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Generalized crested products of Markov chains

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ABSTRACT

We define a finite Markov chain, called generalized crested product, which naturally appears as a generalization of the first crested product of Markov chains. A complete spectral analysis is developed and the k -step transition probability is given. It is important to remark that this Markov chain describes a more general version of the classical Ehrenfest diffusion model.

As a particular case, one gets a generalization of the classical Insect Markov chain defined on the ultrametric space. Finally, an interpretation in terms of representation group theory is given, by showing the correspondence between the spectral decomposition of the generalized crested product and the Gelfand pairs associated with the generalized wreath product of permutation groups.

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1. Introduction

This paper deals with the study of a finite Markov chain, called generalized crested product, which is defined on the product of finite spaces. The generalized crested product is a generalization of the Markov chains introduced and studied in [6,7]. More precisely, in [6] the first crested product of Markov chains is defined, inspired by the analogous definition for association schemes [2], as a sort of mixing between the classical crossed and nested products of Markov chains and it contains, as a particular case, the so-called Insect Markov chain introduced by Figà-Talamanca in [10] in the context of Gelfand pairs theory. In [7] a Markov chain on some special structures called orthogonal block structures is introduced. If the orthogonal block structure is a poset block structure, then that Markov chain can also be defined starting from a finite poset (I, \leq) and it can be interpreted as a slightly different generalization of the classical Insect Markov chain and of the associated Gelfand pairs theory. We noticed that, for some particular posets, the Markov chain in [7] has the same spectral decomposition as the first crested product of Markov chains despite the corresponding operators

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do not coincide. So it is natural to ask if it is possible to define a Markov chain containing the first crested product as a particular case, giving rise to an Insect Markov chain for a particular choice of the parameters involved. This is the reason why this paper aims at introducing a new Markov chain which can be seen as a modification of the Markov chain on the orthogonal block structure and the natural generalization of the first crested product of Markov chains. The idea is to take a finite poset (I, \leq) and a family of Markov operators P_i defined on finite sets X_i indexed by the elements of the poset. Then we consider the sum, over I , of tensor products of Markov chains reflecting, in some sense, the poset hierarchy structure (see Definition 3.2). A necessary and sufficient condition to have reversibility of the Markov chain is proven in Theorem 3.4. In Theorem 3.5, we give a complete spectral analysis of this Markov chain and we show in Proposition 3.8 that it coincides with the first crested Markov chain when the poset (I, \leq) satisfies some particular properties. A formula for the k -step probability is given in Section 3.4. Moreover, we introduce in Section 4 an Insect Markov chain on the product $X = \prod_{i \in I} X_i$, naturally identified with the last level of a graph \mathcal{T} which is the generalization of the rooted tree. This Insect Markov chain is obtained from the generalized crested product of Markov chains for a particular choice of the operators P_i , i.e. $P_i = J_i$, where J_i is the uniform operator on the set X_i . If the poset (I, \leq) is totally ordered, this Insect Markov chain coincides with the classical Insect Markov chain [10]. In Section 5 we highlight the correspondence with the Gelfand pairs theory (for a general theory and applications see [8]): taking the generalized wreath product of permutation groups [3] associated with (I, \leq) and the stabilizer of an element of X under the action of this group, one gets a Gelfand pair [7], and the decomposition of the action of the group on the space $L(X)$ into irreducible submodules is the same as the spectral decomposition of the Insect Markov chain associated with (I, \leq) . This allows us to study many examples of Gelfand pairs only by using basic tools of linear algebra.

It is important to remark that the generalized crested product can be seen as a generalization of a classical diffusion model, the Ehrenfest model, as well as of the (C, N) -Ehrenfest model described in [6] (see [9,11] for more examples and details).

2. Preliminaries

We recall in this section some basic facts about finite Markov chains (see, for instance, [4]). Let X be a finite set, with $|X| = m$. Let $P = (p(x, y))_{x, y \in X}$ be a stochastic matrix, so that

$$\sum_{x \in X} p(x_0, x) = 1,$$

for every $x_0 \in X$. Consider the Markov chain on X with transition matrix P . By abuse of notation, we will denote by P this Markov chain as well as the associated Markov operator.

Definition 2.1. The Markov chain P is reversible if there exists a strict probability measure π on X such that

$$\pi(x)p(x, y) = \pi(y)p(y, x),$$

for all $x, y \in X$.

If this is the case, we say that P and π are in detailed balance [1].

Define a scalar product on $L(X) = \{f : X \rightarrow \mathbb{C}\}$ as

$$\langle f_1, f_2 \rangle_\pi = \sum_{x \in X} f_1(x) \overline{f_2(x)} \pi(x),$$

for all $f_1, f_2 \in L(X)$, and the linear operator $P : L(X) \rightarrow L(X)$ as

$$(Pf)(x) = \sum_{y \in X} p(x, y) f(y). \quad (1)$$

It is easy to verify that π and P are in detailed balance if and only if P is self-adjoint with respect to the scalar product $\langle \cdot, \cdot \rangle_\pi$. Under these hypotheses, it is known that the matrix P can be diagonalized over the reals. Moreover, 1 is always an eigenvalue of P and for any eigenvalue λ one has $|\lambda| \leq 1$.

Let $\{\lambda_z\}_{z \in X}$ be the eigenvalues of the matrix P , with $\lambda_{z_0} = 1$. Then there exists an invertible unitary real matrix $U = (u(x, y))_{x, y \in X}$ such that $PU = U\Delta$, where $\Delta = (\lambda_x \delta_x(y))_{x, y \in X}$ is the diagonal matrix whose entries are the eigenvalues of P . This equation gives, for all $x, z \in X$,

$$\sum_{y \in X} p(x, y)u(y, z) = u(x, z)\lambda_z. \tag{2}$$

Moreover, we have $U^T D U = I$, where $D = (\pi(x) \delta_x(y))_{x, y \in X}$ is the diagonal matrix of coefficients of π . This second equation gives, for all $y, z \in X$,

$$\sum_{x \in X} u(x, y)u(x, z)\pi(x) = \delta_y(z). \tag{3}$$

It follows from (2) that each column of U is an eigenvector of P , and from (3) that these columns are orthogonal with respect to the product $\langle \cdot, \cdot \rangle_\pi$.

Proposition 2.2. *The k th step transition probability is given by*

$$p^{(k)}(x, y) = \pi(y) \sum_{z \in X} u(x, z)\lambda_z^k u(y, z), \tag{4}$$

for all $x, y \in X$.

Proof. The proof is a consequence of (2) and (3). In fact, the matrix $U^T D$ is the inverse of U , so that $U U^T D = I$. In formulæ, we have

$$\sum_{y \in X} u(x, y)u(z, y) = \frac{1}{\pi(z)} \Delta_z(x).$$

From the equation $PU = U\Delta$ we get $P = U\Delta U^T D$, which gives

$$p(x, y) = \pi(y) \sum_{z \in X} u(x, z)\lambda_z u(y, z).$$

Iterating this argument we obtain $P^k = U\Delta^k U^T D$, which is the assertion. \square

Recall that there exists a correspondence between reversible Markov chains and weighted graphs.

