{1}{s}\left(\sum_{N\geqslant 1}\sum_{0\leqslant k\leqslant N-1} a_{k}(N)\frac{t^{N-1}}{(N-1)!}s^k - a_0(1) - \frac{t}{1-t}\right) = \frac{1}{s}\left(g'(t,s) - \frac{1}{1-t}\right).
\end{aligned} \tag{31}$$

By (30) and (31), we get

$$g(t,s) + \frac{t}{s}\left(g'(t,s) - \frac{1}{1-t}\right) = \frac{1}{s}\left(g'(t,s) - \frac{1}{1-t}\right). \tag{32}$$

Thus, by (32), we easily see that

$$0 = g(t,s) + \frac{t-1}{s}g'(t,s) + \frac{1-t}{s(1-t)} = g(t,s) + \frac{t-1}{s}g'(t,s) + \frac{1}{s}. \tag{33}$$

By (33), we get

$$g(t,s) + \frac{t-1}{s}g'(t,s) = -\frac{1}{s}. \tag{34}$$

To solve (34), we consider the solution of the following homogeneous differential equation:

$$0 = g(t,s) + \frac{t-1}{s}g'(t,s). \tag{35}$$

Thus, by (35), we get

$$-g(t,s) = \frac{t-1}{s}g'(t,s). \tag{36}$$

By (33), we get

$$\frac{g'(t,s)}{g(t,s)} = \frac{s}{1-t}. \tag{37}$$

From (37), we have the following equation:

$$\log g(t,s) = -s\log(1-t) + C. \tag{38}$$

By (38), we see that

$$g(t,s) = e^{-s\log(1-t)}\lambda \quad \text{where } \lambda = e^C. \tag{39}$$

By using the variant of constant, we set

$$\lambda = \lambda(t,s). \tag{40}$$

From (39) and (40), we note that

$$\begin{aligned} g'(t,s) = \frac{dg(t,s)}{dt} &= \lambda'(t,s)e^{-s\log(1-t)} + \frac{\lambda(t,s)e^{-s\log(1-t)}}{1-t}s \\ &= \lambda'(t,s)e^{-s\log(1-t)} + \frac{g(t,s)}{1-t}s, \end{aligned} \tag{41}$$

where $\lambda'(t,s) = \frac{d\lambda(t,s)}{dt}$.

Multiplying both sides of Eq. (41) by $\frac{t-1}{s}$, we get

$$\frac{t-1}{s}g'(t,s) + g(t,s) = \lambda'\frac{t-1}{s}e^{-s\log(1-t)}. \tag{42}$$

From (34) and (42), we get

$$-\frac{1}{s} = \lambda'\frac{t-1}{s}e^{-s\log(1-t)}. \tag{43}$$

Thus, by (43), we get

$$\lambda' = \lambda'(t,s) = (1-t)^{s-1}. \tag{44}$$

If we take indefinite integral on both sides of (44), we get

$$\lambda = \int \lambda'\,dt = \int (1-t)^{s-1}\,dt = -\frac{1}{s}(1-t)^s + C_1, \tag{45}$$

where C_1 is constant.

By (39) and (45), we easily see that

$$g(t,s) = e^{-s\log(1-t)}\left(-\frac{1}{s}(1-t)^s + C_1\right). \tag{46}$$

Let us take $t = 0$ in (46). Then, by (27) and (46), we get

$$0=-\frac{1}{s}+C_1,\quad C_1=\frac{1}{s}. \tag{47}$$

Thus, by (46) and (47), we have

$$\begin{aligned} g(t,s)&=e^{-s\log(1-t)}\left(\frac{1}{s}-\frac{1}{s}(1-t)^s\right)=\frac{1}{s}(1-t)^{-s}\big(1-(1-t)^s\big)\\ &=\frac{(1-t)^{-s}-1}{s}=\frac{1}{s}\big(e^{-s\log(1-t)}-1\big). \end{aligned} \tag{48}$$

From (48) and Taylor expansion, we can derive the following equation (49):

$$\begin{aligned} g(t,s)&=\frac{1}{s}\sum_{n\geqslant 1}\frac{s^n}{n!}\big(-\log(1-t)\big)^n=\sum_{n\geqslant 1}\frac{s^{n-1}}{n!}\left(\sum_{l=1}^{\infty}\frac{t^l}{l}\right)^n\\ &=\sum_{n\geqslant 1}\frac{s^{n-1}}{n!}\left(\sum_{l_1=1}^{\infty}\frac{t^{l_1}}{l_1}\times\cdots\times\sum_{l_n=1}^{\infty}\frac{t^{l_n}}{l_n}\right)\\ &=\sum_{n\geqslant 1}\frac{s^{n-1}}{n!}\sum_{N\geqslant n}\left(\sum_{l_1+\cdots+l_n=N}\frac{1}{l_1l_2\cdots l_n}\right)t^N. \end{aligned} \tag{49}$$