Definition 2.3. A weight on a graph $\mathcal{G} = (X, E)$ is a function $w : X \times X \rightarrow [0, +\infty)$ such that

- (1) $w(x, y) = w(y, x)$;
- (2) $w(x, y) > 0$ if and only if $x \sim y$.

If \mathcal{G} is a weighted graph, it is possible to associate with w a stochastic matrix $P = (P(x, y))_{x, y \in X}$ on X by setting

$$p(x, y) = \frac{w(x, y)}{W(x)},$$

with $W(x) = \sum_{z \in X} w(x, z)$. The corresponding Markov chain is called the random walk on \mathcal{G} . It is easy to prove that the matrix P is in detailed balance with the distribution π defined, for every $x \in X$, as

$$\pi(x) = \frac{W(x)}{W},$$

with $W = \sum_{z \in X} W(z)$. Moreover, π is strictly positive if X does not contain isolated vertices. The inverse construction can be done. So, if we have a transition matrix P on X which is in detailed balance with the probability π , then we can define a weight w as $w(x, y) = \pi(x)p(x, y)$. This definition guarantees the symmetry of w and, by setting $E = \{\{x, y\} : w(x, y) > 0\}$, we get a weighted graph.

There are some important relations between the weighted graph associated with a transition matrix P and its spectrum $\sigma(P)$. In fact, it is easy to prove that the multiplicity of the eigenvalue 1 of P equals the number of connected components of \mathcal{G} . Moreover, the following propositions hold.

Proposition 2.4. Let $\mathcal{G} = (X, E, w)$ be a finite connected weighted graph and denote by P the corresponding transition matrix. Then the following are equivalent:

- (1) \mathcal{G} is bipartite;
- (2) the spectrum $\sigma(P)$ is symmetric;
- (3) $-1 \in \sigma(P)$.

Definition 2.5. Let P be a stochastic matrix. P is ergodic if there exists $n_0 \in \mathbb{N}$ such that

$$p^{(n_0)}(x, y) > 0, \quad \text{for all } x, y \in X.$$

Proposition 2.6. Let $\mathcal{G} = (X, E)$ be a finite graph. Then the following conditions are equivalent:

- (1) \mathcal{G} is connected and not bipartite;
- (2) for every weight function on \mathcal{G} , the associated transition matrix P is ergodic.

So we can conclude that a reversible transition matrix P is ergodic if and only if the eigenvalue 1 has multiplicity one and -1 is not an eigenvalue. Note that the condition that 1 is an eigenvalue of P of multiplicity one is equivalent to requiring that the probability P is irreducible, according to the following definition.

Definition 2.7. A stochastic matrix P on a set X is irreducible if, for every $x_1, x_2 \in X$, there exists $n = n(x_1, x_2)$ such that $p^{(n)}(x_1, x_2) > 0$.

3. Generalized crested product

3.1. Definition

Let (I, \leq) be a finite poset, with $I = \{1, \dots, n\}$. The following definitions are given in [3].

Definition 3.1. A subset $J \subseteq I$ is said to be

- *ancestral* if, whenever $i > j$ and $j \in J$, then $i \in J$;
- *hereditary* if, whenever $i < j$ and $j \in J$, then $i \in J$;
- *a chain* if, whenever $i, j \in J$, then either $i \leq j$ or $j \leq i$;
- *an antichain* if, whenever $i, j \in J$ and $i \neq j$, then neither $i \leq j$ nor $j \leq i$.

Given an element $i \in I$, we set $A(i) = \{j \in I : j > i\}$ to be the ancestral set of i and $A[i] = A(i) \sqcup \{i\}$. Analogously we set $H(i) = \{j \in I : j < i\}$ to be the hereditary set of i and $H[i] = H(i) \sqcup \{i\}$. For a subset $J \subseteq I$ we put $A(J) = \bigcup_{j \in J} A(j)$, $A[J] = \bigcup_{j \in J} A[j]$, $H(J) = \bigcup_{j \in J} H(j)$ and $H[J] = \bigcup_{j \in J} H[j]$. Moreover, we denote by \mathbb{S} the set of the antichains of (I, \leq) and we set $\bar{\mathbb{S}} = \{j \in I : A(j) = \emptyset\}$. It is clear that $\bar{\mathbb{S}}$ and the empty set \emptyset belong to \mathbb{S} . Note that $A(\emptyset) = H(\emptyset) = A[\emptyset] = H[\emptyset] = \emptyset$.

For each $i \in I$, let X_i be a finite set, with $|X_i| = m_i$, so that we can identify X_i with the set $\{0, 1, \dots, m_i - 1\}$. Moreover, let P_i be an irreducible Markov chain on X_i and let p_i be the corresponding transition probability. We also denote by P_i the associated Markov operator $P_i : L(X_i) \rightarrow L(X_i)$ defined as in (1). Let I_i be the identity matrix of size m_i and set:

$$J_i = \frac{1}{m_i} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 1 & \dots & \dots & 1 \end{pmatrix}.$$

We also denote by I_i and J_i the associated Markov operator on $L(X_i)$, that we call the identity and the uniform operator, respectively. We are going to define a Markov chain on the set $X = X_1 \times \dots \times X_n$.

Definition 3.2. Let (I, \leq) be a finite poset and let $\{p_i^0\}_{i \in I}$ be a probability distribution on I , i.e. $p_i^0 > 0$ for every $i \in I$ and $\sum_{i=1}^n p_i^0 = 1$. The *generalized crested product* of the Markov chains P_i defined by

(I, \leq) and $\{p_i^0\}_{i \in I}$ is the Markov chain on X whose associated Markov operator is

$$\mathcal{P} = \sum_{i \in I} p_i^0 \left(P_i \otimes \left(\bigotimes_{j \in H(i)} U_j \right) \otimes \left(\bigotimes_{j \in I \setminus H[i]} I_j \right) \right). \tag{5}$$

Remark 3.3. The generalized crested product can be seen as a generalization of the classical diffusion Ehrenfest model. This classical model consists of two urns numbered 0, 1 and n balls numbered $1, \dots, n$. A configuration is given by a placement of the balls into the urns. Note that there is no ordering inside the urns. At each step, a ball is randomly chosen (with probability $1/n$) and it is moved to the other urn. In [6] we generalized it to the (C, N) -Ehrenfest model. Now put $|X_i| = m$, for each $i = 1, \dots, n$: then we have the following interpretation of the generalized crested product. Suppose that we have n balls numbered by $1, \dots, n$ and m urns. Let (I, \leq) be a finite poset with n elements. At each step, we choose a ball i according to a probability distribution p_i^0 : then we move it to another urn following a transition probability P_i and all the other balls numbered by indices j such that $j \leq i$ in the poset (I, \leq) are moved uniformly to a new urn. The balls corresponding to all the other indices are not moved.

From now on, we suppose that each P_i is in detailed balance with the probability measure σ_i .

Theorem 3.4. *The generalized crested product is reversible if and only if P_k is symmetric for every $k \in I \setminus \bar{S}$, i.e. $p_k(x_k, y_k) = p_k(y_k, x_k)$, for all $x_k, y_k \in X_k$. If this is the case, \mathcal{P} is in detailed balance with the strict probability measure π on X given by*

$$\pi(x_1, \dots, x_n) = \frac{\prod_{i \in \bar{S}} \sigma_i(x_i)}{\prod_{i \in I \setminus \bar{S}} m_i}.$$

Proof. We start by proving that the condition $\sigma_k = \frac{1}{m_k}$, for each $k \in I \setminus \bar{S}$, is sufficient. Consider two elements $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in X . First, suppose that there exists $j \in \bar{S}$ such that $x_j \neq y_j$ and $x_i = y_i$ for all $i \in \bar{S} \setminus \{j\}$. In this case, we have

$$\begin{aligned} \pi(x)\mathcal{P}(x, y) &= \frac{\prod_{i \in \bar{S}} \sigma_i(x_i)}{\prod_{i \in I \setminus \bar{S}} m_i} \cdot p_j^0 \cdot \frac{p_j(x_j, y_j)}{\prod_{i \in H(j)} m_i} \cdot \prod_{i \in I \setminus H[j]} \delta_i(x_i, y_i) \\ &= \sigma_j(x_j) p_j(x_j, y_j) \cdot p_j^0 \cdot \frac{\prod_{i \in \bar{S} \setminus \{j\}} \sigma_i(x_i)}{\prod_{i \in I \setminus \bar{S}} m_i} \cdot \frac{\prod_{i \in I \setminus H[j]} \delta_i(x_i, y_i)}{\prod_{i \in H(j)} m_i} \\ &= \sigma_j(y_j) p_j(y_j, x_j) \cdot p_j^0 \cdot \frac{\prod_{i \in \bar{S} \setminus \{j\}} \sigma_i(y_i)}{\prod_{i \in I \setminus \bar{S}} m_i} \cdot \frac{\prod_{i \in I \setminus H[j]} \delta_i(y_i, x_i)}{\prod_{i \in H(j)} m_i} \\ &= \frac{\prod_{i \in \bar{S}} \sigma_i(y_i)}{\prod_{i \in I \setminus \bar{S}} m_i} \cdot p_j^0 \cdot \frac{p_j(y_j, x_j)}{\prod_{i \in H(j)} m_i} \cdot \prod_{i \in I \setminus H[j]} \delta_i(y_i, x_i) \\ &= \pi(y)\mathcal{P}(y, x). \end{aligned}$$

If we suppose that there exist $j_1, j_2 \in \bar{S}$ such that $x_{j_h} \neq y_{j_h}$, for $h = 1, 2$, then $\mathcal{P}(x, y) = \mathcal{P}(y, x) = 0$ and there is nothing to prove.

Suppose now that $x_i = y_i$ for every $i \in \bar{S}$ and there is $j \in I \setminus \bar{S}$ such that $x_j \neq y_j$. We have

$$\begin{aligned} \pi(x)\mathcal{P}(x, y) &= \frac{\prod_{i \in \bar{S}} \sigma_i(x_i)}{\prod_{i \in I \setminus \bar{S}} m_i} \\ &\times \left(\sum_{i \in A(j)} p_i^0 \cdot \frac{p_i(x_i, y_i) \prod_{h \in I \setminus H[i]} \delta_h(x_h, y_h)}{\prod_{h \in H(i)} m_h} + p_j^0 \cdot \frac{p_j(x_j, y_j) \prod_{h \in I \setminus H[j]} \delta_h(x_h, y_h)}{\prod_{h \in H(j)} m_h} \right) \\ &= \frac{\prod_{i \in \bar{S}} \sigma_i(y_i)}{\prod_{i \in I \setminus \bar{S}} m_i} \cdot \left(\sum_{i \in A(j)} p_i^0 \cdot \frac{p_i(y_i, x_i) \prod_{h \in I \setminus H[i]} \delta_h(y_h, x_h)}{\prod_{h \in H(i)} m_h} + p_j^0 \cdot \frac{p_j(y_j, x_j) \prod_{h \in I \setminus H[j]} \delta_h(y_h, x_h)}{\prod_{h \in H(j)} m_h} \right) \\ &= \pi(y)\mathcal{P}(y, x). \end{aligned}$$

On the other hand, we show that the condition $\sigma_k = \frac{1}{m_k}$, for each $k \in I \setminus \bar{S}$, is necessary. Suppose that the equality $\pi(x)\mathcal{P}(x, y) = \pi(y)\mathcal{P}(y, x)$ holds for all $x, y \in X$. Let $i \in \bar{S}$: by irreducibility, we can choose $x, y \in X$ such that $x_i \neq y_i$ and $p_i(x_i, y_i) \neq 0$. Let $x_j = y_j$ for every $j \in \bar{S} \setminus \{i\}$. We have

$$\pi(x)\mathcal{P}(x, y) = \pi(y)\mathcal{P}(y, x) \iff \pi(x)p_i(x_i, y_i) = \pi(y)p_i(y_i, x_i).$$

This gives

$$\frac{\pi(x)}{\pi(y)} = \frac{p_i(y_i, x_i)}{p_i(x_i, y_i)} = \frac{\sigma_i(x_i)}{\sigma_i(y_i)}. \tag{6}$$

Let $\bar{x} \in X$ such that $\bar{x}_j = y_j$ for each $j \in H(i)$ and $\bar{x}_j = x_j$ for each $j \in I \setminus H(i)$. Proceeding as above we get

$$\frac{\pi(\bar{x})}{\pi(y)} = \frac{p_i(y_i, x_i)}{p_i(x_i, y_i)} = \frac{\sigma_i(x_i)}{\sigma_i(y_i)}, \tag{7}$$

so that (6) and (7) imply $\pi(x) = \pi(\bar{x})$, i.e. π does not depend on the coordinates corresponding to indices in $H(i)$. Let $j \in H(i)$ and let $x' \in X$ such that $x'_h \neq x_h$ for each $h \in H[j]$ and $x'_k = x_k$ for each $k \in I \setminus H[j]$. The condition $\pi(x)\mathcal{P}(x, x') = \pi(x')\mathcal{P}(x', x)$ reduces to

$$\mathcal{P}(x, x') = \mathcal{P}(x', x), \tag{8}$$

since x and x' differ only for indices in $H(i)$ and so $\pi(x) = \pi(x')$. Observe that $x_\ell = x'_\ell$ for each $\ell \in I \setminus H[j]$ and so the summands corresponding to these indices are equal on both sides of (8). Moreover, for each $k \in H(j)$, one has $j \in I \setminus H[k]$ and so the summands corresponding to these indices are 0 on both sides of (8), since $x'_k \neq x_k$. Hence, (8) reduces to $p_j(x_j, x'_j) = p_j(x'_j, x_j)$, which implies that $\sigma_j(x_j) = \sigma_j(x'_j)$ and so the hypothesis of irreducibility guarantees that σ_j is uniform on X_j . This completes the proof. \square

3.2. Spectral analysis

The next step is to study the spectral decomposition of the operator \mathcal{P} . Suppose that $L(X_i) = \bigoplus_{j_i=0}^{r_i} V_{j_i}^i$ is the decomposition of $L(X_i)$ into eigenspaces of P_i and that λ_{j_i} is the eigenvalue corresponding to $V_{j_i}^i$. The eigenspace V_0^i is the space of the constant functions over X_i : under our hypothesis of irreducibility, we have $\dim(V_0^i) = 1$.