Thus, by (49), we get

$$\begin{aligned} g(t,s)&=\sum_{k\geqslant 0}\frac{s^k}{(k+1)!}\sum_{N\geqslant k+1}\left(\sum_{l_1+\cdots+l_{k+1}=N}\frac{1}{l_1l_2\cdots l_{k+1}}\right)t^N\\ &=\sum_{N\geqslant 1}\left(\sum_{0\leqslant k\leqslant N-1}\frac{N!}{(k+1)!}\sum_{l_1+\cdots+l_{k+1}=N}\frac{1}{l_1l_2\cdots l_{k+1}}\right)\frac{t^N}{N!}s^k. \end{aligned} \tag{50}$$

From (27) and (50), we can derive the following equation (51):

$$a_k(N)=\frac{N!}{(k+1)!}\sum_{l_1+\cdots+l_{k+1}=N}\frac{1}{l_1l_2\cdots l_{k+1}}. \tag{51}$$

Therefore, by (15) and (51), we obtain the following theorem.

Theorem 1. *For $u\in\mathbb{C}$ with $u\neq 1$, and $N\in\mathbb{N}$, let us consider the following non-linear differential equation with respect to t:*

$$F^N(t)=\frac{N}{u^{N-1}}\sum_{k=0}^{N-1}\frac{1}{(k+1)!}\sum_{l_1+\cdots+l_{k+1}=N}\frac{1}{l_1l_2\cdots l_{k+1}}F^{(k)}(t), \tag{52}$$

where $F^{(k)}(t)=\frac{d^kF(t)}{dt^k}$ and $F^N(t)=\underbrace{F(t)\times\cdots\times F(t)}_{N\text{-times}}$. Then $F(t)=\frac{1}{e^t-u}$ is a solution of (52).

Let us define $F^{(k)}(t,x)=F^{(k)}(t)e^{tx}$. Then we obtain the following corollary.

Corollary 2. *For $N \in \mathbb{N}$, we set*

$$F^N(t,x) = \frac{N}{u^{N-1}} \sum_{k=0}^{N} \frac{1}{(k+1)!} \sum_{l_1+\cdots+l_{k+1}=N} \frac{1}{l_1 l_2 \cdots l_{k+1}} F^{(k)}(t,x). \tag{53}$$

Then $\frac{e^{tx}}{e^t-u}$ is a solution of (53).

From (1) and (6), we note that

$$\frac{1-u}{e^t-u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}, \quad \text{and}$$
$$\underbrace{\left(\frac{1-u}{e^t-u}\right) \times \left(\frac{1-u}{e^t-u}\right) \times \cdots \times \left(\frac{1-u}{e^t-u}\right)}_{N\text{-times}} = \sum_{n=0}^{\infty} H_n^{(N)}(u) \frac{t^n}{n!}, \tag{54}$$

where $H_n^{(N)}(u)$ are called the n-th Frobenius–Euler numbers of order N.
By (7) and (54), we get

$$\begin{aligned} F^N(t) &= \underbrace{\left(\frac{1}{e^t-u}\right) \times \left(\frac{1}{e^t-u}\right) \times \cdots \times \left(\frac{1}{e^t-u}\right)}_{N\text{-times}} \\ &= \frac{1}{(1-u)^N} \underbrace{\left(\frac{1-u}{e^t-u}\right) \times \left(\frac{1-u}{e^t-u}\right) \times \cdots \times \left(\frac{1-u}{e^t-u}\right)}_{N\text{-times}} \\ &= \frac{1}{(1-u)^N} \sum_{l=0}^{\infty} H_l^{(N)}(u) \frac{t^l}{l!}, \quad \text{and} \end{aligned}$$
$$F(t) = \left(\frac{1-u}{e^t-u}\right)\left(\frac{1}{1-u}\right) = \frac{1}{1-u} \sum_{l=0}^{\infty} H_l(u) \frac{t^l}{l!}. \tag{55}$$

From (55), we note that

$$F^{(k)}(t) = \frac{d^k F(t)}{dt^k} = \sum_{l=0}^{\infty} H_{l+k}(u) \frac{t^l}{l!}. \tag{56}$$

Therefore, by (52), (55) and (56), we obtain the following theorem.