For every antichain $S = \{i_1, \dots, i_k\} \in \mathbb{S}$, define

$$\mathcal{J}_S := \{ \underline{j} = (j_{i_1}, \dots, j_{i_k}) : j_{i_h} \in \{1, \dots, r_{i_h}\} \}$$

and, for $S \in \mathbb{S}$ and $\underline{j} \in \mathcal{J}_S$, we put

$$V_{S,\underline{j}} := V_{j_1}^{i_1} \otimes \cdots \otimes V_{j_k}^{i_k}.$$

Moreover, we set

$$W_{S,\underline{j}} := V_{S,\underline{j}} \otimes \left(\bigotimes_{i \in A(S)} L(X_i) \right) \otimes \left(\bigotimes_{i \in I \setminus A[S]} V_0^i \right) \tag{9}$$

and

$$\lambda_{S,\underline{j}} = \sum_{h=1}^k p_{i_h}^0 \lambda_{j_{i_h}} + \sum_{i \in I \setminus A[S]} p_i^0. \tag{10}$$

Theorem 3.5. *Let (I, \leq) be a finite poset and let \mathcal{P} be the generalized crested product defined in (5). The decomposition of $L(X)$ into eigenspaces for \mathcal{P} is*

$$L(X) = \bigoplus_{S \in \mathbb{S}} \left(\bigoplus_{\underline{j} \in \mathcal{J}_S} W_{S,\underline{j}} \right).$$

Moreover, the eigenvalue associated with $W_{S,\underline{j}}$ is $\lambda_{S,\underline{j}}$.

Proof. Let φ be a function in $W_{S,\underline{j}}$. We can represent φ as the tensor product $\varphi_1 \otimes \cdots \otimes \varphi_n$, with $\varphi_i \in V_{j_i}^i$ if $i \in S$, $\varphi_i \in L(X_i)$ if $i \in A(S)$, and $\varphi_i \in V_0^i$ if $i \in I \setminus A[S]$. We have to show that $(\mathcal{P}\varphi)(x) = \lambda_{S,\underline{j}}\varphi(x)$ for every $x = (x_1, \dots, x_n) \in X$. One has:

$$\begin{aligned} (\mathcal{P}\varphi)(x) &= \sum_{(y_1, \dots, y_n) \in X} \sum_{i \in I} p_i^0 \left(p_i(x_i, y_i) \varphi_i(y_i) \frac{\prod_{j \in I \setminus H[i]} \delta_j(x_j, y_j)}{\prod_{j \in H(i)} m_j} \prod_{j \neq i} \varphi_j(y_j) \right) \\ &= \sum_{i \in I} p_i^0 \sum_{y_j: j \in H[i]} \frac{p_i(x_i, y_i) \varphi_i(y_i)}{\prod_{j \in H(i)} m_j} \prod_{j \in I \setminus H[i]} \varphi_j(x_j) \prod_{j \in H(i)} \varphi_j(y_j). \end{aligned}$$

Observe that, if $i \in S$, then $H(i) \subseteq I \setminus A[S]$ and so φ_j is constant for every $j \in H(i)$. Suppose $i = i_h$ for some $h \in \{1, \dots, k\}$. Hence, the term of \mathcal{P} corresponding to i_h is

$$p_{i_h}^0 \prod_{j \in H(i_h)} m_j \cdot \sum_{y_{i_h}} \frac{p_{i_h}(x_{i_h}, y_{i_h}) \varphi_{i_h}(y_{i_h})}{\prod_{j \in H(i_h)} m_j} \prod_{j \neq i_h} \varphi_j(x_j) = (p_{i_h}^0 \lambda_{j_{i_h}}) \varphi(x).$$

On the other hand, if $i \in I \setminus A[S]$, then $S \subseteq I \setminus H[i]$ and so the identity operator I_j , for $j \in S$, acts on the space orthogonal to V_0^j and P_i acts on V_0^i . Note that $H(i) \subseteq I \setminus A[S]$, so that the term corresponding to the index i is

$$p_i^0 \prod_{j \in H(i)} m_j \cdot \sum_{y_i} \frac{p_i(x_i, y_i) \varphi_i(y_i)}{\prod_{j \in H(i)} m_j} \prod_{j \neq i} \varphi_j(x_j) = p_i^0 \prod_{j \in H(i)} m_j \cdot \frac{\varphi_i(y_i)}{\prod_{j \in H(i)} m_j} \prod_{j \neq i} \varphi_j(x_j) = p_i^0 \varphi(x).$$

Finally, if $i \in A(S)$, then there exists $k \in S$ such that $k \in H(i)$. In particular, φ_k is orthogonal to V_0^k , i.e. $\sum_{y_k \in X_k} \varphi_k(y_k) = 0$ and so the term corresponding to the index i is

$$p_i^0 \sum_{y_j: j \in H[i]} \frac{p_i(x_i, y_i) \varphi_i(y_i)}{\prod_{j \in H(i)} m_j} \prod_{j \in I \setminus H[i]} \varphi_j(x_j) \prod_{j \in H(i)} \varphi_j(y_j)$$

$$\begin{aligned}
 &= p_i^0 \left(\sum_{y_j: k \neq j \in H[i]} \frac{p_i(x_i, y_i) \varphi_i(y_i)}{\prod_{k \neq j \in H(i)} m_j} \prod_{j \in I \setminus H[i]} \varphi_j(x_j) \prod_{k \neq j \in H(i)} \varphi_j(y_j) \right) \cdot \frac{1}{m_k} \sum_{y_k \in X_k} \varphi_k(y_k) \\
 &= 0.
 \end{aligned}$$

Hence

$$(\mathcal{P}\varphi)(x) = \left(\sum_{h=1}^k p_{i_h}^0 \lambda_{j_{i_h}} + \sum_{i \in I \setminus A[S]} p_i^0 \right) \varphi(x).$$

and the claim is proven. \square

Corollary 3.6. *If P_i is ergodic for each $i \in I$, then \mathcal{P} is ergodic.*

Proof. The expression of the eigenvalues of \mathcal{P} given in (10) ensures that the eigenvalue 1 is obtained with multiplicity one and the eigenvalue -1 can never be obtained. \square

We are able now to provide the matrices U , D and Δ associated with \mathcal{P} . For every i , let U_i , D_i and Δ_i be the matrices of eigenvectors, of the coefficients of σ_i and of eigenvalues for the probability P_i , respectively. Recall the identification of X_i with the set $\{0, 1, \dots, m_i - 1\}$.

Proposition 3.7. *The matrices U , D and Δ have the following form:*

- $U = \sum_{S \in \mathbb{S}} \left(\otimes_{i \in S} (U_i - A_i) \right) \otimes \left(\otimes_{i \in I \setminus A[S]} A_i \right) \otimes \left(\otimes_{i \in A(S)} I_i^{\sigma_i\text{-norm}} \right)$, where

$$I_i^{\sigma_i\text{-norm}} = \begin{pmatrix} \frac{1}{\sqrt{\sigma_i(0)}} & & & & \\ & \frac{1}{\sqrt{\sigma_i(1)}} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \frac{1}{\sqrt{\sigma_i(m_i - 1)}} \end{pmatrix}.$$

By A_i we denote the matrix of size m_i whose entries on the first column are all 1 and the remaining ones are 0.

- $D = \otimes_{i \in I} D_i$.
- $\Delta = \sum_{i \in I} p_i^0 \Delta_i \otimes \left(\otimes_{j \in I \setminus H[i]} I_j \right) \otimes \left(\otimes_{j \in H(i)} J_j^{\text{diag}} \right)$, where J_j^{diag} is the diagonal matrix $\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$ of size m_j .

Proof. Let us start by proving the statement for the matrix U . By construction, each column of U is an eigenvector of \mathcal{P} . Let us show that the rank of U is maximal. Fix $S \in \mathbb{S}$. Then the matrix

$$\mathcal{M}^S := \left(\otimes_{i \in S} (U_i - A_i) \right) \otimes \left(\otimes_{i \in I \setminus A[S]} A_i \right) \otimes \left(\otimes_{i \in A(S)} I_i^{\sigma_i\text{-norm}} \right) \tag{11}$$

has rank $\prod_{j \in S} (m_j - 1) \prod_{j \in A(S)} m_j$ if $S \neq \emptyset$. If $S = \emptyset$, then $\mathcal{M}^S = \otimes_{i \in I} A_i$ has rank 1. Moreover, eigenvectors arising from different S are independent because they belong to subspaces of $L(X)$ which are orthogonal with respect to the scalar product $\langle \cdot, \cdot \rangle_\pi$.

First, let us show that, if $S \neq S'$, then the sets of indices corresponding to the nonzero columns of \mathcal{M}^S and $\mathcal{M}^{S'}$ are disjoint. Note that $S \neq S'$ implies $I \setminus A[S] \neq I \setminus A[S']$. Hence, we can assume without loss of generality that there exists $h \in I \setminus A[S]$ such that either $h \in S'$ or $h \in A(S')$. Suppose $h \in S'$,

and put $\mathcal{M}_k^S := \bigotimes_{j \geq k} M_j^S$, where $M_j^S = U_j - A_j, A_j$ or $I_j^{\sigma_j - norm}$ according to (11). Under our assumption $M_h^S = A_h$ and $M_h^{S'} = U_h - A_h$, so that our claim is true for \mathcal{M}_h^S and $\mathcal{M}_h^{S'}$. Then the same property can be deduced for \mathcal{M}^S and $\mathcal{M}^{S'}$. Now suppose that $h \in A(S')$. Then there is $h' \in S'$ such that $h \in A(h')$. We claim that $h' \in I \setminus A[S]$. In fact if $h' \in S$ then $h \in A(S)$, which is absurd. If $h' \in A(S)$ then $h \in A(S)$, a contradiction again. Hence, there exists an index $h' \in S'$ such that $h' \in I \setminus A[S]$ and from (11) we deduce that the claim is true for \mathcal{M}^S and $\mathcal{M}^{S'}$.

Hence, we deduce from Theorem 3.5 that the rank of U is $1 + \sum_{S \neq \emptyset} \prod_{j \in S} (m_j - 1) \prod_{j \in A(S)} m_j = \prod_{j \in I} m_j$ and so it is maximal.

In order to get the diagonal matrix D , whose entries are the coefficients of π , it suffices to consider the tensor product of the corresponding matrices associated with the probability P_i , for every $i = 1, \dots, n$.

Finally, to get the matrix Δ of eigenvalues of \mathcal{P} it suffices to replace, in the expression of the matrix \mathcal{P} , the matrix P_i by Δ_i and the matrix J_i by the corresponding diagonal matrix J_i^{diag} . \square

3.3. The case of the first crested product

In [6] the definition of the first crested product of Markov chains is given. More precisely, considering the product $X_1 \times \dots \times X_n$ and a partition

$$\{1, \dots, n\} = C \sqcup N \tag{12}$$

of the set $\{1, \dots, n\}$, given a probability distribution $\{p_i^0\}_{i=1}^n$ on $\{1, \dots, n\}$, the first crested product of the Markov chains P_i with respect to the partition (12) is defined as the Markov chain on $X_1 \times \dots \times X_n$ whose transition matrix is

$$P = \sum_{i \in C} p_i^0 (I_1 \otimes \dots \otimes I_{i-1} \otimes P_i \otimes I_{i+1} \otimes \dots \otimes I_n) + \sum_{i \in N} p_i^0 (I_1 \otimes \dots \otimes I_{i-1} \otimes P_i \otimes J_{i+1} \otimes \dots \otimes J_n).$$

We want to show in this section that, if the poset (I, \leq) satisfies some special conditions, then the generalized crested product defined in (5) reduces to the first crested product. We denote by \leq the usual ordering of natural numbers.

Proposition 3.8. *Suppose that (I, \leq) satisfies the following property: given i such that $H(i) \neq \emptyset$, then $j \in H(i)$ if and only if $i < j$. Then the first crested product of Markov chains is obtained by the operator defined in (5) by putting:*

$$N = \{i : H(i) \neq \emptyset\} \quad \text{and} \quad C = \{1, \dots, n\} \setminus N.$$

Proof. The partition $\{1, \dots, n\} = C \sqcup N$, with

$$N = \{i : H(i) \neq \emptyset\} \quad \text{and} \quad C = \{1, \dots, n\} \setminus N,$$

gives:

$$\mathcal{P} = \sum_{i \in C} p_i^0 I_1 \otimes \dots \otimes I_{i-1} \otimes P_i \otimes I_{i+1} \otimes \dots \otimes I_n + \sum_{i \in N} p_i^0 I_1 \otimes \dots \otimes I_{i-1} \otimes P_i \otimes J_{i+1} \otimes \dots \otimes J_n$$

and this operator coincides with the first crested product associated with the partition $\{1, \dots, n\} = C \sqcup N$. Using the same notations as in [6], we can denote by i_1 the minimal element in N with respect to the ordering \leq of $\{1, \dots, n\}$. The antichains of the poset (I, \leq) in this case are:

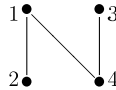
- the empty set \emptyset ;
- the set $\{i\}$, for every $i \in \{1, \dots, n\}$;
- the subsets of C ;
- the sets $D \sqcup \{i\}$, with $D \subseteq C$ and $i \in N$ such that $d < i$ for every $d \in D$.