Theorem 3. *For $N \in \mathbb{N}$, $n \in \mathbb{Z}_+$, we have*

$$H_n^{(N)}(u) = N \left(\frac{1-u}{u}\right)^{N-1} \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{l_1+\cdots+l_{k+1}=N} \frac{H_{n+k}(u)}{l_1 l_2 \cdots l_{k+1}}.$$

From (55), we can derive the following equation:

$$\sum_{n=0}^{\infty} H_n^{(N)}(u)\frac{t^n}{n!} = \underbrace{\left(\frac{1-u}{e^t-u}\right)\times\left(\frac{1-u}{e^t-u}\right)\times\cdots\times\left(\frac{1-u}{e^t-u}\right)}_{N\text{-times}}$$
$$= \left(\sum_{l_1=0}^{\infty} H_{l_1}(u)\frac{t_1^l}{l_1!}\right)\times\cdots\times\left(\sum_{l_N=0}^{\infty} H_{l_N}(u)\frac{t^{l_N}}{l_N!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{l_1+\cdots+l_N=0}\frac{H_{l_1}(u)H_{l_2}(u)\cdots H_{l_N}(u)n!}{l_1!l_1!\cdots l_N!}\right)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{l_1+\cdots+l_N=0}\binom{n}{l_1,\ldots,l_N!}H_{l_1}(u)\cdots H_{l_N}(u)\right)\frac{t^n}{n!}. \qquad (57)$$

Therefore, by (57), we obtain the following corollary.

Corollary 4. *For $N\in\mathbb{N}$, $n\in\mathbb{Z}_+$, we have*

$$\sum_{l_1+\cdots+l_N=n}\binom{n}{l_1,\ldots,l_N!}H_{l_1}(u)H_{l_2}(u)\cdots H_{l_N}(u)$$
$$= N\left(\frac{1-u}{u}\right)^{N-1}\sum_{k=0}^{N-1}\frac{1}{(k+1)!}\sum_{l_1+\cdots+l_{k+1}=N}\frac{H_{n+k}(u)}{l_1l_2\cdots l_{k+1}}.$$

By (53), we obtain the following corollary.

Corollary 5. *For $N\in\mathbb{N}$, $n\in\mathbb{Z}_+$, we have*

$$H_n^{(N)}(x\mid u) = N\left(\frac{1-u}{u}\right)^{N-1}\sum_{k=0}^{N-1}\frac{1}{(k+1)!}\sum_{l_1+\cdots+l_{k+1}=N}\frac{1}{l_1l_2\cdots l_{k+1}}\sum_{m=0}^{n}\binom{n}{m}H_{m+k}(u)x^{n-m}.$$

From (6), we note that

$$\sum_{n=0}^{\infty} H_n^{(N)}(x\mid u)\frac{t^n}{n!} = \underbrace{\left(\frac{1-u}{e^t-u}\right)\times\left(\frac{1-u}{e^t-u}\right)\times\cdots\times\left(\frac{1-u}{e^t-u}\right)}_{N\text{-times}}e^{xt}$$
$$= \left(\sum_{n=0}^{\infty} H_n^{(N)}(u)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} x^m\frac{t^m}{m!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}x^{n-l}H_l^{(N)}(u)\right)\frac{t^n}{n!}. \qquad (58)$$

By comparing coefficients on both sides of (58), we get

$$H_n^{(N)}(x \mid u) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} H_l^{(N)}(u). \tag{59}$$

By the definition of notation, we get

$$F^{(k)}(t,x) = F^{(k)}(t)e^{tx} = \left(\sum_{l=0}^{\infty} H_{l+k}(u)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} \frac{x^m}{m!}t^m\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} H_{l+k}(u)x^{n-l}\right)\frac{t^n}{n!}.$$

From (6), we note that

$$\sum_{n=0}^{\infty} H_n^{(N)}(x \mid u)\frac{t^n}{n!} = \underbrace{\left(\frac{1-u}{e^t-u}\right)\times\cdots\times\left(\frac{1-u}{e^t-u}\right)}_{N\text{-times}} e^{xt}$$
$$= \left(\sum_{l_1=0}^{\infty} H_{l_1}(u)\frac{H_{l_1}(u)}{l_1!}t^{l_1}\right)\times\cdots\times\left(\sum_{l_N=0}^{\infty}\frac{H_{l_N}(u)}{l_N!}t^{l_N}\right)\sum_{m=0}^{\infty}\frac{x^m}{m!}t^m$$
$$= \sum_{n=0}^{\infty}\left(\sum_{l_1+\cdots+l_N+m=n}\frac{H_{l_1}(u)H_{l_2}(u)\cdots H_{l_N}(u)}{l_1!l_1!\cdots l_N!m!}x^m n!\right)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{l_1+\cdots+l_N+m=n}\binom{n}{l_1,\ldots,l_N,m}H_{l_1}(u)\cdots H_{l_N}(u)x^m\right)\frac{t^n}{n!}. \tag{60}$$

By comparing coefficients on both sides of (58), we get

$$H_n^{(N)}(x \mid u) = \sum_{l_1+\cdots+l_N+m=n}\binom{n}{l_1,\ldots,l_N,m}H_{l_1}(u)\cdots H_{l_N}(u)x^m.$$

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