The eigenspaces associated with antichains which are subsets of the set $\{1, 2, \dots, i_1\}$ are exactly the eigenspaces of second type described in Theorem 4.3 of [6]. All the other antichains yield eigenspaces of first type. \square

Example 3.9. Consider the following diagram:



and put on its vertices the labelling



This defines a poset (I, \leq) , with $I = \{1, 2, 3, 4\}, H(1) = \{2, 4\}, H(2) = H(4) = \emptyset, H(3) = \{4\}$, whose associated generalized crested product is (see (5))

$$\mathcal{P} = p_1^0 P_1 \otimes U_2 \otimes I_3 \otimes U_4 + p_2^0 I_1 \otimes P_2 \otimes I_3 \otimes I_4 + p_3^0 I_1 \otimes I_2 \otimes P_3 \otimes U_4 + p_4^0 I_1 \otimes I_2 \otimes I_3 \otimes P_4.$$

On the other hand, there exists no partition $\{1, 2, 3, 4\} = C \sqcup N$ such that the associated first crested product coincides with \mathcal{P} . In fact, it is not difficult to check that no labelling satisfying the conditions of Proposition 3.8 can be given to the vertices of this diagram.

3.4. *k*-step transition probability

We provide here an explicit formula for the *k*-step transition probability. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be two elements of X . From (4) and Proposition 3.7, we get

$$p^{(k)}(x, y) = \pi(y) \cdot \sum_{z \in X} \left[\left(\sum_S \prod_{i \in S} (u_i - a_i)(x_i, z_i) \prod_{i \in A(S)} \delta_{\sigma_i}(x_i, z_i) \prod_{i \in I \setminus A(S)} a_i(x_i, z_i) \right) \times \lambda_z^k \left(\sum_S \prod_{i \in S} (u_i - a_i)(y_i, z_i) \prod_{i \in A(S)} \delta_{\sigma_i}(y_i, z_i) \prod_{i \in I \setminus A(S)} a_i(y_i, z_i) \right) \right].$$

This formula becomes simpler by putting $x = 0 = (0, \dots, 0)$. We get

$$p^{(k)}(0, y) = \pi(y) \cdot \sum_S \sum_{\substack{z \text{ s.t. } z_i \neq 0, \\ z_h = 0, h \notin S}} \left(\frac{\prod_{i \in S} (u_i - a_i)(0, z_i)}{\prod_{j \in A(S)} \sqrt{\sigma_j(0)}} \right) \lambda_z^k \times \left(\sum_S \prod_{i \in S} (u_i - a_i)(y_i, z_i) \prod_{i \in A(S)} \delta_{\sigma_i}(y_i, z_i) \prod_{i \in I \setminus A(S)} a_i(y_i, z_i) \right).$$

Note that this expression consists of no more than $1 + \sum_{S \neq \emptyset} \prod_{i \in S} (m_i - 1)$ terms.

4. Generalized Insect Markov chain

In this section we describe a Markov chain which is a generalization of the so-called Insect Markov chain introduced in [10] and obtained in [6] as a particular case of first crested product. In [7], we extended it to more general structures called orthogonal block structures. In particular, we observed that if the orthogonal block structure is a poset block structure, then the Insect Markov chain can be defined starting from a finite poset (I, \leq) . Our aim is to define a new Insect Markov chain on some structures starting from a finite poset (I, \leq) , and to check that it can be obtained from the operator \mathcal{P} defined in Section 3 for a particular choice of the probability distribution $\{p_i^0\}_{i \in I}$ and of the operators P_i .

Given a finite poset (I, \leq) , one can naturally associate with each antichain $S \in \mathbb{S}$ an ancestral set A_S by setting $A_S := I \setminus H[S]$ and it is not difficult to show that this correspondence is bijective (see [7]). Moreover, we can give to the set of ancestral subsets of (I, \leq) a natural structure of a poset by putting $A_S \leq A_{S'}$ if $A_{S'} \supseteq A_S$. In particular, it is clear that for each i in I the set $\{i\}$ is an antichain. If we only take antichains constituted by singletons, the poset of ancestral subsets that we obtain is naturally isomorphic to (I, \leq) . In fact, it is easy to check that $A_i \supseteq A_j$ if and only if $i \leq j$. We add to this poset the minimal ancestral set I and denote this poset by $(I_{\mathcal{A}}, \leq)$. We now fix our attention on the maximal chains contained in $(I_{\mathcal{A}}, \leq)$. By maximal chain we mean a chain in $(I_{\mathcal{A}}, \leq)$ to which no ancestral set can be added without losing the property of being totally ordered.

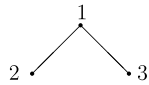
Our aim is to construct a new graph \mathcal{T} obtained by gluing together some trees arising from the construction we are performing. Let $X_i = \{0, 1, \dots, m_i - 1\}$ and put, as usual, $X = X_1 \times \dots \times X_n$. For each ancestral set $A \in (I_{\mathcal{A}}, \leq)$, the ancestral relation \sim_A on X (see [3]) is defined as

$$x \sim_A y \quad \text{if } x_i = y_i \quad \forall i \in A.$$

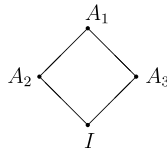
Observe that the cardinality of an equivalence class of the relation \sim_A is $\prod_{i \notin A} m_i$; the cardinality is 1 if $A = I$ (in other words, \sim_I corresponds to the equality relation). Consider a maximal chain $\{I, A_1, \dots, A_k\}$ in $(I_{\mathcal{A}}, \leq)$. Start with the set X corresponding to the relation \sim_I and create a new level corresponding to the ancestral set A_1 , in such a way that all elements of X that are \sim_{A_1} -equivalent have a common father in this new level; iterate this construction for all the ancestral sets in the chain.

For each maximal chain in $(I_{\mathcal{A}}, \leq)$, the arising structure is a disjoint union of finitely many trees; the final step is to glue together the structures arising from different maximal chains, by identifying vertices corresponding to the same ancestral set in $(I_{\mathcal{A}}, \leq)$. We get a graph that we call \mathcal{T} .

Example 4.1. Consider the following simple poset (I, \leq) :



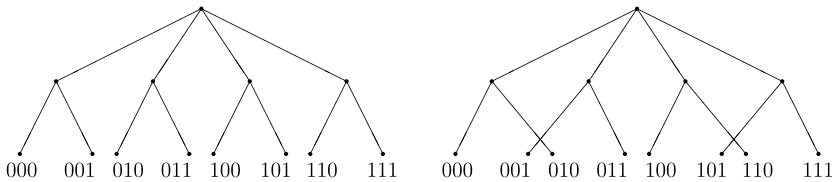
Since $|I| = 3$, the poset contains three antichains which are singleton: $S_1 = \{1\}$, $S_2 = \{2\}$ and $S_3 = \{3\}$. The corresponding ancestral sets are $A_1 = \emptyset$, $A_2 = \{1, 3\}$ and $A_3 = \{1, 2\}$, so that the associated ancestral poset $(I_{\mathcal{A}}, \leq)$ is



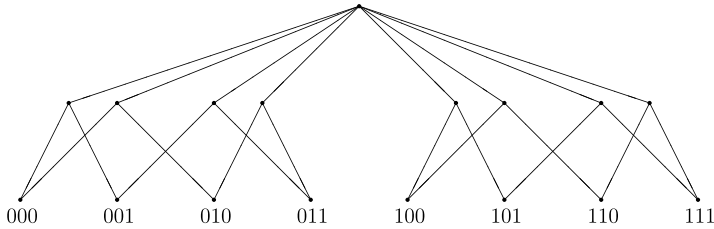
Suppose that $X_1 = X_2 = X_3 = \{0, 1\}$, so that $X = \{000, 001, 010, 011, 100, 101, 110, 111\}$. The partitions of X corresponding to the ancestral equivalence relations defined by A_1, A_2, A_3 are:

- $\sim_{A_1} = X$, since \sim_{A_1} is the universal relation;
- $\sim_{A_2} = \{000, 010\} \sqcup \{001, 011\} \sqcup \{100, 110\} \sqcup \{101, 111\}$;
- $\sim_{A_3} = \{000, 001\} \sqcup \{010, 011\} \sqcup \{100, 101\} \sqcup \{110, 111\}$.

The trees associated with the maximal chains $\{I, A_3, A_1\}$ and $\{I, A_2, A_1\}$ are, respectively,



Finally, the graph \mathcal{T} obtained by gluing them is



Remark 4.2. Compare the graph \mathcal{T} with the more complicated poset in [7], Fig. 3, that can be obtained by considering all the antichains of (I, \leq) . However, Theorem 4.4 implies that the spectral decompositions coincide in the two constructions.

The *generalized Insect Markov chain* is the Markov chain on X obtained by thinking of an insect performing a simple random walk on \mathcal{T} . Starting from an element in X (naturally identified with the bottom level via the identity relation), the next stopping time is when another vertex in X is reached by the insect in the simple random walk on \mathcal{T} . In order to describe this Markov chain we introduce some notations and useful coefficients having a probabilistic meaning.

Observe that moving to an upper level in \mathcal{T} means to pass in $(I_{\mathcal{A}}, \leq)$ from the ancestral set A_i to an ancestral set A_j such that $A_i \triangleleft A_j$, where \triangleleft means that there is no ancestral set between A_i and A_j (we have $|\{A_k \in I_{\mathcal{A}} : A_i \triangleleft A_k\}|$ possibilities), moving to a lower level in \mathcal{T} from the ancestral set A_j means to pass to an ancestral set A_i such that $A_i \triangleleft A_j$ (these are $\sum_{A_i \in I_{\mathcal{A}} : A_i \triangleleft A_j} \prod_{k \in A_i \setminus A_j} m_k$ possibilities).

Let $A_i \triangleleft A_j$ and let $\alpha_{i,j}$ be the probability of moving from the ancestral A_i to the ancestral A_j . The following relation is satisfied:

$$\alpha_{i,j} = \frac{1}{\sum_{A_k \in I_{\mathcal{A}} : A_k \triangleleft A_i} \prod_{h \in A_k \setminus A_i} m_h + |\{A_l \in I_{\mathcal{A}} : A_i \triangleleft A_l\}|} + \sum_{A_k \in I_{\mathcal{A}} : A_k \triangleleft A_i} \frac{\prod_{h \in A_k \setminus A_i} m_h}{\sum_{A_k \in I_{\mathcal{A}} : A_k \triangleleft A_i} \prod_{h \in A_k \setminus A_i} m_h + |\{A_l \in I_{\mathcal{A}} : A_i \triangleleft A_l\}|} \alpha_{k,i} \alpha_{i,j}. \tag{13}$$

In fact, the insect can directly pass from A_i to A_j with probability $\alpha_{i,j}$ or go down to any A_k such that $A_k \triangleleft A_i$, and then come back to A_i with probability $\alpha_{k,i}$, and one starts the recursive argument. From direct computations, one gets

$$\alpha_{i,i} = \frac{1}{|\{A_k \in I_{\mathcal{A}} : I \triangleleft A_k\}|}.$$

Moreover, if $\alpha_{i,i} = 1$ we have, for all A_j such that $A_i \triangleleft A_j$,

$$\alpha_{i,j} = \frac{1}{\sum_{A_k \in I_{\mathcal{A}} : A_k \triangleleft A_i} \prod_{h \in A_k \setminus A_i} m_h + |\{A_l \in I_{\mathcal{A}} : A_i \triangleleft A_l\}|}.$$

If $\alpha_{i,i} \neq 1$, the coefficient $\alpha_{i,j}$ is defined as in (13).

Set

$$p_i = \sum_{\substack{C \subseteq I_{\mathcal{A}} \text{ chain} \\ C = \{I, A_j, \dots, A_k, A_i\}}} \alpha_{I,j} \cdots \alpha_{k,i} \left(1 - \sum_{A_l \triangleleft A_i} \alpha_{i,l} \right) \tag{14}$$

and observe that p_i expresses the probability of reaching the ancestral A_i but not A_l such that $A_i \triangleleft A_l$ in $(I_{\mathcal{A}}, \leq)$. Moreover, we put

$$p_i = \sum_{\substack{C \subseteq I_{\mathcal{A}} \text{ chain} \\ C = \{I, A_j, \dots, A_k, A_i\}}} \alpha_{I,j} \cdots \alpha_{k,i} \tag{15}$$

if A_i is a maximal element of $(I_{\mathcal{A}}, \leq)$.

Proposition 4.3. Let (I, \leq) be a finite poset. The generalized Insect Markov chain coincides with the generalized crested product \mathcal{P} where the coefficients p_i^0 are chosen as in (14) (or (15)) and the operators P_i are the uniform operators J_i .

Proof. Let $x, y \in X$ and let p be the transition probability of the generalized Insect Markov chain. By construction of the poset (I_A, \leq) , we have

$$p(x, y) = \sum_{\substack{A_i \in I_A, \\ x \succ_{A_i} y}} \sum_{\substack{C \subseteq I_A \text{ chain} \\ C = \{I, A_j, \dots, A_k, A_i\}}} \frac{\alpha_{I,j} \cdots \alpha_{k,i} \left(1 - \sum_{A_l \prec A_i} \alpha_{i,l} \right)}{\prod_{h \notin A_i} m_h}.$$

The summand corresponding to A_i can be represented as

$$p_i \cdot \left(\bigotimes_{j \in I \setminus H[i]} J_j(x_j, y_j) \right) \otimes \left(\bigotimes_{j \in H[i]} J_j(x_j, y_j) \right),$$

which is the i th term of the operator \mathcal{P} , where p_i^0 is chosen as in (14) (or (15)) and $P_i = J_i$. \square

If $P_i = J_i$, then the spectral decomposition of $L(X_i)$ is $L(X_i) = V_0^i \oplus V_1^i$, with $V_1^i = \{f \in L(X_i) : \sum_{x_i \in X_i} f(x_i) = 0\}$. Theorem 3.5 implies the following result.

Theorem 4.4. Let (I, \leq) be a finite poset, \mathbb{S} be the set of antichains and p_i be as in (14) (or (15)). Let P be the corresponding generalized Insect Markov chain on X . Then the eigenspaces of P are

$$W_S = \left(\bigotimes_{i \in S} V_1^i \right) \otimes \left(\bigotimes_{i \in A(S)} L(X_i) \right) \otimes \left(\bigotimes_{i \in I \setminus A[S]} V_0^i \right), \quad \text{for each } S \in \mathbb{S},$$

with associated eigenvalue

$$\lambda_S = \sum_{i \in I \setminus A[S]} p_i.$$

Recall that two posets (I, \leq) and (J, \leq) are isomorphic if there exists an order-preserving bijection $\varphi : I \rightarrow J$, i.e.

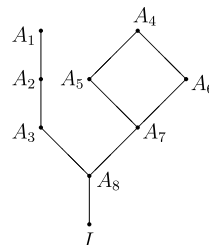
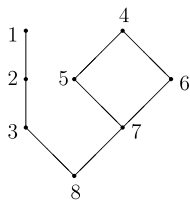
$$x \leq y \iff \varphi(x) \leq \varphi(y) \quad \text{for all } x, y \in I.$$

Now let S, S' be two antichains of (I, \leq) . It is easy to verify that, if there exists an automorphism φ of the poset (I, \leq) such that $\varphi(S) = S'$, then

$$\lambda_S = \lambda_{S'}. \tag{16}$$

In fact, if this is the case, then $i \in I \setminus A[S]$ if and only if $\varphi(i) \in I \setminus A[S']$. Moreover, one has $p_i = p_{\varphi(i)}$ for each $i \in I \setminus A[S]$, since φ is order-preserving and (16) follows. On the other hand, the existence of such an automorphism is not a necessary condition in order to have (16), as the following example shows.

Example 4.5. Consider the poset (I, \leq) and the associated ancestral poset (I_A, \leq) in the pictures below.



Suppose that $|X_i| = m$ for each $i = 1, \dots, 8$. Then it is not difficult to prove that:

- (1) $\lambda_{\{3\}} = \lambda_{\{7\}}$;
- (2) $\lambda_{\{3,5\}} = \lambda_{\{3,6\}}$;
- (3) $\lambda_{\{2,5\}} = \lambda_{\{2,6\}}$;
- (4) $\lambda_{\{1,5\}} = \lambda_{\{1,6\}}$.

Observe that the antichains $\{3\}$ and $\{7\}$ verify $\lambda_{\{3\}} = \lambda_{\{7\}}$ but there exists no automorphism of (I, \leq) mapping $\{3\}$ to $\{7\}$.

5. Gelfand pairs

Consider the generalized crested product defined in (5) obtained by choosing $P_i = J_i$ for each $i \in I$. We know that in this case one has

$$L(X_i) = V_0^i \oplus V_1^i \quad \text{for each } i \in I,$$

where $V_0^i \cong \mathbb{C}$ is the space of constant functions on X_i and $V_1^i = \{f \in L(X_i) : \sum_{x_i \in X_i} f(x_i) = 0\}$. Hence, $\dim(V_0^i) = 1$ and $\dim(V_1^i) = m_i - 1$.

In [6] we made the following remarks: for the crossed product, the eigenspaces of the operator coincide with the irreducible submodules of the representation of the direct product $Sym(m_1) \times \dots \times Sym(m_n)$ over $L(X_1 \times \dots \times X_n)$; for the nested product, the eigenspaces of the operator coincide with the irreducible submodules of the representation of the wreath product $Sym(m_n) \wr \dots \wr Sym(m_1)$ over $L(X_1 \times \dots \times X_n)$.

It is natural to ask if such a correspondence can be extended to the general case. Actually, the answer is positive and it is given by a family of groups containing, as particular cases, both the direct product and the wreath product of permutation groups. These groups are the so-called *generalized wreath product*, introduced in [3] as permutation groups of the so-called poset block structures.

In [3] it is proven that, given n finite spaces X_i , indexed by the elements of a finite poset (I, \leq) , the action of the generalized wreath product of the permutation groups $Sym(X_i)$ on the space $L(X_1 \times \dots \times X_n)$ has the following decomposition into irreducible submodules:

$$L(X_1 \times \dots \times X_n) = \bigoplus_{S \subseteq I \text{ antichain}} W_S,$$

with

$$W_S = \left(\bigotimes_{i \in A(S)} L(X_i) \right) \otimes \left(\bigotimes_{i \in S} V_1^i \right) \otimes \left(\bigotimes_{i \in I \setminus A(S)} V_0^i \right).$$

These irreducible submodules coincide with the eigenspaces that we described in (9) if $P_i = J_i$ and this answers our question.

Finally, we proved in [7] that the action of the generalized wreath product of the groups $Sym(X_i)$ on $L(X_1 \times \dots \times X_n)$ yields symmetric Gelfand pairs (see [4] or [5] for the definition) when one considers the subgroup stabilizing a given element $x_0 = (x_0^1, \dots, x_0^n) \in X_1 \times \dots \times X_n$. Moreover, the spherical function associated with W_S is

$$\phi_S = \bigotimes_{i \in A(S)} \varphi_i \bigotimes_{i \in S} \psi_i \bigotimes_{i \in I \setminus A(S)} \varrho_i, \tag{17}$$

where $\varphi_i, \psi_i \in L(X_i)$ are defined as

$$\varphi_i(x) = \begin{cases} 1 & x = x_0^i \\ 0 & \text{otherwise} \end{cases}, \quad \psi_i(x) = \begin{cases} 1 & x = x_0^i \\ -\frac{1}{m_i - 1} & \text{otherwise} \end{cases}$$

and $\varrho_i \in L(X_i)$ satisfies $\varrho_i(x_i) = 1$ for every $x_i \in X_i$.

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